ROMANIAN MATHEMATICAL MAGAZINE

NEW INEQUALITIES WITH SPIEKER'S CEVIANS-II

By Bogdan Fuștei-Romania

Let be ABC triangle with usual notations. Denote:

$$\mathbf{Q=\min\left\{\frac{m_a m_b}{h_a h_b}, \frac{m_a m_c}{h_a h_c}, \frac{m_c m_b}{h_c h_b}\right\}.}$$

Without loss generality we consider: a≥b≥c. We will prove that:

$$\mathbf{Q=\frac{m_a m_c}{h_a h_c}.\ \frac{m_c m_b}{h_c h_b} \geq \frac{m_a m_c}{h_a h_c} \rightarrow \frac{m_b}{h_b} \geq \frac{m_a}{h_a} \rightarrow h_a m_b \geq m_a h_b.\ 2S=ah_a=bh_b=ch_c \rightarrow \frac{m_b}{a} \geq \frac{m_a}{b} \rightarrow bm_b \geq am_a.}$$

Proof:

$$b^2 m_b^2 \geq a^2 m_a^2 \rightarrow b^2[2(a^2+c^2)-b^2] \geq a^2[2(c^2+b^2)-a^2]$$

$$2a^2b^2+2b^2c^2-b^4 \geq 2a^2c^2+2a^2b^2-a^4$$

$$2b^2c^2-b^4 \geq 2a^2c^2-a^4$$

$$2c^2(b^2-a^2) \geq b^4-a^4=(b^2-a^2)(b^2+a^2)$$

$$(b^2-a^2)(2c^2-b^2-a^2) \geq 0 \text{-true because } a \geq b \geq c.$$

Analog we have:

$$\mathbf{\frac{m_a m_b}{h_a h_b} \geq \frac{m_a m_c}{h_a h_c}}$$

Now we will prove:

$$\mathbf{\frac{R}{2r} \geq \frac{m_a m_c}{h_a h_c}}$$

$$\frac{R}{2r}=\frac{abcp}{8S^2} \geq \frac{acm_a m_c}{4S^2} \rightarrow bp \geq 2m_a m_c \text{(Must be proved !!!)}$$

We consider A_1, B_1, C_1 means of triangle ABC sides. We'll apply Ptolemy's inequality to the quadrilateral AC_1A_1C: $AA_1.CC_1 \leq AC_1.A_1C+C_1A_1.AC$

$$\mathbf{m_a m_c \leq \frac{b}{2}b+\frac{a}{2}\frac{c}{2} \rightarrow 4m_a m_c \leq 2b^2+ac \text{(and analogs)}}$$

Now we will prove that:

$$\mathbf{2b^2+ac \leq 2pb=b(a+b+c)}$$

$$b^2+ac\le ab+bc\to b(b-a)+c(a-b)\le 0$$

$$b(b-a)-c(b-a)\le 0\to(b-a)(b-c)\ \le 0 \text{ ,true because } a\ge b\ge c$$

$$\frac{R}{2r}\ge\min\left\{\frac{m_am_c}{h_ah_c},\frac{m_cm_b}{h_ch_b},\frac{m_am_b}{h_ah_b}\right\}(1)$$

From (1) and Panaitopol inequality: $\frac{R}{2r}\ge\frac{m_a}{h_a}$(and analogs) we obtain:

$$\frac{R}{r}\ge\min\left\{\frac{m_am_c}{h_ah_c},\frac{m_cm_b}{h_ch_b},\frac{m_am_b}{h_ah_b}\right\}+\max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}(2)$$

We use AM-GM and we obtain:

$$\frac{R}{2r}\ge\frac{1}{2}\left(\frac{m_am_c}{h_ah_c}+\frac{m_b}{h_b}\right)\ge\sqrt{\frac{m_am_bm_c}{h_ah_bh_c}}(3)$$

