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**CONTENT**

## THE THREE COSINES IDENTITY,

## USING LINEAR ALGEBRA AND THE SITARU – ABI – KHUZAM LEMMA

***By Martin Celli – Iztapalapa – Mexico***

**ABSTRACT:** We give two new proofs of the classic three cosines identity.The three cosines identity relates the cosines of the three angles of a triangle.

**Theorem (three cosines identity): If $\boldsymbol{A+B+C=\pi[2\pi]}$, then:**

$$\boldsymbol{(\cos(A))^2+(\cos(B))^2+(\cos(C))^2+2\cos(A)\cos(B)\cos(C)=1}.$$

**Proof 1:** Let

$$M=\begin{pmatrix}1 & -\cos(C) & -\cos(B)\\ -\cos(C) & 1 & -\cos(A)\\ -\cos(B) & -\cos(A) & 1\end{pmatrix}, P=\begin{pmatrix}-\cos(B) & -\cos(A) & 1\\ \sin(B) & -\sin(A) & 0\end{pmatrix}.$$

We can check that $P^TP=M$. Moreover, since $P$ has 3 columns and 2 rows, its kernel has dimension $\geq 3-2=1$. Thus, there exists a vector $X\neq 0$ such that $PX=0$. This implies that $MX=0$. Cancelling the determinant of $M$, we obtain the theorem.

**Proof 2:** Taking $X=(\sin(A),\sin(B),\sin(C))$, we can directly check that

$$MX=\begin{pmatrix}1 & -\cos(C) & -\cos(B)\\ -\cos(C) & 1 & -\cos(A)\\ -\cos(B) & -\cos(A) & 1\end{pmatrix}\begin{pmatrix}\sin(A)\\ \sin(B)\\ \sin(C)\end{pmatrix}=0.$$

Then we just have to cancel the determinant of $M$.

**Some remarks** The identity $P^TP=M$ is equivalent to

$$X^TMX=X^TP^TPX=\left||PX|\right|^2 \text{ for all } X=(x,y,z),$$

$$x^2+y^2+z^2-2(yz\cos(A)+zx\cos(B)+xy\cos(C))=$$

$$=(-x\cos(B)-y\cos(A)+z)^2+(x\sin(B)-y\sin(A))^2$$

In [S], this identity is used to prove the inequality

$$x^2+y^2+z^2\geq 2(yz\cos(A)+zx\cos(B)+xy\cos(C)),$$

which is equivalent to the positivity of $M$.

Let $\vec{U},\vec{V},\vec{W}$ be three unit vectors of the plane such that

$$(\vec{V},\vec{W})=\pi-A[2\pi], (\vec{W},\vec{U})\equiv\pi-B[2\pi]$$

Then

$$(\vec{U},\vec{V}) \equiv (\vec{U},\vec{W}) + (\vec{W},\vec{V}) \equiv -(\pi - B) - (\pi - A) \equiv A + B \equiv \pi - C[2\pi]$$

Denoting by $\vec{W}'$ the image of $\vec{W}$ under the rotation of angle $\frac{\pi}{2}$, we can easily see that $P$ is the matrix of $(\vec{U},\vec{V},\vec{W})$ in the basis $(\vec{W},\vec{W}')$. Thus, for every vector $X = (x,y,z)$:

$$X^T M X = ||PX||^2 = \left||x\vec{U} + y\vec{V} + z\vec{W}|\right|^2$$

This gives us a generalization of the inequality of [S]:

$$x^2 + y^2 + z^2 - 2(yz\cos(A) + zx\cos(B) + xy\cos(C)) = \left|x\vec{U} + y\vec{V} + z\vec{W}\right|^2 \geq 0$$

We can check that the identity

$$X^T M X = \left||x\vec{U} + y\vec{V} + z\vec{W}|\right|^2$$

remains valid for any three unit vectors $\vec{U},\vec{V},\vec{W}$ of three – dimensional space, such that

$$(\vec{V},\vec{W}) \equiv \pi - A[2\pi], (\vec{W},\vec{U}) \equiv \pi - B[2\pi], (\vec{U},\vec{V}) \equiv \pi - C[2\pi],$$

even when $A + B + C \not\equiv \pi[2\pi]$. Thus, the inequality of [S] remains valid when

$A + B + C \not\equiv \pi[2\pi]$.

**Reference:**

**[S] D. Sitaru, A simple proof for Abi – Khuzam's inequality, Romanian Mathematical Magazine, April 2021.**

https://www.ssmrmh.ro/wp-content/uploads/2021/04/A-SIMPLE-PROOF-FOR-ABI-KHUZAMS-INEQUALITY.pdf

## IONESCU-WEITZENBÖCK'S TYPE INEQUALITES WITH FIBONACCI NUMBERS

***By D.M. Bătineţu-Giurgiu, Mihaly Bencze, Claudia Nănuți and Neculai Stanciu-Romania***

**ABSTRACT:** In this paper we present some inequalities with Fibonacci numbers related to Ionescu-Weitzenböck's inequality.

Let $m$ be positive real number and $n$ be positive integer number. If $ABC$, is a triangle with area $S$ and usual notations then we have that:

$$(1) \frac{a^{2(m+1)}}{\left(F_n m_a^2 + F_{n+1} m_b^2\right)^m} + \frac{b^{2(m+1)}}{\left(F_n m_b^2 + F_{n+1} m_c^2\right)} + \frac{c^{2(m+1)}}{\left(F_n m_c^2 + F_{n+1} m_a^2\right)} \geq \frac{4^{m+1}\sqrt{3}}{3^m F_{n+2}^m} S$$

(2) $$\frac{m_a^{2(m+1)}}{\left(F_n b^2+F_{n+1}c^2\right)^m}+\frac{m_b^{2(m+1)}}{\left(F_n c^2+F_{n+1}a^2\right)}+\frac{m_c^{2(m+1)}}{\left(F_n a^2+F_{n+1}b^2\right)}\geq\frac{3^{m+1}\sqrt{3}}{4^m F_{n+2}^m}S$$

(3) $$\frac{a^{2(m+1)}}{\left(F_n m_a^2+F_{n+1}m_b^2+F_{n+2}m_c^2\right)^m}+\frac{b^{2(m+1)}}{\left(F_n m_b^2+F_{n+1}m_c^2+F_{n+2}m_a^2\right)}+\frac{c^{2(m+1)}}{\left(F_n m_c^2+F_{n+1}m_a^2+F_{n+2}m_b^2\right)}\geq$$

$$\geq\frac{2^{m+2}\sqrt{3}}{3^m F_{n+2}^m}S$$

(4) $$\frac{m_a^{2(m+1)}}{\left(F_n a^2+F_{n+1}b^2+F_{n+2}c^2\right)^m}+\frac{m_b^{2(m+1)}}{\left(F_n b^2+F_{n+1}c^2+F_{n+2}a^2\right)}+\frac{m_c^{2(m+1)}}{\left(F_n c^2+F_{n+1}a^2+F_{n+2}b^2\right)}\geq$$

$$\geq\frac{3^{m+1}\sqrt{3}}{8^m F_{n+2}^m}S$$

**Proof:** We use Radon' s inequality, the well-known formula

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}\left(a^2+b^2+c^2\right),$$

and Ionescu-Weitzenböck's inequality, i.e.

$$a^2+b^2+c^2\geq 4\sqrt{3}S.$$

Proof of (1) We have:

$$W_n=\sum\frac{a^{2(m+1)}}{\left(F_n m_a^2+F_{n+1}m_b^2\right)^m}=\sum\frac{\left(a^2\right)^{m+1}}{\left(F_n m_a^2+F_{n+1}m_b^2\right)^m},$$

and by J. Radon's inequality we deduce that:

$$W_n\geq\frac{\left(a^2+b^2+c^2\right)^{m+1}}{\left(\sum\left(F_n m_a^2+F_{n+1}m_b^2\right)\right)^m}=\frac{\left(a^2+b^2+c^2\right)^{m+1}}{\left(F_n+F_{n+1}\right)^m\left(m_a^2+m_b^2+m_c^2\right)^m}=$$

$$=\frac{\left(a^2+b^2+c^2\right)^{m+1}}{F_{n+2}^m\left(m_a^2+m_b^2+m_c^2\right)^m}.$$

But, $m_a^2+m_b^2+m_c^2=\frac{3}{4}\left(a^2+b^2+c^2\right)$, so: $W_n\geq\frac{4^m}{3^m F_{n+2}^m}\left(a^2+b^2+c^2\right)$.

By Ionescu-Weitzenböck's inequality, i.e. $a^2+b^2+c^2\geq 4\sqrt{3}S$, we deduce that:

$$W_n \geq \frac{4^{m+1}\sqrt{3}}{3^m F_{n+2}^m} S$$ , so (1) is proved.

Proof of (2). We have:

$$Y_n = \sum \frac{m_a^{2(m+1)}}{\left(F_n b^2 + F_{n+1} c^2\right)^m} = \sum \frac{\left(m_a^2\right)^{m+1}}{\left(F_n b^2 + F_{n+1} c^2\right)^m},$$

and by J. Radon's inequality we deduce that:

$$Y_n \geq \frac{\left(m_a^2 + m_b^2 + m_c^2\right)^{m+1}}{\left(\sum \left(F_n b^2 + F_{n+1} c^2\right)\right)^m} = \frac{\left(m_a^2 + m_b^2 + m_c^2\right)^{m+1}}{\left(F_n + F_{n+1}\right)^m \left(a^2 + b^2 + c^2\right)^m} =$$

$$= \frac{\left(m_a^2 + m_b^2 + m_c^2\right)^{m+1}}{F_{n+2}^m \left(a^2 + b^2 + c^2\right)^m}.$$

Since, $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right)$, we have:

$$Y_n \geq \frac{3^{m+1}}{4^{m+1} F_{n+2}^m}\left(a^2 + b^2 + c^2\right).$$

By Ionescu-Weitzenböck's inequality, i.e. $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$ , we deduce that:

$$Y_n \geq \frac{3^{m+1}\sqrt{3}}{4^m F_{n+2}^m} S$$ , so (2) is proved.

Proof of (3). We have:

$$Z_n = \sum \frac{a^{2(m+1)}}{\left(F_n m_a^2 + F_{n+1} m_b^2 + F_{n+2} m_c^2\right)^m} = \sum \frac{\left(a^2\right)^{m+1}}{\left(F_n m_a^2 + F_{n+1} m_b^2 + F_{n+2} m_c^2\right)^m},$$

and by J. Radon's inequality we deduce that:

$$Z_n \geq \frac{\left(a^2 + b^2 + c^2\right)^{m+1}}{\left(\sum \left(F_n m_a^2 + F_{n+1} m_b^2 + F_{n+2} m_c^2\right)\right)^m} = \frac{\left(a^2 + b^2 + c^2\right)^{m+1}}{\left(F_n + F_{n+1} + F_{n+2}\right)^m \left(m_a^2 + m_b^2 + m_c^2\right)^m} =$$

$$= \frac{\left(a^2 + b^2 + c^2\right)^{m+1}}{2^m F_{n+2}^m \left(m_a^2 + m_b^2 + m_c^2\right)^m}.$$

Since, $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right)$, we have:

$$Z_n \geq \frac{2^m}{3^m F_{n+2}^m}\left(a^2+b^2+c^2\right).$$

By Ionescu-Weitzenböck's inequality, i.e. $a^2+b^2+c^2 \geq 4\sqrt{3}S$ , we deduce that:

$$Z_n \geq \frac{2^{m+2}\sqrt{3}}{3^m F_{n+2}^m}S$$ , so (3) is proved.

Proof of (4). We have:

$$X_n = \sum \frac{m_a^{2(m+1)}}{\left(F_n a^2 + F_{n+1}b^2 + F_{n+2}c^2\right)^m} = \sum \frac{\left(m_a^2\right)^{m+1}}{\left(F_n a^2 + F_{n+1}b^2 + F_{n+2}c^2\right)^m},$$

and by J. Radon's inequality we deduce that:

$$X_n \geq \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{\left(\sum\left(F_n a^2 + F_{n+1}b^2 + F_{n+2}c^2\right)\right)^m} = \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{\left(F_n+F_{n+1}+F_{n+2}\right)^m\left(a^2+b^2+c^2\right)^m} =$$

$$= \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{2^m F_{n+2}^m\left(a^2+b^2+c^2\right)^m}.$$

We know that: $m_a^2+m_b^2+m_c^2 = \frac{3}{4}\left(a^2+b^2+c^2\right)$, so:

$$X_n \geq \frac{3^{m+1}}{2^{3m+2}F_{n+2}^m}\left(a^2+b^2+c^2\right).$$

By Ionescu-Weitzenböck's inequality, i.e. $a^2+b^2+c^2 \geq 4\sqrt{3}S$ , we obtain that:

$$X_n \geq \frac{3^{m+1}\sqrt{3}}{2^{3m}F_{n+2}^m}S$$ , and the proof is complete.

**Reference: Romanian Mathematical Magazine-www.ssmrmh.ro**

## ABOUT AN INEQUALITY BY MARIN CHIRCIU FROM RMM-43

***By Titu Zvonaru and Neculai Stanciu - Romania***

**Here we present 7 methods to solve the problem J.2491 from R.M.M. – 43, Winter Edition, 2024, p. 84.**

**J.2491.**

$$\frac{a}{7a+b+c}+\frac{b}{a+7b+c}+\frac{c}{a+b+7c} \leq \frac{1}{3} \quad (1)$$

Inequality (1) can also be written in the form:

$$\frac{1}{7}-\frac{a}{7a+b+c}+\frac{1}{7}-\frac{b}{a+7b+c}+\frac{1}{7}-\frac{c}{a+b+7c}\geq\frac{3}{7}-\frac{1}{3}$$

$$\frac{b+c}{7a+b+c}+\frac{c+a}{a+7b+c}+\frac{a+b}{a+b+7c}\geq\frac{2}{3}\quad(2)$$

**SOLUTION 1. (REMOVE DENOMINATORS)**

$3a(a+7b+c)(a+b+7c)+3b(a+b+7c)(7a+b+c)+3c(7a+b+c)(a+7b+c)\geq(7a+b+c)(a+7b+c)(a+b+7c)$

$$4(a^3+b^3+c^3)+12(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)\geq 84abc\quad(3)$$

The inequality (3) yields by $AM-GM$: $a^3+b^3+c^3\geq 3abc, a^2b+ab^2+b^2c+bc^2+c^2a+ca^2\geq 6abc$.

**SOLUTION 2. (BREAKING)**

The inequality

$$\frac{a}{7a+b+c}\leq\frac{5a+2b+2c}{27(a+b+c)}\qquad(4)$$

can be written as

$$8a^2-8a(b+c)+2(b+c)^2\geq 0\Leftrightarrow 2\big(2a-(b+c)\big)^2\geq 0.$$

Using the inequality (4) we obtain

$$\frac{a}{7a+b+c}+\frac{b}{a+7b+c}+\frac{c}{a+b+7c}\leq$$

$$\leq\frac{5a+2b+2c}{27(a+b+c)}+\frac{5b+2c+2a}{27(a+b+c)}+\frac{5c+2a+2b}{27(a+b+c)}\leq\frac{9(a+b+c)}{27(a+b+c)}=\frac{1}{3}$$

**SOLUTION 3. (HORNER BREAK)**

Due to the homogeneity we can assume that $a+b+c=1$.
The inequality (2) becomes

$$\frac{a}{6a+1}+\frac{b}{6b+1}+\frac{c}{6c+1}\leq\frac{1}{3}\quad(5)$$

We want to determine $m,n$ such that the inequality

$$\frac{x}{6x+1}\leq mx+n\quad(6)$$

To be true for any $x>0$. The inequality (6) is equivalent to

$$6mx^2+(m+6n-1)x+n\geq 0\quad(7)$$

Bearing in mind that in the inequality $(5)$ we have equality for $a=b=c=\frac{1}{3}$, must as the left side of the relation $(7)$ to admit the double root $\frac{1}{3}$.
Using Horner's scheme we have

| | | | |
|---|---|---|---|
| | $6m$ | $m+6n-1$ | $n$ |
| $\frac{1}{3}$ | $6m$ | $3m+6n-1$ | $m+3n-\frac{1}{3}$ |
| $\frac{1}{3}$ | $6m$ | $5m+6n-1$ | |

By relations $m+3n-\frac{1}{3}=0, 5m+6n-1=0$ yields $m=\frac{1}{9}, n=\frac{2}{27}$. The inequality $(7)$ becomes $2(3x-1)^2\geq 0$. Writing the inequality $(6)$ for $a,b,c$ we obtain

$$\frac{a}{6a+1}+\frac{b}{6b+1}+\frac{c}{6c+1}\leq\frac{3a+2}{27}+\frac{3a+2}{27}+\frac{3c+2}{27}=\frac{3(a+b+c)+6}{27}=\frac{1}{3}.$$

**SOLUTION 4. (TANGENT METHOD)**

Another method to determine $m,n$ such that the inequality $(6)$ is true for any $x>0$. For $x=\frac{1}{3}$ we obtain $\frac{m}{3}+n=\frac{1}{3}$. By derivation yields that $\frac{1}{(6x+1)^2}=m$; for $x=\frac{1}{3}$ we have $m=\frac{1}{9}$. For $m=\frac{1}{9}, n=\frac{2}{27}$, the inequality $(6)$ becomes $2(3x-1)^2\geq 0$, evidently true.

**SOLUTION 5. (BERGSTRÖM)**

Using the form by $(2)$ and applying Bergström's inequality, we obtain

$$\frac{b+c}{7a+b+c}+\frac{c+a}{a+7b+c}+\frac{a+b}{a+b+7c}=$$

$$=\frac{(b+c)^2}{(b+c)(7a+b+c)}+\frac{(c+a)^2}{(c+a)(a+7b+c)}+\frac{(a+b)^2}{(a+b)(a+b+7c)}\geq$$

$$\geq\frac{(a+b+b+c+c+a)^2}{(a+b)(a+b+7c)+(b+c)(b+c+7a)+(c+a)(c+a+7b)}.$$

It remains to prove that

$$6(a+b+c)^2\geq(a+b)(a+b+7c)+(b+c)(b+c+7a)+(c+a)(c+a+7b)$$

$$4(a^2+b^2+c^2)\geq 4(ab+bc+ca),$$

Which is a well-known inequality.

**SOLUTION 6. (SUBSTITUTIONS)**

We denote $x=7a+b+c, y=7b+c+a, z=7c+a+b$. Solving this system, we get $a=\frac{8x-y-z}{54}, b=\frac{8y-z-x}{54}, c=\frac{8z-x-y}{54}$. The inequality $(1)$ becomes

$$\frac{8x-y-z}{54x}+\frac{8y-z-x}{54y}+\frac{8z-x-y}{54z}\leq\frac{1}{3}$$

$$24-\left(\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}\right)\leq 18$$

$\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}\geq 6$, which yields by $\frac{x}{y}+\frac{y}{x}\geq 2$.

**SOLUTION 7. (DELIGATION)**

$$\frac{a}{7a+b+c}+\frac{b}{a+7b+c}+\frac{c}{a+b+7c}-\frac{1}{3}=$$

$$=\frac{a}{7a+b+c}-\frac{1}{9}+\frac{b}{a+7b+c}-\frac{1}{9}+\frac{c}{a+b+7c}-\frac{1}{9}=$$

$$=\frac{2a-b-c}{9(7a+b+c)}+\frac{2b-c-a}{9(a+7b+c)}+\frac{2c-a-b}{9(a+b+7c)}=$$

$$=\frac{a-b}{9(7a+b+c)}+\frac{a-c}{9(7a+b+c)}+\frac{b-c}{9(a+7b+c)}+\frac{b-a}{9(a+7b+c)}+\frac{c-a}{9(a+b+7c)}+\frac{c-b}{9(a+b+7c)}=$$

$$\frac{a-b}{9}\left(\frac{1}{7a+b+c}-\frac{1}{a+7b+c}\right)+\frac{b-c}{9}\left(\frac{1}{a+7b+c}-\frac{1}{a+b+7c}\right)+\frac{c-a}{9}\left(\frac{1}{a+b+7c}-\frac{1}{7a+b+c}\right)=$$

$$=-\frac{6}{9}\left(\frac{(a-b)^2}{(7a+b+c)(a+7b+c)}+\frac{(b-c)^2}{(a+7b+c)(a+b+7c)}+\frac{(c-a)^2}{(a+b+7c)(7a+b+c)}\right)\leq 0.$$

The equality occurs iff $a=b=c$.

**REFERENCE: Romanian Mathematical Magazine-www.ssmrmh.o**

## LALESCU AND EULER-MASCHERONI TYPE NEW LIMITS

**By D.M. Bătineţu-Giurgiu, Mihaly Bencze, Claudia Nănuți and Neculai Stanciu-Romania**

**ABSTRACT:** In this paper we present some new limits with sequences.

**I.** $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\frac{\pi}{\sqrt[n]{n!}}=\frac{2\pi}{e}$.

**Proof.** $\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e$, $\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{2}{e}$.

Denoting $u_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}$ : $\lim_{n\to\infty}u_n=1$, $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$, $\lim_{n\to\infty}u_n^n=e$.

We denote $x_n = (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!})\sqrt[n]{(2n-1)!!}\sin\frac{\pi}{\sqrt[n]{n!}}$ :

$$x_n = (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!})\sqrt[n]{(2n-1)!!}\sin\frac{\pi}{\sqrt[n]{n!}} = \sqrt[n]{n!}\cdot(u_n - 1)\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\sqrt[n]{n!}\cdot\sin\frac{\pi}{\sqrt[n]{n!}} =$$

$$= \frac{\sqrt[n]{n!}}{n}\cdot n\cdot\frac{u_n - 1}{\ln u_n}\cdot\ln u_n\cdot\frac{(2n-1)!!}{n}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\pi\cdot\frac{\sin\frac{\pi}{\sqrt[n]{n!}}}{\frac{\pi}{\sqrt[n]{n!}}} =$$

$$= \pi\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{u_n - 1}{\ln u_n}\cdot\ln u_n^n\cdot\frac{(2n-1)!!}{n}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{\sin\frac{\pi}{\sqrt[n]{n!}}}{\frac{\pi}{\sqrt[n]{n!}}}.$$

Hence, $\lim_{n\to\infty} x_n = \pi\cdot\frac{1}{e}\cdot 1\cdot\ln e\cdot\frac{2}{e}\cdot e\cdot 1 = \frac{2\pi}{e}$.

**II.** $\lim_{n\to\infty}\sqrt{n}(\sqrt{a_{n+1}} - \sqrt{a_n}) = \frac{\sqrt{a}}{2}$, where $(a_n)_{n\geq 1}$ is a positive real sequence such that

$\lim_{n\to\infty}(a_{n+1} - a_n) = a$, $a > 0$.

**Proof:** $\lim_{n\to\infty}\frac{a_n}{n} \overset{C-S}{=} \lim_{n\to\infty}\frac{a_{n+1} - a_n}{(n+1) - n} = a$; denoting $x_n = \sqrt{n}(\sqrt{a_{n+1}} - \sqrt{a_n}) = \sqrt{n}\cdot\frac{a_{n+1} - a_n}{\sqrt{a_{n+1}} - \sqrt{a_n}} =$

$$= (a_{n+1} - a_n)\cdot\frac{1}{\sqrt{\frac{a_{n+1}}{n+1}}\cdot\sqrt{\frac{n+1}{n}} + \sqrt{\frac{a_n}{n}}};\text{ so}$$

$$\lim_{n\to\infty}\sqrt{n}(\sqrt{a_{n+1}} - \sqrt{a_n}) = \lim_{n\to\infty} x_n = a\cdot\frac{1}{\sqrt{a}\cdot 1 + \sqrt{a}} = \frac{\sqrt{a}}{2}.$$

**III.** $\lim_{n\to\infty} e^{-H_n}\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}} = 2\cdot e^{-(\gamma+1)}$, where $(2k-1)!! = 1\cdot 3\cdot\ldots\cdot(2k-1), \forall k\in N^*$

and $H_n = \sum_{k=1}^{n}\frac{1}{k}$.

**Proof:** $n\cdot e^{-H_n} = e^{\ln n}e^{-H_n} = e^{\ln n - H_n} = e^{-\gamma_n}$, where $\lim_{n\to\infty}\gamma_n = \gamma$ is Euler-Mascheroni constant.

$$x_n = e^{-H_n}\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}} = ne^{-H_n}\frac{\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}}{n} =$$

$$= e^{-\gamma_n}\cdot\sqrt[n]{\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}{n^n}};\ \lim_{n\to\infty}\sqrt[n]{\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}{n^n}} \overset{C-D}{=}$$

$$= \lim_{n\to\Re}\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n+1]{(2n+1)!!}}{(n+1)^{n+1}}\cdot\frac{n^n}{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}} = \lim_{n\to\Re}\frac{\sqrt[n+1]{(2n+1)!!}}{n+1} =$$

$$= \lim_{n\to\Re}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\Re}\sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{C-D}{=} \lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\frac{n^n}{(2n-1)!!} = \lim_{n\to\Re}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e}.$$

Hence, $\lim_{n\to\Re} x_n = e^{-\gamma}\cdot\frac{2}{e} = 2\cdot e^{-(\gamma+1)}$.

**IV.** $\lim_{n\to\infty}\left(\sqrt[n+1]{((n+1)!)^a((2n+1)!!)^b} - \sqrt[n]{(n!)^a((2n-1)!!)^b}\right) = \begin{cases} 0, \text{daca } a+b<1 \\ \frac{2^b}{e}, \text{daca } a+b=1 \\ \infty, \text{daca } a+b>1 \end{cases}$, where $a,b\in R$

**Proof:** $\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = e$, $\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \frac{2}{e}$.

Denoting $u_n = \frac{\sqrt[n+1]{((n+1)!)^a((2n+1)!!)^b}}{\sqrt[n]{(n!)^a((2n-1)!!)^b}}$ : $\lim_{n\to\infty} u_n = 1, \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1$;

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\Im}\frac{((n+1)!)^a}{(n!)^a}\cdot\frac{((2n+1)!!)^b}{((2n-1)!!)^b}\cdot\frac{1}{\sqrt[n+1]{((n+1)!)^a(((2n+1)!!)^b}} =$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^a\left(\frac{2n+1}{n+1}\right)^b\left(\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\right)^b = e^a 2^b\left(\frac{e}{2}\right)^b = e^{a+b}.$$

We denote $x_n = \left(\sqrt[n+1]{((n+1)!)^a((2n+1)!!)^b} - \sqrt[n]{(n!)^a((2n-1)!!)^b}\right) =$

$$= \left(\frac{\sqrt[n]{n!}}{n}\right)^a\left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^b\cdot n^{a+b-1}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n.$$

Hence, $\lim\limits_{n\to\infty} x_n = \frac{1}{e^a}\cdot\frac{2^b}{e^b}\cdot 1\cdot \ln e^{a+b}\cdot \lim\limits_{n\to\Re} n^{a+b-1} = \begin{cases} 0, \text{daca } a+b<1 \\ \frac{2^b}{e}, \text{daca } a+b=1. \\ \infty, \text{daca } a+b>1 \end{cases}$

**Reference: Romanian Mathematical Magazine-www.ssmrmh.ro**

## NUMBER BASES

***By Carmen – Victorița Chirfot-Romania***

In everyday life, what coordinates us is the binary system. The other number bases are applications of this binary system, including base 10.

Let be $N = \overline{a_1a_2a_3 \dots a_{p-1}a_pa_{p+1}}_{(b)}$ where $a_1, a_2, a_3, \dots, a_p, a_{p+1}$ are the digits of the number $N$ written in base $b$, namely $a_1, a_2, a_3, \dots, a_p, a_{p+1} \in \{0,1,2,\dots,b-1\}$. Then

$N = a_1 \cdot b^p + a_2 \cdot b^{p-1} + a_3 \cdot b^{p-2} + \dots + a_{p-1} \cdot b^2 + a_p \cdot b^1 + a_{p+1}$ represents the writing in base $b$ of the natural number $N$.

Note: For the transition from a lower base to a higher base, we will use the digits of the higher number base and the index of the base to which we want to convert the number.

For the transition from a higher number base to a lower number base, we will use the digits of the higher number base and then shift them to the lower number base.

Note: In the number base $b$, we keep all the known operations, with the observation that addition and subtraction in the base $b$ will be done by passing over the order or by borrowing to the higher order digit, not using digits higher than the index of the number base in which wants a certain number to pass. Multiplication and division will be done using all digits in base $b$.

To begin, we will convert the first powers of 2 to base 3. We have $1_{(2)} = 2^0 = 1_{(3)}$,

$$10_{(2)} = 2^1 = 2_{(3)}, 100_{(2)} = 2^2 = 2 \cdot 2 = (1+1)(1+1) = 1+1+1+1 =$$

$= 1 \cdot 3^1 + 1 = 11_{(3)}$. But

$$2^2 = 3 + 1| \cdot (1+1) \Rightarrow 1000_{(2)} = 2^3 = (3+1)(1+1) = 3+3+1+1 =$$

$$= 2 \cdot 3^1 + 2 = 22_{(3)}$$

Let's find:

$$2^4 = 2^3 \cdot 2 = (2 \cdot 3 + 2)(1+1) = 2 \cdot 3 + 2 \cdot 3 + 2 + 2 = 2 \cdot 3 + 1 \cdot 3 + 1 \cdot 3 + 2 +$$

$+1 + 1 = (2+1) \cdot 3 + 2 \cdot 3 + 1 = 1 \cdot 3^2 + 2 \cdot 3 + 2 = 121_{(3)}$.

Let's find and $10000_{(2)} = 2^5 = 2^4 \cdot 2 = (3^2 + 2 \cdot 3 + 2) \cdot (1+1) =$

$$= 3^2 + 2 \cdot 3 + 2 + 3^2 + 2 \cdot 3 + 2 = 2 \cdot 3^2 + 2 \cdot 3 + 1 \cdot 3 + 2 + 1 + 1 =$$

$$= 2 \cdot 3^2 + 3 \cdot 3 + 3 + 1 = 2 \cdot 3^2 + 1 \cdot 3^2 + 3 + 1 = 3 \cdot 3^3 + 3 + 1 =$$

$$= 1 \cdot 3^4 + 0 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3 + 1 = 10011_{(3)}$$

In this way, using additions, we can convert any number from base 2 to base 3, using the digits and the base 3 index.

We will also convert the first powers of 3 to base 2. We have $1_{(3)} = 3^0 = 1_{(3)}$,

$$10_{(3)} = 3^1 = 1 \cdot 2 + 1 = 11_{(3)}, 100_{(3)} = 3^2 = (2+1) \cdot (2+1) = 1 + 1 \cdot 2 + 1 \cdot 2 + 2^2 =$$

$$= 1 + (1+1) \cdot 2 + 2^2 =$$

$$= 2^2 + 2^2 + 1 = 2 \cdot 2^2 + 1 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 = 1001_{(2)},$$

$$1000_{(3)} = 3^3 = 3^2 \cdot (2+1) =$$

$$= (2^3 + 1)(2+1) = 2^4 + 2^3 + 2 + 1 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 1 = 11011_{(2)}$$

$$10000_{(3)} = 3^4 = 3^3 \cdot (2+1) = (2^4 + 2^3 + 2 + 1)(2+1) =$$

$$= 2^5 + 2^4 + 2^4 + 2^3 + 2^2 + 2 + 2 + 1 =$$

$$= 2^5 + 2 \cdot 2^4 + 2^3 + 2^2 + 2 \cdot 2 + 1 = 2^5 + 2^5 + 2^3 + 2^2 + 2^2 + 1 =$$

$$= 2 \cdot 2^5 + 2^3 + 2 \cdot 2^2 + 1 =$$

$$= 2^6 + 2^3 + 2^3 + 1 = 2^6 + 2 \cdot 2^3 + 1 = 2^6 + 2^4 + 1 = 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 +$$

$$+ 0 \cdot 2^2 + 0 \cdot 2 + 1 = 1010001_{(2)}$$

For example, the number $N = 1001_{(2)} = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1$ is written in base 2. We propose to pass the number $N$ in base 3. But

$$2^2 = 3 + 1 | \cdot (1+1) \Rightarrow 2^3 = (3+1)(1+1) = 3 + 3 + 1 + 1 = 2 \cdot 3^1 + 2 = 22_{(3)}$$

So, $N = 1 \cdot 2^3 + 1 = (2 \cdot 3 + 2) + 1 = 2 \cdot 3 + 1 \cdot 3 = (2+1) \cdot 3 = 3^2 =$

$$= 1 \cdot 3^2 + 0 \cdot 3^1 + 0 = 100_{(3)}$$

Hence, $1001_{(2)} = 100_{(3)}$

For the reverse transition, from base 3 to base 2, we have

$$100_{(3)} = 1 \cdot 3^2 + 0 \cdot 3^1 + 0 = 3^2 = (2+1)(2+1) =$$

$$= 2^2 + 2 + 2 + 1 = 2^2 + 2 \cdot 2 + 1 = 2^2 + 2^2 + 1 =$$

$$= 2^2 + 2^2 + 1 = 2 \cdot 2^2 + 1 = 2^3 + 1 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2 + 1 = 1001_{(2)}$$

Note. We have used the fact that $3 = 2 + 1$.

Suppose we want to convert a number from base 2 to base 5. Let be

$$N = 101011_{(2)} = 2^5 + 2^3 + 2 + 1 = (5+5+5+5+5+5+2) + (5+3) + 2 + 1 =$$

$$= 5^2 + 5 + 2 + 5 + 3 + 2 + 1 =$$

$$= 5^2 + 5 + 2 + 5 + 3 + 2 + 1 = 1 \cdot 5^2 + 3 \cdot 5 + 3 = 133_{(5)}$$

Suppose we want to transform the number $N = 133_{(5)}$ in base 2. But

$$5 = 1 \cdot 3 + 2, 5^2 = (3+2)(3+2) = 3^2 + 4 \cdot 3 + 2^2 = 3^2 + 1 \cdot 3 + 3^2 + (3+1) =$$

$$= 2 \cdot 3^2 + 2 \cdot 3 + 1$$

Then $N = 133_{(5)} = 1 \cdot 5^2 + 3 \cdot 5 + 3 = (2 \cdot 3^2 + 2 \cdot 3 + 1) + 3 \cdot (3+2) + 3 =$

$$= 2 \cdot 3^2 + 2 \cdot 3 + 1 + 3^2 + 3 \cdot 2 + 3 = 3^3 + (3+2) \cdot 3 + 1 =$$

$$= 3^3 + 3^2 + 2 \cdot 3 + 1 = 1121_{(3)}$$

Let's suppose that we want to pass from base 5 in base 3 the number $N = 133_{(5)}$. But

$$5 = 1 \cdot 3 + 2,$$

$$5^2 = (3+2)(3+2) = 3^2 + 4 \cdot 3 + 2^2 = 3^2 + 1 \cdot 3 + 3^2 + (3+1) =$$

$$= 2 \cdot 3^2 + 2 \cdot 3 + 1$$

Then $N = 133_{(5)} = 1 \cdot 5^2 + 3 \cdot 5 + 3 = (2 \cdot 3^2 + 2 \cdot 3 + 1) + 3 \cdot (3+2) + 3 =$

$$= 2 \cdot 3^2 + 2 \cdot 3 + 1 + 3^2 + 3 \cdot 2 + 3 =$$

$$= 3^3 + (3+2) \cdot 3 + 1 = 3^3 + 3^2 + 2 \cdot 3 + 1 = 1121_{(3)}$$

Note: Conversions from one base to another can also be done with the help of subtraction operations, but the calculations become more cumbersome.

Note: It is known that to convert a number from base 2 to base 4, groups of 2 digits can be made, associating them with a digit from the base 4 system.

For example, for the number $N = 100111_{(2)}$ groupings are made from right to left of 2 digits each, because

$$100111_{(2)} = 1001_{(2)} \cdot 2^2 + 11_{(2)} = \left(10_{(2)} \cdot 2^2 + \overline{01}_{(2)}\right) \cdot 2^2 + 11_{(2)} =$$

$$= 10_{(2)} + 2^4 + \overline{01}_{(2)} \cdot 2^2 + 11_{(2)} = 10_{(2)} \cdot 4^2 + \overline{01}_{(2)} \cdot 4 + 11_{(2)} =$$

$$= 2 \cdot 4^2 + 1 \cdot 4 + 3 = 213_{(4)}$$

The reverse shift is done by associating each digit in the base 4 system with a corresponding number in base 2. The resulting zeros will also be taken into account, minus the first digit of the resulting number.

Note: To convert a number from base 2 to base 8, groups of 3 digits can be made, associating them with a digit from the octal system. It will proceed analogously to the transitions from base 2 to base 4 and vice versa.

Note: To convert a number from base 2 to base 16 (hexadecimal digits are from 0 to 9 and then $A = 10, B = 11, C = 12, D = 13, E = 14, F = 15$) groups of 4 digits can be made, associating them with a digit from the hexadecimal system. So, to move a number from base 2 to base $2^n, n \in \mathbb{N}^*$ , groupings of $n$ digits are made.

To convert a number $N = ABCE_{(16)}$ to base 2, the hexadecimal digits are converted to base 2. We have: $A = 2 + 2 + 2 + 2 + 2 = 2^2 + 2^2 + 2 = 2 \cdot 2^2 + 2 = 2^3 + 1 \cdot 2 =$

$$= 1000_{(2)} + 10_{(2)} = 1010_{(2)}$$

$$B = A + 1 = 1010_{(2)} + 1_{(2)} = 1011_{(2)}, C = B + 1 = 1011_{(2)} + 1_{(2)} = 1100_{(2)},$$

$$D = C + 1 = 1100_{(2)} + 1_{(2)} = 1101_{(2)}, E = D + 1 = 1101_{(2)} + 1_{(2)} = 11110_{(2)},$$

$F = 1111_{(2)}$. So, the number $N$ in base 2 is $N = 1010\ 1011\ 1100\ 1101_{(2)}$.

