ROMANIAN MATHEMATICAL MAGAZINE

JP.563 In acute triangle ABC, A', B', C' are symmetric points of A, B, C to the sides BC, AC and AB respectively. Prove that:

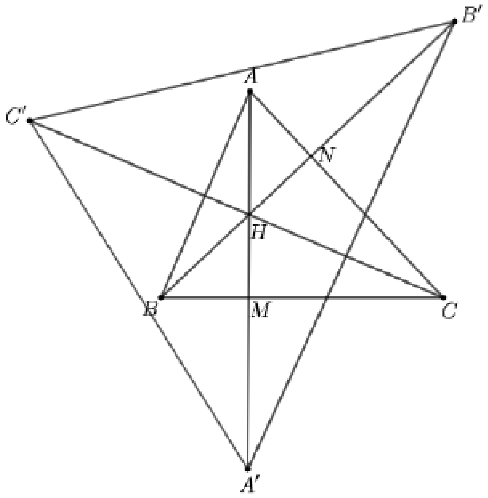
$$\frac{\sigma[A'B'C']}{\sigma[ABC]} = 4\left(\frac{r}{R}\right)^2 + 8\cdot\frac{r}{R} - 1$$

where $\sigma[ABC]$ is area of ΔABC.

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution by proposers

Let be $\{H\} = AA' \cap BB' \cap CC'$. From $h_a = 2R\sin B\sin C$ and $HM = 2R\cos B\cos C$,



then $HA' = 3R\cos(B-C)$ and analogs. Hence, we get:

$$\sigma[A'B'C'] = \frac{1}{2}(HA'\cdot HB'\cdot\sin C + HB'\cdot HC'\cdot\sin A + HC'\cdot HA'\cdot\sin B)$$

$$= 2R^2\sum_{cyc}\cos(B-C)\cos(C-A)\sin C$$

$$= 2R^2\sin A\sin B\sin C\sum_{cyc}\frac{\cos(B-C)\cos(C-A)}{\sin A\sin B} \quad (1)$$

But $\sigma[ABC] = 2R^2\sin A\sin B\sin C$ and

$$\sum_{cyc}\frac{\cos(B-C)\cos(C-A)}{\sin A\sin B} = 3 + 8\cos A\cos B\cos C \quad (2)$$

From (1) and (2), it follows:

$$\frac{\sigma[A'B'C']}{\sigma[ABC]} = 3 + 8\cos A\cos B\cos C \quad (3)$$

But $\cos A\cos B\cos C = \frac{s^2-(2R+r)^2}{4R^2}$; (4)

From (3) and (4), it follows:

$$\frac{\sigma[A'B'C']}{\sigma[ABC]} = 3 + \frac{2(s^2-4R^2-4Rr-r^2)}{R^2} \quad (5)$$

Using Walker's inequality $s^2 \geq 2R^2 + 8Rr + 3r^2$ to obtain:

$$\frac{\sigma[A'B'C']}{\sigma[ABC]} \geq 3 - \frac{4R^2+8Rr+4r^2}{R^2} = 4\left(\frac{r}{R}\right)^2 + 8\cdot\frac{r}{R} - 1$$