ROMANIAN MATHEMATICAL MAGAZINE

JP.569 In ΔABC the following relationship holds:

$$\left|\frac{a^2-b^2}{ab}\right|+\left|\frac{b^2-c^2}{bc}\right|+\left|\frac{c^2-a^2}{ca}\right|<\frac{3R}{r}$$

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Solution by proposer

Lemma 1: $a^2+b^2+c^2=2(p^2-r^2-4Rr)$

Lemma 2: $a^3+b^3+c^3=2(p^3-3pr^2-6pRr)$

$$\left|\frac{a^2-b^2}{ab}\right|=\frac{(a+b)}{ab}\cdot|a-b|<\frac{(a+b)c}{ab}$$

$$LHS<\frac{(a+b)c}{ab}+\frac{(b+c)a}{bc}+\frac{(c+a)b}{ca}=$$

$$=\frac{1}{abc}\cdot[c^2(a+b)+a^2(b+c)+b^2(a+c)]=$$

$$=\frac{1}{abc}\cdot[c^2(2p-c)+a^2(2p-a)+b^2(2p-b)]=$$

$$=\frac{1}{abc}[2p(a^2+b^2+c^2)-(a^3+b^3+c^3)]=$$

$$\underset{Lemma\ 2}{\overset{Lemma\ 1}{=}}\frac{1}{4prR}\cdot[2p\cdot(2p^2-2r^2-8Rr)-(2p^3-6pr^2-12pRr)]=$$

$$=\frac{1}{4pRr}\cdot(4p^3-4pr^2-16pRr-2p^3+6pr^2+12prR)=$$

$$=\frac{1}{4pRr}\cdot(2p^3+2pr^2-4prR)=$$

$$=\frac{1}{4pRr}\cdot 2p(p^2+r^2-2rr)=\frac{1}{2Rr}\cdot(p^2+r^2-2Rr)\le$$

$$\overset{Gerretsen}{\le}\frac{1}{2Rr}\cdot(4R^2+4Rr+3r^2+r^2-2Rr)=$$

$$=\frac{1}{2Rr}\cdot(4R^2+2Rr+4r^2)\overset{Euler}{\le}\frac{1}{2Rr}\cdot(4R^2+R^2+R^2)$$

$$=\frac{1}{2Rr}\cdot 6R^2=\frac{3R}{r}$$