To convert to base 4, convert the hexadecimal digits into numbers in base 4 and write the numbers in the order found.

Let's transform the number $N = 122112_{(3)}$ in base 9. We have that

$$N = 122112_{(3)} = 1221_{(3)} \cdot 3^2 + 12_{(3)} = \left(12_{(3)} \cdot 3^2 + 21_{(3)}\right) \cdot 3^2 + 1 \cdot 3 + 2 =$$

$$= 12_{(3)} \cdot 3^4 + 21_{(3)} \cdot 3^2 + 1 \cdot 3 + 2 = 5 \cdot 9^2 + 7 \cdot 9 + 5 = 575_{(9)}.$$

So, for the transition of a number from base 3 to base 9, groupings of 2 digits are made from right to left, i.e. the exponent of 3 in the writing $9 = 3^2$. To convert a number from base 9 to base 3, each digit in base 9 is written as a number in base 3, also retaining the zeros, less the most significant digit. For example

$$627_{(9)} = 6 \cdot 9^2 + 2 \cdot 9 + 7 = 20_{(3)} \cdot 3^4 + \overline{02}_{(3)} \cdot 3^2 + 21_{(3)} = 200221_{(3)}$$

Note: For the transition from base 2 to base 10, the procedure is the known one. E.g, $N = 1011_{(2)} = 2^3 + 2 + 1 = 11_{(10)}$ . To move from base 10 to base 2, repeated divisions are made, each time retaining the remainders 0 and 1, and the number that is read in reverse order with the remainders found is the number written in base 2.

Solved problems:

1) Prove that $\forall n \in \mathbb{N}, \exists$ unieque numbers $a_0, a_1, a_2, \dots, a_p \in \{0,1,2,\dots,b-1\}, b \in \mathbb{N} - \{0,1\}$ such that $n = a_0 \cdot b^p + a_1 \cdot b^{p-1} + a_2 \cdot b^{p-2} + \dots + a_{p-1} \cdot b + a_p$ and that

$$a_1 \cdot b^{p-1} + a_2 \cdot b^{p-2} + \dots + a_{p-1} \cdot b + a_p < b^p$$

Solution: The writing above represents the writing in base $b$ of the natural number $n$, that is $n = \overline{a_0 a_1 a_2 \dots a_p}_{(b)}$. Since in a numeration base there is only one number corresponding to a number n in base 10 (it is obvious, because the remainders of the repeated divisions by the number $b$ of the number $n$, respectively of the quotients obtained are unique), we conclude that the numbers are unique.

To deduce the inequality, we use that $\overline{1\underbrace{000\ldots000}_{"p"\ times\ 0}}_{(b)} = b^p$. Obviously that $\overline{1\underbrace{000\ldots000}_{"p"\ times\ 0}}_{(b)} > \overline{a_1a_2a_3\ldots a_p}_{(b)}$, because the number on the left of the inequality has

$p+1$ digits, and the one on the right has $p$ digits. But

$$\overline{a_1a_2a_3\ldots a_p}_{(b)} \le \overline{\underbrace{(b-1)(b-1)\ldots(b-1)}_{for\ p\ times\ b-1_{(b)}}}_{(b)} = (b-1)(b^{p-1}+b^{p-2}+\cdots+b^2+b+1)$$

$$= b^p - 1 < b^p \Rightarrow \overline{a_1a_2a_3\ldots a_p}_{(b)} < b^p \Rightarrow a_1\cdot b^{p-1} + a_2\cdot b^{p-2} + \cdots + a_{p-1}\cdot b + a_p < b^p$$

where $a_1, a_2, a_3, \ldots, a_p \in \{0,1,\ldots,b-1\}$ are the numbers found above and $b \in \mathbb{N} - \{0,1\}$

2) Prove that $2^6 + 2^3 + 1 < 2^8 + 2^5 + 2^2$, without making operations with powers.

Solution: $2^6 + 2^3 + 1 = 1001001_{(2)}, 2^8 + 2^5 + 2^2 = 100100100_{(2)}$. Obviously that

$$1001001_{(2)} < 100100100_{(2)} \Rightarrow 2^6 + 2^3 + 1 < 2^8 + 2^5 + 2^2$$

3) Prove that $3^n + 3^{n-1} + 3^{n-2} > 3^{n-3} + 3^{n-4} + 3^{n-5} + \cdots + 3 + 1, n \in \mathbb{N}, n \ge 3$.

Solution: $3^n + 3^{n-1} + 3^{n-2} = \overline{111\underbrace{000\ldots000}_{n-2\ of\ 0}}_{(3)}$,

$3^{n-3} + 3^{n-4} + 3^{n-5} + \cdots + 3 + 1\ \overline{\underbrace{1111..11}_{n-1\ of\ 1}}_{(3)}$. Because $\overline{111\underbrace{000..00}_{n-2\ of\ 0}}_{(3)} > \overline{\underbrace{1111..11}_{n-2\ of\ 1}}_{(3)} \Rightarrow$

$\Rightarrow 3^n + 3^{n-1} + 3^{n-2} > 3^{n-3} + 3^{n-4} + 3^{n-5} + \cdots + 3 + 1, n \in \mathbb{N}, n \ge 3.$

4) Prove that $2\ \cdot 3^n + 2\cdot 3^{n-1} + 2\cdot 3^{n-2} + \cdots + 2\cdot 3 + 2 < 3^{n+1}, n \in \mathbb{N}$

Solution: $2\cdot 3^n + 2\cdot 3^{n-1} + 2\cdot 3^{n-2} + \cdots + 2\cdot 3 + 2 = \overline{\underbrace{222\ldots22}_{n\ digits\ of\ 2}}_{(3)}$,

$3^{n+1} = \overline{1\ \underbrace{000..00}_{n\ digits\ of\ 0}}_{(3)}$. We observe that:

$\overline{\underbrace{222\ldots22}_{n\ digits\ of\ 2}}_{(3)} < \overline{1\ \underbrace{000\ldots0}_{n\ digits\ of\ 0}}_{(3)}$. So, $2\cdot 3^n + 2\cdot 3^{n-1} + 2\cdot 3^{n-2} + \cdots + 2\cdot 3 + 2 < 3^{n+1}$ and
$\overline{\underbrace{222\ldots22}_{n\ digits\ of\ 2}}_{(3)} + 1_{(3)} = \overline{1\ \underbrace{000\ldots00}_{n\ digits\ of\ 0}}_{(3)}$, namely

$2\cdot 3^n + 2\cdot 3^{n-1} + 2\cdot 3^{n-2} + \cdots + 2\cdot 3 + 2 + 1 = 3^{n+1}, n \in \mathbb{N}.$

5) Prove that there exist the natural numbers $a, b, c, d, e, f, g$ for which

$$2^a + 2^b + 2^c + 2^d + 2^e + 2^f + 2^g = 1020$$

Solution: $1024 = 2^{10} = 10000000000_{(2)} \Rightarrow 1023 = 10000000000_{(2)} - 1 =$

$$= 11111111_{(2)} \Rightarrow 1022 = 11111111_{(2)} - 1_{(2)} = 11111110_{(2)} \Rightarrow 1021 =$$

$= 11111110_{(2)} - 1_{(2)} = 11111101_{(2)} \Rightarrow 1020 = 11111101_{(2)} - 1_{(2)} = 11111100_{(2)} =$

$= 2^8 + 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 \Rightarrow a = 8, b = 7, c = 6, d = 5, e = 4, f = 3, g = 2$ and all their permutations.

6) What is the smallest number written in base 3 consisting of 3 different digits? What is the largest base 10 number made up of 3 different digits?

Solutions: $102_{(3)} = 3^2 + 2 = 11$ the smallest number written in base 3 by 3 digits. Also, $210_{(3)} = 2 \cdot 3^2 + 3 = 21$ the largest number written in base 3 of 3 different digits.

7) To count how many three-digit natural numbers $\overline{abc}$ have $ab + c = even$.

Solution: We count how many numbers $\overline{abc}$ has $ab + c = even$. We have

$even \cdot odd + even$. We have $even \cdot odd + even = odd, even \cdot even + even = even, odd \cdot even + even = even, odd \cdot odd + odd = even$. If we associate $even$ with the digit 0 and $odd$ with digit 1, we have to count the combinations $010, 000, 100, 111$, where 0 can take the values 0,2,4,6,8 and 1 can take the values 1,3,5,7,9. The digit $a$ can not be 0, so on the case 010 we will have $5 \cdot 5 \cdot 6 = 150$ possibilities. On the case 000, we will have

$5 \cdot 6 \cdot 6 = 180$ possibilities. On the case 100 we have $150 + 180 + 180 + 125 = 635$ numbers.

Proposed problems:

1) Convert in base 5 the number $25^6 + 5$ from base $10$.

2) Write the number 599 as a sum of powers of 2.

3) Convert the number $1\underbrace{00\ldots000}_{20\ for\ 0}{}_{(2)}$ in base 3 without making the transition to base 10.

4) Find $1111_{(3)} + 1111_{(2)}$ as a base 5 number without making the transition to base 10.

5) To directly convert (without using intermediate base 10) the number $12122_{(3)}$ to base 2.

6) Convert the number $110110111_{(2)}$ to bases 4, 8 and 16 using groupings of digits.

7) Convert the number $ABCD_{(16)}$ in bases $2$ and $4$, without making the transition in base $10$.

8) Convert the number $2221121_{(3)}$ in base 9, using groupings of digits and to do the test, converting the number from base 9 to base 3, then to base 10 for verification.

9) Write the number 1000 as the sum of distinct powers of 3.

10) Show that the natural numbers exist $a, b, c, d$ for which $2^a + 2^b + 2^c + 2^d = 149$.

**Bibliography: The collection of *Romanian Mathematical Magazine.***

# ABOUT THE PROBLEM 12303-AMM

***By Marius Drăgan and Neculai Stanciu***

**ABSTRACT:** This paper presents two refinements of an inequality proposed in The American Mathematical Monthly.

**Keywords:** geometric inequality.
**MSC:** 51M16, 26D05

**In The American Mathematical Monthly (AMM), Vol. 129, Nr. 2, February, 2022, was proposed the following problem:**

**12303.** ***Proposed by George Apostolopoulos, Messolonghi, Greece*. Let $R$ and $r$ be the circumradius and inradius, respectively, of triangle $ABC$. Let $D, E$, and $F$ be chosen on sides $BC$, $CA$, and $AB$ so that $AD$, $BE$, and $CF$ bisect the angles of $ABC$. Prove**

$$\frac{FD}{AB+BC}+\frac{DE}{BC+CA}+\frac{EF}{CA+AB}\le\frac{3}{8}\left(1+\frac{R}{2r}\right).$$

Our purpose is to present two reinforcements of the above inequality.

**I.** From bisector theorem we have $\frac{BD}{DC}=\frac{c}{b}$, so $BD=\frac{ac}{b+c}$. From cosine law we deduce that

$$FD=\sqrt{BF^2+BD^2-2BF\cdot BD\cdot\cos B}=\sqrt{\left(\frac{ac}{a+b}\right)^2+\left(\frac{ac}{b+c}\right)^2-\frac{2a^2c^2}{(a+b)(b+c)}\cdot\frac{a^2+c^2-b^2}{2ac}}=$$

$$=\sqrt{\frac{abc\cdot(-a^3+b^3-c^3-a^2b+a^2c+ab^2+ac^2+b^2c-bc^2+3abc)}{(a+b)^2(b+c)^2}}.$$

Since $\sqrt{x}+\sqrt{y}+\sqrt{z}\le\sqrt{3(x+y+z)}$, $\forall x,y,z>0$ we get

$$\sum_{cyc}\frac{FD}{AB+BC}=\frac{\sqrt{abc}}{\prod_{cyc}(a+b)}\sum_{cyc}\sqrt{-a^3+b^3-c^3-a^2b+a^2c+ab^2+ac^2+b^2c-bc^2+3abc}\le$$

$$\le\frac{\sqrt{abc}}{\prod_{cyc}(a+b)}\sqrt{3(-a^3-b^3-c^3+a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+9abc}=$$

$$=\frac{\sqrt{abc}}{\prod_{cyc}(a+b)}\sqrt{3(-\sum_{cyc}a^3+\prod_{cyc}(a+b)+7abc)}$$. We denote $2s=a+b+c$.

Since, $\sum_{cyc}ab=s^2+r^2+4Rr$, $\prod_{cyc}(a+b)=\prod_{cyc}(2s-c)=8s^3-2s\cdot 4s^2+\sum_{cyc}ab\cdot 2s-abc=$

$=\sum_{cyc}ab\cdot 2s-4Rrs=2s(s^2+r^2+4Rr-2Rr)=2s(s^2+2Rr+r^2)$ and

$\sum_{cyc}a^3=2s(s^2-3r^2-6Rr)$, then by the last inequality we get:

$$\sum_{cyc}\frac{FD}{AB+BC}\le\frac{\sqrt{4Rrs}}{2s(s^2+2Rr+r^2)}\sqrt{3[-2s(s^2-6Rr-3r^2)+2s(s^2+2Rr+r^2)+28Rrs]}=$$

$=\frac{\sqrt{12Rr^2(11R+2r)}}{s^2+2Rr+r^2}$. Using Gerretsen inequality, i.e. $s^2\ge 16Rr-5r^2$ we obtain:

$$\sum_{cyc}\frac{FD}{AB+BC}\le\frac{\sqrt{12Rr^2(11R+2r)}}{18Rr-4r^2}=\frac{\sqrt{3R(11R+2r)}}{9R-2r}.$$

We will prove that the inequality from above improves the inequality from the problem 12303. Indeed, if we denote $x=R/r$, $x\ge 2$ we have successively that

$$\frac{\sqrt{3R(11R+2r)}}{9R-2r}\le\frac{3}{8}\left(1+\frac{R}{2r}\right)\Leftrightarrow\frac{\sqrt{3x(11x+2)}}{9x-2}\le\frac{3}{8}\left(1+\frac{x}{2}\right)\Leftrightarrow$$

$\Leftrightarrow 3\cdot 256\cdot(11x+2)\le 9(x+2)^2(9x-2)^2\Leftrightarrow 3(x-2)(243x^3+1350x^2+436x-24)\ge 0$, true.

Hence, we obtained the following strengthening of the inequality from AMM:

$$\sum_{cyc}\frac{FD}{AB+BC}\le\frac{\sqrt{12Rr^2(11R+2r)}}{s^2+2Rr+r^2}\le\frac{\sqrt{3R(11R+2r)}}{9R-2r}\le\frac{3}{8}\left(1+\frac{R}{2r}\right),\ (*).$$

**II.** Next we will get another reinforcement of inequality from the AMM problem.

Let $s_1=\sqrt{2R^2+10Rr-r^2-2\sqrt{R(R-2r)^3}}$. By Blundon theorem we know that $s_1\le s$, so

$$\sum_{cyc}\frac{FD}{AB+BC}\le\frac{\sqrt{12Rr^2(11R+2r)}}{s^2+2Rr+r^2}\le\frac{\sqrt{12Rr^2(11R+2r)}}{2R^2+12Rr-2\sqrt{R(R-2r)^3}}.$$

Now, we shall prove that

$$\frac{\sqrt{12Rr^2(11R+2r)}}{2R^2+12Rr-2\sqrt{R(R-2r)^3}}\le\frac{3}{4}\Leftrightarrow\frac{\sqrt{12x(11x+2)}}{2x^2+12x-2\sqrt{x(x-2)^3}}\le\frac{3}{4}\Leftrightarrow$$

$\Leftrightarrow 16\cdot 12x(11x+2)\le 9\left(2x^2+12x-2\sqrt{x(x-2)^3}\right)^2$, or after some algebra equivalent to

$3x(x-2)\left(3x^2+15x+14-3(x+6)\sqrt{x(x-2)}\right)\ge 0$, $\forall x\ge 2$, which is true since

$$(3x^2+15x+14)^2-9x(x-2)(x+6)^2\ge 0,\forall x\ge 2\Leftrightarrow 201x^2+1068x+196\ge 0,\forall x\ge 2.$$

Therefore, we obtain the following refinement

$$\sum_{cyc}\frac{FD}{AB+BC}\le\frac{\sqrt{12Rr^2(11R+2r)}}{s^2+2Rr+r^2}\le\frac{\sqrt{12Rr^2(11R+2r)}}{2R^2+12Rr-2\sqrt{R(R-2r)^3}}\le\frac{3}{4}\le\frac{3}{8}\left(1+\frac{R}{2r}\right),\ (**).$$

## PRODUCT OF TWO NUMBERS IN MATRIX (TABLE)

***By Carmen – Victorița Chirfot-Romania***

In this article I will present the product of two unstandardized numbers, using a different method of multiplication. Suppose we have two numbers $A=\overline{a_1a_2\dots a_m}$ and $B=\overline{b_1b_2\dots b_n}, m,n\in\mathbb{N}$, $m,n\ge 1$. One method of calculating this product is to distribute the digits of number $A$ as the head of the column and of number $B$ on one line (we put the digits of number $B$ on the lines in reverse order, descending to the left). Suppose we want to calculate $A\cdot B$, where $A=123456$ and $B=57$. We will distribute the digits in an array, making the appropriate traversals.

| | | | **1** | **2** | **3** | **5** | **6** |
|---|---|---|---|---|---|---|---|
| | **7** | | $^{0}$ 7 | $^{1}$ 4 | $^{2}$ 1 | $^{3}$ 5 | $^{4}$ 2 |
| **5** | | $^{0}$ 5 | $^{1}$ 0 | $^{1}$ 5 | $^{2}$ 5 | $^{3}$ 0 | |
| **5** | **7** | $^{0}$ 5 | $^{1}$ 7 | $^{2}$ 9 | $^{4}$ 6 | $^{6}$ 5 | $^{4}$ 2 |
| **Sum** | | $^{0}$ 5 | $^{1}$ 7 | $^{2}$ 9 | $^{4}$ 6 | $^{6}$ 9 | $^{0}$ 2 |
| **The overorder sum** | | $^{0}$ 5 | $^{1}$ 7 | $^{2}$ 9 | $^{5}$ 2 | $^{0}$ 9 | $^{0}$ 2 |
| **The overorder sum** | | $^{0}$ 5 | $^{1}$ 7 | $^{3}$ 4 | $^{0}$ 2 | $^{0}$ 9 | $^{0}$ 2 |
| **The overorder sum** | | $^{0}$ 5 | $^{2}$ 0 | $^{0}$ 4 | $^{0}$ 2 | $^{0}$ 9 | $^{0}$ 2 |
| **Product** | | $^{0}$ 7 | $^{0}$ 0 | $^{0}$ 4 | $^{0}$ 2 | $^{0}$ 9 | $^{0}$ 2 |

Using matrix notation $A \cdot B = 123456 \cdot 57 = 704292$. To calculate this product, we will make the partial products $c_i c_j$, in the form ${}^{z}u$ , moved to the left by one position (depending on the order of the second digit with which we do the multiplication), $z$ being the tens digit, and $u$ the units digit, for the product between one digit $c_i$ of one number and one digit $c_j$ of the other number. On the line labeled Sum, we'll do the appropriate additions one column at a time (that is, the sum of the array's column elements, without using the header with the digits of the first number). Unit to unit and tens digit to tens digit will be added. On the lines Sum with overorder, we will correct the overorders, so that, if they exist, we will add them to the units digit of the number in the matrix cell, changing the content of the cell. If we need more calculation time, we will complete with other lines labeled Amount with overorder (for particular cases).

For example, for the adjacent cells in the table, we will perform $5 +^{4} = 9$ , we will pass the 9 to the units below the cell corresponding to the addition operation, and because we do not have tens, we will not change the 6 from the tens. We will proceed in this way until we get the product of the two numbers on the Product line. So $123456 \cdot 57 = 704292$ . However we do the additions, if we calculate correctly, the product is the same. We will stop the algorithm when the tens are zero, that is, we have no overorder.

| | |
|---|---|
| ${}^{6}\ 5$ | ${}^{4}\ 2$ |
| ${}^{6}\ 9$ | ${}^{0}\ 2$ |

For generalization, we will consider the following table, which we will analyze later.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $a_1$ | $a_2$ | $a_3$ | $\cdots$ | $a_{m-1}$ | $a_m$ |
| | | $b_n$ | | | | $b_n a_1$ | $b_n a_2$ | $b_n a_3$ | $\cdots$ | $b_n a_{m-1}$ | $b_n a_m$ |
| | | | | | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | |
| | $b_2$ | | | $b_2 a_1$ | $b_2 a_2$ | $b_2 a_3$ | $b_2 a_4$ | $b_2 a_5$ | $\cdots$ | | |
| $b_1$ | | | $b_1 a_1$ | $b_1 a_2$ | $b_1 a_3$ | $b_1 a_4$ | $b_1 a_5$ | $b_1 a_6$ | | | |
| $b_1$ | $b_2$ | $b_n$ | $\sum_{\substack{i=\overline{1,n} \\ j=\overline{1,1}}} b_i a_j$ | $\sum_{\substack{i=\overline{1,n} \\ j=\overline{1,2}}} b_i a_j$ | $\sum_{\substack{i=\overline{1,n} \\ j=\overline{1,3}}} b_i a_j$ | $\sum_{\substack{i=\overline{1,n} \\ j=\overline{1,4}}} b_i a_j$ | $\sum_{\substack{i=\overline{1,n} \\ j=\overline{2,5}}} b_i a_j$ | $\sum_{\substack{i=\overline{1,n} \\ j=\overline{3,5}}} b_i a_j$ | $\cdots$ | $\sum_{\substack{i=\overline{1,n} \\ j=\overline{m-1,m}}} b_i a_j$ | $\sum_{\substack{i=\overline{1,n} \\ j=\overline{m,m}}} b_i a_j$ |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **The overorder sum** | $z=\left[\frac{\sum_{\substack{i=\overline{1,n} \\ j\,variable}} b_i a_j}{10}\right]$ $u=\sum_{\substack{i=\overline{1,n} \\ j\,variable}} b_i a_j-10\cdot\left[\frac{\sum_{\substack{i=\overline{1,n} \\ j\,variable}} b_i a_j}{10}\right]$ | | | | | | | | |
| **The product** | | | | | | | | | |

On the line $b_i$, we will write the partial products $b_i a_j$, with $1 \le i \le n, 1 \le j \le m$, using the notation ${}^{z}u$. On the row $(b_1, b_2, b_3, \dots, b_n)$ and column $a_i$, we will calculate the sum corresponding to the numbers on each column $a_i$ or unlabeled (when we have no more digits in number $A$). On line $(b_1, b_2, b_3, \dots, b_n)$ , we will have all partial sums calculated.

But, $10k \le \sum_{i=\overline{1.n}} b_i a_i < 10(k+1), k \in \mathbb{N}, k$ digits

$$\Rightarrow k \le \frac{\sum_{\substack{i=\overline{1,n} \\ j\ variable}} b_i a_j}{10} < k+1 \Rightarrow k=\left[\frac{\sum_{\substack{i=\overline{1,n} \\ j\ variable}} b_i a_j}{10}\right]$$

The number $\left[\frac{\sum_{\substack{i=\overline{1,n} \\ j\,variable}} b_i a_j}{10}\right]$ is the transition (the number of tens) over the units order of the number $\sum_{\substack{i=\overline{1,n} \\ j\ variable}} b_i a_j$. To bring the number to the order of units, we subtract the tens of the number, i.e

$$\sum_{\substack{i=\overline{1,n} \\ j\ variable}} b_i a_j - 10 \cdot \left[\frac{\sum_{\substack{i=\overline{1,n} \\ j\ variable}} b_i a_j}{10}\right]$$

In this way

$$0 \le \sum_{\substack{i=\overline{1,n} \\ j\ variable}} b_i a_j - 10 \cdot \left[\frac{\sum_{\substack{i=\overline{1,n} \\ j\ variable}} b_i a_j}{10}\right] < 10$$

and the number was converted to unity. After calculating the partial sums with crossing over the order, we will find the product of the two numbers.

Note: The calculation of the partial sums with crossing the order can be done from right to left or from left to right, no matter how we proceed we will get the product of the two numbers.

Applications:

1) Calculate the product in the table $2367 \cdot 1234$.

2) Calculate the product in the table $1111 \cdot 11111$. What is observed?

3) Calculate in the table $156{,}23 \cdot 23{,}156$.

4) Calculate in the table $25^6$.

5) Find the digits $a$ and $b$ knowing that it respects the given table.

Find the two factors that generate such a product.

| | | | | |
|---|---|---|---|---|
| | | $^0$ $b$ | $^2$ $a$ | $^1$ 4 |
| *Product* | 1 | 7 | 0 | 4 |

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## CONNECTIONS BETWEEN FAMOUS CEVIANS-I

***By Bogdan Fuștei-Romania***

We consider the triangle ABC with notations:

$p_a, p_b, p_c$-Spieker's cevians, $n_a, n_b, n_c$-Nagel's cevians, $g_a, g_b, g_c$-Gergonne's cevians

It is known that:

$$\mathbf{n_a m_a \geq p_a^2} \textbf{ (and analogs) [1]}$$

$$\mathbf{n_a g_a \geq m_a l_a} \textbf{(and analogs) [2]}$$

We will find an interesting connection between $n_a, g_a, p_a, l_a$:

From those relations we obtain a new one:

$$\mathbf{n_a\sqrt{g_a} \geq p_a\sqrt{l_a}} \textbf{ (and analogs) (1)}$$

(1) Is a refinement for $n_a \geq p_a$ because $l_a \geq g_a$.

$$\frac{a}{2r}=\frac{n_a}{h_a}+2\frac{r_a}{n_a+p} \text{ (and analogs) [3],}$$

and using (1) we obtain:

$$\frac{a}{2r}\geq\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}+2\frac{r_a}{n_a+p} \text{ (and analogs)}(2)$$

From $(2)$ after summation:

$$\frac{p}{r}\geq\sum\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}+2\sum\frac{r_a}{n_a+p} \text{ (3)}$$

From (1):

$$\sum n_a\sqrt{g_a}\geq\sum p_a\sqrt{l_a} \text{ (4)}$$

$$\sum\frac{n_a}{p_a}\geq\sum\sqrt{\frac{l_a}{g_a}} \text{ (5)}$$

$$n_a+n_b+n_c\geq p_a\sqrt{\frac{l_a}{g_a}}+p_b\sqrt{\frac{l_b}{g_b}}+p_c\sqrt{\frac{l_c}{g_c}} \text{ (6)}$$

a=2Rsin A (Sine Theorem) $\rightarrow\frac{a}{2r}=\frac{R}{r}\sin A$ (and analogs)

$$\frac{R}{r}\geq\left(\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}+2\frac{r_a}{n_a+p}\right)\frac{1}{\sin A} \text{ (and analogs) (7)}$$

From $\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogs) $[4]\rightarrow 2\frac{r_a}{n_a+p}=\frac{a-\sqrt{4r^2+(b-c)^2}}{2r}$(and analogs)

$$\rightarrow\left(p_a\sqrt{\frac{l_a}{g_a}}+p\right)\left(a-\sqrt{4r^2+(b-c)^2}\right)\leq 4rr_a \text{ (and analogs) (8)}$$

From $4rr_a=4(p-b)(p-c)$(and analogs) and (8) $\rightarrow$

$$\left(p_a\sqrt{\frac{l_a}{g_a}}+p\right)\left(a-\sqrt{4r^2+(b-c)^2}\right)\leq 4(p-b)(p-c) \text{ (9)}$$

From (8) and $r_a+r_b+r_c=4R+r$ after summation we obtain:

$$\sum\left(p_a\sqrt{\frac{l_a}{g_a}}+p\right)\left(a-\sqrt{4r^2+(b-c)^2}\right)\leq 4r(r_a+r_b+r_c)=4r(4R+r)(10)$$

From $\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}=\frac{h_a}{r}$ (and analogs)[4]; $\frac{h_a}{r}=1+\frac{b+c}{a}$(and analogs) and (1):

$$\frac{h_a}{r}\geq\frac{2p_a}{\sqrt{4r^2+(b-c)^2}}\sqrt{\frac{l_a}{g_a}} \text{ (and analogs)}(11)$$

From (11) after summation:

$$\frac{h_a+h_b+h_c}{2r} \geq \sum \frac{p_a}{\sqrt{4r^2+(b-c)^2}}\sqrt{\frac{l_a}{g_a}} \ (12)$$

From (11) →

$$\frac{b+c}{a} \geq \frac{2p_a}{\sqrt{4r^2+(b-c)^2}}\sqrt{\frac{l_a}{g_a}} - 1 \text{ (and analogs) } (13)$$

From $l_a=\frac{2\sqrt{bc}}{b+c}\sqrt{r_b r_c}$ (and analogs) → $l_a l_b l_c=\frac{8abc}{(a+b)(b+c)(c+a)} r_a r_b r_c$

$$8\frac{r_a r_b r_c}{l_a l_b l_c}=\frac{(a+b)(b+c)(c+a)}{abc} \geq \prod\left(\frac{2p_a}{\sqrt{4r^2+(b-c)^2}}\sqrt{\frac{l_a}{g_a}} - 1\right)(14)$$

We know: $\frac{R}{r}-1=\frac{n_a^2+r_a^2}{2h_a r_a}$ (and analogs)[5] we obtain:

$2r_a h_a\left(\frac{R}{r}-1\right)=n_a^2+r_a^2$ (and analogs) →$2\frac{h_a}{n_a}\left(\frac{R}{r}-1\right)=\frac{n_a}{r_a}+\frac{r_a}{n_a}$ (and analogs) using (1) we obtain:

$$2\left(\frac{R}{r}-1\right) \geq \frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)(15)$$

From: $\frac{R}{r} \geq 1+\frac{n_a}{h_a}$(and analogs) →

$$\frac{R}{r} \geq 1+\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}} \text{ (and analogs) } (16)$$

From $\frac{R}{r} \geq 1+\frac{n_a}{h_a}$(and analogs) →$\left(\frac{R}{r}-1\right)^3 \geq \frac{n_a n_b n_c}{h_a h_b h_c}$ → $\frac{R}{r} \geq 1+\sqrt[3]{\frac{n_a n_b n_c}{h_a h_b h_c}}$ we obtain: $\frac{R}{r} \geq$

$$1+\sqrt[3]{\frac{p_a}{h_a}\frac{p_b}{h_b}\frac{p_c}{h_c}\sqrt{\frac{l_a}{g_a}\frac{l_b}{g_b}\frac{l_c}{g_c}}} \ (17)$$

From $p^2=n_a^2+2h_a r_a$ (and analogs) →$p^2-n_a^2=2h_a r_a$

$(p-n_a)(p+n_a)=2h_a r_a$ , and $\frac{p}{h_a}=\frac{a}{2r}$ (and analogs) →$\frac{a}{2r}+\frac{n_a}{h_a}=\frac{2r_a}{p-n_a}$ (and analogs). Using $\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogs) and $\frac{a}{2r}+\frac{n_a}{h_a}=\frac{2r_a}{p-n_a}$ we obtain:

$$\frac{2r_a}{p-n_a}=\frac{a+\sqrt{4r^2+(b-c)^2}}{2r} \text{ (and analogs)}$$

$$\frac{4r_a r}{p-n_a}= a+\sqrt{4r^2+(b-c)^2} \text{ (and analogs)}$$

$$p= n_a+\frac{4r_a r}{a+\sqrt{4r^2+(b-c)^2}} \text{ (and analogs)(*)}$$

$$p= n_a+\frac{4(p-b)(p-c)}{a+\sqrt{4r^2+(b-c)^2}} \text{ (and analogs) (**)}$$

From those we obtain:

$$\mathbf{p} \geq \mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \frac{\mathbf{4(p-b)(p-c)}}{\mathbf{a+\sqrt{4r^2+(b-c)^2}}} \ \textbf{(and analogs)(18)}$$

From (18) after summation we obtain:

$$\mathbf{3p} \geq \sum\left(\mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \frac{\mathbf{4(p-b)(p-c)}}{\mathbf{a+\sqrt{4r^2+(b-c)^2}}}\right) \ \textbf{(19)}$$

From $\frac{a}{2r} + \frac{n_a}{h_a} = \frac{2r_a}{p-n_a}$ (and analogs) and (1) we obtain:

$$\frac{\mathbf{2r_a}}{\mathbf{p-n_a}} \geq \frac{\mathbf{p_a}}{\mathbf{h_a}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \frac{\mathbf{a}}{\mathbf{2r}} \textbf{(and analogs)(20)}$$

From (20) after summation we obtain:

$$\sum\frac{\mathbf{2r_a}}{\mathbf{p-n_a}} \geq \frac{\mathbf{p}}{\mathbf{r}} + \sum\frac{\mathbf{p_a}}{\mathbf{h_a}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} \ \textbf{(21)}$$

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## CONNECTIONS BETWEEN FAMOUS CEVIANS-II

***By Bogdan Fuștei-Romania***

We consider triangle ABC with notations:

$p_a, p_b, p_c$-Spieker's cevians, $n_a, n_b, n_c$-Nagel's cevians, $g_a, g_b, g_c$-Gergonne's cevians

1.) $n_a m_a \geq p_a^2$ (and analogs) [1]

2.) $n_a g_a \geq m_a l_a$(and analogs) [2]

3. ) $m_a^2 \geq l_a p_a$(and analogs) [3]

From 1) and 3) we obtain:

$$\mathbf{m_a^3 n_a \geq p_a^3 l_a}\textbf{(and analogs) (1)}$$

We obtain:

$$\sqrt[3]{\frac{\mathbf{n_a}}{\mathbf{l_a}}} \geq \frac{\mathbf{p_a}}{\mathbf{m_a}}\textbf{ (and analogs) (2)}$$

From (2) after summation we obtain:

$$\sum \mathbf{m_a}\sqrt[3]{\frac{\mathbf{n_a}}{\mathbf{l_a}}} \geq \mathbf{p_a+p_b+p_c}\textbf{ (3)}$$

$$\sum \sqrt[3]{\frac{\mathbf{n_a}}{\mathbf{l_a}}} \geq \sum \frac{\mathbf{p_a}}{\mathbf{m_a}}\textbf{ (4)}$$

$$\sum \mathbf{m_a^3 n_a} \geq \sum \mathbf{p_a^3 l_a}\textbf{ (5)}$$

From 2) and 3) →

$$\mathbf{n_a g_a m_a \geq l_a^2 p_a}\textbf{ (and analogs) (6)}$$

From (6) after summation we obtain:

$$\sum \mathbf{n_a g_a m_a} \geq \sum \mathbf{l_a^2 p_a}\textbf{ (7)}$$

$$\sum \sqrt{\mathbf{n_a g_a m_a}} \geq \sum \mathbf{l_a}\sqrt{\mathbf{p_a}}\textbf{ (8)}$$

From (6) and $l_a = 2\frac{\sqrt{bc}}{b+c}\sqrt{r_b r_c}$ (and analogs)

$$\sqrt{\frac{\mathbf{b}}{\mathbf{c}}}+\sqrt{\frac{\mathbf{c}}{\mathbf{b}}} \geq \mathbf{2}\sqrt{\frac{\mathbf{p_a r_b r_c}}{\mathbf{n_a g_a m_a}}}\textbf{ (and analogs) (9)}$$

From: a $h_a = b\, h_b = c\, h_c$ and (9) we obtain:

$$\sqrt{\frac{\mathbf{h_b}}{\mathbf{h_c}}}+\sqrt{\frac{\mathbf{h_c}}{\mathbf{h_b}}} \geq \mathbf{2}\sqrt{\frac{\mathbf{p_a r_b r_c}}{\mathbf{n_a g_a m_a}}}\textbf{ (and analogs) (10)}$$

From (9) we obtain:

$$\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}} \geq \mathbf{2}\left(\mathbf{2}\frac{\mathbf{p_a r_b r_c}}{\mathbf{n_a g_a m_a}}-\mathbf{1}\right)\textbf{ (and analogs) (11)}$$

ω=Brocard angle, $\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs)(Traian Lalescu)[4] and using (11):

$$\frac{\mathbf{sin(A+\omega)}}{\mathbf{sin\,\omega}} \geq \mathbf{2}\left(\mathbf{2}\frac{\mathbf{p_a r_b r_c}}{\mathbf{n_a g_a m_a}}-\mathbf{1}\right)\textbf{(and analogs) (12)}$$

$m_a \geq \frac{b^2+c^2}{4R}$ (Tereshin) and bc=2R$h_a$ (and analogs) → $2\frac{m_a}{h_a} \geq \frac{b}{c}+\frac{c}{b}$(and analogs) and (11) we obtain: $2\frac{m_a}{h_a} \geq 2\left(2\frac{p_a r_b r_c}{n_a g_a m_a}-1\right)$

$$\frac{m_a}{h_a} \geq 2\frac{p_a r_b r_c}{n_a g_a m_a} - 1 \text{(and analogs) (13)}$$

We proved: $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) [5] and using (2) we obtain:

$$n_a\sqrt{\frac{g_a}{l_a}} + m_a\sqrt[3]{\frac{n_a}{l_a}} \geq 2p_a \text{ (and analogs)(14)}$$

From (14) after summation we obtain:

$$\sum\left(n_a\sqrt{\frac{g_a}{l_a}} + m_a\sqrt[3]{\frac{n_a}{l_a}}\right) \geq 2(p_a + p_b + p_c) \text{ (15)}$$

From (2) and (6) after summation we obtain:

$$m_a\left(\sqrt[3]{\frac{n_a}{l_a}} + \frac{n_a g_a}{l_a^2}\right) \geq 2p_a \text{(and analogs) (16)}$$

From (2) and $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) we obtain:

$$n_a\left(\sqrt{\frac{g_a}{l_a}} + \frac{g_a m_a}{l_a^2}\right) \geq 2p_a \text{(and analogs) (17)}$$

From (2) and $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$ we obtain:

$$a^2 + b^2 + c^2 \geq \frac{4}{3}\sum p_a^2\sqrt[3]{\left(\frac{l_a}{n_a}\right)^2} \text{(18)}$$

From 1.) and $m_a l_a \geq r_b r_c$(and analogs)(Panaitopol) we obtain:

$$\sqrt{\frac{n_a l_a}{r_b r_c}} \geq \frac{p_a}{m_a} \text{ (and analogs) (19)}$$

From (19) after summation we obtain:

$$\sum\sqrt{\frac{n_a l_a}{r_b r_c}} \geq \sum\frac{p_a}{m_a} \text{(20)}$$

$$\sum m_a\sqrt{\frac{n_a l_a}{r_b r_c}} \geq p_a + p_b + p_c \text{ (21)}$$

From (2) and (19) we obtain:

$$m_a\left(\sqrt[3]{\frac{n_a}{l_a}} + \sqrt{\frac{n_a l_a}{r_b r_c}}\right) \geq 2p_a \text{(and analogs)(22)}$$

From (22) we obtain:

$$\prod\left(\sqrt[3]{\frac{n_a}{l_a}} + \sqrt{\frac{n_a l_a}{r_b r_c}}\right) \geq 8\frac{p_a p_b p_c}{m_a m_b m_c} \text{(23)}$$

From 1.) and $m_a l_a \geq r_b r_c$(and analogs)(Panaitopol) we obtain:

$$\mathbf{m_a(n_a+l_a) \geq p_a^2 + r_b r_c \text{(and analogs)} \ (24)}$$

From (24) after summation we obtain:

$$\sum \mathbf{m_a(n_a+l_a) \geq p_a^2 + p_b^2 + p_c^2 + p^2 (25)}$$

$$\mathbf{m_a + m_b + m_c \geq \sum \frac{p_a^2 + r_b r_c}{n_a + l_a} \ (26)}$$

From (22) after summation we obtain:

$$\sum \frac{m_a}{p_a}\left(\sqrt[3]{\frac{n_a}{l_a}} + \sqrt{\frac{n_a l_a}{r_b r_c}}\right) \geq 6 \ (27)$$

From (6) after summation we obtain:

$$\mathbf{m_a + m_b + m_c \geq \sum \frac{l_a^2 p_a}{n_a g_a} \ (28)}$$

From (26) and (28) we obtain:

$$\mathbf{m_a + m_b + m_c \geq \sqrt{\sum \frac{l_a^2 p_a}{n_a g_a} \sum \frac{p_a^2 + r_b r_c}{n_a + l_a}} \ (29)}$$

From (3) and (21) we obtain:

$$\sqrt{\sum m_a \sqrt[3]{\frac{n_a}{l_a}} \sum m_a \sqrt{\frac{n_a l_a}{r_b r_c}}} \geq p_a + p_b + p_c \ (30)$$

From (6) after summation we obtain:

$$n_a + n_b + n_c \geq \sum \frac{l_a^2 p_a}{g_a m_a} \ (31)$$

From (31) and $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) we obtain:

$$\mathbf{n_a + n_b + n_c \geq \sqrt{\left(p_a\sqrt{\frac{l_a}{g_a}} + p_b\sqrt{\frac{l_b}{g_b}} + p_c\sqrt{\frac{l_c}{g_c}}\right)\sum \frac{l_a^2 p_a}{g_a m_a}} \ (32)}$$

From (6) and $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) after summation we obtain:

$$\mathbf{2n_a \geq p_a\sqrt{\frac{l_a}{g_a}}\left(1 + \frac{l_a}{m_a}\sqrt{\frac{l_a}{g_a}}\right) \text{(and analogs)} \ (33)}$$

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## CONNECTIONS BETWEEN FAMOUS CEVIANS-III

***By Bogdan Fuștei-Romania***

In Δ ABC we have:

$$m_a^2=p(p-a)+\frac{1}{4}(b-c)^2 \text{ (and analogs)}$$

$$l_a^2=p(p-a)-\frac{p(p-a)}{(b+c)^2}(b-c)^2 \text{ (and analogs) [1]}$$

$$p_a^2= p(p-a)+p\frac{(3p+a)}{(2p+a)^2}(b-c)^2 \text{(and analogs) [2]}$$

Will prove that: $l_a^2+p_a^2\le 2m_a^2$(and analogs)

After we simplify with 2p(p-a) we obtain:

$$(b-c)^2\left[p\frac{(3p+a)}{(2p+a)^2}-\frac{p(p-a)}{(b+c)^2}\right]\le\frac{1}{2}(b-c)^2$$

If b=c we obtain equality.

