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JP.562 If $a,b,c\in(0,1)$ and $x,y,z>0$ such that $a=(bc)^x, b=(ca)^y, c=(ab)^z$ and $xyz=1$ then holds:

$$\sqrt[n]{\sum_{cyc} a^n(a^n+y+2)^{2n-1}}\ge 6\cdot\sqrt[3]{abc}, n\in\mathbb{N}^*, n\ge 2$$

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution by proposers

$$\sum_{cyc} a^n(y+z+2)^{2n-1}=\sum_{cyc}\frac{a^n(y+z+2)^n}{\frac{1}{(y+z+2)^{n-1}}}\overset{Radon}{\ge}$$

$$\ge\frac{[a(y+z+2)+b(z+x+2)+c(x+y+2)\,)]^n}{\left(\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{x+y+2}\right)^{n-1}}=$$

$$=\frac{[(b+c)x+(c+a)y+(a+b)x]^n}{\left(\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{x+y+2}\right)^{n-1}}\overset{AM-GM}{\ge}$$

$$\ge\frac{\left(3\sqrt[3]{(a+b)(b+c)(c+a)xyz}\right)^n}{\left(\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{x+y+2}\right)}\ge$$

$$\ge\frac{\left(6\sqrt[3]{(a+b)(b+c)(c+a)xyz}\right)^n}{\left(\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{x+y+2}\right)^{n-1}}\ge$$

$$\ge\frac{\left(6\sqrt[3]{abc}\right)^n}{\left(\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{x+y+2}\right)^{n-1}};\qquad (1)$$

Now, from $a=(bc)^x, b=(ca)^y, c=(ab)^z$, we get $x=\frac{-\ln a}{-\ln b-\ln c}, y=\frac{-\ln b}{-\ln c-\ln a}$

and $z=\frac{-\ln b}{-\ln a-\ln c}$, Let us denote $\begin{cases}-\ln a-\ln c=q\\-\ln b-\ln c=p\\-\ln a-\ln b=r\end{cases}$ with $p,q,r>0$, then

$\begin{cases}x=\frac{-p+q+r}{2p}\\ y=\frac{p-q+r}{2q}\\ z=\frac{p+q-r}{2r}\end{cases}$ and using $\frac{pq}{p+q}\le\frac{p+q}{2}$, we get:

$$\frac{1}{x+y+2}=\frac{2pq}{(p+q+r)(p+q)}\le\frac{p+q}{2(p+q+r)}\quad \text{(and analogs).}$$

Hence, it follows that:

$$\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{z+x+2}\le\frac{p+q+q+r+r+p}{2(p+q+r)}=1\quad (2)$$

From (1) and (2) we obtain:

$$a^n(y+z+2)^{2n-1}+b^n(z+x+2)^{2n-1}+c^n(x+y+z)^{2n-1}\ge\left(6\sqrt[3]{abc}\right)^n$$

Finally,

$$\sqrt[n]{\sum_{cyc}a^n(y+z+2)^{2n-1}}\ge 6\cdot\sqrt[3]{abc}, n\in\mathbb{N}^*, n\ge 2$$