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JP.561 Solve for real numbers:

$$\log_{2\sqrt{8+2\sqrt{15}}}(x^2+x+2)=\log_{\sqrt{4+\sqrt{15}}}(x^2+x+1)$$

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Solution by proposers

Let be $\log_{2\sqrt{8+2\sqrt{15}}}(x^2+x+2)=\log_{\sqrt{4+\sqrt{15}}}(x^2+x+1)=y\Rightarrow$

$$\begin{cases} x^2+x+1=\left(\sqrt{y+\sqrt{15}}\right)^y \\ x^2+x+2=\left(2\sqrt{8+2\sqrt{15}}\right)^y\end{cases}\Leftrightarrow \left(4+\sqrt{15}\right)^y+1=\left(2\sqrt{2}\cdot\sqrt{4+\sqrt{15}}\right)^y$$

We observe that $y=2$ is a solution, $\left(4+\sqrt{15}\right)^2+1=\left(2\sqrt{2}\cdot\sqrt{4+\sqrt{15}}\right)^y\Leftrightarrow$

$32+8\sqrt{15}=32+8\sqrt{15}$ true. Let us prove that $y=$ is the unique solution:

$$1+\left(\frac{1}{4+\sqrt{15}}\right)^y=\left(\frac{2\sqrt{2}}{\sqrt{4+\sqrt{15}}}\right)^y$$

Let be the function $f(y)=1+\left(\frac{1}{4+\sqrt{15}}\right)^y$ strictly decreases because $\frac{1}{4+\sqrt{15}}<1$, and

$g(y)=\left(\frac{2\sqrt{2}}{\sqrt{4+\sqrt{15}}}\right)^y$ strictly increases function because $\frac{2\sqrt{2}}{\sqrt{4+\sqrt{15}}}>1$.

Hence, $y=2$ is the unique solution i.e. $x^2+x+1=4+\sqrt{15}\Leftrightarrow x^2+x-3-\sqrt{15}=0$

with $x_{1,2}=\frac{-1\pm\sqrt{3+5\sqrt{15}}}{2}$