If b≠c we obtain: $p\frac{(3p+a)}{(2p+a)^2}-\frac{p(p-a)}{(b+c)^2}<\frac{1}{2}$

$$p\frac{(3p+a)}{(2p+a)^2}=\frac{1}{4}+\frac{8p^2-a^2}{4(2p+a)^2}\rightarrow\frac{1}{4}+\frac{8p^2-a^2}{4(2p+a)^2}<\frac{1}{2}\rightarrow\frac{8p^2-a^2}{4(2p+a)^2}<\frac{1}{4}+\frac{p(p-a)}{(b+c)^2}$$

$$8p^2-a^2<(2p+a)^2+p(p-a)\left(\frac{4p+2a}{b+c}\right)^2=(2p+a)^2+p(p-a)\left(\frac{2a+2(b+c)+2a}{b+c}\right)^2$$

$$8p^2-a^2<(2p+a)^2+p(p-a)\left(2+\frac{4a}{b+c}\right)^2\rightarrow\text{TRUE !!!}$$

We obtain:

$$\mathbf{l_a^2+p_a^2\le 2m_a^2\text{(and analogs) (1)}}$$

$l_a^2 + p_a^2 \geq 2p_a l_a$ (and analogs) →

$$\mathbf{m_a^2 \geq p_a l_a (and\ analogs)(2)}$$

From (1) and (2) after summation we obtain:

$$\mathbf{2m_a \geq p_a + l_a \text{ (and analogs) (3)}}$$

From (1) and $4m_a^2 = n_a^2 + g_a^2 + 2r_b r_c$ (and analogs) [3]

$$\mathbf{n_a^2 + g_a^2 \geq 2(\ l_a^2 + p_a^2 - r_b r_c)(and\ analogs) \text{ (4)}}$$

From $(b-c)^2 = n_a^2 + g_a^2 - 2r_b r_c$ (and analogs) [3]

We obtain:

$$\mathbf{(b-c)^2 \geq 2(l_a^2 + p_a^2 - 2r_b r_c)(and\ analogs)(5)}$$

From (5) and $l_a^2 + p_a^2 \geq 2p_a l_a$ (and analogs) we obtain:

$$\mathbf{(b-c)^2 \geq 4(p_a l_a - r_b r_c)(and\ analogs)(6)}$$

From $p_a \geq m_a$(and analogs)

$m_a l_a \geq r_b r_c$(and analogs) (Panaitopol)

we obtain:

$$\mathbf{(b-c)^2 \geq 2(l_a^2 + p_a^2 - 2m_a l_a) \text{ (and analogs) (7)}}$$

$$\mathbf{|b\text{-}c| \geq \sqrt{2}\,(p_a - l_a) \text{ (and analogs) (8)}}$$

We know that $n_a + g_a \geq 2m_a$ (and analogs) [4] and using (3) we obtain:

$$\mathbf{n_a + g_a \geq 2m_a \geq p_a + l_a \text{(and analogs) (9)}}$$

From $|b\text{-}c| \geq n_a - g_a$ (and analogs) [4] and (8) after summation we obtain:

$$\mathbf{2|\ b\text{-}c| \geq n_a + \sqrt{2}\,(p_a - l_a) - g_a \text{(and analogs) (10)}}$$

We know that $2\sum\ |b-c| = 4[\max(a,b,c)\ - \min(a,b,c)]$ and using (10) we obtain:

$$\mathbf{\max(a,b,c)\ - \min(a,b,c)\ \geq \frac{1}{4}\sum\left(n_a + \sqrt{2}\,(p_a - l_a)\ - g_a\right)(11)}$$

From $\sum \frac{m_a^2}{h_a^2} = 1 + \frac{1}{2\sin^2\omega}$ and (1) we obtain:

$$\mathbf{1 + \frac{1}{2\sin^2\omega} \geq \frac{1}{2}\sum \frac{l_a^2 + p_a^2}{h_a^2} \text{ (12)}}$$

$$2 \geq \sum \frac{l_a^2 + p_a^2}{h_a^2} - \frac{1}{\sin^2\omega}$$

From $2m_a = \sqrt{2(b^2 + c^2) - a^2}$ (and analogs) and (3)

$$\sqrt{2(b^2+c^2)-a^2} \geq p_a + l_a \text{ (and analogs) (13)}$$

From $2\frac{m_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R} \leq \frac{b}{c}+\frac{c}{b}$ (and analogs) [5] and (3) we obtain

$$\frac{b}{c}+\frac{c}{b} \geq \frac{p_a+l_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R} \text{ (and analogs) (14)}$$

$\frac{\sin(A+\omega)}{\sin\omega} = \frac{b}{c}+\frac{c}{b}$ (and analogs) (Traian Lalescu) , $\omega =$ Brocard angle in triangle ABC

$$\frac{\sin(A+\omega)}{\sin\omega} \geq \frac{p_a+l_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R} \text{ (and analogs)(15)}$$

From (15) we obtain:

$$\frac{1}{\sin\omega} \geq \frac{p_a+l_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R} \text{ (and analogs)(16)}$$

From (10) and $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) [6] we obtain:

$$2|b-c| \geq p_a\sqrt{\frac{l_a}{g_a}} + \sqrt{2}(p_a - l_a) - g_a \text{ (and analogs) (17)}$$

From (1) and $n_a g_a \geq m_a l_a$ (and analogs) [1] we obtain:

$$n_a \geq \frac{l_a}{g_a}\sqrt{\frac{l_a^2+p_a^2}{2}} \text{ (and analogs) (18)}$$

From (18) after summation we obtain:

$$n_a + n_b + n_c \geq \sum \frac{l_a}{g_a}\sqrt{\frac{l_a^2+p_a^2}{2}} \text{ (19)}$$

From (1) and $m_a n_a \geq p_a^2$ (and analogs) [7] we obtain:

$$2n_a \geq \frac{(l_a^2+p_a^2)}{m_a^3}p_a^2 \text{ (and analogs)(20)}$$

From (20) after summation we obtain:

$$2(n_a+n_b+n_c) \geq \sum \frac{(l_a^2+p_a^2)}{m_a^3}p_a^2 \text{(21)}$$

From (20) we obtain:

$$m_a \geq \sqrt[3]{(l_a^2+p_a^2)\frac{p_a^2}{2n_a}} \text{ (22)}$$

After summation from (22) we obtain:

$$m_a+m_b+m_c \geq \sum \sqrt[3]{(l_a^2+p_a^2)\frac{p_a^2}{2n_a}} \text{ (23)}$$

From (22) we obtain: $m_a^2 \geq \sqrt[3]{\left((l_a^2+p_a^2)\ \frac{p_a^2}{2n_a}\right)^2}$

$m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$ and after summation we obtain:

$$\boldsymbol{a^2+b^2+c^2 \geq \frac{4}{3}\sum\sqrt[3]{\left((l_a^2+p_a^2)\ \frac{p_a^2}{2n_a}\right)^2}}\ \textbf{(24)}$$

From (20) we obtain $2n_a\frac{m_a^3}{p_a^3} \geq p_a+\frac{l_a^2}{p_a}$ (and analogs) and after summation we obtain:

$$\mathbf{2\sum n_a\frac{m_a^3}{p_a^3} \geq p_a+p_b+p_c+\sum\frac{l_a^2}{p_a}}\ \textbf{(25)}$$

From (25) and using Bergstrom inequality we obtain:

$$\mathbf{2\sum n_a\frac{m_a^3}{p_a^3} \geq p_a+p_b+p_c+\frac{(l_a+l_b+l_c)^2}{p_a+p_b+p_c}}\ \textbf{(26)}$$

From $2n_a\frac{m_a^3}{p_a^3} \geq p_a+\frac{l_a^2}{p_a}$ (and analogs) we obtain:

$$\mathbf{2\frac{n_a p_a}{l_a^2+p_a^2} \geq \frac{p_a^3}{m_a^3}}\ \textbf{(and analogs) (27)}$$

From (27) after summation we obtain:

$$\mathbf{2\sum\frac{n_a p_a}{l_a^2+p_a^2} \geq \sum\frac{p_a^3}{m_a^3}}\ \textbf{(28)}$$

$$\mathbf{\sum\sqrt[3]{2\frac{n_a p_a}{l_a^2+p_a^2}} \geq \frac{p_a}{m_a}+\frac{p_b}{m_b}+\frac{p_c}{m_c}}\ \textbf{(29)}$$

$$\mathbf{\sum m_a\sqrt[3]{2\frac{n_a p_a}{l_a^2+p_a^2}} \geq p_a+p_b+p_c}\ (30)$$

From (27) and $p_a \geq m_a$ we obtain:

$$\mathbf{2n_a p_a \geq l_a^2+p_a^2}\ \textbf{(and analogs) (31)}$$

$p_a(2n_a-p_a) \geq l_a^2$ (and analogs) and after summation we obtain:

$$\mathbf{\sum\sqrt{p_a(2n_a-p_a)} \geq l_a+l_b+l_c}\ \textbf{(32)}$$

From $p_a(2n_a-p_a) \geq l_a^2$ (and analogs) and Bergstrom inequality we obtain:

$$\mathbf{p_a+p_b+p_c \geq \frac{(l_a+l_b+l_c)^2}{2(n_a+n_b+n_c)-(p_a+p_b+p_c)}}\ \textbf{(33)}$$

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## FUNDAMENTAL SYMMETRIC POLYNOMIALS IN ALGEBRAIC INEQUALITIES-I

***By Marius Drăgan and Neculai Stanciu-Romania***

**Definition: Polinomyal** $p \in R[x, y, z]$ **it is symmetric in the variables** $x, y, z$ **if for any permutation** $\sigma$ **of the variables** $x, y, z$ **we have:** $\sigma(p) = p$ **, i.e. the polynomial** $p$ **satisfy** $p(x, y, z) = p(x, z, y) = p(y, x, z) = p(y, z, x) = p(z, x, y) = p(z, y, x)$ **.**
**Polynomials:** $\sigma_1 \in R[x, y, z]$ **,** $\sigma_1 = x + y + z$ **;** $\sigma_2 \in R[x, y, z]$ **,** $\sigma_2 = xy + yz + zx$ **;** $\sigma_3 \in R[x, y, z]$ **,** $\sigma_3 = xyz$ **they are called fundamental symmetric polynomials.**

**Lemma. Polynomials** $S_k = x^k + y^k + z^k$ **can be expressed using polynomials** $\sigma_1, \sigma_2, \sigma_3$ **.**

***Proof.*** We have that:

$$S_k = \sigma_1 S_{k-1} - \sigma_2 S_{k-2} + \sigma_3 S_{k-3},\ \forall k \geq 3\,(S_0 = 3, S_1 = \sigma_1, S_2 = \sigma_2),\ (1).$$

Indeed, substituting $S_{k-3}, S_{k-2}, S_{k-1}, \sigma_1, \sigma_2, \sigma_3$ through their expressions we obtain

$$(x + y + z)(x^{k-1} + y^{k-1} + z^{k-1}) - (xy + yz + zx)(x^{k-2} + y^{k-2} + z^{k-2}) +$$

$$+ xyz(x^{k-3} + y^{k-3} + z^{k-3}) = x^k + y^k + z^k.$$

**Theorem 1. Any symmetric polynomial in three variables** $x, y, z$ **can be expressed uniquely with the help of fundamental symmetric polynomials.Waring's formula is deduced from formula (1):**

$$\frac{S_k}{k} = \sum \frac{(-1)^{k-\lambda_1-\lambda_2-\lambda_3}(\lambda_1 + \lambda_2 + \lambda_3 - 1)!\,\sigma_1^{\lambda_1}\sigma_2^{\lambda_2}\sigma_3^{\lambda_3}}{\lambda_1!\lambda_2!\lambda_3!}$$ **, unde** $\lambda_1 + 2\lambda_2 + 3\lambda_3 = k$ **, (2).**

**By (2), we obtain** $S_k\ (k = \overline{3,6}$ **):**

**1.** $S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$ **;**

**2.** $S_4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3$;

**3.** $S_5 = \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 + 5\sigma_1^2\sigma_3 - 5\sigma_2\sigma_3$;

**4.** $S_6 = \sigma_1^6 - 6\sigma_1^4\sigma_2 + 9\sigma_1^2\sigma_2^2 - 2\sigma_2^3 + 6\sigma_1^3\sigma_3 - 12\sigma_1\sigma_2\sigma_3 + 3\sigma_3^2$.

**Example.** If $p \in R[x,y,z]$, $p = x^n(y^m + z^m) + y^n(x^m + z^m) + z^n(x^m + y^m)$, $m,n \in N$, then

$$p = x^n(y^m + z^m + x^m) - x^{m+n} + y^n(x^m + y^m + z^m) - y^{m+n} + z^n(x^m + y^m + z^m) - z^{m+n} =$$

$$= S_m S_n - S_{m+n}.$$

**Theorem 2. If** $x>0, y>0, z>0$**, then** $\sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0$.

**If** $x,y,z \in R$ **and** $\sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0$**, then** $x>0, y>0, z>0$.

***Proof.*** By equation $u: u^3 - \sigma_1 u^2 + \sigma_2 u - \sigma_3 = 0$, if $u<0$, then $u^3 - \sigma_1 u^2 + \sigma_2 u - \sigma_3 < 0$. So, $u>0$ and $x>0, y>0, z>0$.

**Theorem 3. If** $x>0, y>0, z>0$**, then we have the following inequalities:**

**(1)** $\sigma_1^2 - 3\sigma_2 \ge 0$; **(2)** $\sigma_1^3 - 3\sigma_1\sigma_2 \ge 0$; **(3)** $\sigma_2^2 \ge 3\sigma_1\sigma_3$; **(4)** $\sigma_1\sigma_2 \ge 9\sigma_3$;

**(5)** $\sigma_1^3 - 4\sigma_1\sigma_2 + 9\sigma_3 \ge 0$ (***Schur***);

**(6)** $\sigma_2^3 \ge 27\sigma_3^2$; **(7)** $\sigma_1^3 \ge 27\sigma_3$; **(8)** $2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3 \ge 0$; **(9)** $8\sigma_1^3 - 27\sigma_1\sigma_2 + 27\sigma_3 \ge 0$.

***Proof.***

**(1)** By $(x-y)^2 + (y-z)^2 + (z-x)^2 \ge 0$, we have $(x+y+z)^2 \ge 3(xy+yz+zx)$, i.e. $\sigma_1^2 - 3\sigma_2 \ge 0$.

**(2)** Yields by (1) by multiplying with $\sigma_1$.

**(3)** Yields by $(a+b+c)^2 \ge 3(ab+bc+ca)$ with $a=xy, b=yz, c=zx$.

**(4)** $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge 9$ it is $\sigma_1\sigma_2 \ge 9\sigma_3$.

**(5)** In $(a+b)(b+c)(c+a) \ge 8abc$ we take $a+b=x, b+c=y, c+a=z$

$xyz \ge (-x+y+z)(x-y+z)(x+y-z)$, then $\sigma_1^3 - 4\sigma_1\sigma_2 + 9\sigma_3 \ge 0$;

**(6)** $\sigma_2^3 = \sigma_2\sigma_2^2 \overset{(3)}{\geq} \sigma_2 \cdot 3\sigma_1\sigma_3 = 3\sigma_3(\sigma_1\sigma_2) \overset{(4)}{\geq} 27\sigma_3^2$;

**(7)** $\sigma_1^4 = \sigma_1^2\sigma_1^2 \overset{(1)}{\geq} \sigma_1^2 \cdot 3\sigma_2 = 3\sigma_1(\sigma_1\sigma_2) \overset{(4)}{\geq} 3\sigma_1 9\sigma_3$, so $\sigma_1^3 \geq 27\sigma_3$.

**(8)** $2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3 = (\sigma_1^3 - 4\sigma_1\sigma_2 + 9\sigma_3) + (\sigma_1^3 - 3\sigma_1\sigma_2) \underset{(5)}{\overset{(2)}{\geq}} 0$.

**(9)** $8\sigma_1^3 - 27\sigma_1\sigma_2 + 27\sigma_3 = 4(2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3) + (\sigma_1\sigma_2 - 9\sigma_3) \underset{(8)}{\overset{(4)}{\geq}} 0$.

**Applications**

**1)** $(x+y+z)(x^2+y^2+z^2) \geq 9xyz$, $\forall x,y,z>0$.

***Solution.*** The inequality becomes $\sigma_1(\sigma_1^2 - 2\sigma_2) \geq 9\sigma_3 \Leftrightarrow \sigma_1^3 \geq 2\sigma_1\sigma_2 + 9\sigma_3$.

By (1) we have $\sigma_1^2 \geq 3\sigma_2 \Leftrightarrow \sigma_1^3 \geq 3\sigma_1\sigma_2 \Leftrightarrow \sigma_1^3 \geq 2\sigma_1\sigma_2 + \sigma_1\sigma_2$, than by (4) we deduce

$$\sigma_1^3 \geq 2\sigma_1\sigma_2 + \sigma_1\sigma_2 \Rightarrow \sigma_1^3 \geq 2\sigma_1\sigma_2 + 9\sigma_3.$$

**2)** $2(x^3+y^3+z^3) \geq x^2(y+z) + y^2(z+x) + z^2(x+y)$, $\forall x,y,z>0$.

***Solution.*** The inequality becomes

$$2(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3) \geq \sigma_1\sigma_2 - 3\sigma_3 \Leftrightarrow 2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3 \geq 0, \text{ true by (8).}$$

**3)** $\dfrac{2}{x+y} + \dfrac{2}{y+z} + \dfrac{2}{z+x} \geq \dfrac{9}{x+y+z}$, $\forall x,y,z>0$.

***Solution.*** The inequality becomes

$$2(x+y+z)\big((x+y)(x+z) + (y+x)(y+z) + (z+x)(z+y)\big) \geq 9(x+y)(y+z)(z+x) \text{ or}$$

$$2\sigma_1(S_2 + 3\sigma_2) \geq 9(\sigma_1\sigma_2 - \sigma_3) \Leftrightarrow 2\sigma_1(\sigma_1^2 - 2\sigma_2 + 3\sigma_2) \geq 9(\sigma_1\sigma_2 - \sigma_3) \Leftrightarrow$$

$$\Leftrightarrow 2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3 \geq 0, \text{ true by (8).}$$

**4)** $\dfrac{x^3+y^3+z^3}{x^2+y^2+z^2} \geq \dfrac{x+y+z}{3}$, $\forall x,y,z>0$.

***Solution.*** The inequality becomes successively

$$3S_3 \geq S_1S_2 \Leftrightarrow 3(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3) \geq \sigma_1(\sigma_1^2 - 2\sigma_2) \Leftrightarrow 2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3 \geq 0,$$

true by (8).

**5)** $(x+y+z)^3 \geq 9(x^3+y^3+z^3)$, $\forall x,y,z>0$.

***Solution.*** The inequality becomes successively $\sigma_1^3 \geq 9S_3 \Leftrightarrow 8\sigma_1^3 - 27\sigma_1\sigma_2 + 27\sigma_3 \geq 0$,

true by (9).

**6) If** $x,y,z>0$ **such that** $x+y+z=1$**, then**

$$\frac{xy}{1+z}+\frac{yz}{1+x}+\frac{xz}{1+y} \leq \frac{1}{4}-\frac{m}{4}(1-3(xy+yz+zx)) \leq \frac{1}{4}$$, where $m=\min(x,y,z)$.

***Solution.*** $\frac{1}{1+z}=1-\frac{z}{1+z} \Rightarrow \frac{xy}{1+z}=xy\left(1-\frac{z}{1+z}\right)=xy-\frac{xyz}{1+z}$. Analogusly:

$$\frac{yz}{1+x}=yz-\frac{xyz}{1+x} \text{ şi } \frac{xz}{1+x}=xz-\frac{xyz}{1+y}\text{; then } \sum\frac{xy}{1+z}=\sum xy - xyz\sum\frac{1}{1+z}.$$

$$\left(\sum\frac{1}{1+z}\right)\left(\sum(1+z)\right) \geq 9 \Rightarrow \sum\frac{1}{1+z} \geq \frac{9}{4}\text{; so, } \sum\frac{xy}{1+z}=\sum xy - xyz\sum\frac{1}{1+z} \leq \sigma_2 - \frac{9}{4}\sigma_3.$$

From $\sigma_1 = 1 \Rightarrow m(1-3\sigma_2) \leq 1-4\sigma_2+9\sigma_3 \Rightarrow \sigma_2 - \frac{9}{4}\sigma_3 \leq \frac{1}{4}-\frac{m}{4}(1-3\sigma_2)$.

Therefore, $\frac{xy}{1+z}+\frac{yz}{1+x}+\frac{xz}{1+y} \leq \frac{1}{4}-\frac{m}{4}(1-3\sigma_2)$.

$$xy+yz+zx \leq \frac{(x+y+z)^2}{3} \Rightarrow \sigma_2 \leq \frac{1}{3} \Rightarrow 1-3\sigma_2 \geq 0 \Rightarrow -\frac{m}{4}(1-3\sigma_2) \leq 0$$, so

$$\frac{xy}{1+z}+\frac{yz}{1+x}+\frac{xz}{1+y} \leq \frac{1}{4}-\frac{m}{4}(1-3(xy+yz+zx)) \leq \frac{1}{4}.$$

Equality occurs iff $x=y=z=\frac{1}{3}$.

**7) Prove that in any triangle $ABC$ holds:**

**a)** $2(ab+bc+ca) > a^2+b^2+c^2$**;**

**b)** $(a^2+b^2+c^2)(a+b+c) > 2(a^3+b^3+c^3)$**.**

***Solution.*** a) We denote $x=a+b-c, y=a-b+c, z=-a+b+c$, where $x,y,z>0$.

We have $a=\frac{x+y}{2}, b=\frac{x+z}{2}, c=\frac{y+z}{2}$ and the inequality becomes

$$\frac{(x+y)(x+z)+(y+x)(y+z)+(z+x)(z+y)}{2}>\frac{(x+y)^2+(y+z)^2+(z+x)^2}{4}$$

$$\Leftrightarrow 2(S_2+3\sigma_2)>2S_2+2\sigma_2 \Leftrightarrow \sigma_2>0,\ \text{true.}$$

b) The inequality becomes successively

$$\left[\left(\frac{x+y}{2}\right)^2+\left(\frac{y+z}{2}\right)^2+\left(\frac{z+x}{2}\right)^2\right](x+y+z)>2\left[\left(\frac{x+y}{2}\right)^3+\left(\frac{y+z}{2}\right)^3+\left(\frac{z+x}{2}\right)^3\right]$$

$$\Leftrightarrow \sigma_1\sigma_2+3\sigma_3>0,\ \text{true.}$$

**8)** Prove that among all triangles with the same perimeter, the equilateral triangle has the maximum area.

***Solution.*** Let $a,b,c$ be the sides of triangle and $p$ the semiperimeter, so $2p=\sigma_1$.

The area of triangle is $S=\sqrt{\frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{16}}$.

We denote $x=a+b-c, y=a-b+c, z=-a+b+c$, then:

$$S=\sqrt{\frac{(x+y+z)xyz}{16}}=\frac{1}{4}\sqrt{\sigma_1\sigma_3}\overset{(7)}{\leq}\frac{1}{4}\sqrt{\sigma_1\cdot\frac{\sigma_1^3}{27}}=\frac{\sigma_1^2\sqrt{3}}{36}.$$

Hence, $\frac{\sigma_1^2\sqrt{3}}{36}$ represents the area of an equilateral triangle with the side $l=\frac{\sigma_1}{3}$.

**9) (IMO, 1977)** $x^3+y^3+z^3+3xyz\geq x^2(y+z)+y^2(x+z)+z^2(x+y), \forall x,y,z>0$.

***Solution.*** The inequality becomes successively

$\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3+3\sigma_3\geq\sigma_1\sigma_2-3\sigma_3 \Leftrightarrow \sigma_1^3-4\sigma_1\sigma_2+9\sigma_3\geq 0$, i.e. the inequality of *Schur.*

**10) (IMO, 1964) Prove that in any triangle $ABC$ is true the following inequality:**

$$a^2(b+c-a)+b^2(a+c-b)+c^2(a+b-c)\leq 3abc.$$

***Solution.*** Denoting $x=b+c-a, y=a-b+c, z=a+b-c$ ( $x,y,z>0$ )the inequality becomes:

$$\sum (y+z)^2 \frac{x}{4} \le \frac{3}{8}(y+z)(z+x)(x+y) \Leftrightarrow 2\sum (y^2+z^2+2yz)x \le 3\left(\sum x^2 y + 2xyz\right) \Leftrightarrow$$

$$\Leftrightarrow \sum (y+z)^2 \frac{x}{4} \le \frac{3}{8}(y+z)(z+x)(x+y) \Leftrightarrow \sigma_1\sigma_2 - 3\sigma_3 \ge 6\sigma_3 \Leftrightarrow \sigma_1\sigma_2 \ge 9\sigma_3,$$

true by (4).

**11)** $x^4+y^4+z^4+xyz(x+y+z) \ge 2(x^2y^2+y^2z^2+z^2x^2)$, $\forall x,y,z>0$.

***Solution.*** The inequality becomes successively

$$S_4+\sigma_1\sigma_3 \ge 2(\sigma_2^2-3\sigma_3\sigma_1) \Leftrightarrow \sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2+4\sigma_1\sigma_3+\sigma_1\sigma_3 \ge 2\sigma_2^2-4\sigma_3\sigma_1 \Leftrightarrow$$

$$\Leftrightarrow \sigma_1^3-4\sigma_1\sigma_2+9\sigma_3 \ge 0,$$ i.e. *Schur's* inequality.

**12) If** $x,y,z>0$ **such that** $x+y+z=1$ **, then** $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \ge 64$ **.**

***Solution.*** The inequality becomes successively

$1+x+y+z+xy+yz+zx \ge 63xyz \Leftrightarrow 2\sigma_1+\sigma_2 \ge 63\sigma_3$, true since $6\sigma_2 \le 2 = 2\sigma_1$, $2\sigma_1^3 \le 54\sigma_3$ and $\sigma_2 \le 9\sigma_3$.

13) ( Moldavia Republic , National Math Olympiad 1993)

$$x(x-z)^2+y(y-z)^2 \ge (x-z)(y-z)(x+y+z), \forall x,y,z>0.$$

*Solution.* The inequality becomes successively

$$x^3+y^3+z^3+3xyz \ge xy^2+yx^2+yz^2+zy^2+zx^2+xz^2 \Leftrightarrow S_3+3\sigma_3 \ge \sigma_1\sigma_2-3\sigma_3 \Leftrightarrow$$

$$\Leftrightarrow \sigma_1^3-4\sigma_1\sigma_2+9\sigma_3 \ge 0,$$ true (*Schur*).

**14) If** $x,y,z>0$ **such that** $xyz=1$**, then**

$$2(x+y+z)^3+(xy+yz+zx)^3+27 \ge 18(x+y+z)(xy+yz+zx).$$

***Solution.*** The inequality becomes successively $2\sigma_1^3+3\sigma_2^3+27 \ge 18\sigma_1\sigma_2$.

$\sigma_1^3-4\sigma_1\sigma_2+9\sigma_3 \ge 0 \Leftrightarrow 2\sigma_1^3+18 \ge 8\sigma_1\sigma_2$ ; $\sigma_2^3+9\sigma_3^2 \ge 4\sigma_1\sigma_2 \Leftrightarrow 3\sigma_2^3+27 \ge 12\sigma_1\sigma_2$ and $\sigma_1\sigma_2 \ge 9\sigma_3 \Leftrightarrow 2\sigma_1\sigma_2 \ge 18$, which by adding up yields to the desired inequality la inegalitatea.

**15)** $a^3+b^3+c^3-3abc \geq (a^2+b^2+c^2)\sqrt{a^2+b^2+c^2}$ , $\forall a,b,c \in R$

***Solution.*** Denoting $x=\frac{a}{\sqrt{a^2+b^2+c^2}}$, $y=\frac{b}{\sqrt{a^2+b^2+c^2}}$, $z=\frac{c}{\sqrt{a^2+b^2+c^2}}$,

( $a \neq 0, b \neq 0, c \neq 0$ ) and we have $x^2+y^2+z^2=1$. The case $a=b=c=0$ it is obvious.

The inequality becomes successively :

$$x^3+y^3+z^3-3xyz \leq 1 \Leftrightarrow (x+y+z)(x^2+y^2+z^2-xy-yz-zx) \leq 1 \Leftrightarrow \sigma_1(3-\sigma_1^2) \leq 2 \Leftrightarrow$$

$$\Leftrightarrow (\sigma_1-1)^2(\sigma_1+2) \geq 0, \text{ true.}$$

Equality occurs if $\sigma_1 = 1 \Leftrightarrow ab+bc+ca=0$ or if

$$\sigma_1=-2 \Leftrightarrow 2\sum ab = 3\sum a^2 \Leftrightarrow 2\sum(a-b)^2+\sum a^2 = 0 \Leftrightarrow a=b=c=0.$$

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## FUNDAMENTAL SYMMETRIC POLYNOMIALS IN ALGEBRAIC INEQUALITIES-II

***By Marius Drăgan and Neculai Stanciu-Romania***

**Abstract:** In this article we will present a method for proving symmetric inequalities in two and three variables.

**Preliminary considerations:** In what follows, we will demonstrate some symmetric inequalities with the help of fundamental symmetric polynomials

**Definition. Polinomial** $p \in R[x,y]$ **it is symmetrical if** $p(x,y)=p(y,x), \forall x,y \in R$.

**Very important to remember: polynomials** $\sigma_1 = x+y, \sigma_1 \in R[x,y]$ **and** $\sigma_2 = xy, \sigma_2 \in R[x,y]$ **they are called fundamental symmetric polynomials. Fundamental symmetric polynomials are applied to prove some types of inequalities. The solution of these inequalities is based on Lemma 1.**

**Lemma 1. Let** $x+y=\sigma_1$; $xy=\sigma_2$. **Both numbers** $x,y$ **are real and positive iff:** $\sigma_1^2-4\sigma_2 \geq 0, \sigma_1 \geq 0, \sigma_2 \geq 0$.

**Proof**. $x, y$ are the roots of equation $z^2 - \sigma_1 z + \sigma_2 = 0$, i.e.

$$x = \frac{\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2}}{2}; y = \frac{\sigma_1 - \sqrt{\sigma_1^2 - 4\sigma_2}}{2} \text{ or } x = \frac{\sigma_1 - \sqrt{\sigma_1^2 - 4\sigma_2}}{2}; y = \frac{\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2}}{2}.$$

$x, y$ are real if $\sigma_1^2 - 4\sigma_2 \geq 0$. If $\sigma_1^2 - 4\sigma_2 = 0$, then $x = y$. So, there is $z > 0$, such that $\sigma_1^2 - 4\sigma_2 = z$. We have $\sigma_1^2 = 4\sigma_2 + z$ or $\sigma_2 = \frac{\sigma_1^2 - z}{4}$, $x \geq 0, y \geq 0$, hence: $\sigma_1 \geq 0, \sigma_2 \geq 0$.

***Reciprocal:*** If $\sigma_1^2 - 4\sigma_2 \geq 0$, then $x, y \in R$ and if $\sigma_1 \geq 0, \sigma_2 \geq 0$, then $x \geq 0$ and $y \geq 0$.

Indeed, if $x < 0$, then $z^2 - \sigma_1 z + \sigma_2 > 0$ for $z = x$ and also $z^2 - \sigma_1 z + \sigma_2 > 0$ for $z = y$. So, $x \geq 0$ and $y \geq 0$.

**Lemma 2. We denote $S_k = x^k + y^k$, $\forall k \in N$. We have:**

$$S_{k+1} = \sigma_1 S_k - \sigma_2 S_{k-1}, \ (S_0 = 2, S_1 = \sigma_1), \forall k \geq 1, \text{ (i).}$$

**Proof**. By some algebra we obtain:

$$x^{k+1} + y^{k+1} = (x+y)(x^k + y^k) - xy(x^{k-1} + y^{k-1}) = x^{k+1} + y^{k+1} + xy^k + yx^k - x^k y - xy^k =$$

$$= x^{k+1} + y^{k+1}, \text{ q.e.d.}$$

By (i) we deduce Waring's formula:

$$\frac{1}{k} S_k = \sum_m \frac{(-1)^m (k-m-1)!}{m!(k-2m)!} \sigma_1^{k-2m} \sigma_2^m, \text{ (ii).}$$

By (ii) we compute $S_k$ ($k \in \{3,4,5,6\}$) and we obtain:

$$S_3 = \sigma_1^3 - 3\sigma_1\sigma_2; \ S_4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2; S_5 = \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2;$$

$$S_6 = \sigma_1^6 - 6\sigma_1^4\sigma_2 + 9\sigma_1^2\sigma_2^2 - 2\sigma_2^3.$$

**Theorem. Any symmetric polynomial $p \in R[x, y]$ can be expressed uniquely with the help of fundamental symmetric polynomials**.