$\frac{R}{2r}=\frac{r_ar_br_c}{h_ah_bh_c}$ from (3) we obtain: $\frac{r_ar_br_c}{h_ah_bh_c}\ge\sqrt{\frac{m_am_bm_c}{h_ah_bh_c}}\to$

$$\frac{R}{2r}\ge\frac{m_am_bm_c}{r_ar_br_c}(4)$$

From [1]: $\frac{m_am_bm_c}{h_ah_bh_c}\ge\frac{R}{2r}\sqrt[4]{\frac{p_ap_bp_c}{l_al_bl_c}}$ and (1) we obtain:

$$\frac{m_a}{h_a}\ge\sqrt[4]{\frac{p_ap_bp_c}{l_al_bl_c}}\text{(and analogs)}(5)$$

From [1]: $\frac{m_am_bm_c}{h_ah_bh_c}\ge\frac{R}{2r}\sqrt{\frac{(l_a+p_a)(l_b+p_b)(l_c+p_c)}{8l_al_bl_c}}$ and (1) we obtain:

$$\frac{m_a}{h_a}\ge\sqrt{\frac{(l_a+p_a)(l_b+p_b)(l_c+p_c)}{8l_al_bl_c}}\text{(and analogs)}(6)$$

From [2]: $\frac{R}{r}\ge\frac{m_b}{h_c}+\frac{m_c}{h_b}$ (and analogs) and (1) we obtain:

$$\frac{3R}{2r}\ge\max\left\{\frac{m_b}{h_c}+\frac{m_c}{h_b},\frac{m_b}{h_a}+\frac{m_a}{h_b},\frac{m_a}{h_c}+\frac{m_c}{h_a}\right\}+\min\left\{\frac{m_am_c}{h_ah_c},\frac{m_cm_b}{h_ch_b},\frac{m_am_b}{h_ah_b}\right\}(7)$$

From [3] : $\frac{R}{r}\ge\max\left\{1+\frac{n_a}{h_a},1+\frac{n_b}{h_b},1+\frac{n_c}{h_c}\right\}$ and (1) we obtain:

$$\frac{3R}{2r}\ge\max\left\{1+\frac{n_a}{h_a},1+\frac{n_b}{h_b},1+\frac{n_c}{h_c}\right\}+\min\left\{\frac{m_am_c}{h_ah_c},\frac{m_cm_b}{h_ch_b},\frac{m_am_b}{h_ah_b}\right\}(8)$$

From [4]: $\frac{1}{\sin\omega} \geq \frac{m_b}{h_b} + \frac{m_c}{h_c}$ (and analogs) and (5) and (6) we obtain:

$$\frac{1}{2\sin\omega} \geq \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}} \quad (9)$$

$$\frac{1}{2\sin\omega} \geq \sqrt{\frac{(l_a+p_a)(l_b+p_b)(l_c+p_c)}{8 l_a l_b l_c}} \quad (10)$$

From [2]: $\frac{m_b}{h_c} + \frac{m_c}{h_b} \geq 2\frac{m_a}{h_a}$ and (5) and (6) we obtain:

$$\frac{1}{2}\left(\frac{m_b}{h_c} + \frac{m_c}{h_b}\right) \geq \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}} \quad (11)$$

$$\frac{1}{2}\left(\frac{m_b}{h_c} + \frac{m_c}{h_b}\right) \geq \sqrt{\frac{(l_a+p_a)(l_b+p_b)(l_c+p_c)}{8 l_a l_b l_c}} \quad (12)$$

From (3) we obtain:

$$\frac{1}{2} R p^2 \geq m_a m_b m_c \quad (13)$$

From $p^2 = n_a^2 + 2 r_a h_a$ (and analogs) and (13) we obtain:

$$n_a^2 \geq 2\left(\frac{m_a m_b m_c}{R} - r_a h_a\right) \text{(and analogs)} \quad (14)$$

$h_a = 2\frac{r_b r_c}{r_b + r_c}$ (and analogs) and (14) we obtain:

$$n_a^2 \geq 2\left(\frac{m_a m_b m_c}{R} - 2\frac{r_a r_b r_c}{r_b + r_c}\right) \text{(and analogs)} \quad (15)$$

