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NEW INEQUALITIES WITH SPIEKER'S CEVIAN

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Let be ABC triangle with p_a, p_b, p_c-Spieker's cevians. We proved in [1]: $2m_a \geq l_a + p_a$(and analogs)

$m_a l_a \geq r_b r_c$=p(p-a)(and analogs)(Panaitopol),from this two inequalitys we obtain:

$$m_a^2 \geq r_b r_c \frac{l_a+p_a}{2l_a} \text{(and analogs)(1)}$$

From (1) we obtain:

$$\frac{m_a m_b m_c}{r_a r_b r_c} \geq \sqrt{\frac{(l_a+p_a)(l_b+p_b)(l_c+p_c)}{8l_a l_b l_c}} \text{ (2)}$$

From (1) and $l_a + p_a \geq 2\sqrt{l_a p_a}$(AM-GM inequality) we obtain:

$$m_a^2 \geq r_b r_c \sqrt{\frac{p_a}{l_a}} \text{(and analogs)(3)}$$

We use identity: $\frac{r_a r_b r_c}{h_a h_b h_c} = \frac{R}{2r}$ from (2) and (3) we obtain:

$$\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r} \sqrt[4]{\frac{p_a p_b p_c}{l_a l_b l_c}} \text{(4)}$$

$$\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r} \sqrt{\frac{(l_a+p_a)(l_b+p_b)(l_c+p_c)}{8l_a l_b l_c}} \text{(5)}$$

Now we use: $l_a = 2\frac{\sqrt{bc}}{b+c}\sqrt{r_b r_c}$ (and analogs) we obtain: $2\frac{\sqrt{r_b r_c}}{l_a} = \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}$(and analogs) and from (1) we obtain:

$$2\frac{m_a}{l_a} \geq \sqrt{\frac{l_a+p_a}{2l_a}}\left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right) \text{ (and analogs)(6)}$$

From 2S=ah_a=bh_b=$ch_c \rightarrow \frac{c}{b} = \frac{h_b}{h_c}$(and analogs). From(6) we obtain:

$$\mathbf{2\frac{m_a}{l_a}\geq\sqrt{\frac{l_a+p_a}{2l_a}}\left(\sqrt{\frac{h_b}{h_c}}+\sqrt{\frac{h_c}{h_b}}\right)(7)}$$

From $l_a=2\frac{\sqrt{bc}}{b+c}\sqrt{r_br_c}$ we obtain: $\frac{r_ar_br_c}{l_al_bl_c}=\frac{(a+b)(b+c)(c+a)}{8abc}$ and we use (1) and we obtain:

$$\mathbf{\frac{m_am_bm_c}{l_al_bl_c}\geq\frac{(a+b)(b+c)(c+a)}{8abc}\sqrt{\frac{(l_a+p_a)(l_b+p_b)(l_c+p_c)}{8l_al_bl_c}}(8)}$$

From (6),(7),(8) and and $l_a+p_a\geq2\sqrt{l_ap_a}$,we obtain:

$$\mathbf{2\frac{m_a}{l_a}\geq\sqrt[4]{\frac{p_a}{l_a}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\text{(and analogs)}(9)}$$

$$\mathbf{2\frac{m_a}{l_a}\geq\sqrt[4]{\frac{p_a}{l_a}}\left(\sqrt{\frac{h_b}{h_c}}+\sqrt{\frac{h_c}{h_b}}\right)\text{(and analogs)}(10)}$$

$$\mathbf{\frac{m_am_bm_c}{l_al_bl_c}\geq\frac{(a+b)(b+c)(c+a)}{8abc}\sqrt[4]{\frac{p_ap_bp_c}{l_al_bl_c}}(11)}$$