**Proof**. We will use (ii) by Lemma 2 and the facts: $ax^k y^y = a\sigma_2^k$ or

$b(x^k y^l + x^l y^k) = bx^k y^k (x^{l-k} + y^{l-k})$, $l \geq k$ şi $b(x^k y^l + x^l y^k) = bx^l y^l (x^{k-l} + y^{k-l})$, $k \geq l$,

(iii).

By example: $p = xy(x^3 + y^3) + x^2 y^2 + x^3 y^3 (x + y) = \sigma_2 S_3 + \sigma_2^2 + \sigma_2^3 S_1 =$

$$= \sigma_2(\sigma_2 - 3\sigma_1\sigma_2) + \sigma_2^2 + \sigma_2^2\sigma_1 = \sigma_1^3\sigma_2 - 3\sigma_1\sigma_2^2 + \sigma_2^2 + \sigma_1\sigma_2^2 =$$

$$= \sigma_1^3\sigma_2 - 2\sigma_1\sigma_2^2 + \sigma_2^2 = \sigma_2(\sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_2).$$

In what follows, we will use the above results to prove a number of inequalities in two variables highlighting, for each individual case, the essence of the solution method.

**Observation.** The theorem can also be extended for all rational algebraic expressions, as will be seen in the following examples.

**Applications:**

**1) If $a, b$ are positive real numbers, then prove that :**

$$\frac{a^3 + b^3}{2} \geq \left(\frac{a+b}{2}\right)^3.$$

***Solution***. Putting $a + b = \sigma_1$, $ab = \sigma_2$, we obatin:

$$\frac{a^3 + b^3}{2} - \left(\frac{a+b}{2}\right)^3 = \frac{S_3}{2} - \frac{\sigma_1^3}{8} = \frac{1}{2}\left(\sigma_1^3 - 3\sigma_1\sigma_2\right) - \frac{\sigma_1^3}{8} = \frac{3}{8}\sigma_1 z \geq 0,$$ true because $z \geq 0$.

**2) If $a, b, c$ are positive real numbers such that $a + b \geq c$, then prove that:**

$$a^2 + b^2 \geq \frac{c^2}{2},\ a^4 + b^4 \geq \frac{c^4}{8},\ a^8 + b^8 \geq \frac{c^8}{128}.$$

***Generalization:*** $a^{2^n} + b^{2^n} \geq \dfrac{c^{2^n}}{2^{n+1}}$.

***Solution.*** We let $a + b = \sigma_1$, $ab = \sigma_2$, so we have:

$$S_2 = a^2 + b^2 = \sigma_1^2 - 2\sigma_2 = \sigma_1^2 - 2 \cdot \frac{\sigma_1^2 - z}{4} = \frac{\sigma_1^2 + z}{2} \geq \frac{\sigma_1^2}{2} \Leftrightarrow a^2 + b^2 \geq \frac{c^2}{2}.$$

Analogously:

$$a^4 + b^4 = (a^2)^2 + (b^2)^2 \geq \frac{1}{2}\left(\frac{c^2}{2}\right)^2 = \frac{c^4}{8};\ a^8 + b^8 = (a^4)^2 + (b^4)^2 \geq \frac{1}{2}\left(\frac{c^4}{8}\right)^2 = \frac{c^8}{128}.$$

By induction yields that $a^{2^n}+b^{2^n}\geq \frac{c^{2^n}}{2^{n+1}}$.

**3) If $a,b$ are positive real numbers, then prove that:**

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}\geq \sqrt{a}+\sqrt{b}.$$

***Solution.*** We denote $\sqrt{a}=u,\sqrt{b}=v$ and the inequality to prove becomes

$$\frac{u^2}{v}+\frac{v^2}{u}\geq u+v \Leftrightarrow u^3+v^3\geq uv(u+v).$$

Since, $u^3+v^3-uv(u+v)=S_3-\sigma_2\sigma_1=\sigma_1(\sigma_1^2-4\sigma_2)\geq 0$, true.

**4) If $a,b$ are positive real numbers, then: $a^4+b^4\geq a^3b+ab^3$.**

***Solution.*** $a^4+b^4-a^3b-ab^3=a^4+b^4-ab(a^2+b^2)=S_4-\sigma_2S_2=$

$$=\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2-\sigma_2(\sigma_1^2-2\sigma_2)=\sigma_1^4-5\sigma_1^2\sigma_2+4\sigma_2^2=\frac{3z\sigma_1^2+z^2}{4}\geq 0,$$

where $z=\sigma_1^2-4\sigma_2\geq 0$.

**5) If $a,b$ are positive real numbers, then**

$$a^4+2a^3b+2ab^3+b^4\geq 6a^2b^2.$$

***Solution.*** Let $a+b=\sigma_1, ab=\sigma_2, z=\sigma_1^2-4\sigma_2\geq 0$, then:

$$a^4+2a^3b+2ab^3+b^4-6a^2b^2=S_4+2\sigma_2S_2-6\sigma_2^2=\sigma_1^4-2s_1^2\sigma_2-8\sigma_2^2=$$

$$=(z+4\sigma_2)^2-2(z+4\sigma_2)\sigma_2-8\sigma_2^2=z^2+6\sigma_2z\geq 0, \text{ true.}$$

**6) ( *I.V. Maftei*) If $a,b$ are positive real numbers such that $a+b=1$, then**

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\geq \frac{25}{2}.$$

***Generalization.*** $\left(a+\frac{1}{a}\right)^n+\left(b+\frac{1}{b}\right)^n\geq \frac{5^n}{2^{n-1}}, \forall n\in N.$

***Solution.*** $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2-\frac{25}{2}=a^2+b^2+\frac{a^2+b^2}{a^2b^2}-\frac{17}{2}=S_2+\frac{S_2}{\sigma_2^2}-\frac{17}{2}=$

$=\frac{1}{2\sigma_2^2}(-4\sigma_2^3-15\sigma_2^2-4\sigma_2+2)$ and it remains to prove that

$$4\sigma_2^3+15\sigma_2^2+4\sigma_2\le 2.$$

Since: $\sigma_2\ge 0$ and $z=\sigma_1^2-4\sigma_2\ge 0$ , and using $\sigma_1=1$ yields that $0\le\sigma_2\le\frac{1}{4}$.

So, $4\sigma_2^3+15\sigma_2^2+4\sigma_2\le\frac{1}{16}+\frac{15}{16}+1=2$.We have equality for $a=b=2$.

To prove the generalization we consider the convex function $f:[0,\infty)\to R$, $f(x)=x^n$, $n\in N^*$ and by Jensen' s inequality $f(x)+f(y)\ge 2f\left(\frac{x+y}{2}\right)$.

If we take $x=a+\frac{1}{a}, y=b+\frac{1}{b}$, then we obatin

$$\left(a+\frac{1}{a}\right)^n+\left(b+\frac{1}{b}\right)^n\ge 2\left[\frac{1}{2}\left(\sigma_1+\frac{\sigma_1}{\sigma_2}\right)\right]^n=\frac{1}{2^{n-1}}\left(1+\frac{1}{\sigma_2}\right)^n.$$

Using: $\sigma_1^2\ge 4\sigma_2\Leftrightarrow\frac{1}{\sigma_2}\ge 4\Leftrightarrow\frac{1}{\sigma_2}+1\ge 5$, we will obtain $\left(a+\frac{1}{a}\right)^n+\left(b+\frac{1}{b}\right)^n\ge\frac{5^n}{2^{n-1}}$.

**7) (*I.V. Maftei*) If $a,b$ are positive real numbers such that $a^2+b^2=1$, then:**

$$\frac{1}{1-a}+\frac{1}{1-b}\ge 4+2\sqrt{2}.$$

***Solution.*** We have $\frac{1}{1-a}+\frac{1}{1-b}=\frac{2-\sigma_1}{1-\sigma_1+\frac{\sigma_1^2-1}{2}}=\frac{2(2-\sigma_1)}{(\sigma_1-1)^2}$.

From, $\sigma_1^2-2\sigma_2=1$ şi $\sigma_1^2\ge 4\sigma_2$ we have $\sigma_2\le\frac{1}{2}$ and $\sigma_1\le\sqrt{2}$.

Then, $2-\sigma_1\ge 2-\sqrt{2}$ şi $(\sigma_1-1)^2\le(\sqrt{2}-1)^2$.

Hence, $\frac{2(2-\sigma_1)}{(\sigma_1-1)^2}\ge\frac{2(2-\sqrt{2})}{(\sqrt{2}-1)^2}=4+2\sqrt{2}$.

**8) (AoPS, *Daniel Sitaru*) If** $a,b>0$**, then prove that**

$$9\le\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right)\le 5+2\left(\frac{a}{b}+\frac{b}{a}\right).$$

***Solution.*** LHS becomes

$$\frac{2\sqrt{\sigma_2}}{\sigma_1}+\frac{2\sqrt{\sigma_2}}{\sigma_1}+\frac{\sigma_1}{2\sqrt{\sigma_2}}+\frac{\sigma_1}{2\sqrt{\sigma_2}}+\frac{4\sigma_2}{\sigma_1^2}+\frac{\sigma_1^2}{4\sigma_2}\ge 6,$$

which yields by AM-GM inequality. RHS becomes

$2+\frac{4\sqrt{\sigma_2}}{\sigma_1}+\frac{\sigma_1}{\sqrt{\sigma_2}}+\frac{4\sigma_2}{\sigma_1^2}\le\frac{7\sigma_1^2}{4\sigma_2}$, and if we denote $t=\frac{\sigma_1}{\sqrt{\sigma_2}}\ge 2$, then the last inequality becomes

$$7t^4-4t^3-8t^2-16t-16\ge 0\Leftrightarrow (t-2)(7t^3+10t^2+12t+8)\ge 0,\text{ true.}$$

**9) (Olympiad, URSS, 1984) If** $a,b>0,$ **then:**

$$\frac{(a+b)^2}{2}+\frac{a+b}{4}\ge a\sqrt{b}+b\sqrt{a}.$$

***Solution.*** Denoting $\sqrt{a}=x,\sqrt{b}=y,x+y=\sigma_1=S,xy=\sigma_2=P$ :

$$\frac{(x^2+y^2)^2}{2}+\frac{x^2+y^2}{4}\ge xy(x+y)\Leftrightarrow 2(S^2-2P)^2+S^2-2P\ge 4SP\Leftrightarrow$$

$$\Leftrightarrow 2S^4-8S^2P+8P^2+S^2-2P\ge 4SP\Leftrightarrow 8P^2-(8S^2+4S+2)P+2S^4+S^2\ge 0.$$

We consider the function $f:\left[0,\frac{S^2}{4}\right]\to R,\ f(P)=8P^2-(8S^2+4S+2)P+2S^4+S^2.$

$\Delta=4(S^3+4S^2+4S+1)>0$ and let $P_1,P_2$ the roots of equation $f(P)=0$.

We prove that $f(P)\ge 0$ , so it is suffices to prove

$$P_1>\frac{S^2}{4}\Leftrightarrow\frac{8S^2+4S+2-2\sqrt{16S^3+4S^2+4S+1}}{16}>\frac{S^2}{4}\Leftrightarrow$$

$$\Leftrightarrow(2S^2+2S+1)^2\ge 16S^3+4S^2+4S+1\Leftrightarrow 4S^2(S-1)^2\ge 0.$$

So, $f(P)\ge 0$, $\forall P\in\left[0,\frac{S^2}{4}\right]$, q.e.d.

**10) If** $x,y,z>0,$ **such that** $x+y+z=1$**, then** $0\le xy+yz+zx-2xyz\le\frac{7}{27}$ **(IMO, 1984).**

***Solution.*** Since $x+y+z=1$, so $z=1-\sigma_1$, where $z\in[0,1]$. The inequality becomes:

$$0\le\sigma_2+\sigma_1(1-\sigma_1)-2\sigma_2(1-\sigma_1)\le\frac{7}{27}.$$

We consider the function: $f:\left[0,\frac{\sigma_1^2}{4}\right]\to R$, $f(\sigma_2)=25(2\sigma_1-1)\sigma_2+\sigma_1-\sigma_1^2$, $\sigma_1$ fixed.

We have $f(0)=\sigma_1(1-\sigma_1)\ge 0$, $f\left(\frac{\sigma_1^2}{4}\right)=\frac{\sigma_1(2\sigma_1^2-5\sigma_1+4)}{4}>0$.

The equality for $\sigma_1=0$ or $\sigma_1=1$, i.e. for $\{x,y,z\}=\{1,0,0\}$ .Also we consider the function:

$$g:\left[0,\frac{\sigma_1^2}{4}\right]\to R,\ g(\sigma_2)=27(2\sigma_1-1)\sigma_2-27\sigma_1(\sigma_1-1)-7,\ g(0)=27\sigma_1^2+27\sigma_1-7\le 0.$$

Since, $\Delta=-27<0$, $g\left(\frac{\sigma_1^2}{4}\right)=\frac{1}{4}(3\sigma_1-2)^2(6\sigma_1-7)\le 0$ , so $g(\sigma_2)\ge 0,\forall\sigma_2\in\left[0,\frac{\sigma_1^2}{4}\right]$.

Equality occurs for $\sigma_1=\frac{2}{3}$, i.e. $\{x,y,z\}=\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\}$.

**11) If** $x,y,z>0,$ **such that** $x^2+y^2+z^2+2xyz=1$**, then:**
$2(xy+yz+zx)\le x+y+z$ **.**

***Solution.*** Let $x+y=\max\{x+y,y+z,z+x\}$, so

$$\sigma_1^2-2\sigma_2+z^2+2\sigma_2 z=1\Leftrightarrow z=\sqrt{\sigma_2^2+2\sigma_2+1-\sigma_1^2}-\sigma_2.$$

The inequality becomes:

$$2\sigma_2+2\sigma_1(\sqrt{\sigma_2^2+2\sigma_2+1-\sigma_1^2}-\sigma_2)\le\sigma_1+\sqrt{\sigma_2^2+2\sigma_2+1-\sigma_1^2}-\sigma_2$$

$$\Leftrightarrow\sqrt{\sigma_2^2+2\sigma_2+1-\sigma_1^2}(2\sigma_1-1)\le 2\sigma_1\sigma_2-3\sigma_2+\sigma_1$$

$$\Leftrightarrow(8\sigma_1-8)\sigma_2^2+(4\sigma_1^2-2\sigma_1+2)\sigma_2-4\sigma_1^2+4\sigma_1^3+2\sigma_1^2-4\sigma_1+1\le 0.$$

We consider the function

$$f:\left[0,\frac{\sigma_1^2}{4}\right]\to R,\ f(\sigma_2)=(8\sigma_1-8)\sigma_2^2+(4\sigma_1^2-2\sigma_1+2)\sigma_2-4\sigma_1^2+4\sigma_1^3+2\sigma_1^2-4\sigma_1+1.$$

We have: $1=x^2+y^2+z^2+2xyz\le(x+y)^2+z^2$ and since $x+y\ge y+z\ge z$ we deduce

$2(xy)^2\ge(x+y)^2+z^2\ge1\Leftrightarrow x+y\ge\frac{1}{\sqrt{2}}$. Because $x\le1, y\le1$ we have $\frac{1}{\sqrt{2}}\le\sigma_1\le2$.

We will prove that $f(\sigma_2)\le0, \forall\sigma_2\in\left[0,\frac{\sigma_1^2}{4}\right]$.

*Case 1.* $\sigma_1\in(0,2]$. It is suffices to prove that $f(0)\le0$ and $f\left(\frac{\sigma_1^2}{4}\right)\le0$.

$f(0)=-4\sigma_1^2+4\sigma_1^3+2\sigma_1^2-4\sigma_1+1$ and we consider the function

$g(\sigma_1)=-4\sigma_1^2+4\sigma_1^3+2\sigma_1^2-4\sigma_1+1$ with

$g'(\sigma_1)=4(-4\sigma_1^3+3\sigma_1^2+\sigma_1-1)$, $g''(\sigma_1)=4(-12\sigma_1^2+6\sigma_1+1)\le0$ since $\sigma_1>1>\frac{3+\sqrt{21}}{12}$.

So $g'$ is decreasing on $(1,2]$, i.e. $g'(\sigma_1)\le g'(1)<0$. Therefore, $g$ is decreasing on $(1,2]$, i.e. $g(\sigma_1)\le g(1)<0$, hence $f(0)\le0$.

$$f\left(\frac{\sigma_1^2}{4}\right)=\frac{(\sigma_1+1)(\sigma_1-1)(\sigma_1^2-6\sigma_1+2)}{2}<0, \text{ since } \sigma_1^2-6\sigma_1+2<-3<0.$$

*Case 2.* $\sigma_1\in\left[\frac{1}{\sqrt{2}},1\right]$.

We prove that $\frac{\sigma_1^2}{4}<\sigma_{2V}\Leftrightarrow\frac{\sigma_1^2}{4}<\frac{4\sigma_1^2-2\sigma_1+2}{16(1-\sigma_1)}\Leftrightarrow2\sigma_1^2(1-\sigma_1)<2\sigma_1^3-\sigma_1+1\Leftrightarrow$

$\Leftrightarrow(2\sigma_1^2-1)\sigma_1+1>0$, true because $2\sigma_1^2-1>0$.

$$f\left(\frac{\sigma_1^2}{4}\right)=\frac{(\sigma_1+1)(\sigma_1-1)(\sigma_1^2-6\sigma_1+2)}{2}<0, \text{ since } \sigma_1^2-6\sigma_1+2<\frac{5-6\sqrt{2}}{2}<0.$$

Therefore, $f(\sigma_2)<0,\ \forall\sigma_2\in\left[0,\frac{\sigma_1^2}{4}\right]$, then $f(\sigma_2)\le0, \forall\sigma_2\in\left[0,\frac{\sigma_1^2}{4}\right]$ şi $\sigma_1\in\left[\frac{1}{\sqrt{2}},2\right]$.

**12) If** $x, y, z > 0$**, then**

$$(x+y+z)^3 \geq 27xyz + \frac{x(y+z-2x)^2}{4}.$$

***Solution.*** $\frac{y}{x}+\frac{z}{x}=\sigma_1, \frac{y}{x}\cdot\frac{z}{x}=\sigma_2$. The inequality is equivalent to

$$(1+\sigma_1)^3 \leq 27\sigma_2 + \frac{(\sigma_1-2)^2}{4}.$$

Since $\frac{\sigma_1^2}{4} \geq \sigma_2$ it suffices to prove that

$$(1+\sigma_1)^3 \geq \frac{27\sigma_1^2}{4} + \frac{(\sigma_1-2)^2}{4} \Leftrightarrow 4\sigma_1(\sigma_1-2)^2 \geq 0, \text{ true.}$$

**Remark.** A stronger inequality it is:

$$(x+y+z)^3 - 27xyz \geq \max\left\{\frac{x(y+z-2x)^2}{4}, \frac{y(x+z-2y)^2}{4}, \frac{z(x+y-2z)^2}{4}\right\}.$$

**13) If** $x, y, z > 0$**, then:**

$$(x+y+z)^3 - 27xyz \geq \frac{(x+y-2z)^2(4x+4y+z)}{4}.$$

***Solution.*** $\frac{y}{x}+\frac{z}{x}=\sigma_1, \frac{y}{x}\cdot\frac{z}{x}=\sigma_2$, and the inequality becomes

$(1+\sigma_1)^3 - 27\sigma_2 \geq \frac{(\sigma_1-2)^2(4\sigma_1+1)}{4}$. Since $\frac{\sigma_1^2}{4} \geq \sigma_2$ it is suffices to prove that

$$(1+\sigma_1)^3 - \frac{27\sigma_1^2}{4} \geq \frac{(\sigma_1-2)^2(4\sigma_1+1)}{4}, \text{ which is true.}$$

**Remark.** A stronger inequality it is:

$$(x+y+z)^3 - 27xyz \geq$$

$$\ge \max\left\{\frac{(x+y-2z)^2(4x+4y+4z)}{4},\frac{(y+z-2x)^2(4x+4y+4z)}{4},\frac{(z+x-2y)^2(4x+4y+4z)}{4}\right\}$$

**14) (*Marius Drăgan*) If** $x,y,z>0$**, then** $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge\left(\frac{y+z}{\sqrt{yz}}+1\right)^2$**.**

***Solution***. $y+z=\sigma_1, yz=\sigma_2$. The inequality becomes

$$(x+\sigma_1)\left(\frac{1}{x}+\frac{\sigma_1}{\sigma_2}\right)\ge\left(\frac{\sigma_1}{\sqrt{\sigma_2}}+1\right)^2 \Leftrightarrow 1+\frac{\sigma_1 x}{\sigma_2}+\frac{\sigma_1}{x}+\frac{\sigma_1^2}{\sigma_2}\ge\frac{\sigma_1^2}{\sigma_2}+\frac{2\sigma_1}{\sqrt{\sigma_2}}+1\Leftrightarrow$$

$$\Leftrightarrow \sigma_1\left(\frac{x}{\sigma_2}+\frac{1}{x}\right)\ge\frac{2\sigma_1}{\sqrt{\sigma_2}}\Leftrightarrow\sigma_1\left(\frac{x}{\sigma_2}+\frac{1}{x}-\frac{2}{\sqrt{\sigma_2}}\right)\ge 0\Leftrightarrow\sigma_1\left(\frac{\sqrt{x}}{\sqrt{\sigma_2}}-\frac{1}{\sqrt{x}}\right)\ge 0,$$ which is true.

**15) If** $a,b,c\in R$ **with** $a+b+c\ge 0$ **, then** $a^3+b^3+c^3\ge 3abc$**.**

***Solution***. $b+c=\sigma_1, bc=\sigma_2$. The inequality becomes

$$a^3+\sigma_1^3-3\sigma_2(\sigma_1+a)\ge 0\Leftrightarrow(\sigma_1+a)(\sigma_1^2-a\sigma_1+a^2-3\sigma_2)\ge 0.$$

Since $\sigma_1+a\ge 0$, it is remains to prove that $\sigma_1^2-a\sigma_1+a^2-3\sigma_2\ge 0$ and since $\frac{\sigma_1^2}{4}\ge\sigma_2$ it suffices to show $\sigma_1^2-a\sigma_1+a^2-3\cdot\frac{\sigma_1^2}{4}\ge 0\Leftrightarrow\frac{(\sigma_1-2a)^2}{4}\ge 0$, true.

**16) If** $a,b,c>0$ **, then** $a^4+b^4+c^4\ge abc(a+b+c)$**.**

***Solution.*** $\frac{b}{a}+\frac{c}{a}=\sigma_1, \frac{b}{a}\cdot\frac{c}{a}=\sigma_2$. The inequality becomes

$$1+(\sigma_1^2-2\sigma_2)^2-2\sigma_2^2\ge\sigma_2(1+\sigma_1)\Leftrightarrow\Leftrightarrow 2\sigma_2^2-(4\sigma_1^2+\sigma_1+1)\sigma_2+\sigma_1^4+1\ge 0.$$

Considering the function: $f:\left[0,\frac{\sigma_1^2}{4}\right]\to R$, $f(\sigma_2)=2\sigma_2^2-(4\sigma_1^2+\sigma_1+1)\sigma_2+\sigma_1^4+1\ge 0.$

$$f(0)=\sigma_1^4+1>0,\ f\left(\frac{\sigma_1^2}{4}\right)=\frac{(\sigma_1-2)^2(25^2+25+2)}{4}\ge 0$$

Since: $\frac{\sigma_1^2}{4}<\sigma_{2V}=\frac{4\sigma_1^2+\sigma_1+1}{4}$ yields that $f(\sigma_2)\ge 0,\ \forall\sigma_2\in\left[0,\frac{\sigma_1^2}{4}\right]$.

**17) (*I.V. Maftei, L. Toderiuc*, IMAC, 2007) If** $a,b,c>0$**, then:**

$$8(a^3+b^3+c^3)^2 \ge 9(a^2+bc)(b^2+ac)(c^2+ab).$$

***Solution***. $\frac{b}{a}=x, \frac{c}{a}=y$, $x^3+y^3=\sigma_1, xy=\sigma_2$. The inequality becomes

$$8(x^3+y^3+1)^2 \ge 9(x^2+y)(y^2+x)(1+xy) \Leftrightarrow 8(\sigma_1+1)^2 \ge 9(\sigma_1+\sigma_2^2+\sigma_2)(\sigma_2+1) \Leftrightarrow$$

$$\Leftrightarrow 9\sigma_2^3+18\sigma_2^2+(9+9\sigma_1)\sigma_2-8\sigma_1^2-7\sigma_1-8 \le 0.$$

Because $x^3+y^3 \ge 2\sqrt{x^3y^3} \Leftrightarrow \sigma_1^2 \ge 4\sigma_2^3$; denoting $u^3=\frac{\sigma_1}{2}$, then $\sigma_2 \le u^2$. So,

$$9\sigma_2^3+18\sigma_2^2+(9+9\sigma_1)\sigma_2-8\sigma_1^2-7\sigma_1-8 \le 9u^6+18u^4+(9-18u^3)u^2-32u^6-14u^3-8.$$

It remains to prove that $23u^6-18u^5-18u^4+14u^3-9u^2+8 \ge 0 \Leftrightarrow$

$\Leftrightarrow (u-1)^2(23u^4+28u^3+15u^2+16u+8) \ge 0$, true.

The equality occurs for $u=1$ and $x=y$, i.e. $x=y=1$, so $a=b=c$.

**18) (Moldavian Math Olympiad, 1993) If** $x,y,z>0$**, then**

$$x(x-z)^2+y(y-z)^2 \ge (x-z)(y-z)(x+y-z).$$

***Solution***. $\frac{x}{z}+\frac{y}{z}=\sigma_1$, $\frac{x}{z}\cdot\frac{y}{z}=\sigma_2$. The inequality becomes

$$\sigma_1(\sigma_1^2-3\sigma_2)+\sigma_1-2(\sigma_1^2-2\sigma_2) \ge (\sigma_2+1-\sigma_1)(\sigma_1-1) \Leftrightarrow$$

$$\Leftrightarrow (5-3\sigma_1)\sigma_2+(\sigma_2+1)(\sigma_1-1)^2 \ge 0.$$

Considering the function $f:\left[0,\frac{\sigma_1^2}{4}\right] \to R$, $f(\sigma_2)=(5-3\sigma_1)\sigma_2+(\sigma_2+1)(\sigma_1-1)^2$, since $f\left(\frac{\sigma_1^2}{4}\right)=\frac{(\sigma_1-2)^2}{4} \ge 0$, yields $f(\sigma_2) \ge 0$, $\forall \sigma_2 \in \left[0,\frac{\sigma_1^2}{4}\right]$. Equality iff $\sigma_1=2$, i.e. $x=y=z$.

**Reference: Romanian Mathematical Magazine-www.ssmrmh.ro**

## SOME GENERALIZATIONS FOR LANGLEY'S PROBLEM

***By Marius Drăgan and Neculai Stanciu-Romania***

**Abstract: This paper presents other generalizations for Langley's problem.** **Keywords: isosceles triangle, adventitious angles.MSC: 51M04**

**Langley's Adventitious Angles is a puzzle in which one must infer an angle in a geometric diagram from other given angles. It was posed by Edward Mann Langley in *The Mathematical Gazette* in 1922.**

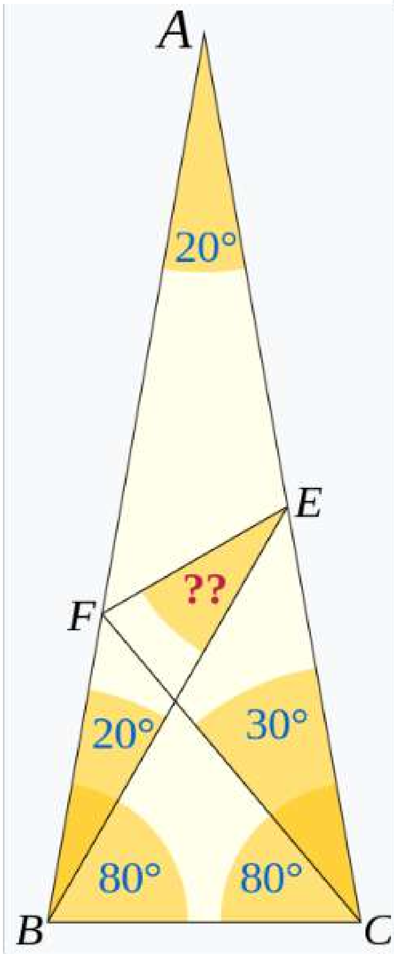
In its original form the problem was as follows:

$ABC$ is an isosceles triangle with $\angle CBA = \angle ACB = 80^\circ$.

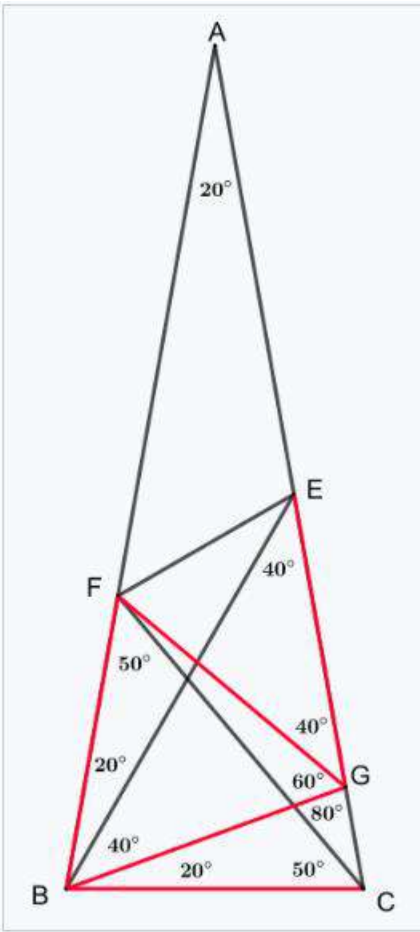
$CF$ at $30^\circ$ to $AC$ cuts $AB$ in $F$.

$BE$ at $20^\circ$ to $AB$ cuts $AC$ in $E$.

Prove $\angle BEF = 30^\circ$. [1][2][3]




Langley's Adventitious Angles



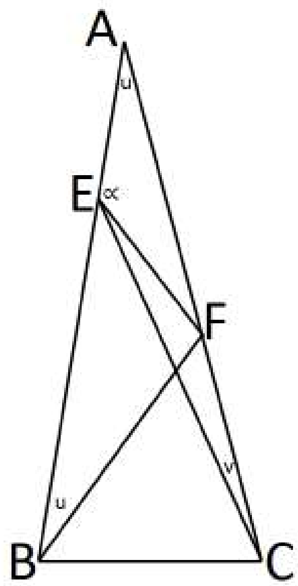

Solution to Langley's 80-80-20 triangle problem

**Generalizations of Langley's problem. If** $ABC$ **is a isosceles triangle with** $AB = AC$**,** $E \in (AB)$**,** $F \in (AC)$ **such that** $\angle ABF = \angle FAB = u$ **and** $\angle ACE = v$**, then:**

$\alpha = \angle AEF = \arcsin f(u,v)$ **or** $\alpha = \angle AEF = 180^0 - \arcsin f(u,v)$**, where**

$$f(u,v)=\frac{\sin u\sin(u+v)}{\sqrt{4\cos^2 u\sin^2 v-4\cos^2 u\sin v\sin(u+v)+\sin^2(u+v)}}.$$

**Proof.** $AB=AC\overset{not.}{=}y$, $BF=AF\overset{not.}{=}x$, $\angle AEF\overset{not.}{=}\alpha$.




**By Sine Law in** $\Delta AEC$ **:** $\frac{y}{\sin(u+v)}=\frac{AE}{\sin v}\Leftrightarrow AE=\frac{y\sin v}{\sin(u+v)}$.

**By Cosine Law in** $\Delta AEF : EF^2=AE^2+AF^2-2\cdot AE\cdot AF\cdot\cos u\Leftrightarrow$

$$\Leftrightarrow EF^2=\frac{y^2\sin^2 v}{\sin^2(u+v)}+x^2-\frac{2xy\sin v\cos u}{\sin(u+v)}\Leftrightarrow\frac{EF^2}{x^2}=\left(\frac{y}{x}\right)^2\frac{\sin^2 v}{\sin^2(u+v)}+1-\frac{2y}{x}\cdot\frac{\sin v\cos u}{\sin(u+v)}.$$

**By Sine Law in** $\Delta ABF$ **:** $\frac{y}{\sin 2u}=\frac{x}{\sin u}\Leftrightarrow\frac{y}{x}=2\cos u$.

$$\frac{EF^2}{x^2}=\frac{4\cos^2 u\sin^2 v}{\sin^2(u+v)}+1-\frac{4\sin v\cos^2 u}{\sin(u+v)}=\frac{4\cos^2 u\sin^2 v+\sin^2(u+v)-4\sin v\cos^2 u\sin(u+v)}{\sin^2(u+v)}$$

.

**By Sine Law in** $\Delta AEF$ **:** $\frac{EF^2}{x^2}=\frac{\sin^2 u}{\sin^2\alpha}$.

$$\frac{\sin^2 u}{\sin^2\alpha}=\frac{4\cos^2 u\sin^2 v+\sin^2(u+v)-4\sin v\cos^2 u\sin(u+v)}{\sin^2(u+v)}.$$

**Hence:** $\alpha=\angle AEF=\arcsin f(u,v)$ **or** $\alpha=\angle AEF=180^0-\arcsin f(u,v)$**, where**

$$f(u,v)=\frac{\sin u\sin(u+v)}{\sqrt{4\cos^2 u\sin^2 v-4\cos^2 u\sin v\sin(u+v)+\sin^2(u+v)}}.$$

**Next: If** $u$ **and** $v$ **are by the form** $u=\overline{m0}^{0}$**,** $v=\overline{n0}^{0}$**;** $m,n\in\{1,2,\dots,8\}$ **we will determine the angles** $\alpha$ **which satisfy the Langley's generalizations from above, where** $\alpha$ **is of the form** $\overline{k0}^{0}$**,** $k\in\{1,2,\dots,8\}$**. Since** $u<\frac{180^{0}-u}{2}$**, yields that** $u<60^{0}$**, so** $m\in\{1,2,3,4,5\}$**.**

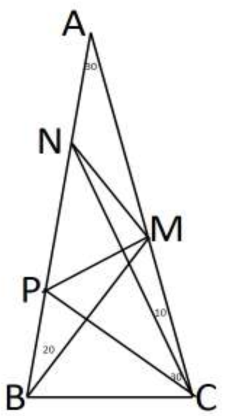
**We will obtain the following cases:**

1. $(u,v,\alpha)=(10^{0},10^{0},85^{0})$ **- trivial;**
2. $(u,v,\alpha)=(20^{0},10^{0},180^{0}-\arcsin f(20^{0},10^{0}))=(20^{0},10^{0},130^{0})$**;**
3. $(u,v,\alpha)=(20^{0},30^{0},\arcsin f(20^{0},30^{0}))=(20^{0},30^{0},50^{0})$ **- the problem of Langley;**
4. $(u,v,\alpha)=(20^{0},50^{0},\arcsin f(20^{0},30^{0}))=(20^{0},50^{0},30^{0})$ **;**
5. $(u,v,\alpha)=(40^{0},40^{0},\arcsin f(40^{0},40^{0}))=(40^{0},40^{0},70^{0})$ **- trivial;**
6. $(u,v,\alpha)=(50^{0},50^{0},\arcsin f(50^{0},50^{0}))=(50^{0},50^{0},65^{0})$ **- trivial.**

**For other cases** $k\notin N^{*}$ **, and** $(u,v,\alpha)=(20^{0},80^{0},\alpha)$ **and** $(u,v,\alpha)=(40^{0},70^{0},\alpha)$ **doesn't work.**

**Problem. If** $ABC$ **is an isosceles triangle with** $\angle A=20^{0}$**,** $N,P\in(AB)$**,** $M\in(AC)$**,** $\angle ABM=20^{0}$**,** $\angle PCA=30^{0}$ **şi** $\angle NCA=10^{0}$**, then** $MN=MP$**.**

**Solution.**




**By the case 2, yields** $\angle ANM=130^{0}$**, and by the case 3, yields** $\angle APM=50^{0}$**.**

**Hence,** $\angle MNP=\angle APM=50^{0}$**, so** $MN=MP$**.**

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**WIKIPEDIA:**

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## SOME INEQUALITIES SOLVED BY BW METHOD

***By Marius Drăgan and Neculai Stanciu-Romania***

**Abstract. This paper presents some inequalities solved by Buffalo Way (BW) method.**
**Keywords: algebraic inequalities, problem solving.**
**MSC: 26D05.**

**I was going through an "article" on the "Buffalo Way", where the author said that one should *NEVER* use the Buffalo Way for proving inequalities in actual real-time contests as it is "highly inelengant". What is the reason behind this notion ? In Mathematics, there are a whole lot of ways to attempt a given question. If the BW provides a proof for some inequality, then why it is given the downvote ?[1]**
**As an answer to this question we will solve some inequalities by ''*Buffalo Way*'' (BW) method.**

**APPLICATION I. In [2], [3] and [5] was proved on many pages the following inequality:**

**If** $x, y, z, t \geq 0$**, then:**

$$(x+y+z+t)(xyz+xyt+xzt+yzt) \geq (-x+y+z+t)(x-y+z+t)(x+y-z+t)(x+y+z-t).$$

***Solution:*** Here we proved the inequality in few lines ! We use ''Buffalo Way'' (**BW**) method – see also [5]. We make substitutions:

(1) $$x = \min\{x, y, z, t\},\ y = x+u, z = x+v, t = x+w, \text{ wher } u, v, w \geq 0.$$

The inequality to prove becomes:

$$(3\sum u^2 - 2\sum uv)x^2 + \left(4\sum u^3 - 2\sum uv(u+v) + 2uvw\right)x + \sum u^4 - 2\sum u^2v^2 + uvw(u+v+w) \geq 0$$

We denote: $A = 3\sum u^2 - 2\sum uv$, $B = 4\sum u^3 - 2\sum uv(u+v) + 2uvw$,

$$C = \sum u^4 - 2\sum u^2v^2 + uvw(u+v+w);\ A = \sum u^2 + \sum (u-v)^2 \geq 0.$$

Since $\sum u^3 + 3uvw - \sum uv(u+v) \geq 0$ (*Schur*) and $\sum \frac{u+v}{w} = \sum\left(\frac{u}{v} + \frac{v}{u}\right) \geq 6 > 5 \Rightarrow$

$\Rightarrow \sum uv(u+v) - 5uvw \geq 0$, we obtain

$$B = 4(\sum u^3 + 3uvw - \sum uv(u+v)) + 2(\sum uv(u+v) - 5uvw) > 0.$$

By $\sum u^4 + uvw(u+v+w) \overset{Schur}{\geq} \sum uv(u^2+v^2) \overset{MA-MG}{\geq} 2\sum u^2v^2$ we deduce $C \geq 0$.