From (5), (6) and $\frac{m_a^2}{h_a^2} = 1 + \frac{(b^2-c^2)^2}{16S^2}$ (and analogs) we obtain:

$$1 + \frac{(b^2-c^2)^2}{16S^2} \geq \frac{(l_a+p_a)(l_b+p_b)(l_c+p_c)}{8 l_a l_b l_c} \text{(and analogs)} \quad (16)$$

$$1 + \frac{(b^2-c^2)^2}{16S^2} \geq \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}} \text{(and analogs)} \quad (17)$$

From (13) and $p^2 = r_a r_b + r_b r_c + r_a r_c$ we obtain:

$$r_a r_b + r_b r_c + r_a r_c \geq 2\frac{m_a m_b m_c}{R} \quad (18)$$

From $m_a l_a \geq r_b r_c = p(p-a)$ (and analogs)(Panaitopol) and (18) we obtain:

$$\sum \mathbf{m_a l_a} \geq 2\,\frac{\mathbf{m_a m_b m_c}}{R}\ \mathbf{(19)}$$

From $n_a g_a \geq m_a l_a$(and analogs) and (19) we obtain:

$$\sum \mathbf{n_a g_a} \geq 2\,\frac{\mathbf{m_a m_b m_c}}{\mathbf{R}}\ \mathbf{(20)}$$

From(4) and [3]: $\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)$(and analogs),we obtain:

$$\frac{\mathbf{R}}{\mathbf{r}} \geq \sqrt{\frac{\mathbf{m_a m_b m_c (r_a+r_b+r_c)}}{\mathbf{r_a r_b r_c (h_a+h_b+h_c)}}}\left(\sqrt{\frac{\mathbf{b}}{\mathbf{c}}}+\sqrt{\frac{\mathbf{c}}{\mathbf{b}}}\right)\mathbf{(and\ analogs)(21)}$$

From(4) and [3]: $\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}$(and analogs) we obtain:

$$\frac{\mathbf{R}}{\mathbf{r}} \geq \sqrt{\frac{\mathbf{m_a m_b m_c}}{\mathbf{r_a r_b r_c}}}\left(\sqrt{\frac{\mathbf{n_a}}{\mathbf{r_a}}}+\sqrt{\frac{\mathbf{r_a}}{\mathbf{n_a}}}\right)\mathbf{(and\ analogs)(22)}$$

From(4) and [3]: $\frac{R}{r} \geq 1+\sqrt{\frac{R}{2r}}\,\frac{\sqrt{n_a n_b+n_b n_c+n_a n_c}}{p}$ we obtain:

$$\frac{\mathbf{R}}{\mathbf{r}} \geq 1+\sqrt{\frac{\mathbf{m_a m_b m_c}}{\mathbf{r_a r_b r_c}}}\,\frac{\sqrt{\mathbf{n_a n_b+n_b n_c+n_a n_c}}}{\mathbf{p}}\ \mathbf{(23)}$$

From(4) and [3]: $\sqrt{\frac{2R}{r}} \geq \frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\frac{p\sqrt{3}}{h_a+h_b+h_c}$ we obtain:

$$\frac{\mathbf{R}}{\mathbf{r}} \geq \sqrt{\frac{\mathbf{m_a m_b m_c}}{\mathbf{r_a r_b r_c}}}\left(\frac{\mathbf{r_a+r_b+r_c}}{\mathbf{m_a+m_b+m_c}}+\frac{\mathbf{p\sqrt{3}}}{\mathbf{h_a+h_b+h_c}}\right)\mathbf{(24)}$$

From $\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r}\sqrt{\frac{(l_a+p_a)(l_b+p_b)(l_c+p_c)}{8 l_a l_b l_c}}$ and [4]:

$\frac{R}{r} \geq \frac{5R-r+\frac{n_a g_a}{h_a}+\frac{n_b g_b}{h_b}+\frac{n_c g_c}{h_c}}{h_a+h_b+h_c}$ we obtain:

$$2\,\frac{\mathbf{m_a m_b m_c}}{\mathbf{h_a h_b h_c}} \geq \frac{\mathbf{5R-r+\frac{n_a g_a}{h_a}+\frac{n_b g_b}{h_b}+\frac{n_c g_c}{h_c}}}{\mathbf{h_a+h_b+h_c}}\sqrt{\frac{\mathbf{(l_a+p_a)(l_b+p_b)(l_c+p_c)}}{\mathbf{8 l_a l_b l_c}}}\ \mathbf{(25)}$$

From $\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r} \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}}$ and $\frac{R}{r} \geq \frac{5R - r + \frac{n_a g_a}{h_a} + \frac{n_b g_b}{h_b} + \frac{n_c g_c}{h_c}}{h_a + h_b + h_c}$ we obtain:

$$\mathbf{2\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{5R - r + \frac{n_a g_a}{h_a} + \frac{n_b g_b}{h_b} + \frac{n_c g_c}{h_c}}{h_a + h_b + h_c} \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}}} \quad \textbf{(26)}$$

From $\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r} \sqrt{\frac{(l_a + p_a)(l_b + p_b)(l_c + p_c)}{8 l_a l_b l_c}}$ and (21) we obtain:

$$\mathbf{2\frac{m_a m_b m_c}{h_a h_b h_c} \geq \sqrt{\frac{m_a m_b m_c (r_a + r_b + r_c)}{r_a r_b r_c (h_a + h_b + h_c)}} \sqrt{\frac{(l_a + p_a)(l_b + p_b)(l_c + p_c)}{8 l_a l_b l_c}} \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right)} \textbf{(and analogs) (27)}$$

From $\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r} \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}}$ and (21) we obtain:

$$\mathbf{2\frac{m_a m_b m_c}{h_a h_b h_c} \geq \sqrt{\frac{m_a m_b m_c (r_a + r_b + r_c)}{r_a r_b r_c (h_a + h_b + h_c)}} \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}} \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right)} \textbf{(and analogs)(28)}$$

From $\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{n_a}{r_a}} + \sqrt{\frac{r_a}{n_a}}$(and analogs),

$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a + r_b + r_c}{h_a + h_b + h_c}} \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right)$(and analogs)

and $\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r} \sqrt{\frac{(l_a + p_a)(l_b + p_b)(l_c + p_c)}{8 l_a l_b l_c}}$ we obtain:

$$\mathbf{4\frac{m_a m_b m_c}{h_a h_b h_c} \geq \sqrt{\frac{(l_a + p_a)(l_b + p_b)(l_c + p_c)(r_a + r_b + r_c)}{8 l_a l_b l_c (h_a + h_b + h_c)}} \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right) \left(\sqrt{\frac{n_a}{r_a}} + \sqrt{\frac{r_a}{n_a}}\right)} \textbf{(and analogs)(29)}$$

From $\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{n_a}{r_a}} + \sqrt{\frac{r_a}{n_a}}$(and analogs),

$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a + r_b + r_c}{h_a + h_b + h_c}} \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right)$(and analogs)

and $\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r} \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}}$ we obtain:

$$\mathbf{4\frac{m_a m_b m_c}{h_a h_b h_c} \geq \sqrt{\frac{r_a + r_b + r_c}{h_a + h_b + h_c}} \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}} \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right) \left(\sqrt{\frac{n_a}{r_a}} + \sqrt{\frac{r_a}{n_a}}\right)} \textbf{(and analogs)(30)}$$

References:

[1]. Bogdan Fuștei- NEW INEQUALITIES WITH SPIEKER'S CEVIANS-www.ssmrmh.ro

[2]. Bogdan Fuștei, Mohamed Amine Ben Ajiba- NEW TRIANGLE INEQUALITIES WITH BROCARD'S ANGLE-www.ssmrmh.ro

[3]. Bogdan Fuștei- THE AVALANCHE OF GEOMETRIC INEQUALITIES-www.ssmrmh.ro

[4]. Bogdan Fuștei- 150 TRIANGLE IDENTITIES AND INEQUALITIES INVOLVING BROCARD'S ANGLE-www.ssmrmh.ro