From $\sum m_a^2=\frac{3}{4}\sum a^2$ and (1),(3) we obtain:

$$\mathbf{\sum a^2\geq\frac{2}{3}\sum r_br_c\frac{l_a+p_a}{l_a}(12)}$$

$$\mathbf{\sum a^2\geq\frac{4}{3}\sum r_br_c\sqrt{\frac{p_a}{l_a}}(13)}$$

From(1), $\frac{\sqrt{r_br_c}}{2l_a}=\frac{1}{4}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)$; $m_a^2=r_br_c+\frac{1}{4}(b-c)^2$(and analogs),we obtain:

$$r_br_c+\frac{1}{4}(b-c)^2\geq r_br_c\frac{l_a+p_a}{2l_a}$$

$$\frac{1}{4}(b-c)^2\geq r_br_c\left(\frac{l_a+p_a}{2l_a}-1\right)$$

$$\frac{1}{4}(b-c)^2\geq\frac{\sqrt{r_br_c}}{2l_a}\sqrt{r_br_c}(p_a-l_a)=\frac{1}{4}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt{r_br_c}(p_a-l_a)$$

We will obtain:

$$\mathbf{(b-c)^2\geq\sqrt{r_br_c}(p_a-l_a)\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\text{(and analogs)}(14)}$$

$$(\mathbf{b}-\mathbf{c})^2 \geq \sqrt{\mathbf{r_b r_c}}(\mathbf{p_a}-\mathbf{l_a})\left(\sqrt{\frac{\mathbf{h_b}}{\mathbf{h_c}}}+\sqrt{\frac{\mathbf{h_c}}{\mathbf{h_b}}}\right)(\textbf{and analogs})\textbf{(15)}$$

From (3) and $m_a^2 = r_b r_c + \frac{1}{4}(b-c)^2$(and analogs),we obtain:

$$\frac{\mathbf{1}}{\mathbf{4}}(\mathbf{b}-\mathbf{c})^2 \geq \mathbf{r_b r_c}\left(\sqrt{\frac{\mathbf{p_a}}{\mathbf{l_a}}}-\mathbf{1}\right)\textbf{(and analogs)(16)}$$

From (14) we obtain:

$b^2+c^2 \geq 2bc+\sqrt{r_b r_c}(p_a-l_a)\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)$,we know that

$\cos\frac{A}{2}=\sqrt{\frac{r_b r_c}{bc}}$ (and analogs), we can write:

$$\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\geq\mathbf{2}+\mathbf{cos}\frac{\mathbf{A}}{\mathbf{2}}\left(\sqrt{\frac{\mathbf{b}}{\mathbf{c}}}+\sqrt{\frac{\mathbf{c}}{\mathbf{b}}}\right)\frac{(\mathbf{p_a}-\mathbf{l_a})}{\sqrt{\mathbf{bc}}}\textbf{(and analogs)(17)}$$

ω-is Brocard angle in ABC triangle, and $\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs). From (17) we obtain:

$$\frac{\mathbf{sin}(\boldsymbol{A}+\boldsymbol{\omega})}{\mathbf{sin}\,\boldsymbol{\omega}}\geq\mathbf{2}+\mathbf{cos}\frac{\mathbf{A}}{\mathbf{2}}\left(\sqrt{\frac{\mathbf{b}}{\mathbf{c}}}+\sqrt{\frac{\mathbf{c}}{\mathbf{b}}}\right)\frac{(\mathbf{p_a}-\mathbf{l_a})}{\sqrt{\mathbf{bc}}}\textbf{(18)}$$

From $\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}\rightarrow\frac{1}{\sin\omega}\geq\frac{b}{c}+\frac{c}{b}$ (and analogs). Next result is:

$$\frac{\mathbf{1}}{\mathbf{sin}\,\boldsymbol{\omega}}\geq\mathbf{2}+\mathbf{cos}\frac{\mathbf{A}}{\mathbf{2}}\left(\sqrt{\frac{\mathbf{b}}{\mathbf{c}}}+\sqrt{\frac{\mathbf{c}}{\mathbf{b}}}\right)\frac{(\mathbf{p_a}-\mathbf{l_a})}{\sqrt{\mathbf{bc}}}\textbf{(and analogs)(19)}$$

From [1] and [2] we end our paper with:

$$\boldsymbol{n_a}+\boldsymbol{g_a}\geq\mathbf{2m_a}\geq\mathbf{l_a}+\mathbf{p_a}\textbf{(and analogs)(20)}$$

References:

[1]. Bogdan Fuștei- CONNECTIONS BETWEEN FAMOUS CEVIANS-III-www.ssmrmh.ro

[2]. Bogdan Fuștei- ABOUT NAGEL AND GERGONNE CEVIANS (III)-www.ssmrmh.ro