**APPLICATION II. If** $x, y, z, t \geq 0$**, then:**

$$4(x^3+y^3+z^3+t^3)+15(xyz+xyt+xzt+yzt) \geq (x+y+z+t)^3,$$

**(see the inequality 3 from [4], pag. 271).**

***Solution:*** Using (1) the inequality becomes

$12x^3+9\sum uv\cdot x^2+6\sum uv\cdot x+3\left(\sum u^3-\sum(u+v)uv+3uvw\right)\geq 0$, which is true because

$$\sum u^3-\sum uv(u+v)+3uvw \geq 0, \text{ (\textit{Schur})}.$$

**APPLICATION III. If** $x, y, z, t \geq 0$ **şi** $\lambda \geq \frac{11}{2}$ **, then**

$$\lambda(x^3+y^3+z^3+t^3)+(16-\lambda)(xyz+xyt+xzt+yzt) \geq (x+y+z+t)^3.$$

***Solution:*** Using (1) the inequality becomes

$$(\lambda-4)\sum uv(u+v)-(4\lambda-13)uvw = uvw\left[(\lambda-4)\sum\left(\frac{u}{v}+\frac{v}{u}\right)+13-4\lambda\right] \geq$$

$$\geq uvw(6\lambda-24+13-4\lambda)=(2\lambda-11)uvw \geq 0, \text{ the inequality is proved.}$$

**Remark.** For $\lambda=\frac{11}{2}$ we obtain a problem proposed by *Vasile Cârtoaje* (M.S. 2006).

**APPLICATION IV. If** $x, y, z, t \geq 0$ **şi** $\alpha, \beta \in R$ **such that** $\alpha \geq 4, \alpha+\beta \geq 16, 3\alpha+\beta \geq 27$ **, then**

$$\alpha(x^3+y^3+z^3+t^3)+\beta(xyz+xyt+xzt+yzt) \geq (x+y+z+t)^3.$$

**(see the inequality 41 from [4], pag. 374).**

***Solution:*** Using (1) the inequality becomes

$$4(\alpha+\beta-16)x^3+3(\alpha+\beta-16)\sum u\cdot x^2+\left[(3\alpha-12)\sum u^2+(2\beta-24)\sum uv\right]\cdot x+$$

$$+(\alpha-1)\sum u^3+(\beta-6)uvw-3\sum uv(u+v) \geq 0.$$

Since $3\alpha+2\beta=\frac{6\alpha+4\beta}{2}=\frac{3\alpha+\beta+3(\alpha+\beta)}{2} \geq \frac{75}{2} \geq 36$, we deduce:

$$(3\alpha-12)\sum u^2+(2\beta-24)\sum uv \geq (3\alpha+2\beta-36)\sum uv \geq 0.$$

Hence, from above yields the given inequality.On **BW** method see also [6].

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**pag. 15-16.**

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**[5]** https://artofproblemsolving.com/community/c6h605279

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## SPECIAL DIFFERENTIAL EQUATIONS

***By Samir Cabiyev-Azerbaijan***

**Abstract:** An initial condition for the wave equation is a problem of finding a solution that satisfies a given function in the form of an integral statement. Bringing the integral expression to the Bessel's function with the help of the Euler's integral. Using a new theorem, we determine the unknown coefficient and consider it in the starting problem.

**Key words and Phrases:** differential equation, Bessel's function , D'Alembert's formula, complex numbers, gamma function

### 1. INTRODUCTION

**In this article, we will look at the solution of the problem where the initial condition is an integral condition. Here, the initial condition has undefined coefficients, which are determined by an inequality that ensures that all the roots of the new equation are real. We have used time analysis, Bessel's function, Gamma function, and D'Alembert's formula, which simplify this starting condition. The D'Alembert's formula is defined as:**

$$u(x,t)=\frac{\varphi_0(x-at)+\varphi_0(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\varphi_1(z)dz,$$ here $u_{tt}+a^2u_{xx}=0$ *and* initial condition

$$u\Big|_{t=0}=\varphi_0(x),\quad \frac{\partial u}{\partial t}\Big|_{t=0}=\varphi_1(x)$$ [1]

## 2. A NEW THEOREM ABOUT THE DIFFERENTIAL EQUATIONS

**Theorem .** Suppose the following differential equation is given:

$ay^{IV} + by^{III} - cy^{I} - dy = 0$ here a$> 0$ $b >$ $c < 0$ $d < 0$ . For the following numbers, one condition is satisfied (b,c,d are taken positive):

$$b: c: d = a: 2a: a$$

A necessary and sufficient condition for satisfying all the roots of the characteristic equation corresponding to the differential equation is that the following inequality is true:

$$0 < \frac{a}{b} \leq 0.3125$$

**Proof:** The proof of the new theorem is given in the solution of the problem:

## 3. MAIN PROBLEM AND ITS SOLUTIONS

Consider the following issue:

$$u_{tt} + u_{xx} = 0 \quad (1)$$

for the wave equation

$$u(x,0) = \left(\frac{\pi x}{\alpha}\right)^{\frac{1}{\alpha}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\alpha}{\Gamma\left(k+\frac{1}{\alpha}\right)\Gamma\left(\nu+\frac{1}{\alpha}\right)} \left(\frac{x}{\alpha}\right)^{\alpha k+\nu} \int_0^{\frac{\pi}{2}} cos^{\alpha\nu} x sin^{\alpha k} x dx$$

here $\nu = \frac{1}{\alpha}$ (2)

$$u_t(x,0) = 0 \quad (3)$$

Note that α here is a positive number in the interval that ensures that all the roots of the characteristic equation corresponding to the following differential equation are reals:

$$ay^{IV} + by^{III} - cy^{I} - dy = 0 \quad \text{here} \quad a > 0 \ b > 0 \ c < 0 \ d < 0 \ . \quad (4)$$

The following condition is satisfied for the coefficients:

(b,c,d are taken positive)

$$b\colon c\colon d = a\colon 2a\colon a \quad \frac{a}{b}\epsilon(0;\frac{5}{4^{\alpha}}] \qquad (5)$$

**Solution of the problem:**

First, let's find α, which belongs to the part of the differential equation. Taking into account the coefficients parameter, let's construct the following characteristic equation:

$$\beta^4 + \frac{b}{a}\beta^3 - 2\frac{b}{a}\beta - \frac{b}{a} = 0 \qquad (6)$$

Let's solve this algebraic equation:

$$\frac{b}{a}(\beta^3 - 2\beta - 1) = -\beta^4$$

If we apply Bezu's theorem known from the algebra course, we get:

$$(\beta + 1)(\beta^2 - \beta - 1) = -\frac{a}{b}\beta^4$$

With technical conversions and this substitution, we find the following roots: $\frac{1}{\beta} + \frac{1}{\beta^2} = z$

$$\beta_{12} = \frac{1 \pm \sqrt{3+2\sqrt{1+\frac{4a}{b}}}}{1+\sqrt{1+\frac{4a}{b}}},$$ here roots always are positive.

$$\beta_{34} = \frac{1 \pm \sqrt{3 - 2\sqrt{1 + \frac{4a}{b}}}}{1 - \sqrt{1 + \frac{4a}{b}}}$$

where the following condition must be satisfied to ensure that all the roots are positive:

$3 - 2\sqrt{1 + \frac{4a}{b}} \geq 0$ from here $\frac{a}{b} \leq \frac{5}{16}$ (7) when a and b are positive $\frac{a}{b} > 0$ (8) 7-8 from inequalities we get that $\frac{a}{b}\epsilon(0;\frac{5}{16}]$ we will buy.That is, $a$=2. If we consider this in the initial condition (2), we get the integral expression as:

$$u(x\ 0) = \sqrt{\frac{\pi x}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{2}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{2k+\nu}\int_0^{\frac{\pi}{2}} cos^{2\nu}x sin^{2k} x dx 2k \qquad (9)$$

To bring this to the Bessel's function, let's get the following formula with the help of the Euler integrals known from the course of mathematical analysis [2]:

$$\int_0^{\frac{\pi}{2}} cos^p x sin^q x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (sin^2 x)^{\frac{q-1}{2}} (1 - sin^2 x)^{\frac{p-1}{2}} 2sinxcosxdx \xrightarrow{sin^2 x = u}$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} u^{\frac{q-1}{2}} (1-u)^{\frac{p-1}{2}} du = \frac{1}{2} B(\frac{q+1}{2} \quad \frac{p+1}{2}) = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} \qquad (10)$$

If we take p= 2ν q= 2k here and consider equation (10) in (9):

$$u(x\ 0) = \sqrt{\frac{\pi x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{2}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{2k+\nu} \frac{1}{2} \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(k+\nu+1)} = \sqrt{\frac{\pi x}{2}} J_\nu(x)$$

As we mentioned above, since we take $\nu = \frac{1}{2}$ for the half-power Bessel's function

$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} sinx$ taking into account that, we get the following problem for the wave equation:

$$u_{tt} + u_{xx} = 0 \quad u(x,0) = sinx \qquad u_t(x\ ,0) = 0$$

To solve this problem, if we take into account that the coefficient a is complex in the D'Alembert's formula and the wave equation in the finite domain and apply Euler's formula known from the course of complex analysis (according to condition 3, the integral part in the formula will be 0.

$$u(x,t) = \frac{sin(x-it) + sin(x+it)}{2} =$$

$$= \frac{\frac{e^{i(x-it)} - e^{-i(x-it)}}{2i} + \frac{e^{i(x+it)} - e^{-i(x+it)}}{2i}}{2} = \frac{e^{ix} - e^{-ix}}{2i} \frac{e^t + e^{-t}}{2} = sinxcht$$

**References:**

**[1]A.N. Tikhonov and A.A Samarskii "Equations of Mathematical Physics "**

**[2] Azerbaijani mathematicians – "Differential equation course"**

## SPIEKER'S CEVIANS IN THE GEOMETRY OF TRIANGLE

***By Bogdan Fuștei-Romania, Mohamed Amine Ben Ajiba-Morocco***

**Abstract :**

In this article, we establish new geometric inequalities in triangle involving Spieker's Cevians.

We consider $\Delta ABC$ with usual notations. Let $p_a, p_a, p_a$ be the Spieker's Cevians in $\Delta ABC$.

**Lemma 1.**

**Given two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in normalized barycentric coordinates in the plane of the triangle $ABC$. Then :**

$$PQ^2 = -a^2(y_1 - y_2)(z_1 - z_2) - b^2(z_1 - z_2)(x_1 - x_2) - c^2(x_1 - x_2)(y_1 - y_2).$$

The above formulae is well known (see [1, pp. 11]).

Note that the barycentric coordinates of $A$ and $S_p$ are

$$\text{A} = (1 : 0 : 0) \text{ and } S_p = (b + c : c + a : a + b).$$

If $D$ is the point of intersection of the lines $AS_p$ and $BC$, then we have

$D = (0 : c + a : a + b)$, and by using the Lemma 1, we have

$$p_a{}^2 = DA^2 = -a^2.\frac{c+a}{2a+b+c}.\frac{a+b}{2a+b+c} - b^2.\frac{a+b}{2a+b+c}.(-1) - c^2(-1).\frac{c+a}{2a+b+c}$$

$$\Rightarrow p_a{}^2 = \frac{b^3 + c^3 + a(b^2 + c^2)}{2a + b + c} - \frac{a^2(a+b)(a+c)}{(2a+b+c)^2}. \qquad (1')$$

**Lemma 2. For any triangle $ABC$, we have**

$$p_a{}^2 = s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}. \qquad (2')$$

**Proof.** Using the identity $(1')$, we have

$$p_a{}^2 = \frac{(b+c+2a)(b+c)^2 + [3(b+c)+2a](b-c)^2}{4(2a+b+c)} - \frac{a^2[(2a+b+c)^2 - (b-c)^2]}{4(2a+b+c)^2}$$

$$= \frac{(b+c)^2 - a^2}{4} + \frac{(2a+b+c)[3(b+c)+2a] + a^2}{4(2a+b+c)^2}.(b-c)^2$$

$$= s(s-a) + \frac{(a+b+c)(5a+3b+3c)}{4(2a+b+c)^2}.(b-c)^2 = s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}.$$

**Lemma 3. For any triangle $ABC$, we have**

$$m_a \le p_a \le n_a. \qquad (3')$$

**Equality holds if and only if $b = c$.**

**Proof.** We have the following known formulas

$$m_a{}^2 = s(s-a) + \frac{(b-c)^2}{4}, \qquad n_a{}^2 = s(s-a) + \frac{s(b-c)^2}{a}.$$

Since $\frac{s(3s+a)}{(2s+a)^2} = \frac{1}{4} + \frac{8s^2 - a^2}{4(2s+a)^2} \geq \frac{1}{4}$ and $\frac{3s+a}{(2s+a)^2} =$

$= \frac{1}{a} - \frac{4s^2 + sa}{a(2s+a)^2} \leq \frac{1}{a}$, then by using (2),

we obtain : $m_a \leq p_a \leq n_a$.

**Lemma 4. For any triangle $ABC$, we have**

$$(2s+a)(2s+b)(2s+c) = 2s(9s^2 + r^2 + 6Rr). \tag{4'}$$

$$\frac{1}{2s+a} + \frac{1}{2s+b} + \frac{1}{2s+c} = \frac{21s^2 + r^2 + 4Rr}{2s(9s^2 + r^2 + 6Rr)}. \tag{5'}$$

$$\sum_{cyc} (2s+a) p_a{}^2 = 2s.\frac{21s^4 - s^2(92Rr + 30r^2) - 64R^2r^2 - 28Rr^3 - 3r^4}{9s^2 + r^2 + 6Rr}. \tag{6'}$$

$$\sum_{cyc} (2s+a)^2 p_a{}^2 = 4s^2(3s^2 - r^2 - 16Rr). \tag{7'}$$

$$\sum_{cyc} (2s+a)^3 p_a{}^2 = 32s^3(s^2 + r^2 - 6Rr). \tag{8'}$$

**Proof**. With known identities [2, pp. 52],

$$a+b+c = 2s, \qquad abc = 4Rsr$$

$$a^2+b^2+c^2 = 2(s^2 - r^2 - 4Rr), \qquad ab+bc+ca = s^2 + r^2 + 4Rr$$

$$a^3+b^3+c^3 = 2s(s^2 - 3r^2 - 6Rr),$$

$$a^4+b^4+c^4 = 2s^4 - 4(4Rr + 3r^2)s^2 + 2(4R+r)^2r^2,$$

we obtain

$$(2s+a)(2s+b)(2s+c) = 8s^3 + 4s^2(a+b+c) + 2s(ab+bc+ca) + abc$$

$$= 2s(9s^2 + r^2 + 6Rr).$$

$$\frac{1}{2s+a} + \frac{1}{2s+b} + \frac{1}{2s+c} = \frac{12s^2 + 4s(a+b+c) + (ab+bc+ca)}{(2s+a)(2s+b)(2s+c)}$$
$$= \frac{21s^2 + r^2 + 4Rr}{2s(9s^2 + r^2 + 6Rr)}.$$

Now, since we have

$$a^2(a+b)(a+c) = a^2(2sa+bc) = a(2sa^2+4sRs)$$

$$= [(2s+a)-2s].2s[4s^2+2Rr-(2s+a)(2s-a)]$$

$$= 2s(2s+a)(a^2+4s^2+2Rr-2sa)-8s^2(2s^2+Rr).$$

then, by using the identity $(1')$, we obtain

$$(2s+a){p_a}^2 = b^3+c^3+a(b^2+c^2)-\frac{a^2(a+b)(a+c)}{2s+a}$$

$$= b^3+c^3+a(b^2+c^2)-2s(a^2+4s^2+2Rr-2sa)+\frac{8s^2(2s^2+Rr)}{2s+a}.$$

Adding this identity with its similar ones and using the identity $(5')$, we obtain

$$\sum_{cyc}(2s+a){p_a}^2 = \sum a^3+\sum a^2.\sum a-2s\sum a^2-6s(4s^2+2Rr)+4s^2\sum a+$$

$$+8s^2(2s^2+Rr)\left(\frac{1}{2s+a}+\frac{1}{2s+b}+\frac{1}{2s+c}\right)$$

$$= 2s.\frac{21s^4-s^2(92Rr+30r^2)-64R^2r^2-28Rr^3-3r^4}{9s^2+r^2+6Rr}.$$

Now, by using the identity $(2')$, we have

$$(2s+a)^2{p_a}^2 = s(s-a)(2s+a)^2+s(3s+a)(b-c)^2$$

$$= 4s^4-2sa^3+3s^2(b^2+c^2-a^2-2bc)+s(a^2+b^2+c^2)a-2sabc,$$

Adding this identity with its similar ones, we obtain

$$\sum_{cyc}(2s+a)^3{p_a}^2 = 12s^4-2s\sum a^3+3s^2\sum(a^2-2bc)+s\sum a^2.\sum a-6sabc$$

$$= 4s^2(3s^2-r^2-16Rr).$$

Also, by using the identity $(2')$, we have

$$(2s+a)^3{p_a}^2 = s(s-a)(2s+a)^3+s(3s+a)(2s+a)(b-c)^2$$

$$= 8s^5+2s(2s^3-abc)a-10s^2a^3-2sa^4-12s^3(a^2+bc)$$
$$+(6s^3+5s^2a+sa^2)(a^2+b^2+c^2)-10s^2abc,$$

Adding this identity with its similar ones, we obtain

$$\sum_{cyc}(2s+a)^3{p_a}^2$$
$$=24s^5+2s(2s^3-abc)\sum a-10s^2\sum a^3-2s\sum a^4$$
$$-12s^3\sum(a^2+bc)+\sum a^2.\left(18s^3+5s^2\sum a+s\sum a^2\right)-30s^2abc$$
$$=32s^3(s^2+r^2-6Rr).$$

**Theorem 1. For any triangle $ABC$, we have**

$$\boldsymbol{p_a+p_b+p_c\le\frac{14R-r}{3}}.\qquad(9')$$

**Equality holds if and only if the triangle $ABC$ is equilateral**.

**Proof**. By using the CBS inequality and the results $(5')$ and $(6')$, we have

$$(p_a+p_b+p_c)^2$$
$$\le\left((2s+a){p_a}^2+(2s+b){p_b}^2+(2s+c){p_c}^2\right)\left(\frac{1}{2s+a}+\frac{1}{2s+b}+\frac{1}{2s+c}\right)$$
$$=\frac{[21s^4-s^2(92Rr+30r^2)-64R^2r^2-28Rr^3-3r^4](21s^2+r^2+4Rr)}{(9s^2+r^2+6Rr)^2}$$
$$=(21s^2+r^2+4Rr).f(s^2),$$

where $f(s^2)=\dfrac{21s^4-(92Rr+30r^2)s^2-64R^2r^2-28Rr^3-3r^4}{(9s^2+r^2+6Rr)^2}$.

We have $f'(s^2)=\dfrac{24(45Rr+13r^2)s^2+8(75R^2+29Rr+3r^2)r^2}{(9s^2+r^2+6Rr)^3}$
$>0$, then $f$ is increasing,

and by using Gerretsen′s inequality $s^2\le4R^2+4Rr+3r^2$, we obtain

$$(p_a+p_b+p_c)^2\le(21s^2+r^2+4Rr).f(s^2)$$
$$\le(84R^2+88Rr+64r^2).f(4R^2+4Rr+3r^2)$$
$$=\frac{16(441R^6+861R^5r+1132R^4r^2+805R^3r^3+524R^2r^4+212Rr^5+96r^6)}{(18R^2+21Rr+14r^2)^2}$$
$$=\left(\frac{14R-r}{3}\right)^2$$
$$-\frac{(R-2r)(15120R^4r+31608R^3r^2+36840R^2r^3+21121Rr^4+6814r^5)}{9(18R^2+21Rr+14r^2)^2}$$
$$\le\left(\frac{14R-r}{3}\right)^2,$$

the last line is true by Euler′s inequality

$R \geq 2r$, with equality if and only if $ABC$ is equilateral.

This completes the proof of Theorem 1.

**Theorem 2. For any triangle $ABC$, we have**

$$\boldsymbol{p_a p_b p_c \leq \frac{(8R-7r)s^2}{9}}. \qquad (10')$$

**Equality holds if and only if the triangle $ABC$ is equilateral**.

**Proof**. By using the AM − GM inequality, we have

$$(2s+a)^2{p_a}^2+(2s+b)^2{p_b}^2+(2s+c)^2{p_c}^2 \geq 3\sqrt[3]{\left((2s+a)(2s+b)(2s+c)p_a p_b p_c\right)^2}.$$

Using the identities (4′) and (7′), we obtain

$$(p_a p_b p_c)^2 \leq \frac{[(2s+a)^2{p_a}^2+(2s+b)^2{p_b}^2+(2s+c)^2{p_c}^2]^3}{27[(2s+a)(2s+b)(2s+c)]^2} = \frac{[4s^2(3s^2-r^2-16Rr)]^3}{27[2s(9s^2+r^2+6Rr)]^2}$$

$$= \frac{16s^4(3s^2-r^2-16Rr)}{243}.\left(1-\frac{54Rr+4r^2}{9s^2+r^2+6Rr}\right)^2,$$

and by using Gerretsen′s inequality $s^2 \leq 4R^2+4Rr+3r^2$, we obtain

$$(p_a p_b p_c)^2 \leq \frac{16s^4(12R^2-4Rr+8r^2)}{243}.\left(1-\frac{54Rr+4r^2}{36R^2+42Rr+28r^2}\right)^2$$

$$= \frac{256(3R^2-Rr+2r^2)^3}{27(18R^2+21Rr+14r^2)^2}.s^4.$$

To complete the proof it is enough to prove that

$$768(3R^2-Rr+2r^2)^3 \leq (8R-7r)^2(18R^2+21Rr+14r^2)^2,$$

which is equivalent to

$$r(R-2r)(32832R^4+8964R^3r+15180R^2r^2-8903Rr^3-1730r^4) \geq 0,$$

which is true by Euler′s inequality $R \geq 2r$, with equality if and only if $ABC$ is equilateral.

This completes the proof of Theorem 2.

**Theorem 3. For any triangle $ABC$, we have**

$$\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}\geq\frac{2}{R}. \qquad (11')$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Proof.** By using Hölder's inequality, we have

$$\left(\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}\right)^2\geq\frac{[(2s+a)+(2s+b)+(2s+c)]^3}{(2s+a)^3{p_a}^2+(2s+b)^3{p_b}^2+(2s+c)^3{p_c}^2}$$

$$=\frac{(8s)^3}{32s^3(s^2+r^2-6Rr)}=\frac{16}{s^2+r^2-6Rr},$$

and by using Gerretsen's inequality $s^2\leq 4R^2+4Rr+3r^2$,

and Euler's inequality $R\geq 2r$, we get

$$\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}\geq\frac{4}{\sqrt{4R^2-2r(R-2r)}}\geq\frac{2}{R},$$

with equality if and only if $ABC$ is equilateral. This completes the proof of Theorem 3.

The main aim of the following part is to establish a geometric inequality involving Speiker's

cev**ians angle bisectors and medians of a triangle**, **and applications of this inequality**.

**Theorem 4. For any triangle $ABC$, we have**

$$p_aw_a\leq{m_a}^2. \qquad (12')$$

**Equality holds if and only if $b=c$.**

**Proof.** By the formulas for median and angle bisector of triangle $ABC$,

$${m_a}^2=\frac{1}{4}(2b^2+2c^2-a^2) \text{ and } w_a=\frac{2\sqrt{bcs(s-a)}}{b+c}, \text{ we can easily get}$$

$${m_a}^2=s(s-a)+\frac{(b-c)^2}{4} \text{ and } {w_a}^2=s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}.$$

and by using the identity $(2')$, we can get

$${p_a}^2={m_a}^2+\frac{(8s^2-a^2)(b-c)^2}{4(2s+a)^2} \text{ and } {w_a}^2={m_a}^2-\frac{(8s^2-8sa+a^2)(b-c)^2}{4(2s-a)^2}.$$

Based on these results, we have

$$p_a{}^2 w_a{}^2 = \left(m_a{}^2 + \frac{(8s^2-a^2)(b-c)^2}{4(2s+a)^2}\right)\left(m_a{}^2 - \frac{(8s^2-8sa+a^2)(b-c)^2}{4(2s-a)^2}\right)$$

$$\leq m_a{}^4 - m_a{}^2\left(\frac{8s^2-8sa+a^2}{(2s-a)^2} - \frac{8s^2-a^2}{(2s+a)^2}\right)\frac{(b-c)^2}{4}$$

$$= m_a{}^4 - \frac{m_a{}^2[4sa(s-a)(4s+a)+a^4](b-c)^2}{2(4s^2-a^2)^2} \leq m_a{}^4,$$

the last line is true because $s > a$, with equality if and only if $b = c$.

This completes the proof of Lemma 5.

By Lemma 5 and some known identities a inequalities, we can get the following results

**Corollary 1. For any triangle $ABC$, we have**

$$p_a w_a + p_b w_b + p_c w_c \leq \frac{3}{4}(s^2 - r^2 - 4Rr). \tag{13'}$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 2. For any triangle $ABC$, we have**

$$p_a w_a + p_b w_b + p_c w_c \leq \frac{3}{2}(2R^2 + r^2). \tag{14'}$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 3. For any triangle $ABC$, we have**

$$a p_a w_a + b p_b w_b + c p_c w_c \leq \frac{s}{2}(s^2 + 5r^2 + 2Rr). \tag{15'}$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 4. For any triangle $ABC$, we have**

$$(b+c)p_a w_a + (c+a)p_b w_b + (a+b)p_c w_c \leq \frac{s}{2}(5s^2 - 11r^2 - 26Rr). \tag{16'}$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 5. For any triangle $ABC$, we have**

$$\sqrt{p_a w_a} + \sqrt{p_b w_b} + \sqrt{p_c w_c} \leq m_a + m_b + m_c. \tag{17'}$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 6. For any triangle $ABC$, we have**

$$\sqrt{p_a w_a}+\sqrt{p_b w_b}+\sqrt{p_c w_c}\le 4R+r. \qquad (18')$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 7. For any triangle $ABC$, we have**

$$w_a+w_b+w_c\le \frac{{m_a}^2}{p_a}+\frac{{m_b}^2}{p_b}+\frac{{m_c}^2}{p_c}\le m_a+m_b+m_c. \qquad (19')$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 8. For any triangle $ABC$, we have**

$$p_a+p_b+p_c\le \frac{{m_a}^2}{w_a}+\frac{{m_b}^2}{w_b}+\frac{{m_c}^2}{w_c}. \qquad (20')$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 9. For any triangle $ABC$, we have**

$$\frac{p_a}{m_a}+\frac{p_b}{m_b}+\frac{p_c}{m_c}\le \frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}. \quad (\textit{Soumava Chakraborty}) \qquad (21')$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 10. For any triangle $ABC$, we have**

$$\frac{p_a w_a}{h_a m_a}+\frac{p_b w_b}{h_b m_b}+\frac{p_c w_c}{h_c m_c}\le \frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\le \frac{3R}{2r}. \quad (\textit{Soumava Chakraborty}) \qquad (22')$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

**Corollary 11. For any triangle $ABC$, we have**

$$\frac{\sqrt{p_a w_a}}{h_a}+\frac{\sqrt{p_b w_b}}{h_b}+\frac{\sqrt{p_c w_c}}{h_c}\le \frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\le \frac{3R}{2r}. \qquad (23')$$

**Equality holds if and only if the triangle $ABC$ is equilateral.**

In this part, we will use the Theorem 4. to obtain new results;

After summation we obtain $\sum p_a l_a \le \sum m_a^2$

$$\sum m_a^2=\frac{3}{4}(a^2+b^2+c^2)\rightarrow \frac{4}{3}\sum p_a l_a \le a^2+b^2+c^2 \ (1)$$

We use well-known identity: $m_a^2=r_b r_c+\frac{1}{4}(b-c)^2$ (and analogs) and we obtain:

$$\frac{1}{2}\,|b\text{-}c\,|\ge\sqrt{p_a l_a-r_b r_c} \text{ (and analogs) } (2)$$

We use $4m_a^2 = n_a^2 + g_a^2 + 2r_br_c$ (and analogs) [4] and obtain:

$$\mathbf{n_a^2 + g_a^2 \geq 4p_al_a - 2r_br_c} \textbf{ (and analogs) (3)}$$

From (3) we obtain :

$$\mathbf{n_a + g_a \geq \sqrt{4p_al_a + 2n_ag_a - 2r_br_c}} \textbf{ (and analogs) (4)}$$

But $n_ag_a \geq m_al_a$ (and analogs) and from (4) we obtain:

$$\mathbf{n_a + g_a \geq \sqrt{4p_al_a + 2m_al_a - 2r_br_c}} \textbf{ (and analogs) (5)}$$

But $(b-c)^2=(n_a-g_a)^2+2(n_ag_a - r_br_c)$ (and analogs) and (2) we obtain:

$$\mathbf{n_a - g_a \geq \sqrt{|4p_al_a - 2n_ag_a - 2r_br_c|}} \textbf{ (and analogs) (6)}$$

From (4) and (6) after summation we obtain:

$$\mathbf{2n_a \geq \sqrt{4p_al_a + 2n_ag_a - 2r_br_c} + \sqrt{|4p_al_a - 2n_ag_a - 2r_br_c|}} \textbf{ (and analogs) (7)}$$

$$\mathbf{4m_a^2 = 2(b^2+c^2) - a^2 \geq 4p_al_a \to b^2+c^2 \geq \frac{1}{2}(4p_al_a + a^2)} \textbf{ (and analogs) (8)}$$

From (8) we obtain : $\frac{b}{c} + \frac{c}{b} \geq \frac{4p_al_a+a^2}{2bc}$ (and analogs)

But $\frac{\sin(A+\omega)}{\sin\omega} = \frac{b}{c} + \frac{c}{b}$ (and analogs)(Traian Lalescu)[5]

ω-Brocard angle in ΔABC, We obtain:

$$\frac{\sin(A+\omega)}{\sin\omega} \geq \frac{4p_al_a+a^2}{2bc} \textbf{ (and analogs)(9)}$$

From (9) we obtain:

$$\frac{1}{\sin\omega} \geq \frac{4p_al_a+a^2}{2bc} \textbf{ (and analogs)(10)}$$

Triangle ABC with sides a, b, c and triangle with sides $m_a, m_b, m_c$ have same Brocard angle; [6] From $\frac{1}{\sin\omega} \geq \frac{\sin(A+\omega)}{\sin\omega} = \frac{b}{c} + \frac{c}{b}$ and theorem above we obtain:

$\frac{1}{\sin\omega} \geq \frac{m_b}{m_c} + \frac{m_c}{m_b}$ (and analogs) and from (10) we obtain:

$$\frac{1}{\sin\omega} \geq \frac{1}{2}\left(\frac{4p_al_a+a^2}{2bc} + \frac{m_b}{m_c} + \frac{m_c}{m_b}\right) \textbf{ (and analogs) (11)}$$

$$\frac{1}{\sin\omega} \geq \frac{1}{2}\left(\frac{4p_al_a+a^2}{2bc} + \frac{m_b}{m_a} + \frac{m_a}{m_b}\right) \textbf{ (and analogs) (12)}$$

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{4p_al_a+a^2}{2bc}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right)} \textbf{ (and analogs) (13)}$$

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{4p_a l_a + a^2}{2bc}\left(\frac{m_b}{m_a} + \frac{m_a}{m_b}\right)} \textbf{(and analogs) (14)}$$

From $p_a l_a \leq m_a^2$ (and analogs) and $\frac{m_a^2}{h_a^2}=1+\frac{(b^2-c^2)^2}{16S^2}$(and analogs) and

$2S=ah_a = bh_b = ch_c$ , $b^2 - c^2 = (b+c)(b-c)$ we obtain:

$$\frac{b+c}{2a}\,|b\text{-}c\,| \geq \sqrt{p_a l_a - h_a^2} \textbf{ (and analogs) (15)}$$

From $p_a l_a \leq m_a^2$ (and analogs) and $\sum\frac{m_a^2}{h_a^2}=1+\frac{1}{2\sin^2\omega}$ we obtain:

$$1+\frac{1}{2\sin^2\omega} \geq \sum\frac{p_a l_a}{h_a^2} \textbf{ (16)}$$

From Catalan inequality:

$$a^2b\,(a-b)+b^2c\,(b-c) + c^2a\,(c-a) \geq 0$$

$$\rightarrow a^3b + b^3c + c^3a \geq a^2b^2 + b^2c^2 + c^2a^2 \rightarrow \frac{a^3b+b^3c+c^3a}{4S^2} \geq \frac{a^2b^2+b^2c^2+c^2a^2}{4S^2} = \frac{1}{\sin^2\omega}$$

$$\rightarrow \frac{ab}{h_a^2} + \frac{bc}{h_b^2} + \frac{ca}{h_c^2} \geq \frac{1}{\sin^2\omega} \text{ and (16) we obtain:}$$

$$1+\frac{1}{2}\left(\frac{ab}{h_a^2} + \frac{bc}{h_b^2} + \frac{ca}{h_c^2}\right) \geq \sum\frac{p_a l_a}{h_a^2} \textbf{(17)}$$

From $4m_a=2\sqrt{2(b^2+c^2) - a^2}$ (and analogs) and using AM-GM we obtain:

$4m_a\sqrt{a^2+b^2+c^2} \leq 2(b^2+c^2) - a^2 + a^2 + b^2 + c^2 = 3(b^2+c^2)$ and

$m_a \geq \sqrt{p_a l_a}$(and analogs) we obtain:

$$\frac{4}{3}\sqrt{p_a l_a(a^2+b^2+c^2)} \leq b^2 + c^2 \textbf{ (and analogs)(18)}$$

We know that $bc=2Rh_a$(and analogs) and from (18) we obtain:

$$2\,\frac{\sqrt{p_a l_a}}{h_a}\,\frac{\sqrt{a^2+b^2+c^2}}{3R} \leq \frac{\sin(A+\omega)}{\sin\omega} = \frac{b}{c} + \frac{c}{b} \textbf{ (19)}$$

From (19) we obtain:

$$2\max\left\{\frac{\sqrt{p_a l_a}}{h_a}, \frac{\sqrt{p_b l_b}}{h_b}, \frac{\sqrt{p_c l_c}}{h_c}\right\}\frac{\sqrt{a^2+b^2+c^2}}{3R} \leq \frac{1}{\sin\omega} \textbf{ (20)}$$

From (19) after summation we obtain:

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\sum\frac{\sqrt{p_a l_a}}{h_a} \leq \frac{b+c}{a} + \frac{a+c}{b} + \frac{a+b}{c} \textbf{ (21)}$$

From (21) and $2S=ah_a = bh_b = ch_c$ , $\frac{b}{c} = \frac{h_c}{h_b}$ (and analogs) we obtain:

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\sum\frac{\sqrt{p_a l_a}}{h_a}\le\sum\frac{h_b+h_b}{h_a}\ (22)$$

From (19) we obtain:

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\sum\frac{\sqrt{p_a l_a}}{h_a}\le\sum\frac{\sin(A+\omega)}{\sin\omega}\ (23)$$

We proved $\frac{1}{\sin\omega}\ge\frac{m_b}{h_b}+\frac{m_c}{h_c}$ (and analogs) [7,8] and from $m_a\ge\sqrt{p_a l_a}$ (and analogs) we obtain:

$$\frac{1}{\sin\omega}\ge\frac{\sqrt{p_b l_b}}{h_b}+\frac{\sqrt{p_c l_c}}{h_c}\ \text{(and analogs) (24)}$$

From $l_a=\frac{2\sqrt{bc}}{b+c}\sqrt{p(p-a)}=\frac{2\sqrt{bc}}{b+c}\sqrt{r_b r_c}\to\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)=\frac{\sqrt{r_b r_c}}{l_a}$ (and analogs)

$4\frac{r_b r_c}{l_a^2}=2+\frac{b}{c}+\frac{c}{b}=2+\frac{\sin(A+\omega)}{\sin\omega}$ and using (9) we obtain:

$$4\frac{r_b r_c}{l_a^2}\ge2+\frac{4p_a l_a+a^2}{2bc}\text{(and analogs) (25)}$$

From $m_a l_a\ge p(p-a)=r_b r_c$(and analogs)(Panaitopol) and (25) we obtain:

$$4\frac{m_a}{l_a}\ge2+\frac{4p_a l_a+a^2}{2bc}\text{(and analogs) (26)}$$

$(a^2+b^2)(a^2+c^2)\ge(ab+ac)^2$(C.B.S inequality)

$m_b\ge\frac{a^2+c^2}{4R}$(and analogs) (Tereshin Inequality), $ac=2Rh_b$(and analogs)

After simple manipulations we obtain:

$2\sqrt{m_b m_c}\ge h_b+h_c$ (and analogs) and using $m_a\ge\sqrt{p_a l_a}$ (and analogs) we obtain after summation:

$$3\sqrt{m_b m_c}\ge h_b+h_c+\sqrt[4]{p_c l_c p_b l_b}\ \text{(and analogs) (27)}$$

From $n_a+g_a\ge2m_a$(and analogs)[9]

$2m_a\ge2\sqrt{p_a l_a}$ (and analogs) we obtain:

$$n_a+g_a\ge2\sqrt{p_a l_a}\text{(and analogs)(28)}$$

From (28) and $n_a+g_a\ge2\sqrt{n_a g_a}$(and analogs) after summation we obtain:

$$n_a+g_a\ge\sqrt{p_a l_a}+\sqrt{n_a g_a}\text{(and analogs) (29)}$$

From $\frac{1}{2}(|b-c|+|a-c|+|b-a|)=\max(a,b,c)-\min(a,b,c)$ and (2) we obtain:

$$\max(a,b,c)-\min(a,b,c)\ge\sum\sqrt{p_a l_a-r_b r_c}\text{(30)}$$

From $m_a l_a \ge p(p-a) = r_b r_c$(and analogs)(Panaitopol) and (2) we obtain:

$$\frac{1}{2}\,|b\text{-}c\,| \ge \sqrt{l_a(p_a - m_a)}\text{(and analogs) (31)}$$

From (31) after summation we obtain:

$$\max(a,b,c) - \min(a,b,c) \ge \sum \sqrt{l_a(p_a - m_a)}\text{(32)}$$

We use the identity: $\frac{n_a^2}{h_a^2} = 1 + \frac{(b-c)^2}{4r^2}$(and analogs) and (2) and obtain:

$$\frac{n_a}{h_a} \ge \frac{\sqrt{r^2 + p_a l_a - r_b r_c}}{r}\text{(and analogs) (33)}$$

From (33) and $m_a l_a \ge p(p-a) = r_b r_c$(and analogs)(Panaitopol) we obtain:

$$\frac{n_a}{h_a} \ge \frac{\sqrt{r^2 + l_a(p_a - m_a)}}{r}\text{(and analogs) (34)}$$

From (33) and $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$ after summation we obtain:

$$1 \ge \sum \frac{\sqrt{r^2 + p_a l_a - r_b r_c}}{n_a}\text{ (35)}$$

From (35) and $m_a l_a \ge p(p-a) = r_b r_c$(and analogs)(Panaitopol) we obtain:

$$1 \ge \sum \frac{\sqrt{r^2 + l_a(p_a - m_a)}}{n_a}\text{(36)}$$

From (33) $\rightarrow \frac{n_a}{\sqrt{r^2 + p_a l_a - r_b r_c}} \ge \frac{h_a}{r}$ and after summation we obtain:

$$\sum \frac{n_a}{\sqrt{r^2 + p_a l_a - r_b r_c}} \ge \frac{h_a + h_b + h_c}{r}\text{ (37)}$$

2S=2pr=$ah_a \rightarrow (a+b+c)r = ah_a \rightarrow \frac{h_a}{r} = 1 + \frac{b+c}{a}$ (and analogs)

$$\rightarrow \frac{n_a}{\sqrt{r^2 + p_a l_a - r_b r_c}} \ge 1 + \frac{b+c}{a}\text{ (and analogs)(38)}$$

From (38) we obtain:

$$\prod\left(\frac{n_a}{\sqrt{r^2 + p_a l_a - r_b r_c}} - 1\right) \ge \frac{(a+b)(b+c)(a+c)}{abc}\text{(39)}$$

From $l_a = \frac{2\sqrt{bc}}{b+c}\sqrt{p(p-a)} = \frac{2\sqrt{bc}}{b+c}\sqrt{r_b r_c} \rightarrow l_a l_b l_c = \frac{8abc}{(a+b)(b+c)(a+c)} r_a r_b r_c$ and using (39)

we obtain:

$$\frac{1}{8}\prod\left(\frac{n_a}{\sqrt{r^2 + p_a l_a - r_b r_c}} - 1\right) \ge \frac{r_a r_b r_c}{l_a l_b l_c}\text{(40)}$$

$\frac{R}{2r} = \frac{r_a r_b r_c}{h_a h_b h_c} = \frac{r_a r_b r_c}{l_a l_b l_c}\,\frac{l_a l_b l_c}{h_a h_b h_c}$ and using (40) we obtain:

$$\frac{\mathbf{l_a l_b l_c}}{\mathbf{h_a h_b h_c}}\prod\left(\frac{\mathbf{n_a}}{\sqrt{\mathbf{r^2+p_a l_a-r_b r_c}}}-\mathbf{1}\right)\geq\frac{\mathbf{4R}}{\mathbf{r}}\ \mathbf{(41)}$$

We proved $\frac{R}{r}\geq\frac{m_b}{h_c}+\frac{m_c}{h_b}$ (and analogs)[8] and using (41) we obtain:

$$\frac{\mathbf{l_a l_b l_c}}{\mathbf{h_a h_b h_c}}\prod\left(\frac{\mathbf{n_a}}{\sqrt{\mathbf{r^2+p_a l_a-r_b r_c}}}-\mathbf{1}\right)\geq \mathbf{4max}\left\{\frac{\mathbf{m_b}}{\mathbf{h_c}}+\frac{\mathbf{m_c}}{\mathbf{h_b}},\frac{\mathbf{m_b}}{\mathbf{h_a}}+\frac{\mathbf{m_a}}{\mathbf{h_b}},\frac{\mathbf{m_a}}{\mathbf{h_c}}+\frac{\mathbf{m_c}}{\mathbf{h_a}}\right\}\mathbf{(42)}$$

From $r_a=\frac{S}{p-a}$ ; $p-a=\frac{b+c-a}{2}\rightarrow r_a=\frac{2S}{b+c-a}$ ;$2S=ah_a\rightarrow\frac{r_a}{h_a}=\frac{a}{b+c-a}$ (and analogs)

$\frac{h_a}{r_a}=\frac{b+c-a}{a}\rightarrow\frac{b+c}{a}=1+\frac{h_a}{r_a}$ (and analogs) and using (38) we obtain:

$$\frac{\mathbf{n_a}}{\sqrt{\mathbf{r^2+p_a l_a-r_b r_c}}}-\mathbf{2}\geq\frac{\mathbf{h_a}}{\mathbf{r_a}}\ \textbf{(and analogs)(43)}$$

$$\frac{\mathbf{r_a}}{\mathbf{h_a}}\geq\left[\frac{\mathbf{n_a}}{\sqrt{\mathbf{r^2+p_a l_a-r_b r_c}}}-\mathbf{2}\right]^{-1}\textbf{(and analogs)}\rightarrow\frac{\mathbf{R}}{\mathbf{2r}}\geq\prod\left[\frac{\mathbf{n_a}}{\sqrt{\mathbf{r^2+p_a l_a-r_b r_c}}}-\mathbf{2}\right]^{-1}\mathbf{(44)}$$

$\sin^2\frac{A}{2}=\frac{r}{2R}\frac{r_a}{h_a}=\frac{r_a-r}{4R}$ (and analogs) $\rightarrow\frac{r_a}{h_a}=\frac{r_a-r}{2r}$ (and analogs)

$r_a+r_b+r_c=4R+r\rightarrow\sum\frac{r_a}{h_a}=\frac{2R-r}{r}$ ,we obtain:

$$\frac{\mathbf{2R}}{\mathbf{r}}\geq\mathbf{1}+\sum\left[\frac{\mathbf{n_a}}{\sqrt{\mathbf{r^2+p_a l_a-r_b r_c}}}-\mathbf{2}\right]^{-1}\mathbf{(45)}$$

From (44) and (45) we obtain:

$\left(\frac{R}{r}\right)^2\geq q_1q_2$ (46) where $q_1=\prod\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}$

$$q_2=1+\sum\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}$$

From $\frac{r_a}{h_a}\geq\left[\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-2\right]^{-1}$ (and analogs) and $r_a+r_b+r_c=4R+r$ we obtain: $4R+$

$$\mathbf{r}\geq\sum\mathbf{h_a}\left[\frac{\mathbf{n_a}}{\sqrt{\mathbf{r^2+p_a l_a-r_b r_c}}}-\mathbf{2}\right]^{-1}\mathbf{(47)}$$

Can be proved that: $\frac{b+c}{a}=\frac{r_a+r}{r_a-r}$ (and analogs) and using (38) we obtain:

$$\left(\frac{\mathbf{n_a}}{\sqrt{\mathbf{r^2+p_a l_a-r_b r_c}}}-\mathbf{1}\right)(\mathbf{r_a}-\mathbf{r})\geq\mathbf{r_a}+\mathbf{r}\ \textbf{(and analogs) (48)}$$

From (48) after summation we obtain:

$$\sum\left(\frac{\mathbf{n_a}}{\sqrt{\mathbf{r^2+p_a l_a-r_b r_c}}}-\mathbf{1}\right)(\mathbf{r_a}-\mathbf{r})\geq\mathbf{4(R+r)}\ \mathbf{(49)}$$

Now we use the well-known relation:

$\cos\frac{B-C}{2}=\frac{h_a}{l_a}=\frac{b+c}{a}\sin\frac{A}{2}$(and analogs) and using (38) we obtain:

$$\mathbf{\sin\frac{A}{2}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\frac{h_a}{l_a}\ (and\ analogs)\ (50)}$$

From (50) after summation we obtain:

$$\mathbf{\sum\sin\frac{A}{2}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\sum\frac{h_a}{l_a}(51)}$$

We use $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs) [10] and the inequality:

$\sqrt{\frac{2R}{r}}\geq\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\frac{p\sqrt{3}}{h_a+h_b+h_c}$ [11], we obtain:

$$\mathbf{\frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\sqrt{\frac{h_a}{r_a}}\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\frac{p\sqrt{3}}{h_a+h_b+h_c}\right)(and\ analogs)\ (52)}$$

From (52) after summation we obtain:

$$\mathbf{\sum\frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\frac{p\sqrt{3}}{h_a+h_b+h_c}\right)\sum\sqrt{\frac{h_a}{r_a}}(53)}$$

Using (38), $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs),

$\sqrt{\frac{2R}{r}}\geq\frac{p\sqrt{3}}{h_a+h_b+h_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_a r_c}}{l_a+l_b+l_c}$ [11] we obtain:

$$\mathbf{\sum\frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\left(\frac{p\sqrt{3}}{h_a+h_b+h_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_a r_c}}{l_a+l_b+l_c}\right)\sum\sqrt{\frac{h_a}{r_a}}(54)}$$

Using (38), $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs),

$\sqrt{\frac{2R}{r}}\geq\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_a r_c}}{l_a+l_b+l_c}$ [11], we obtain:

$$\mathbf{\sum\frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_a r_c}}{l_a+l_b+l_c},\right)\sum\sqrt{\frac{h_a}{r_a}}(55)}$$

Using (38), $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs), $\sqrt{\frac{2R}{r}}\geq\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}$(and analogs) [11] we obtain:

$$\mathbf{\sum\frac{l_a}{h_a}\left(\frac{n_a}{\sqrt{r^2+p_a l_a-r_b r_c}}-1\right)\geq\sum\sqrt{\frac{h_a}{r_a}}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)(56)}$$

From (15), $\frac{b+c}{a}=\sqrt{\frac{2R}{r}\frac{h_a}{l_a}}\sqrt{\frac{h_a}{r_a}}$ (and analogs) we obtain:

$$\sqrt{\frac{R}{2r}}\ |b\text{-}c\ |\geq\frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}(p_a l_a-h_a^2)}\text{(and analogs) (57)}$$

From (57) after summation we obtain:

$$\sqrt{\frac{R}{2r}}(\ |b-c\ |+\ |a-c\ |+\ |b-a\ |)\geq\sum\frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}(p_a l_a-h_a^2)}(58)$$

$\frac{1}{2}(\ |b-c\ |+\ |a-c\ |+\ |b-a\ |)=\max(a,b,c)-\min(a,b,c)$ we obtain:

$$\sqrt{\frac{2R}{r}}[\max(a,b,c)\ -\min(a,b,c)\ ]\geq\sum\frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}(p_a l_a-h_a^2)}(59)$$

From (9) and $2\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right)\geq\frac{\sin(A+\omega)}{\sin\omega}$(and analogs)[7] we obtain:

$$2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}\geq 1+\frac{4p_a l_a+a^2}{4bc}\text{(and analogs) (60)}$$

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## SOME SPECIAL DEFINITE INTEGRALS

***By D.M. Bătineţu-Giurgiu, Mihály Bencze, Daniel Sitaru and Neculai Stanciu-Romania***

**ABSTRACT:** In this paper we present some special definite integrals.

**Theorem 1.** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(1+\ln x)\sin x - x\cos x \ln x}{\sin^2 x + x^2 \ln^2 x} dx = \arctan \frac{3\sqrt{3}+2\pi \ln \frac{\pi}{3}}{3\sqrt{3}-2\pi \ln \frac{\pi}{3}} - \arctan \frac{2\sqrt{2}+\pi \ln \frac{\pi}{4}}{2\sqrt{2}-\pi \ln \frac{\pi}{4}}$.

***Proof.***

$$\frac{(1+\ln x)\sin x - x\cos x \ln x}{\sin^2 x + x^2 \ln^2 x} = \frac{(\cos x + \ln x + 1)(\sin x - x\ln x)}{2\sin^2 x + 2x^2 \ln^2 x} - \frac{(\cos x + \ln x + 1)(\sin x + x\ln x)}{2\sin^2 x + 2x^2 \ln^2 x} =$$

$$= \frac{\left(\frac{\sin x + x\ln x}{\sin x - x\ln x}\right)'}{1+\left(\frac{\sin x + x\ln x}{\sin x - x\ln x}\right)^2}.$$

**Hence,**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(1+\ln x)\sin x - x\cos x \ln x}{\sin^2 x + x^2 \ln^2 x} dx = \left(\arctan \frac{\sin x + x\ln x}{\sin x - x\ln x}\right)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \arctan \frac{\frac{\sqrt{3}}{2}+\frac{\pi}{3}\ln\frac{\pi}{3}}{\frac{\sqrt{3}}{2}-\frac{\pi}{3}\ln\frac{\pi}{3}} - \arctan \frac{\frac{\sqrt{2}}{2}+\frac{\pi}{4}\ln\frac{\pi}{4}}{\frac{\sqrt{2}}{2}-\frac{\pi}{4}\ln\frac{\pi}{4}} =$$

$$= \arctan \frac{3\sqrt{3}+2\pi \ln \frac{\pi}{3}}{3\sqrt{3}-2\pi \ln \frac{\pi}{3}} - \arctan \frac{2\sqrt{2}+\pi \ln \frac{\pi}{4}}{2\sqrt{2}-\pi \ln \frac{\pi}{4}}.$$

**Theorem 2.** $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(1+\ln x)\cos x + x\sin x\ln x}{\cos^2 x + x^2\ln^2 x}dx = \arctan\frac{2\sqrt{2}+\pi\ln\frac{\pi}{4}}{2\sqrt{2}-\pi\ln\frac{\pi}{4}} - \arctan\frac{3+\pi\ln\frac{\pi}{6}}{3-\pi\ln\frac{\pi}{6}}$.

**Proof:**

$$\frac{(1+\ln x)\cos x + x\sin x\ln x}{\cos^2 x + x^2\ln^2 x} = \frac{(-\sin x+\ln x+1)(\cos x - x\ln x)}{2\cos^2 x + 2x^2\ln^2 x} + \frac{(\sin x+\ln x+1)(\cos x + x\ln x)}{2\cos^2 x + 2x^2\ln^2 x} =$$

$$= \frac{\left(\dfrac{\cos x + x\ln x}{\cos x - x\ln x}\right)'}{1+\left(\dfrac{\cos x + x\ln x}{\cos x - x\ln x}\right)^2}.$$

**Hence,**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(1+\ln x)\cos x + x\sin x\ln x}{\cos^2 x + x^2\ln^2 x}dx = \left(\arctan\frac{\cos x + x\ln x}{\cos x - x\ln x}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \arctan\frac{\frac{\sqrt{2}}{2}+\frac{\pi}{4}\ln\frac{\pi}{4}}{\frac{\sqrt{2}}{2}-\frac{\pi}{4}\ln\frac{\pi}{4}} - \arctan\frac{\frac{1}{2}+\frac{\pi}{6}\ln\frac{\pi}{6}}{\frac{1}{2}-\frac{\pi}{6}\ln\frac{\pi}{6}} =$$

$$= \arctan\frac{2\sqrt{2}+\pi\ln\frac{\pi}{4}}{2\sqrt{2}-\pi\ln\frac{\pi}{4}} - \arctan\frac{3+\pi\ln\frac{\pi}{6}}{3-\pi\ln\frac{\pi}{6}}.$$

**Theorem 3.** $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(1+\ln x)\sin x + x\cos x\ln x}{\sin^2 x + x^2\ln^2 x}dx = \arctan\frac{3\sqrt{3}+2\pi\ln\frac{\pi}{3}}{3\sqrt{3}-2\pi\ln\frac{\pi}{3}} - \arctan\frac{3+\pi\ln\frac{\pi}{6}}{3-\pi\ln\frac{\pi}{6}}$.

**Proof:**

$$\frac{(1+\ln x)\sin x + x\cos x\ln x}{\sin^2 x + x^2\ln^2 x} = \frac{(\cos x+\ln x+1)(\sin x + x\ln x)}{2\cos^2 x + 2x^2\ln^2 x} - \frac{(\cos x+\ln x+1)(\sin x + x\ln x)}{2\sin^2 x + 2x^2\ln^2 x} =$$

$$= \frac{\left(\dfrac{\sin x + x\ln x}{\sin x - x\ln x}\right)'}{1+\left(\dfrac{\sin x + x\ln x}{\sin x - x\ln x}\right)^2}.$$

**Hence,**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(1+\ln x)\sin x + x\cos x \ln x}{\sin^2 x + x^2 \ln^2 x} dx = \left( \arctan \frac{\sin x + x\ln x}{\sin x - x\ln x} \right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \arctan \frac{\frac{\sqrt{3}}{2} + \frac{\pi}{3}\ln\frac{\pi}{3}}{\frac{\sqrt{3}}{2} - \frac{\pi}{3}\ln\frac{\pi}{3}} - \arctan \frac{\frac{1}{2} + \frac{\pi}{6}\ln\frac{\pi}{6}}{\frac{1}{2} - \frac{\pi}{6}\ln\frac{\pi}{6}} =$$

$$= \arctan \frac{3\sqrt{3} + 2\pi \ln\frac{\pi}{3}}{3\sqrt{3} - 2\pi\ln\frac{\pi}{3}} - \arctan \frac{3 + \pi\ln\frac{\pi}{6}}{3 - \pi\ln\frac{\pi}{6}}.$$

**Theorem 4. If** $f: R \to R$**, satisfy** $f(0) = 2019$ **and** $3f(x) = f(x+y) + 2f(x-y) + y$ **for any** $x, y \in R$**, then** $\int_e^{\pi} f(x)dx = \frac{\pi - e}{2}(\pi + e + 4038)$.

**Proof: If we take** $y = x$**, then** $3f(x) = f(2x) + 2f(0) + x$**, (1).**

**If we take** $y = -x$**, then** $3f(x) = f(0) + 2f(x) - x$**, (2). From (1) and (2) we obtain that** $f(x) = x + f(0)$**, so** $f: R \to R$**,** $f(x) = x + 2019$**. Then,**

$$\int_e^{\pi} f(x)dx = \int_e^{\pi}(x+2019)dx = \left(\frac{x^2}{2} + 2019x\right)\Bigg|_e^{\pi} = \frac{1}{2}(\pi - e)(\pi + e) + 2019(\pi - e) =$$

$$= \frac{\pi - e}{2}(\pi + e + 4038).$$

**Theorem 5. If** $f: R \to R$**, satisfy** $f(x+2019) \le x \le f(x) + 2019$ **for any** $x \in R$**, then**

$$\int_e^{\pi} f(x^2)dx = \frac{\pi - e}{3}(\pi^2 + \pi e + e^2 - 6057).$$

**Proof: We have** $f(x) + 2019 \ge x \Leftrightarrow f(x) \ge x - 2019,\ \forall x \in R$**, (1).**

**Also we have** $f(x+2019) \le x$**, where we take** $x + 2019 = y$**, thus** $f(y) \le y - 2019, \forall y \in R$ **so** $f(x) \le x - 2019, \forall x \in R$**, (2). From (1) and (2) we obtain** $f(x) = x - 2019,\ \forall x \in R$**.**

**Therefore,** $\int_e^{\pi} f(x^2)dx =$

$$\int_e^{\pi}(x^2-2019)dx=\left(\frac{x^3}{3}-2019x\right)\Big|_e^{\pi}=\frac{1}{3}\left(\pi^3-e^3\right)-2019(\pi-e)=$$

$$=\frac{\pi-e}{3}\left(\pi^2+\pi e+e^2-6057\right).$$

**Theorem 6. If** $a\in R$, $f:R\to R$, **is a strictly increasing function and** $g:R\to R$ **be a function which satisfy** $f(g(x+a))\le f(x)\le f(g(x)+a)$ **for any** $x\in R$, **then**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} g(x)dx=\frac{\pi}{12}\left(\frac{7\pi}{24}-a\right).$$

**Proof: We have** $g(x)+a\ge x\Leftrightarrow g(x)\ge x-a\ \ \forall x\in R$, **(1).**

**Also we have** $g(x+a)\le x$, **where we take** $x+a=y$, **thus** $g(y)\le y-a,\forall y\in R$ **so** $g(x)\le x-a,\forall x\in R$, **(2). From (1) and (2) we obtain** $g(x)=x-a,\ \forall x\in R$.

**Hence,** $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} g(x)dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(x-a)dx=\left(\frac{x^2}{2}-ax\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\frac{\pi}{12}\left(\frac{7\pi}{24}-a\right).$

**Theorem 7.** $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos x+1-x^2}{(1+x\sin x)\sqrt{1-x^2}}dx=2\arcsin\frac{2(\pi+2\sqrt{2})}{8+\sqrt{2}\pi}.$

**Proof. We have** $\displaystyle\frac{\cos x+1-x^2}{(1+x\sin x)\sqrt{1-x^2}}=\frac{\cos^2 x+(1-x^2)\cos x}{(1+x\sin x)\sqrt{1-x^2}\cdot\cos x}=$

$$=\frac{1+x\sin x+\cos x+x\sin x\cos x-x\sin x-x^2\cos x-\sin^2 x-x\sin x\cos x}{(1+x\sin x)\sqrt{1+2x\sin x+x^2\sin^2 x-x^2-2x\sin x-\sin^2 x}}=$$

$$=\frac{(1+\cos x)(1+x\sin x)-(x+\sin x)(\sin x+x\cos x)}{\sqrt{(1+x\sin x)^2-(x+\sin x)^2}}=\frac{\left(\frac{x+\sin x}{1+x\sin x}\right)'}{\sqrt{1-\left(\frac{x+\sin x}{1+x\sin x}\right)^2}}$$ **, (1).**

**Also we have** $\left|\frac{x+\sin x}{1+x\sin x}\right|<1\Leftrightarrow (x+\sin x)^2<(1+x\sin x)^2\Leftrightarrow$

$\Leftrightarrow x^2+2x\sin x+x^2\sin^2 x<1+2x\sin x+x^2\sin^2 x\Leftrightarrow x^2<1\Leftrightarrow -1<x<1$**, (2).**

**From (1) and (2) we obtain**

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos x+1-x^2}{(1+x\sin x)\sqrt{1-x^2}}dx=\left(\arcsin\frac{x+\sin x}{1+x\sin x}\right)\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\arcsin\frac{\frac{\pi}{4}+\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}\cdot\frac{\pi}{4}}-\arcsin\frac{-\frac{\pi}{4}-\sin\frac{\pi}{4}}{1-\frac{\pi}{4}\left(\sin\left(-\frac{\pi}{4}\right)\right)}=$$

$$=2\arcsin\frac{\frac{\pi}{4}+\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}\cdot\frac{\pi}{4}}=2\arcsin\frac{2(\pi+2\sqrt{2})}{8+\sqrt{2}\pi}.$$

**Theorem 8. If** $f:R\to R$ **be a continuous function such that** $f(0)=2019$ **and there exists** $t\in(0,1)$ **such that** $f(x)-f(tx)=x^2+x$**, for any real** $x$**. Then,**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}f(x)dx=\frac{\pi^3}{648(1-t^2)}+\frac{5\pi^2}{288(1-t)}+\frac{2019\pi}{12}.$$

**Proof. If we take** $x=t^k y, y\in R, k\in N$**, then by** $f(x)-f(tx)=x^2+x$ **we obtain that** $f(t^k y)-f(t^{k+1}y)=t^{2k}y+t^k y$**. So,** $\sum_{k=0}^{n-1}\left(f(t^k x)-f(t^{k+1}x)\right)=\sum_{k=0}^{n-1}\left(t^{2k}x^2+t^k x\right)\Leftrightarrow$

$$\Leftrightarrow f(x)-f(t^n x)=x^2\cdot\frac{1-t^{2n}}{1-t^2}+x\cdot\frac{1-t^n}{1-t}\Leftrightarrow f(x)-\lim_{n\to\infty}f(t^n x)=x^2\lim_{n\to\infty}\frac{1-t^{2n}}{1-t^2}+x\lim_{n\to\infty}\frac{1-t^n}{1-t}\Leftrightarrow$$

$$\Leftrightarrow f(x)-f(0)=x^2\cdot\frac{1}{1-t^2}+x\cdot\frac{1}{1-t}\Leftrightarrow f(x)=x^2\cdot\frac{1}{1-t^2}+x\cdot\frac{1}{1-t}+f(0).$$

**Hence,** $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}f(x)dx=\left(\frac{1}{1-t^2}\cdot\frac{x^3}{3}+\frac{1}{1-t}\cdot\frac{x^2}{2}+2019x\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{4}}=$

$$=\frac{1}{3(1-t^2)}\left(\frac{\pi^3}{4^4}-\frac{\pi^3}{6^3}\right)+\frac{1}{2(1-t)}\left(\frac{\pi^2}{4^2}-\frac{\pi^2}{6^2}\right)+2019\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=$$

$$=\frac{\pi^3}{648(1-t^2)}+\frac{5\pi^2}{288(1-t)}+\frac{2019\pi}{12}.$$

**Theorem 9. If** $f:(0,\pi)\to R$ **with** $f'(x)=\frac{\cos 2020x}{\sin x}$ **for any real** $x\in(0,\pi)$**, then**

$$f(x)=\frac{2}{2019}\cos 2019x+\frac{2}{2017}\cos 2017x+...+\frac{2}{3}\cos 3x+2\cos x+\ln\left|\tan\frac{x}{2}\right|+C$$

**Proof. Let** $f_n:(0,\pi)\to R$**,** $f_n(x)=\int\frac{\cos nx}{\sin x}dx, \forall x\in(0,\pi)$**. We have that**

$$f_{n+2}(x)-f_n(x)=\int\frac{\cos(n+2)x-\cos nx}{\sin x}dx=-2\int\sin(n+1)x\,dx=\frac{2}{n+1}\cos(n+1)x+C \text{, so}$$

$$f_{k+2}(x)=f_k(x)+\frac{2}{k+1}\cos(k+1)x, \forall k\in N \text{, (1).}$$

$$f_0(x)=\int\frac{1}{\sin x}dx=\ln\left|\tan\frac{x}{2}\right|+C \text{, (2).}$$

**From (1) and (2) we obtain**

$$\sum_{k=0}^{2018}\left(f_{k+2}(x)-f_k(x)\right)=2\sum_{k=0}^{2018}\frac{1}{k+1}\cos(k+1)x \Leftrightarrow f_{2020}(x)-f_0(x)=2\sum_{k=0}^{2018}\frac{1}{k+1}\cos(k+1)x \Leftrightarrow$$

$$\Leftrightarrow f_{2020}(x)=f_0(x)+2\sum_{k=0}^{2018}\frac{1}{k+1}\cos(k+1)x=$$

$$=\frac{2}{2019}\cos 2019x+\frac{2}{2017}\cos 2017x+...+\frac{2}{3}\cos 3x+2\cos x+\ln\left|\tan\frac{x}{2}\right|+C.$$

**Hence,**

$$f(x)=\frac{2}{2019}\cos 2019x+\frac{2}{2017}\cos 2017x+...+\frac{2}{3}\cos 3x+2\cos x+\ln\left|\tan\frac{x}{2}\right|+C,$$

**where** $C$ **is an arbitrary real constant.**

## SOME LIMITS OF SEQUENCES OF BĂTINEŢU AND LALESCU TYPE

***By D.M. Bătineţu-Giurgiu, Mihály Bencze, Daniel Sitaru and Neculai Stanciu-Romania***

**ABSTRACT:** In this paper we present some limits of Bătineţu and Lalescu sequences.

**Theorem 1.** **If** $(a_n)_{n\geq1}$ **is a positive real sequence such that** $\lim\limits_{n\to\infty}\frac{a_n}{n}=a\in R_+^*$ **and** $\lim\limits_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n=b\in R_+^*$**, then** $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a\ln b$ **.**

***Proof.*** **We have** $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{a_{n+1}}{n+1}\cdot\frac{n}{a_n}\cdot\frac{n+1}{n}=a\cdot\frac{1}{a}\cdot1=1$**;**

**We denote** $u_n=\frac{a_{n+1}}{a_n}$ **and** $a_{n+1}-a_n=a_n\left(\frac{a_{n+1}}{a_n}-1\right)=a_n(u_n-1)=a_n\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=$

$$=\frac{a_n}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n,\ \forall n\in N^*, (1).$$

**We have** $\lim\limits_{n\to\infty}u_n=1$**, so** $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$**;** $\lim\limits_{n\to\aleph}u_n^n=\lim\limits_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n=b$**, (2).**

**From (1) and (2) we obtain** $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a\cdot1\cdot\ln b=a\ln b$ **.**

**Example. If** $a_n=\sqrt[n]{n!}$ **, then** $\lim\limits_{n\to\infty}\frac{a_n}{n}=\frac{1}{e}$**,** $\lim\limits_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n=e$ **and**

$\lim\limits_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\frac{1}{e}\ln e=\frac{1}{e}$**, i.e. Traian Lalescu limit.**

**Theorem 2. If** $(a_n)_{n\geq1}$ **and** $(b_n)_{n\geq1}$ **are positive real sequences such that**

$\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{n}=a\in R_+^*$ **and** $\lim\limits_{n\to\infty}\frac{b_{n+1}}{nb_n}=b\in R_+^*$**, then** $\lim\limits_{n\to\infty}\left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}-\frac{a_n}{\sqrt[n]{b_n}}\right)=\frac{ae}{2b}$ **.**

***Proof.*** **We have** $\lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{b_n}}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{n^n}{b_n}}=\lim\limits_{n\to\infty}\frac{(n+1)^{n+1}}{b_{n+1}}\cdot\frac{b_n}{n^n}=\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{n+1}\frac{b_n n}{b_{n+1}}=\frac{e}{b}$**;**

$$\lim_{n\to\infty}\frac{a_n}{n^2}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)^2-n^2}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n}\cdot\frac{n}{2n+1}=\frac{a}{2}.$$

$$\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}-\frac{a_n}{\sqrt[n]{b_n}}=\frac{a_n}{\sqrt[n]{b_n}}(u_n-1)=\frac{a_n}{\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=\frac{a_n}{n\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n=$$

$$=\frac{a_n}{n^2}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n$$ **; where we denote** $$u_n=\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{\sqrt[n]{b_n}}{a_n}=$$

$$=\frac{a_{n+1}}{(n+1)^2}\cdot\frac{n^2}{a_n}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{\sqrt[n]{b_n}}{n}\cdot\frac{n+1}{n}.$$

**Therefore,** $\lim_{n\to\infty}u_n=\frac{a}{2}\cdot\frac{2}{a}\cdot\frac{e}{b}\cdot\frac{b}{e}\cdot 1=1$**, so** $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$**,** $\lim_{n\to\aleph}u_n^n=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n\frac{b_n}{b_{n+1}}\sqrt[n+1]{b_{n+1}}=$

$$\lim_{n\to\infty}\left(\frac{b_n n}{b_{n+1}}\cdot\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n+1}{n}\cdot\left(\frac{a_{n+1}}{a_n}\right)^n\right)=\frac{1}{b}\cdot\frac{b}{e}\cdot 1\cdot\lim_{n\to\infty}\left(\left(\left(1+\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right)^{\frac{a_{n+1}-a_n}{n}\cdot\frac{n^2}{a_n}}\right)=$$

$$=\frac{1}{e}\cdot e^{a\cdot\frac{2}{a}}=e.$$ **Hence,** $$\lim_{n\to\infty}\left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}-\frac{a_n}{\sqrt[n]{b_n}}\right)=\frac{a}{2}\cdot\frac{e}{b}\cdot 1\cdot\ln e=\frac{ae}{2b}.$$

**Theorem 3.** $$\lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}-\frac{\sqrt[n]{n!}}{n}\right)=0.$$

***Proof.*** **It is well-hnown that** $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$ .

**If we denote** $u_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\left(\frac{n}{n+1}\right)$**, then** $\lim_{n\to\infty}u_n=1$**,** $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$ **and**

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!}{n!}\left(\frac{n}{n+1}\right)^n\frac{n+1}{\sqrt[n+1]{(n+1)!}}=\frac{1}{e}\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}=\frac{1}{e}\cdot e=1.$$

$$\lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}-\frac{\sqrt[n]{n!}}{n}\right)=\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)^2\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n=\frac{1}{e^2}\cdot 1\cdot\ln 1=0.$$

**Theorem 4.** $\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^2}-\frac{\sqrt[n]{n!}}{n^2}\right)=-\frac{4}{e^3}$.

***Proof.*** **It is well-hnown that** $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$ **and respectively** $\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{2}{e}$.

**If we denote** $u_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\left(\frac{n}{n+1}\right)^2$**, then** $\lim_{n\to\infty}u_n=1$**,** $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$ **and**

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!}{n!}\left(\frac{n}{n+1}\right)^{2n}\frac{1}{\sqrt[n+1]{(n+1)!}}=\frac{1}{e^2}\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}=\frac{1}{e}$$

**Hence,**

$$\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^2}-\frac{\sqrt[n]{n!}}{n^2}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^2\frac{\sqrt[n]{n!}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n=\frac{4}{e^2}\cdot\frac{1}{e}\cdot 1\cdot\ln\left(\frac{1}{e}\right)=-\frac{4}{e^3}.$$

**Theorem 5. If** $(a_n)_{n\geq 1}$ **is a positive real sequence such that** $\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}=a\in R_+^*$**, then** $\lim_{n\to\infty}(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n})=\frac{a}{e^2}$.

***Proof.*** $\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^n}\cdot\frac{n^n}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}\cdot\left(\frac{n}{n+1}\right)^{n+1}\cdot\frac{\sqrt[n]{n!}}{n}=$

$$=\lim_{n\to\infty}\frac{a}{e}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'A}{=}\frac{a}{e}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\frac{a}{e}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{a}{e^2}.$$

$$\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}=\sqrt[n]{a_n}\cdot(u_n-1)=\frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\text{, where}$$

$$u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}=\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n+1}{n}\cdot\frac{n}{\sqrt[n]{a_n}},\forall n\geq 2.$$

$$\lim_{n\to\infty}u_n=1,\ \lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1,\ \lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{n}{n+1}\cdot\frac{\sqrt[n]{n!}}{n}=$$

$$= a\cdot\frac{e^2}{a}\cdot 1\cdot\frac{1}{e} = e$$. **Hence,** $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\frac{a}{e^2}\cdot 1\cdot\ln e=\frac{a}{e^2}$.

**Theorem 6. If** $(a_n)_{n\ge 1}$ **is a positive real sequence such that** $\lim\limits_{n\to\infty}\frac{a_{n+1}}{na_n\sqrt[n]{n!}}=a\in R_+^*$**, then:**

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\frac{2a}{e^3}.$$

***Proof.*** $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n n^2}=\lim\limits_{n\to\infty}\frac{a_{n+1}}{na_n\sqrt[n]{n!}}\cdot\frac{\sqrt[n]{n!}}{n}=a\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{a}{e}$.

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^2}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{2n}}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{2(n+1)}}\cdot\frac{n^{2n}}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n n^2}\cdot\left(\frac{n}{n+1}\right)^{2n+2}=\frac{a}{e}\cdot\frac{1}{e^2}=\frac{a}{e^3}.$$

$$\frac{1}{n}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\frac{\sqrt[n]{a_n}}{n}\cdot(u_n-1)=\frac{\sqrt[n]{a_n}}{n^2}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n$$ **, where**

$$u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}=\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^2}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{n^2}{\sqrt[n]{a_n}},\ \forall n\ge 2.$$

$\lim\limits_{n\to\infty}u_n=1$, $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$, $\lim\limits_{n\to\infty}u_n^n=\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}=\lim\limits_{n\to\infty}\frac{a_{n+1}}{n^2a_n}\cdot\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}\cdot\left(\frac{n}{n+1}\right)^2=$

$$=\frac{a}{e}\cdot\frac{e^3}{a}\cdot 1=e^2$$. **Hence,** $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\frac{a}{e^3}\cdot 1\cdot\ln e^2=\frac{2a}{e^3}$.

**Theorem 7. If** $(a_n)_{n\ge 1}$ **is a positive real sequence such that** $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}=a\in R_+^*$ **and**

$x_n=a_1\cdot\sqrt{a_2}\cdot\sqrt[3]{a_3}\cdot\ldots\cdot\sqrt[n]{a_n}$ **, then** $\lim\limits_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\frac{a}{e^3}$.

***Proof.*** $\lim\limits_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{x_n}{n^n}}\overset{C-D'A}{=}\lim\limits_{n\to\infty}\frac{x_{n+1}}{(n+1)^n}\cdot\frac{n^n}{x_n}=\lim\limits_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}\lim\limits_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=$

$$=\frac{1}{e}\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C-D'A}{=}\frac{1}{e}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^n}\cdot\frac{n^n}{a_n}=\frac{1}{e}\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}\left(\frac{n}{n+1}\right)^{n+1}\frac{\sqrt[n]{n!}}{n}=$$

$$=\frac{1}{e}\cdot\frac{a}{e}\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'A}{=}\frac{a}{e^2}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\frac{a}{e^2}\cdot\frac{1}{e}=\frac{a}{e^3}.$$

**Theorem 8. If** $(a_n)_{n\geq 1}$ **is a positive real sequence such that** $\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}=a\in R_+^*$**,**

**then** $\lim_{n\to\infty}(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n})=\frac{2a}{e^2}$**.**

***Proof.***

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^n}\cdot\frac{n^n}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}\cdot\left(\frac{n}{n+1}\right)^{n+1}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}=$$

$$=\lim_{n\to\infty}\frac{a}{e}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D'A}{=}\frac{a}{e}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=\frac{a}{e}\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n=\frac{2a}{e^2}.$$

$$\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}=\sqrt[n]{a_n}\cdot(u_n-1)=\frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\text{, where}$$

$$u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}=\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n+1}{n}\cdot\frac{n}{\sqrt[n]{a_n}},\ \forall n\geq 2.$$

$$\lim_{n\to\infty}u_n=1,\ \lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1,$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{n}{n+1}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}=$$

$$=a\cdot\frac{e^2}{2a}\cdot 1\cdot\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{e^2}{2}\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}\overset{C-D'A}{=}\frac{e^2}{2}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$=\frac{e^2}{2}\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{e^2}{2}\cdot 2\cdot\frac{1}{e}=e.$$

**Hence,** $\lim_{n\to\infty}(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n})=\frac{2a}{e^2}\cdot 1\cdot\ln e=\frac{2a}{e^2}$**.**

**Theorem 9. If** $(a_n)_{n\geq 1}$ **is a positive real sequence such that** $\lim_{n\to\infty}\frac{a_{n+1}}{na_n\sqrt[n]{(2n-1)!!}}=a\in R_+^*$**,**

**then** $\lim_{n\to\infty}\frac{1}{n}(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n})=\frac{4a}{e^3}$**.**

***Proof.*** $\lim_{n\to\infty}\frac{a_{n+1}}{a_n n^2}=\lim_{n\to\infty}\frac{a_{n+1}}{na_n\sqrt[n]{(2n-1)!!}}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}=a\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=$

$$= a\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{C-D'A}{=} a\lim_{n\to\infty}\frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{n^n}{(n+1)^{n+1}} = a\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2a}{e}.$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^2} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{2n}}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{2(n+1)}}\cdot\frac{n^{2n}}{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n n^2}\cdot\left(\frac{n}{n+1}\right)^{2n+2} = \frac{2a}{e}\cdot\frac{1}{e^2} = \frac{2a}{e^3}.$$

$$\frac{1}{n}(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}) = \frac{\sqrt[n]{a_n}}{n}\cdot(u_n - 1) = \frac{\sqrt[n]{a_n}}{n^2}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n \text{, where}$$

$$u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} = \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^2}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{n^2}{\sqrt[n]{a_n}}, \forall n\ge 2.$$

$$\lim_{n\to\infty} u_n = 1,\ \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1,\ \lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}} = \lim_{n\to\infty}\frac{a_{n+1}}{n^2 a_n}\cdot\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}\cdot\left(\frac{n}{n+1}\right)^2 =$$

$$= \frac{2a}{e}\cdot\frac{e^3}{2a}\cdot 1 = e^2. \text{ Hence, } \lim_{n\to\infty}\frac{1}{n}(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}) = \frac{2a}{e^3}\cdot 1\cdot\ln e^2 = \frac{4a}{e^3}.$$

**Theorem 10. If $(a_n)_{n\ge 1}$ is a positive real sequence such that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}} = a \in R_+^*$, then**

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right) = \frac{e^2}{2a}.$$

***Proof.***

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^n}\cdot\frac{n^n}{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}\cdot\left(\frac{n}{n+1}\right)^{n+1}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n} =$$

$$= \lim_{n\to\infty}\frac{a}{e}\sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{C-D'A}{=} \frac{a}{e}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!} = \frac{a}{e}\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2a}{e^2}.$$

$$\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right) = \frac{n^2}{\sqrt[n]{a_n}}(u_n - 1) = \frac{n^2}{\sqrt[n]{a_n}}\cdot\frac{u_n-1}{\ln u_n} = \frac{n}{\sqrt[n]{a_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n \text{, where}$$

$$u_n = \left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}},\ \forall n\ge 2;\ \lim_{n\to\infty} u_n = 1,\ \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1,$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{a_n}{a_{n+1}}\cdot\sqrt[n+1]{a_{n+1}} = e^2\lim_{n\to\infty}\frac{na_n}{a_{n+1}}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n+1}{n} = e^2\cdot\frac{e}{2a}\cdot\frac{2a}{e^2}\cdot 1 = e.$$

**Hence,** $\lim\limits_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)=\frac{e^2}{2a}\cdot 1\cdot \ln e=\frac{e^2}{2a}$.

## ABOUT AN EQUATION BY JALIL HAJIMIR-I

***By Marin Chirciu – Romania***

**1) Solve in $\mathbb{R}$: $2^{x^2-3x}+2^{x-x^2}=2^{1-x}$**

***Jalil Hajimir – Canada***

**Solution:** Using the means inequality we obtain:

$LHS=2^{x^2-3x}+2^{x-x^2}\geq 2\sqrt{2^{x^2-3x}\cdot 2^{x-x^2}}=2\cdot 2^{-x}=2^{1-x}=RHS$, with equality if and only if $2^{x^2-3x}=2^{x-x^2}\Leftrightarrow x^2-3x=x-x^2\Leftrightarrow x(x-2)=0\Leftrightarrow x\in\{0,2\}$.

The set of equation's solutions is $S=\{0,2\}$

**Remark:** The problem can be developed.

**1.2 Let $\lambda\in\mathbb{R}$ fixed. Solve in $\mathbb{R}$: $2^{x^2-3\lambda x}+2^{\lambda x-x^2}=2^{1-\lambda x}$**

***Marin Chirciu – Romania***

**Solution:** Using the means inequality we obtain:

$LHS=2^{x^2-3\lambda x}+2^{\lambda x-x^2}\geq 2\sqrt{2^{x^2-3\lambda x}\cdot 2^{\lambda x-x^2}}=2\cdot 2^{-\lambda x}=2^{1-\lambda x}=RHS$, with equality if and only if $2^{x^2-3\lambda x}=2^{\lambda x-x^2}\Leftrightarrow x^2-3\lambda x=\lambda x-x^2\Leftrightarrow x(x-2\lambda)=0\Leftrightarrow x\in\{0,2\lambda\}$

The set of equation's solutions is $S=\{0,2\lambda\}$.

**Note:** For $\lambda=1$ we obtain the proposed problem by Jalil Hajimir, Canada, in Pascal Academy 10/2019.**Remark:** The problem can be developed.

**1.3 Let $a,b\in\mathbb{R}$ fixed. Solve in $\mathbb{R}$: $2^{x^2-2ax}+2^{bx-x^2}=2^{1+(b-a)x}$**

***Marin Chirciu – Romania***

**Solution:** Using the means inequality we obtain:

$LHS=2^{x^2-2ax}+2^{2bx-x^2}\geq 2\sqrt{2^{x^2-2ax}\cdot 2^{2bx-x^2}}=2\cdot 2^{(b-a)x}=2^{1+(b-a)x}=RHS$, with equality if and only if $2^{x^2-2ax}=2^{2bx-x^2}\Leftrightarrow x^2-2ax=2bx-x^2\Leftrightarrow x(x-a-b)=0\Leftrightarrow$

$\Leftrightarrow x\in\{0,a+b\}$.The set of equation's solutions is $S=\{0,a+b\}$.

**Note:** For $a=\frac{3}{2}$ and $b=\frac{1}{2}$ we obtain the proposed problem by Jalil Hajimir, Canada, in Pascal Academy 10/2019. **Remark:** The problem can be developed.

**2) Let $\lambda \in \mathbb{R}$ fixed. Solve in $\mathbb{R}$: $3^{2x^2-3\lambda x}+3^{3\lambda x-x^2}+3^{3\lambda x-x^2}=3^{1+\lambda x}$**

***Marin Chirciu – Romania***

**Solution:** Using the means inequality we obtain:

$$LHS=3^{2x^2-3\lambda x}+3^{3\lambda x-x^2}+3^{3\lambda x-x^2}\geq 3\sqrt[3]{3^{2x^2-3\lambda x}\cdot 3^{3\lambda x-x^2}\cdot 3^{3\lambda x-x^2}}=3\cdot 3^{\lambda x}=$$

$=3^{1+\lambda x}=RHS$, with equality if and only if $3^{2x^2-3\lambda x}=3^{3\lambda x-x^2}=3^{3\lambda x-x^2}\Leftrightarrow$

$\Leftrightarrow 2x^2-3\lambda x=3\lambda x-x^2\Leftrightarrow 3x(x-2\lambda)=0\Leftrightarrow x\in\{0,2\lambda\}$.

The set of equation's solutions is $S=\{0,2\lambda\}$. **Remark:** The problem can be developed.

**3) Let $\lambda \in \mathbb{R}$ fixed. Solve in $\mathbb{R}$:**

$$4^{3x^2-2\lambda x}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}=4^{1+\lambda x}$$

***Marin Chirciu – Romania***

**Solution:** Using the means inequality we obtain:

$$LHS=4^{3x^2-2\lambda x}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}\geq$$

$\geq 4\sqrt[4]{4^{3x^2-2\lambda x}\cdot 4^{2\lambda x-x^2}\cdot 4^{2\lambda x-x^2}\cdot 4^{2\lambda x-x^2}}=4\cdot 4^{\lambda x}=4^{1+\lambda x}=RHS$, with equality if and only if $4^{3x^2-2\lambda x}=4^{2\lambda x-x^2}=4^{2\lambda x-x^2}=4^{2\lambda x-x^2}\Leftrightarrow 3x^2-2\lambda x=2\lambda x-x^2\Leftrightarrow$

$\Leftrightarrow 4x(x-\lambda)=0\Leftrightarrow x\in\{0,\lambda\}$. The set of equation's solution is $S=\{0,\lambda\}$.

**Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro**

## ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-X

***By Marin Chirciu-Romania***

**1) In $\Delta ABC$ the following relationship holds:**

$$m_a^3+m_b^3+m_c^3\geq 3F\sqrt{m_a^2+m_b^2+m_c^2}$$

***Proposed by Adil Abdullayev-Baku-Azerbaijan***

***Solution.*** **Lemma . 2) In $\Delta ABC$ the following relationship holds:**

$$\left(\boldsymbol{m_a^3+m_b^3+m_c^3}\right)^2 \geq \frac{\mathbf{1}}{\mathbf{3}}\left(\boldsymbol{m_a^2+m_b^2+m_c^2}\right)^3$$

**Proof.** Using Power Means Inequality: If $x_1, x_2, \dots, x_n > 0$ and $r \geq s > 0$ then:

$$\left(\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}\right)^{\frac{1}{r}} \geq \left(\frac{x_1^s+x_2^s+\cdots+x_n^s}{n}\right)^{\frac{1}{s}} \geq \sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}} \geq \sqrt[s]{\frac{x_1^s+x_2^s+\cdots+x_n^s}{n}} \geq \sqrt[n]{x_1x_2\dots x_n}, ; r,s \in \mathbb{N}, r \geq s \geq 2.$$

We consider the particular case $r=3, s=2, n=3$, then we have:

$\sqrt[3]{\frac{x^3+y^3+z^3}{3}} \geq \sqrt{\frac{x^2+y^2+z^2}{3}}$ and putting $x=m_a, y=m_b, z=m_c$, we get:

$$\sqrt[3]{\frac{m_a^3+m_b^3+m_c^3}{3}} \geq \sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}} \Leftrightarrow \left(\frac{m_a^3+m_b^3+m_c^3}{3}\right)^2 \geq \left(\frac{m_a^2+m_b^2+m_c^2}{3}\right)^3$$

$\Leftrightarrow (m_a^3+m_b^3+m_c^3)^2 \geq \frac{1}{3}(m_a^2+m_b^2+m_c^2)^3$.Let's get back to the main problem.

$$m_a^3+m_b^3+m_c^3 \geq 3F\sqrt{m_a^2+m_b^2+m_c^2} \Leftrightarrow (m_a^3+m_b^3+m_c^3)^2 \geq 9F^2(m_a^2+m_b^2+m_c^2)$$

Which follows from Lemma and Ionescu-Weitzenbock inequality:

$LHS = (m_a^3+m_b^3+m_c^3)^2 \geq \frac{1}{3}(m_a^2+m_b^2+m_c^2)^3 \overset{(1)}{\geq} 9F^2(m_a^2+m_b^2+m_c^2)$, where

$(1) \Leftrightarrow (\sum m_a^2)^3 \geq 27F^2(\sum m_a^2) \Leftrightarrow (\sum m_a^2)^2 \geq 27F^2$, which follows from

$\sum m_a^2 = \frac{3}{4}\sum a^2$ and $\sum a^2 \geq 4\sqrt{3}F$, therefore

$\sum m_a^2 = \frac{3}{4}\sum a^2 \geq \frac{3}{4}4\sqrt{3}F = 3\sqrt{3}F$, and from $\sum m_a^2 \geq 3\sqrt{3}F$, we get $(\sum m_a^2)^3 \geq 27F^2$.

Equality holds if and only if triangle is equilateral. **Remark.** In same class of problems.

**3) In $\Delta ABC$ the following relationship holds:**

$$\boldsymbol{m_a^k+m_b^k+m_c^k \geq 3^{\frac{k+1}{2}}F^{\frac{k-1}{2}}\sqrt{m_a^2+m_b^2+m_c^2}, k \geq 2}$$

***Proposed by Marin Chirciu-Romania***

***Solution.*** **Lemma. 4) In $\Delta ABC$ the following relationship holds:**

$$\left(m_a^k+m_b^k+m_c^k\right)^2\geq\frac{1}{3^{k-2}}\left(m_a^2+m_b^2+m_c^2\right)^k, k\geq 2$$

**Proof.** Using Power Means Inequality: If $x_1,x_2,\dots,x_n>0$ and $r\geq s>0$ then:

$$\left(\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}\right)^{\frac{1}{r}}\geq\left(\frac{x_1^s+x_2^s+\cdots+x_n^s}{n}\right)^{\frac{1}{s}}\geq\sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}}\geq\sqrt[s]{\frac{x_1^s+x_2^s+\cdots+x_n^s}{n}}\geq\sqrt[n]{x_1x_2\dots x_n};r,s\in\mathbb{N},r\geq s\geq 2.$$

Putting $x=m_a,y=m_b,z=m_c$, we get: $\sqrt[k]{\frac{m_a^k+m_b^k+m_c^k}{3}}\geq\sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}}$.

$$\sqrt[k]{\frac{m_a^k+m_b^k+m_c^k}{3}}\geq\sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}}\Leftrightarrow\left(\frac{m_a^k+m_b^k+m_c^k}{3}\right)^2\geq\left(\frac{m_a^2+m_b^2+m_c^2}{3}\right)^k\Leftrightarrow$$

$\left(m_a^k+m_b^k+m_c^k\right)^2\geq\frac{1}{3^{k-2}}(m_a^2+m_b^2+m_c^2)^k$. Let's get back to the main problem.

$$m_a^k+m_b^k+m_c^k\geq 3^{\frac{k+1}{2}}F^{\frac{k-1}{2}}\sqrt{m_a^2+m_b^2+m_c^2}\Leftrightarrow$$

$\left(m_a^k+m_b^k+m_c^k\right)^2\geq 3^{\frac{k+1}{2}}F^{k-1}(m_a^2+m_b^2+m_c^2)$, which follows from Lemma and Ionescu-Weitzenbock inequality.

$$LHS=\left(m_a^k+m_b^k+m_c^k\right)^2\geq\frac{1}{3^{k-2}}(m_a^2+m_b^2+m_c^2)^k\overset{(1)}{\geq}3^{\frac{k+1}{2}}F^{k-1}(m_a^2+m_b^2+m_c^2)=RHS,$$

where $(1)\Leftrightarrow\frac{1}{3^{k-2}}(m_a^2+m_b^2+m_c^2)^k\geq 3^{\frac{k+1}{2}}F^{k-1}(m_a^2+m_b^2+m_c^2)\Leftrightarrow$

$(m_a^2+m_b^2+m_c^2)^k\geq 3^{\frac{3k-3}{2}}F^{k-1}(m_a^2+m_b^2+m_c^2)\Leftrightarrow(m_a^2+m_b^2+m_c^2)^{k-1}\geq\left(3\sqrt{3}\right)^{k-1}F^{k-1}\Leftrightarrow\sum m_a^2\geq 3\sqrt{3}F$, which follows from $\sum m_a^2=\frac{3}{4}\sum a^2$ and

$$\sum a^2\geq 4\sqrt{3}F,(I-W)\Rightarrow\sum m_a^2=\frac{3}{4}\sum a^2\geq\frac{3}{4}4\sqrt{3}F=3\sqrt{3}F.$$

Equality holds if and only if triangle is equilateral.

**Note.** For $k=3$, we get proposed problem by Adil Abdullayev-Baku-Azerbaijan-R.M.M.-4/2020.

**5) In $\Delta ABC$ the following relationship holds:**

$$m_a^7+m_b^7+m_c^7\ge 9F^3\sqrt{m_a^2+m_b^2+m_c^2}$$

***Proposed by Marin Chirciu-Romania***

***Solution.*** **Lemma. In $\Delta ABC$ the following relationship holds:**

$$m_a^7+m_b^7+m_c^7\ge \frac{1}{3^5}\left(m_a^2+m_b^2+m_c^2\right)^7$$

**Proof.** Using Power Means Inequality: If $x_1,x_2,\dots,x_n>0$ and $r\ge s>0$ then:

$$\left(\frac{x_1^r+x_2^r+\dots+x_n^r}{n}\right)^{\frac{1}{r}}\ge\left(\frac{x_1^s+x_2^s+\dots+x_n^s}{n}\right)^{\frac{1}{s}}\ge\sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\dots+x_n^r}{n}}\ge\sqrt[s]{\frac{x_1^s+x_2^s+\dots+x_n^s}{n}}\ge\sqrt[n]{x_1x_2\dots x_n},;r,s\in\mathbb{N},r\ge s\ge 2.$$

$$r=7,s=2,n=3:\ \sqrt[7]{\frac{x^7+y^7+z^7}{3}}\ge\sqrt{\frac{x^2+y^2+z^2}{3}}$$

Putting $x=m_a,y=m_b,z=m_c$, we get: $\sqrt[7]{\frac{m_a^7+m_b^7+m_c^7}{3}}\ge\sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}}$.

$$\sqrt[7]{\frac{m_a^7+m_b^7+m_c^7}{3}}\ge\sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}}\Leftrightarrow\left(\frac{m_a^7+m_b^7+m_c^7}{3}\right)^2\ge\left(\frac{m_a^2+m_b^2+m_c^2}{3}\right)^7\Leftrightarrow$$

$(m_a^7+m_b^7+m_c^7)^2\ge\frac{1}{3^5}(m_a^2+m_b^2+m_c^2)^7$. Let's get back to the main problem.

$$m_a^7+m_b^7+m_c^7\ge 9F^3\sqrt{m_a^2+m_b^2+m_c^2}\Leftrightarrow(m_a^7+m_b^7+m_c^7)^2\ge 81F^6(m_a^2+m_b^2+m_c^2)$$

Which follows from Lemma and Ionescu-Weitzenbock inequality:

$$LHS=(m_a^7+m_b^7+m_c^7)^2\ge\frac{1}{3^5}(m_a^2+m_b^2+m_c^2)^7\overset{(1)}{\ge}81F^6(m_a^2+m_b^2+m_c^2)=RHS,$$

$(1)\Leftrightarrow\frac{1}{3^5}(m_a^2+m_b^2+m_c^2)^7\ge 81F^6(m_a^2+m_b^2+m_c^2)\Leftrightarrow(m_a^2+m_b^2+m_c^2)^7\ge$ $3^9F^6(m_a^2+m_b^2+m_c^2)\Leftrightarrow(m_a^2+m_b^2+m_c^2)^6\ge\left(3\sqrt{3}\right)^6F^6\Leftrightarrow\sum m_a^2\ge 3\sqrt{3}F$, which follows from

$\sum m_a^2 = \frac{3}{4}\sum a^2$ and $\sum a^2 \geq 4\sqrt{3}F, (I-W) \Rightarrow \sum m_a^2 = \frac{3}{4}\sum a^2 \geq \frac{3}{4}4\sqrt{3}F = 3\sqrt{3}F.$

Equality holds if and only if triangle is equilateral.

**6) In $\Delta ABC$ the following relationship holds:**

$$m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1} \geq 3^k F^{2k-1}\sqrt{m_a^2+m_b^2+m_c^2}, k \geq \frac{3}{4}$$

***Proposed by Marin Chirciu-Romania***

***Solution.*** **Lemma. 8) In $\Delta ABC$ the following relationship holds:**

$$\left(m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}\right)^2 \geq \frac{1}{3^{4k-3}}\left(m_a^2+m_b^2+m_c^2\right)^{4k-1}, k \geq \frac{3}{4}.$$

Using Power Means Inequality: If $x_1, x_2, \dots, x_n > 0$ and $r \geq s > 0$ then:

$$\left(\frac{x_1^r+x_2^r+\dots+x_n^r}{n}\right)^{\frac{1}{r}} \geq \left(\frac{x_1^s+x_2^s+\dots+x_n^s}{n}\right)^{\frac{1}{s}} \geq \sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\dots+x_n^r}{n}} \geq \sqrt[s]{\frac{x_1^s+x_2^s+\dots+x_n^s}{n}} \geq \sqrt[n]{x_1x_2\dots x_n}, ; r,s \in \mathbb{N}, r \geq s \geq 2.$$

$$r = 4k-1, s = 2, n = 3: \sqrt[4k-1]{\frac{x^{4k-1}+y^{4k-1}+z^{4k-1}}{3}} \geq \sqrt{\frac{x^2+y^2+z^2}{3}}$$

Putting $x = m_a, y = m_b, z = m_c$, we get:

$$\sqrt[4k-1]{\frac{m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}}{3}} \geq \sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}} \Leftrightarrow$$

$$\left(\frac{m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}}{3}\right)^2 \geq \left(\frac{m_a^2+m_b^2+m_c^2}{3}\right)^{4k-1}$$

$$\left(m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}\right)^2 \geq \frac{1}{3^{4k-3}}(m_a^2+m_b^2+m_c^2)^{4k-1}$$

Let's get back to the main problem.

$$m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1} \geq 3^k F^{2k-1}\sqrt{m_a^2+m_b^2+m_c^2} \Leftrightarrow$$

$\left(m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}\right)^2 \geq 3^{2k}F^{4k-2}(m_a^2+m_b^2+m_c^2)$, which follows from Lemma and Ionescu-Weitzenbock (I-W):

$$LHS=\left(m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}\right)^2 \geq \frac{1}{3^{4k-3}}(m_a^2+m_b^2+m_c^2)^{4k-1} \overset{(1)}{\geq}$$

$$\overset{(1)}{\geq} 3^{2k}F^{4k-2}(m_a^2+m_b^2+m_c^2)=RHS$$

$$(1)\Leftrightarrow \frac{1}{3^{4k-3}}(m_a^2+m_b^2+m_c^2)^{4k-1} \geq 3^{2k}F^{4k-2}(m_a^2+m_b^2+m_c^2) \Leftrightarrow$$

$$(m_a^2+m_b^2+m_c^2)^{4k-1} \geq 6^{6k-3}F^{4k-2}(m_a^2+m_b^2+m_c^2) \Leftrightarrow$$

$$(m_a^2+m_b^2+m_c^2)^{4k-2} \geq 3^{6k-3}F^{4k-2} \Leftrightarrow \left(\sum m_a^2\right)^2 \geq 27F^2 \Leftrightarrow \sum m_a^2 \geq 3\sqrt{3}F,$$

which follows from $\sum m_a^2=\frac{3}{4}\sum a^2$ and $\sum a^2 \geq 4\sqrt{3}F, (I-W) \Rightarrow$

$$\sum m_a^2=\frac{3}{4}\sum a^2 \geq \frac{3}{4}4\sqrt{3}F=3\sqrt{3}F.$$

Equality holds if and only if triangle is equilateral.

**Note.** For $k=1$, we get proposed problem by Adil Abdullayev-Baku-Azerbaijan-R.M.M.-4/2020.

**Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro.**

## ABOUT AN INEQUALITY BY DRAGOLJUB MILOSEVIC-I

***By Marin Chirciu – Romania***

**1) In $\Delta ABC$ the following relationship holds:**

$$\sum \frac{h_a^2}{r_a} \geq 4r\left(2-\frac{r}{R}\right)^2$$

***Proposed by Dragoljub Milosevic – Serbia***

**Solution:** We prove the following lemma:

**Lemma:**

**2) In $\Delta ABC$ the following relationship holds:**

$$\sum \frac{\mathbf{h_a^2}}{\mathbf{r_a}} = \frac{\mathbf{s^4 + s^2(2r^2 - 12Rr) + r^3(4R + r)}}{\mathbf{4R^2r}}$$

**Proof:**

Using $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{s-a}$ we obtain:

$\sum \frac{h_a^2}{r_a} = \sum \frac{\left(\frac{2S}{a}\right)^2}{\frac{S}{s-a}} = 4S\sum \frac{s-a}{a^2} = \frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{4R^2r}$, which follows from:

$$\sum \frac{s-a}{a^2} = \frac{s^4 + s^2(2r^2 - 12Rr) + r^3(4R + r)}{16R^2r^2s}$$

Let's get back to the main problem:

Using the Lemma we write the inequality:

$$\frac{s^4 + s^2(2r^2 - 12Rr) + r^3(4R + r)}{4R^2r} \geq 4r\left(2 - \frac{r}{R}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow s^2(s^2 + 2r^2 - 12Rr) + r^3(4R + r) \geq 16r^2(2R - r)^2,$$

Which follows from Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$. It remains to prove that:

$$(16Rr - 5r^2)(16Rr - 5r^2 + 2r^2 - 12Rr) + r^3(4R + r) \geq 16r^2(2R - r)^2 \Leftrightarrow$$

$\Leftrightarrow (2R - r)^2 \geq (2R - r)^2$ obviously with equality.

Equality holds if and only if the triangle is equilateral.

**Remark:** Let's emphasize and inequality having an opposite sense.

**3)In ΔABC the following relationship holds:**

$$\sum \frac{\mathbf{h_a^2}}{\mathbf{r_a}} \leq \frac{\mathbf{1}}{\mathbf{r}}(\mathbf{2R - r})^2$$

***Proposed by Marin Chirciu – Romania***

**Solution:** Using the Lemma we write the inequality:

$$\frac{s^4 + s^2(2r^2 - 12Rr) + r^3(4R + r)}{16R^2r^2s} \leq \frac{1}{r}(2R - r)^2 \Leftrightarrow$$

$$\Leftrightarrow s^2(s^2 + 2r^2 - 12Rr) + r^3(4R + r) \leq 4r\text{^}2(2R - r)^2,$$

which follows from Gerretsen's: $s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$(4R^2 + 4Rr + 3r^2)(4R^2 + 4Rr + 3r^2 + 2r^2 - 12Rr) + r^3(4R + r) \leq 4r^2(2R - r)^2 \Leftrightarrow$$

$\Leftrightarrow 4R^2r^2 \geq 16r^4$, which follows from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

**Remark:** We can write the double inequality:

**4) In $\Delta ABC$ the following relationship holds:**

$$4r\left(2-\frac{r}{R}\right)^2 \leq \sum \frac{h_a^2}{r_a} \leq \frac{1}{r}(2R-r)^2$$

**Solution:** See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.

**Remark:** Switching between them $h_a$ and $r_a$ we propose:

**5) In $\Delta ABC$ the following relationship holds:**

$$\frac{1}{r}(2R-r)^2 \leq \sum \frac{r_a^2}{h_a} \leq \frac{9R^4}{16r^3}$$

***Proposed by Marin Chirciu – Romania***

**Solution:** We prove the following lemma:

**Lemma:**

**6) In $\Delta ABC$ the following relationship holds:**

$$\sum \frac{r_a^2}{h_a} = \frac{8R^2+2Rr-s^2}{r}$$

**Proof:** Using $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{s-a}$ we obtain:

$\sum \frac{r_a^2}{h_a} = \sum \frac{\left(\frac{S}{s-a}\right)^2}{\frac{2S}{a}} = \frac{S}{2}\sum \frac{a}{(s-a)^2} = \frac{8R^2+2Rr-s^2}{r}$, which follows from:

$$\sum \frac{a}{(s-a)^2} = \frac{2(8R^2+2Rr-s^2)}{r^2s}$$

Let's get back to the main problem: LHS inequality:

Using Lemma, LHS inequality can be written:

$$\frac{8R^2+2Rr-s^2}{r} \geq \frac{1}{r}(2R-r)^2 \Leftrightarrow s^2 \leq 4R^2+6Rr-r^2$$

which follows from Gerretsen's inequality: $s^2 \leq 4R^2+4Rr+3r^2$

It remains to prove that:

$4R^2+4Rr+3r^2 \leq 4R^2+6Rr-r^2 \Leftrightarrow R \geq 2r$ (Euler's inequality)

Equality holds if and only if the triangle is equilateral

RHS inequality: Using Lemma the RHS inequality can be written:

$$\frac{8R^2+2Rr-s^2}{r}\le\frac{9R^4}{16r^3}\Leftrightarrow 16r^2(8R^2+2Rr-s^2)\le 9R^4$$

which follows from Gerretsen's inequality: $s^2\ge 16Rr-5r^2$.

It remains to prove that:

$$16r^2\big(8R^2+2Rr-(16Rr-5r^2)\big)\le 9R^4\Leftrightarrow 9R^4-128R^2r^2+224Rr^3-80r^4\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)^2(9R^2+36Rr-20r^2)\ge 0$, obviously. Equality holds if and only if the triangle is equilateral.

**Remark:** Between the sums $\sum\frac{h_a^2}{r_a}$ and $\sum\frac{r_a^2}{h_a}$ we can write the relationship:

**7) In $\Delta ABC$ the following relationship holds:**

$$\sum\frac{h_a^2}{r_a}\le\sum\frac{r_a^2}{h_a}$$

***Proposed by Marin Chirciu – Romania***

**Solution** Using the sums:

$\sum\frac{h_a^2}{r_a}=\frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{4R^2r}$ and $\sum\frac{r_a^2}{h_a}=\frac{8R^2+2Rr-s^2}{r}$ the inequality can be written:

$$\frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{4R^2r}\le\frac{8R^2+2Rr-s^2}{r}\Leftrightarrow$$

$$\Leftrightarrow s^2(s^2+2r^2-12Rr+4R^2)+r^3(4R+r)\le 8R^3(4R+r),$$

which follows from Gerretsen's inequality: $s^2\le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr+4R^2)+r^3(4R+r)\le$$

$$\le 8R^3(4R+r)\Leftrightarrow 2R^3-3R^2r-4r^3\ge 0\Leftrightarrow (R-2r)(2R^2+Rr+2r^2)\ge 0$$

which follows from Euler's inequality $R\ge 2r$. Equality holds if and only if the triangle is equilateral.

**Remark:** We can write the sequence of inequalities:

**8) In $\Delta ABC$ the following inequality holds:**

$$4r\left(2-\frac{r}{R}\right)^2\le\sum\frac{h_a^2}{r_a}\le\frac{1}{r}(2R-r)^2\le\sum\frac{r_a^2}{h_a}\le\frac{9R^4}{16r^3}$$

**Solution:** See inequalities 4) and 5). Equality holds if and only if the triangle is equilateral.

**Reference: Romanian Mathematical Magazine-www.ssmrmh.ro**

## ABOUT AN INEQUALITY BY ELDENIZ HESENOV-V

***By Marin Chirciu – Romania***

**1) $I_a, I_b, I_c$ – excenters in $\Delta ABC$. Prove that:**

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\ge\frac{3}{S}$$

***Eldeniz Hesenov – Georgia***

**Solution***:* We prove: **Lemma:**

**2) If $I_a, I_b, I_c$ – excenters in $\Delta ABC$, then:**

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}$$

**Proof:** Using $[BCI_a]=\frac{a\cdot r_a}{2}$ we obtain:

$$E=\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\sum\frac{1}{[BCI_a]}=\sum\frac{1}{\frac{a\cdot r_a}{2}}=2\sum\frac{1}{a\cdot r_a}=2\cdot\sum\frac{1}{a\cdot\frac{S}{s-a}}=$$

$$=\frac{2}{S}\sum\frac{s-a}{a}=\frac{2}{sr}\cdot\frac{s^2+r^2-8Rr}{4Rr}=\frac{s^2+r^2-8Rr}{2sRr^2}$$

Let's get back to the main problem. Using the Lemma the inequality can be written:

$\frac{s^2+r^2-8Rr}{2sRr^2}\ge\frac{3}{sr}\Leftrightarrow s^2\ge 14Rr-r^2$, which follows from Gerretsen's inequality

$s^2\ge 16Rr-5r^2$. It remains to prove that: $16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow R\ge 2r$ (Euler's inequality). Equality holds if and only if the triangle is equilateral.

**Remark:** The inequality can be strengthened.

**3) $I_a, I_b, I_c$ – excenters in $\Delta ABC$. Prove that:**

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\ge\frac{2}{S}\left(2-\frac{r}{R}\right)$$

***Marin Chirciu***

**Proof:** Using the Lemma we obtain:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2+r^2-8Rr}{2sRr^2}=$$

$$=\frac{8Rr-4r^2}{2sRr^2}=\frac{4r(2R-r)}{2sRr^2}=\frac{2(2R-r)}{sRr}=\frac{2}{S}\left(2-\frac{r}{R}\right)$$

Equality holds if and only if the triangle is equilateral.

**Remark:** Inequality is stronger than the inequality

**4) $I_a, I_b, I_c$ – excenters in $\Delta ABC$. Prove that:**

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\geq\frac{2}{S}\left(2-\frac{r}{R}\right)\geq\frac{3}{S}$$

**Solution:** See inequality 3) and $\frac{2}{S}\left(2-\frac{r}{R}\right)\geq\frac{3}{S}\Leftrightarrow R\geq 2r$, (Euler's inequality). Equality holds if and only if the triangle is equilateral.**Remark:** Let's find an inequality of opposite sense.

**5) $I_a, I_b, I_c$ – excenters in $\Delta ABC$. Prove that:**

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\leq\frac{2}{S}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

***Marin Chirciu***

**Proof:** Using the Lemma we obtain:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2+r^2-8Rr}{2sRr^2}=$$

$$=\frac{4R^2-4Rr+4r^2}{2sRr^2}=\frac{2(R^2-Rr+r^2)}{sRr^2}=\frac{2}{S}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Equality holds if and only if the triangle is equilateral.**Remark:**

We can write the double inequality:

**6) $I_a, I_b, I_c$ – excenters in $\Delta ABC$. Prove that:**

$$\frac{2}{S}\left(2-\frac{r}{R}\right)\leq\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\leq\frac{2}{S}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

***Marin Chirciu***

**Solution:** We prove:**Lemma:**

**7) If $I_a, I_b, I_c$ – excenters in $\Delta ABC$, then:**

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}$$

**Proof:** Using $[BCI_a]=\frac{a\cdot r_a}{2}$ we obtain:

$$E=\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\sum\frac{1}{[BCI_a]}=\sum\frac{1}{\frac{a\cdot r_a}{2}}=2\sum\frac{1}{a\cdot r_a}=2\cdot\sum\frac{1}{a\cdot\frac{S}{s-a}}=$$

$$=\frac{2}{S}\sum\frac{s-a}{a}=\frac{2}{sr}\cdot\frac{s^2+r^2-8Rr}{4Rr}=\frac{s^2+r^2-8Rr}{2sRr^2}$$

Let's get back to the main problem.RHS inequality.Using the Lemma we obtain:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2+r^2-8Rr}{2sRr^2}=$$

$$=\frac{4R^2-4Rr+4r^2}{2sRr^2}=\frac{2(R^2-Rr+r^2)}{sRr^2}=\frac{2}{S}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Equality holds if and only if the triangle is equilateral.LHS inequality. Using the Lemma we obtain:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2+r^2-8Rr}{2sRr^2}=$$

$$=\frac{8Rr-4r^2}{2sRr^2}=\frac{4r(2R-r)}{2sRr^2}=\frac{2(2R-r)}{sRr}=\frac{2}{S}\left(2-\frac{r}{R}\right)$$

Equality holds if and only if the triangle is equilateral.

**Note:**The inequality strengthen the proposed problem by Eldeniz Hesenov in RMM 12/2020.

**Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro**

## ABOUT AN INEQUALITY BY ERKAN OZAL-I

***By Marin Chirciu-Romania***

**1) In $\Delta ABC$ the following relationship holds:**

$$\frac{64}{9R^3}\leq\frac{1}{r^3}-\frac{1}{r_a^3}-\frac{1}{r_b^3}-\frac{1}{r_c^3}\leq\frac{8}{9r^3}$$

***Proposed by Erkan Ozal-Turkiye***

***Solution.*** **Lemma. 2) In $\Delta ABC$ the following relationship holds:**

$$\frac{1}{r^3}-\frac{1}{r_a^3}-\frac{1}{r_b^3}-\frac{1}{r_c^3}=\frac{12R}{F^2}\ (Toscano\ Identity)$$

**Proof.** Using identities $r=\frac{F}{s}$ and $r_a=\frac{F}{s-a}$, we get:

$$\frac{1}{r^3}-\frac{1}{r_a^3}-\frac{1}{r_b^3}-\frac{1}{r_c^3}=\frac{1}{\left(\frac{F}{s}\right)^3}-\sum_{cyc}\frac{1}{\left(\frac{F}{s-a}\right)^3}=\frac{s^3-\sum_{cyc}(s-a)^3}{F^3}=$$

$$=\frac{3s^2(a+b+c)-3s(a^2+b^2+c^2)+a^3+b^3+c^3-2s^3}{F^3}=$$

$$=\frac{3s^2\cdot 2s-3s\cdot 2(s^2-r^2-4Rr)+2s(s^2-3r^2-6Rr)-2s^3}{F^3}=$$

$$=\frac{6s^3-6s^3+6sr^2+24Rrs+2s^3-6sr^2-12rrs-2s^3}{F^3}=$$

$=\frac{12Rrs}{F^3}=\frac{12RF}{F^3}=\frac{12R}{F^2}$. Let's get back to the main problem. For RHS using Lemma, we have:

$\frac{12R}{F^2}\geq\frac{64}{9R^3}\Leftrightarrow\frac{3R}{s^2r^2}\geq\frac{16}{9R^3}\Leftrightarrow 27R^4\geq 16s^2r^2$, which follows from

$s^2\leq 4R^2+4Rr+3r^2(Gerretsen)$ and $R\geq 2r(Euler)$. Remains to prove that:

$$27R^4\geq 16r^2(4R^2+4Rr+3r^2)\Leftrightarrow 27R^4-64R^2r^2-64Rr^3-48r^4\geq 0\Leftrightarrow$$

$(R-2)(27R^3+54R^2R+44Rr^2+24r^3)\geq 0$, which follows from $R\geq 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

**Remark.** In same class of problems.

**3) In $\Delta ABC$ the following relationship holds:**

$$\frac{8}{3R^2}\leq\frac{1}{r^2}-\frac{1}{r_a^2}-\frac{1}{r_b^2}-\frac{1}{r_c^2}\leq\frac{2}{3r^2}$$

***Proposed by Marin Chirciu-Romania***

**Solution: Lemma. 4) In $\Delta ABC$ the following relationship holds:**

$$\frac{1}{r^2}-\frac{1}{r_a^2}-\frac{1}{r_b^2}-\frac{1}{r_c^2}=\frac{2r(4R+r)}{F^2}\ (\textit{Toscano Identity})$$

**Proof.** Using identities $r=\frac{F}{s}$ and $r_a=\frac{F}{s-a}$, we get:

$$\frac{1}{r^2}-\frac{1}{r_a^2}-\frac{1}{r_b^2}-\frac{1}{r_c^2}=\frac{1}{\left(\frac{F}{s}\right)^2}-\sum_{cyc}\frac{1}{\left(\frac{F}{s-a}\right)^2}=\frac{s^2-\sum(s-a)^2}{F^2}=$$

$$=\frac{2s(a+b+c)-a^2-b^2-c^2-2s^2}{F^2}=\frac{2s\cdot 2s-2(s^2-r^2-4Rr)-2s^2}{F^2}=$$

$$=\frac{4s^2-2s^2+2r^2+8rr-2s^2}{F^2}=\frac{8Rr+2r^2}{F^2}=\frac{2r(4R+r)}{F^2}$$

Let’s get back to the main problem. For RHS using Lemma, we get:

$$\frac{2r(4R+r)}{F^2}\geq\frac{8}{3R^2}\Leftrightarrow\frac{2r(4R+r)}{s^2r^2}\geq\frac{8}{3R^2}\Leftrightarrow 3R^2(4R+r)\geq 4s^2r$$

Which follows from $s^2\leq 4R^2+4Rr+3r^2$ and $R\geq 2r(Euler)$.

Equality holds if and only if triangle is equilateral. **Remark.** In same class of problem.

**5) In $\Delta ABC$ the following relationship holds:**

$$\frac{112}{27R^2r^2}\leq\frac{1}{r^4}+\frac{1}{r_a^4}+\frac{1}{r_b^4}+\frac{1}{r_c^4}\leq\frac{7R^2}{27R^6}$$

***Proposed by Marin Chirciu-Romania***

***Solution.*** **6)** In $\Delta ABC$ the following relationship holds:

$$\frac{1}{r^4}+\frac{1}{r_a^4}+\frac{1}{r_b^4}+\frac{1}{r_c^4}=\frac{1}{\left(\frac{F}{s}\right)^4}+\sum_{cyc}\frac{1}{\left(\frac{F}{s-a}\right)^4}=\frac{s^4+\sum(s-a)^4}{F^4}=$$

$$=\frac{s^4+\sum(s-a)^4}{F^4}=\frac{s^4+\sum(s^4-4s^3a+6s^2a^2-4sa^3+a^4)}{F^4}=$$

$$=\frac{s^4+3s^4-4s^3\cdot 2s+6s^2\cdot 2(s^2-r^2-4Rr)-4s\cdot 2s(s^2-3r^2-6Rr)+\sum a^4}{F^4}=$$

$$=\frac{12s^2r^2+\sum a^4}{F^4}=\frac{12F^2+\sum a^4}{F^4}=\frac{12s^2r^2+2(s^4-s^2(8Rr+6r^2)+r^2(4R+r)^2)}{F^4}=$$

$$=\frac{12s^2r^2+2s^4-2s^2(8Rr+6r^2)+2r^2(4R+r)^2}{F^4}=\frac{2s^4-16s^2Rr+2r^2(4R+r)^2}{F^4}$$

$$\frac{1}{r^4}+\frac{1}{r_a^4}+\frac{1}{r_b^4}+\frac{1}{r_c^4}=\frac{(a^2+b^2+c^2)^2+8F^2}{2F^4}=\frac{12F^2+\sum a^4}{F^4}=$$

$$=\frac{2s^4-16s^2Rr+2r^2(4R+r)^2}{F^2}.$$

Let’s get back to the main problem. For RHS using Lemma, we get:

$$\frac{2s^4-16s^2Rr+2r^2(4R+r)^2}{F^2}=\frac{2s^4-16s^2Rr+2r^2(4R+r)^2}{s^4r^4}=$$

$$=\frac{2}{r^4}\left[1-\frac{8Rr}{s^2}+\frac{r^2(4R+r)^2}{s^4}\right]\leq\frac{2}{r^4}\left[1-\frac{8Rr}{\frac{27R^2}{4}}+\frac{r^2(4R+r)^2}{27r^2\cdot\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{2}{r^4}\left[1-\frac{32r}{27R}+\frac{R+r}{27r}\right]=\frac{2(R^2+28Rr-32r^2)}{27Rr^5}\overset{(1)}{\le}\frac{7R^2}{27R^6}$$

$$(1)\Leftrightarrow 2r(R^2+28Rr-32r^2)\le 7R^3\Leftrightarrow 7R^3-2R^2r-56Rr^2-64r^3\ge 0\Leftrightarrow$$

$(R-2r)(7R^2+2Rr-32r^2)\ge 0$, which follows from $R\ge 2e(Euler)$.

Equality holds if and only if triangle is equilateral.

$\because s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}(Gerretsen)$ and $27r^2\le s^2\le\frac{27R^2}{4}(Mitrinovic)$.

Equality holds if and only if triangle is equilateral. For LHS using Lemma, we get:

$$\frac{2s^4-16s^2Rr+2r^2(4R+r)^2}{F^2}=\frac{2s^4-16s^2Rr+2r^2(4R+r)^2}{s^4r^4}\overset{(2)}{\ge}\frac{112}{27R^2r^2}$$

$$(2)\Leftrightarrow\frac{2s^4-16s^2Rr+2r^2(4R+r)^2}{s^4r^4}\ge\frac{112}{27R^2r^2}\Leftrightarrow$$

$$\frac{s^4-8s^2Rr+r^2(4R+r)^2}{s^4r^2}\ge\frac{56}{27R^2}\Leftrightarrow 27R^2s^4-27R^2\cdot 8s^2Rr+27R^2r^2(4R+r)^2$$

$$\ge 56s^4r^2\Leftrightarrow s^4(27R^2-56r^2)-216s^2R^3r+27R^2r^2(4R+r)^2\ge 0\Leftrightarrow$$

$$s^2[s^2(27R^2-56r^2)-216R^3r]+27R^2r^2(4R+r)^2\ge 0$$

Distinguish the cases:

Case I) If $[s^2(27R^2-56r^2)-216R^3r]\ge 0$, inequality is obviously true.

Case II) If $[s^2(27R^2-56r^2)-216R^3r]<0$, inequality can be written as:

$27R^2r^2(4R+r)^2\ge s^2[216R^3r-s^2(27R^2-56r^2)]$, which follows from

$$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2(B-Gerretsen)$$

Remains to prove that:

$$27R^2r^2(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}[216R^3r-(16Rr-5r^2)(27R^2-56r^2)]\Leftrightarrow$$

$$54Rr(2R-r)\ge(-216R^3+135R^2r-896Rr^2+280r^3)\Leftrightarrow$$

$$108R^2r-54Rr^2\ge -216R^3+135R^2r+896Rr^2-280r^3\Leftrightarrow$$

$$216R^3-27R^2r-950Rr^2+280r^3\ge 0\Leftrightarrow(R-2r)(216R^2+405Rr-140r^2)\ge 0,$$

which follows from $R\ge 2r(Euler)$. Equality holds if and only if triangle is equilateral.

**Reference:**

**ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro**

## ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-VI

***By Marin Chirciu – Romania***

**1) Prove that in any acute-angled triangle the following inequality holds:**

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{12r^2}{R^2}$$

***Proposed by Marian Ursărescu – Romania***

**Solution:** We prove the following lemma:

**2) In $\Delta ABC$ the following relationship holds:**

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{(\sum bc)^2}{4R^2s^2}$$

**Proof:** Using the following formulas: $h_a=\frac{2S}{a}$ and $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$, we obtain:

$$\frac{h_a^2}{w_a^2}=\frac{\left(\frac{2S}{a}\right)^2}{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}=\frac{S^2}{a^2b^2c^2}\cdot\frac{bc(b+c)^2}{s(s-a)}\ge\frac{S^2}{16R^2S^2}\cdot\frac{bc\cdot 4bc}{s(s-a)}=\frac{b^2c^2}{4R^2s(s-a)}$$

It follows $\sum\frac{h_a^2}{w_a^2}\ge\frac{1}{4R^2}\sum\frac{b^2c^2}{s(s-a)}\overset{\text{Bergstrom}}{\ge}\frac{1}{4R^2}\cdot\frac{(\sum bc)^2}{\sum s(s-a)}=\frac{(\sum bc)^2}{4R^2s^2}$

Let's get back to the main problem:

Using the Lemma it suffices to prove that:

$$\frac{(\sum bc)^2}{4R^2s^2}\ge\frac{12r^2}{R^2}\Leftrightarrow\frac{(s^2+r^2+4Rr)^2}{4R^2s^2}\ge\frac{12r^2}{R^2}\Leftrightarrow s^2(s^2+8Rr-46r^2)+r^2(4R+r)^2\ge 0$$

We distinguish the following cases:

Case 1). If $(s^2+8Rr-46r^2)\ge 0$, the inequality is obvious.

Case 2). If $(s^2+8Rr-46r^2)<0$, the inequality can be rewritten:

$r^2(4R+r)^2\ge s^2(46r^2-8Rr-s^2)$, which follows from Blundon-Gerretsen's inequality:

$16Rr-5r^2\le s^2\le\frac{R(4R+R)^2}{2(2R-r)}$. It remains to prove that:

$$r^2(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}(46r^2-8Rr-16Rr+5r^2)\Leftrightarrow 24R^2-47Rr-2r^2\ge 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(4R+r)\ge 0,$$ obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

**Remark:** From the above proof, the condition of acute-angled triangle it is not necessary.

**Remark:** Inequality can be strengthened:

**3) In $\Delta ABC$ the following inequality holds:**

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\geq\frac{6r}{R}$$

***Proposed by Marin Chirciu – Romania***

**Solution:** Using Lemma, it suffices to prove that:

$$\frac{(\sum bc)^2}{4R^2s^2}\geq\frac{6r}{R}\Leftrightarrow\frac{(s^2+r^2+4Rr)^2}{4R^2s^2}\geq\frac{6r}{R}\Leftrightarrow s^2(s^2+2r^2-16Rr)+r^2(4R+r)^2\geq 0$$

We distinguish the following cases:

Case 1). If $(p^2+2r^2-16Rr)\geq 0$, the inequality is obvious.

Case 2). If $(s^2+2r^2-16Rr)<0$, the inequality can be rewritten:

$r^2(4R+r)^2\geq s^2(16Rr-2r^2-s^2)$, which follows from Blundon-Gerretsen's inequality:

$16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}$. It remains to prove that:

$$r^2(4R+r)^2\geq\frac{R(4R+R)^2}{2(2R-r)}(16Rr-2r^2-16Rr+5r^2)\Leftrightarrow R\geq 2r \text{ (Euler)}$$

Equality holds if and only if the triangle is equilateral.

**Remark:** Inequality 3) is stronger than inequality 1)

**4) In $\Delta ABC$ the following relationship holds:**

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\geq\frac{6r}{R}\geq\frac{12r^2}{R^2}$$

**Solution:** See inequality 3) and $\frac{6r}{R}\geq\frac{12r^2}{R^2}\Leftrightarrow R\geq 2r$ (Euler) Equality holds if and only if the triangle is equilateral.

**Remark:** If we replace $h_a$ with $r_a$ we propose:

**5) In $\Delta ABC$ the following relationship holds:**

$$\frac{r_a^2}{w_a^2}+\frac{r_b^2}{w_b^2}+\frac{r_c^2}{w_c^2}\geq\frac{3R}{2r}$$

***Proposed by Marin Chirciu – Romania***

**Solution:** We prove the following lemma:

**Lemma:**

**6) In $\Delta ABC$ the following relationship holds:**

$$\frac{r_a^2}{w_a^2}+\frac{r_b^2}{w_b^2}+\frac{r_c^2}{w_c^2}\geq\sum\frac{r^2s}{(s-a)^3}$$

**Proof:** Using the following formulas: $r_a=\frac{S}{s-a}$ and $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$, we obtain:

$$\frac{r_a^2}{w_a^2}=\frac{\left(\frac{S}{s-a}\right)^2}{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}=\frac{S^2}{4s}\cdot\frac{(b+c)^2}{bc(s-a)^3}\geq\frac{r^2s^2}{4s}\cdot\frac{4bc}{bc(s-a)^3}=\frac{r^2s}{(s-a)^3}$$

It follows $\sum\frac{r_a^2}{w_a^2}\geq\sum\frac{r^2s}{(s-a)^3}$.

Let's get back to the main problem.

Using Lemma it suffices to prove that: $\sum\frac{r^2s}{(s-a)^3}\geq\frac{3R}{2r}$

Using the identity in triangle: $\sum\frac{1}{(s-a)^3}=\frac{(4R+r)^3-12s^2R}{r^3s^3}$ the inequality holds:

$\sum\frac{r^2s}{(s-a)^3}\geq\frac{3R}{2r}$ we write $r^2s\cdot\frac{(4R+r)^3-12s^2R}{r^3s^3}\geq\frac{3R}{2r}\Leftrightarrow 2(4R+r)^3-24s^2R\geq 3s^2R\Leftrightarrow$

$\Leftrightarrow 2(4R+r)^3\geq 27s^2R$, it follows from Blundon-Gerretsen's inequality.

$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}$. It remains to prove that:

$$2(4R+r)^3\geq 27R\cdot\frac{R(4R+r)^2}{2(2R-r)}\Leftrightarrow 5R^2-8Rr-4r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(5R+2r)\geq 0$, obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

**Reference: Romanian Mathematical Magazine-www.ssmrmh.ro**

## PROPOSED PROBLEMS
## PROBLEMS FOR JUNIORS

**J.2975** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$a^2b^2+b^2c^2+c^2a^2 \geq 16\cdot F^2+\frac{1}{2}((ab-bc)^2+(bc-ca)^2+(ca-ab)^2)$$

***Proposed by D.M. Bătinețu – Giurgiu– Romania***

**J.2976** If $u,v,w,x,y,z>0$ then in any $\Delta ABC$ with the area $F$ the following inequaity holds:

$$(u+x)^2a^4+(v+y)^2b^4+(w+z)^2c^4 \geq \frac{16}{3}\left(\sqrt{uv+vw+wu}+\sqrt{xy+yz+zx}\right)^2\cdot F^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**J.2977** If $m,n\geq 0, m+n,x,y,z>0$ then in any $\Delta ABC$ with the area $F$ the following inequality holds:

$$\frac{mx+ny}{nx+my+(m+n)z}a^4+\frac{my+nz}{ny+mz+(m+n)x}b^4+\frac{m+nx}{my+mx+(m+n)y}c^4 \geq 8F^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**J.2978** If $a,b,c,t,x,y>0$ then:

$$\left(\frac{a^2}{(bx+cy)^2}+2t^2\right)\cdot\left(\frac{b^2}{(cx+ay)^2}+2t^2\right)\cdot\left(\frac{c^2}{(ax+by)^2}+2t^2\right) \geq \frac{27\cdot t^4}{(x+y)^2}$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.2979** If $a,b,c,m,n\in\mathbb{R}_+^*$ and $m+n\in[1,\infty)$, then:

$$\prod_{cyc}\left(\left(a^{2m}+\frac{a^{2n}}{(b+c)^{2(m+n)}}\right)^2+2\right) \geq \frac{27}{4^{m+n-1}}$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.2980** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$r_a^2+r_b^2+r_c^2 \geq 3\sqrt{3}\cdot F+\frac{1}{2}\cdot((r_a-r_b)^2+(r_b-r_c)^2+(r_c-r_a)^2)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.2981** If $a, b, c, x, y > 0$, then:

$$\left(\frac{a^4}{(bx+cy)^8}+2\right)\cdot\left(\frac{b^4}{(cx+ay)^8}+2\right)\cdot\left(\frac{c^4}{(ax+by)^8}+2\right)\geq\frac{27}{(x+y)^4\cdot(a+b+c)^2}$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.2982** If $m \geq 0, n \in \mathbb{N}^* - \{1\}, x_k \in \mathbb{R}_+^*, p_k = \frac{x_1 \cdot x_2 \cdot \ldots \cdot x_n}{x_k}, k = \overline{1,n}$ and $\sum_{k=1}^{n} x_k = s$, then:

$$\sum_{k=1}^{n}\frac{x_k}{(s+p_k)^m}\geq\frac{s^{1-m}\cdot n^{m(n-1)}}{(n^{n-1}+s^{n-2})^m}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania***

**J.2983** If $m \geq 0$ and $t, u, x, y, z > 0$ and $x + y + z = s$ then:

$$\frac{x}{(ts+uyz)^m}+\frac{y}{(ts+uzx)^m}+\frac{z}{(ts+uxy)^m}\geq\frac{9^m\cdot s^{1-m}}{(9t+us)^m}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.2984** If $m \geq 0, x, y, z > 0$ then:

$$\frac{(x+y)^{m+1}}{(x+y+2z)^{2m+1}}+\frac{(y+z)^{m+1}}{(y+z+2x)^{2m+1}}+\frac{(z+x)^{m+1}}{(z+x+2y)^{2m+1}}\geq\frac{3^{m+1}}{2^{3m+1}(x+y+z)^m}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania***

**J.2985** If $m \geq 0, u, v, x, y, z > 0$ then:

$$\frac{x^{m+1}}{(uy+vz)^{2m+1}}+\frac{y^{2m+1}}{(uz+vx)^{2m+1}}+\frac{z^{m+1}}{(ux+vy)^{2m+1}}\geq\frac{3^{m+1}}{(u+v)^{2m+1}(x+y+z)^m}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.2986** If $x, y > 0$ and $ABC, A_1B_1C_1$ are two triangles with the area $F$ respectively $F_1$, then:

$$x^2(a^2+b^2+c^2)+y^2(a_1^2+b_1^2+c_1^2)\geq$$

$$\geq 8xy\cdot\sqrt{3}\cdot\sqrt{F\cdot F_1}+(xa-ya_1)^2+(xb-yb_1)^2+(xc-yc_1)^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.2987** Let be $x, y > 0$, then in any $\Delta ABC$ with the area $F$ the following inequality holds:

$$\frac{a^3}{xbR+ycr}+\frac{b^3}{xcR+yar}+\frac{c^3}{xaR+ybr}\geq\frac{4\sqrt{3}}{xR+yr}\cdot F$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania***

**J.2988** If $m \geq 0$ and $x, y > 0$, then in any $\Delta ABC$ with the area $F$ the following inequality holds:

$$\frac{a^{m+2}}{(bxR+cry)^m}+\frac{b^{m+2}}{(cxR+ayr)^m}+\frac{c^{m+2}}{(axR+byr)^m}\geq\frac{4\sqrt{3}}{(xR+yr)^mF}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.2989** If $m\geq 0$ and $a,b,c\in(1,\infty)$, then:

$$\frac{\left(\log_a\frac{a}{b}\right)^m}{(\log_a ab)^m}+\frac{\left(\log_a\frac{a}{c}\right)^{m+1}}{(\log_a ac)^m}+\frac{(\log_a bc)^{m+1}}{\left(\log_a\frac{a^2}{bc}\right)^m}\geq 2^{1-m}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.2990** If $a,b,c>0$ then:

$$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\cdot\left(\frac{a^2b^2}{(a+b)^2}+\frac{b^2c^2}{(b+c)^2}+\frac{c^2a^2}{(c+a)^2}\right)\geq\frac{9}{4}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania***

**J.2991** If $a,b,c,x,y>0$, then:

$$\left(a^{2(x+y)}+b^{2(x+y)}+c^{2(x+y)}\right)\cdot\left(\frac{1}{a^{2y}}+\frac{1}{b^{2y}}+\frac{1}{c^{2y}}\right)\geq(a^x+b^x+c^x)^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania***

**J.2992** If $A_1B_1C_1, A_2B_2C_2$ are two triangles with the area $F_1$ respectively $F_2$, then:

$$a_1^2\cdot a_2^2+b_1^2\cdot b_2^2+c_1^2\cdot c_2^2\geq 16\cdot F_1\cdot F_2+\frac{1}{2}\cdot\sum_{cyc}(a_1a_2-b_1b_2)^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți– Romania***

**J.2993** If $n_a,n_b,n_c$ are Nagel's cevians of $\Delta ABC$ with the area $F$ and $m\geq 0$, then:

$$\frac{n_a}{a^m}+\frac{n_b}{b^m}+\frac{n_c}{c^m}\geq\frac{2F\left(\sqrt{3}\right)^{1-m}}{R^{m+1}}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania***

**J.2994** In any $\Delta ABC$ the following inequality holds:

$$\left(\frac{IA^4}{a^4}+2\right)\cdot\left(\frac{IB^4}{b^4}+2\right)\cdot\left(\frac{IC^4}{c^4}+2\right)\geq 3$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania***

**J.2995** If $a,b,c\geq 0$ then prove that:

$$(a+bc)^2+(b+ca)^2+(c+ab)^2\geq\sqrt{2}(a+b)(b+c)(c+a)$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**J.2996** If $a,b,c>0$ and $a+b+c=a^2b^2+b^2c^2+c^2a^2$ then prove that:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 3$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**J.2997** In any $\Delta ABC$ with $n_a,n_b,n_c$ – nagel's cevian the following relationship holds:

$$\frac{n_a^2n_b^2}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{n_b^2n_c^2}{\sin\frac{C}{2}\sin\frac{A}{2}}+\frac{n_c^2n_a^2}{\sin\frac{A}{2}\sin\frac{B}{2}}\geq 972r^4$$

***Proposed by Zaza Mzhavanadze – Georgia***

**J.2998** If $a,b,c>0$ and $a^3b^3+b^3c^3+c^3a^3=(abc)^4$ then prove that:

$$\frac{1}{a^6}+\frac{1}{b^6}+\frac{1}{c^6}\geq 1$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**J.2999** If $a,b,c>0$ and $a^2+b^2+c^2=1$ then prove that:

$$\frac{a^5-2a^3+a}{b^2+c^2}+\frac{b^5-2b^3+b}{c^2+a^2}+\frac{c^5-2c^3+c}{a^2+b^2}\leq\frac{2\sqrt{3}}{3}$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**J.3000** If $x,y,z>0$ and $x^2+y^2+z^2=3$ then prove that:

$$\frac{x^2+xy}{5-z^2}+\frac{y^2+yz}{5-x^2}+\frac{z^2+zx}{5-y^2}\leq\frac{3}{2}$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**J.3001** If $a,b,c>0$ then:

$$\frac{ab+2a+b}{a+2b+1}+\frac{ac+2a+c}{a+2c+1}\geq\frac{4a}{a+1}$$

***Proposed by Daniel Sitaru – Romania***

**J.3002** If $a,b,c>0$, then prove that: $\sum_{cyc}\frac{(b+c)\sqrt{a+b}+(a+c)\sqrt{b+c}}{(a+b)\sqrt{b+c}+(a+b)\sqrt{a+c}}\geq 3$

***Proposed by Zaza Mzhavanadze – Georgia***

**J.3003** If $x,y,z>0$ and $x\geq z$ then prove that:

$$\frac{xz}{y^2+yz}+\frac{y^2}{xz+yz}+\frac{x+2z}{x+z}\geq\frac{5}{2}$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**J.3004** In $\Delta ABC$ the following relationship holds:

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq 2s$$

***Proposed by Daniel Sitaru – Romania***

**J.3005** In any triangles $\Delta ABC$ and $\forall n\in\mathbb{N}$, the following relationship holds:

$$\frac{h_a^n}{h_b^n+h_c^n}\cdot\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{h_b^n}{h_c^n+h_a^n}\cdot\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{h_c^n}{h_a^n+h_b^n}\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq\sqrt{3}$$

***Proposed by Zaza Mzhavanadze – Georgia***

**J.3006** If $a,b,c<0$ and $a^2+b^2+c^2\leq 3$ then prove that:

$$2(a+b+c)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq -3$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**J.3007** In $\Delta ABC$ the following relationship holds:

$$a\sqrt{4a^2+9b^2}+b\sqrt{4b^2+9c^2}+c\sqrt{4c^2+9a^2}\geq 10\sqrt{6}F$$

***Proposed by Daniel Sitaru – Romania***

**J.3008** Let $-1\leq x,y\leq 2$. Find the maximum and minimum the values of

$$P=\frac{1}{x^2+y^2+2022}+x^2+y^2+x+y+2023$$

***Proposed by Nguyen Van Canh-Vietnam***

**J.3009** Let be the real numbers $x\neq y\neq z\neq x$. Prove that:

$$E(x,y,z)=(x-y)^5+(y-z)^5+(z-x)^5\neq 0$$

***Proposed by Sorin Botea, Nicoleta Dincă – Craiova – Romania***

**J.3010** Let be $x\geq 1$ and $y\geq 1$. Prove that $x^2y^2+16(x+y)\geq 16xy+16$. When does the equality occurs?

***Proposed by Lucian Tuțescu, Sorin Botea – Romania***

**J.3011** Let be $ABCD$ trapezoid and $E$ the middle of the base $AD$. If $BD\cap CE=\{F\}$ and $BC=CF$. Prove that $AF\perp BD$

***Proposed by Dan Grigorie, Gigi Zaharia – Romania***

**J.3012** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$(a^2r_a^2+2)(b^2r_b^2+2)(c^2r_c^2+2)\geq 12\cdot F^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3013** Given three positive real numbers $x, y, z$ such that $x + y + z = x^2 + y^2 + z^2$. Prove that:

$$\sqrt{\frac{xy}{z}} + \sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + 15\sqrt{xyz} \geq 2\left(\sqrt{x^3 + 8xyz} + \sqrt{y^3 + 8xyz} + \sqrt{z^3 + 8xyz}\right)$$

***Proposed by Phan Ngoc Chau -Vietnam***

**J.3014** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$(m_a^4 + 2)(m_b^4 + 2)(m_c^4 + 2) \geq 81 \cdot F^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3015** In $\Delta ABC$ the following relationship holds:

$$2 + \frac{1}{\sin^2 \omega} \geq \frac{2}{3}\left(\sum_{cyc} \frac{m_a}{h_a}\right)^2 + \max\left\{\left(\frac{m_a}{h_a} - \frac{m_b}{h_b}\right)^2, \left(\frac{m_b}{h_b} - \frac{m_c}{h_c}\right)^2, \left(\frac{m_c}{h_c} - \frac{m_a}{h_a}\right)^2\right\}$$

***Proposed by Bogdan Fuștei – Romania***

**J.3016** If $a, b, c \geq 0$ and $a + b + c = \frac{3}{2}$ then:

$$\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} + ab + bc + ca \geq \frac{15}{4}$$

***Proposed by Sidi Abdullah Lemrabott-Mauritania***

**J.3017** If $a, b, c \leq 3$ and $a + b + c = 6$ then: $\frac{a}{b^3} + \frac{b}{c^3} + \frac{c}{a^3} + \frac{3}{8} \geq \frac{3(a^2+b^2+c^2)}{32}$

***Proposed by Sidi Abdullah Lemrabott-Mauritania***

**J.3018** If $a, b, c > 0, (n \geq 1)$ and $(a + b)(b + c)(c + a) = 2(a + b + c + 1)$ then:

$$a^2\sqrt[n]{a + b + 1} + b^2\sqrt[n]{b + c + 1} + c^2\sqrt[n]{c + a + 1} \geq \sqrt[n]{3}.(ab + bc + ca)$$

***Proposed by Sidi Abdullah Lemrabott-Mauritania***

**J.3019** If $a, b > 0, (n \geq 1)$ and $(a + b)(ab - 2) = 2$ then:

$$a^2\sqrt[n]{a + 1} + (ab)^2\sqrt[n]{a + b + 1} + b^2\sqrt[n]{b + 1} \geq 3^{\frac{1}{n}-1}(a + ab + b)^2$$

When does equality holds?

***Proposed by Sidi Abdullah Lemrabott-Mauritania***

**J.3020** Find $x, y, z \in \mathbb{Z}$ such that:

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} = \frac{3}{2}xyz$$

***Proposed by Kerimov Elsen-Azerbaijan***

**J.3021** If $(s \in \mathbb{R})$ and $k, m, a_i > 0, i \in \{1,2,\dots,n\}$ then:

$$a_1^m + a_2^m + \cdots + a_n^m \geq (k^s - 1)^{s-1}\left(\frac{a_1^{\frac{m}{s}} + a_2^{\frac{m}{s}}}{k} + \cdots + \frac{a_n^{\frac{m}{s}}}{k^n}\right)^s$$

***Proposed by Sidi Abdullah Lemrabott-Mauritania***

**J.3022** If $x, y, z > 0$ then in any $\Delta ABC$ with the area $F$ the following inequality holds:

$$xa^2 + yb^2 + zc^2 \geq 2\sqrt{2} \cdot \sqrt{xy \cdot \cot\frac{C}{2} + yz \cdot \cot\frac{A}{2} + zx \cdot \cot\frac{B}{2}} \cdot F$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3023** Let be $M$ an interior point in $\Delta ABC$ with the area $F$ and $d_a, d_b, d_c$ the distances of point $M$ to the sides $BC, CA, AB$, then:

$$(a^2 d_a^2 + 2) \cdot (b^2 d_b^2 + 2) \cdot (c^2 d_c^2 + 2) \geq 12 \cdot F^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3024** If $x, y, z > 0$ then in any $\Delta ABC$ with the area $F$ we have:

$$\frac{\frac{x}{h_a} + \frac{y}{h_b} + \frac{z}{h_c}}{xy + yz + zx} \geq \frac{R}{2F}\sqrt{\frac{x+y+z}{xyz}}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3025** If $a, b, c > 0$ and $x, y, z \in \mathbb{R}$ then:

$$\left(a^2 + 4(\sin^4 x + \cos^4 x)\right) \cdot \left(b^2 + 4(\sin^4 y + \cos^4 y)\right) \cdot \left(c^2 + 4(\sin^4 z + \cos^4 z)\right) \geq 9(ab + bc + ca)$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3026** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$\frac{a(a^2 + b^2)}{a + b} + \frac{b(b^2 + c^2)}{b + c} + \frac{c(c^2 + a^2)}{c + a} \geq 4\sqrt{3} \cdot F$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3027** In any $\Delta ABC$ the following inequality holds:

$$\frac{a^2}{h_a^2} + \frac{b^2}{h_b^2} + \frac{c^2}{h_c^2} \geq 4$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3028** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c}\geq 2\sqrt{3}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3029** Let be $M$ an interior point in $\Delta ABC$ and $x=MA,y=MB,z=MC$ and $u,v,w$ the distances of point $M$ to the sides $BC,CA,AB$ then:

$$\left(\left(\frac{x}{v+w}+\frac{y^2}{wu}\right)^2+2\right)\cdot\left(\left(\frac{y}{w+u}+\frac{z^2}{uv}\right)^2+2\right)\cdot\left(\left(\frac{z}{u+v}+\frac{x^2}{vw}\right)^2+2\right)\geq 675$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3030** In any $\Delta ABC$ the following inequality holds:

$$\left(n_a^2\cdot n_b^2+2\right)\cdot\left(n_b^2\cdot n_c^2+2\right)\cdot(n_c^2\cdot n_a^2+2)\geq 3\cdot(m_a s_a+m_b s_b+3m_c s_c)^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3031** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$(a^3+b^3+c^3)^2\geq 64\cdot\sqrt{3}\cdot F^3$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3032** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$w_a^2+w_b^2+w_c^2\geq\frac{6\sqrt{3}\cdot r}{R}\cdot F$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**J.3033** In any $\Delta ABC$ the following inequality holds:

$$a\cdot w_a+b\cdot w_b+c\cdot w_c\geq 18\sqrt{3}\cdot r^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**J.3034** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$(h_a\cdot b^2+h_b\cdot c^2+h_c\cdot a^2)^2\geq 32\cdot\frac{s}{R}\cdot F^3$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**J.3035** If $a,x,y,t>0$ then:

$$(a^2x^2+t^2)\cdot(a^2y^2+t^2)\geq\frac{3}{4}\cdot(a^2(x+y)^2+t^2)t^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3036** If $x,y,z>0$ then:

$$(x^2+2(y+z)^2)\cdot(y^2+2(z+x)^2)\cdot(z^2+2(x+y)^2)\geq\frac{27}{4}(x+y)^2(y+z)^2(z+x)^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3037** Let be $M$ an interior point in $\Delta ABC, x=MA, y=MB, z=MC$ and $u,v,w$ the distances of point $M$ to the sides $BC, CA, AB$, then:

$$\left(\left(\frac{x}{v+w}+\frac{y}{\sqrt{wu}}\right)^2+2\right)\cdot\left(\left(\frac{y}{w+u}+\frac{z}{\sqrt{uv}}\right)^2+2\right)\cdot\left(\left(\frac{z}{u+c}+\frac{x}{\sqrt{vw}}\right)^2+2\right)\geq 243$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3038** Let be $m,n\geq 0; m+n>0$ and $M$ an interior point in $\Delta ABC$ and $x=MA$,

$y=MB, z=MC$ and $u,v,w$ the distances of point $M$ to the sides $BC, CA, AB$, then:

$$\left(\left(\frac{mx^2}{vw}+\frac{ny^2}{wu}\right)^2+2\right)\cdot\left(\left(\frac{my^2}{wu}+\frac{nz^2}{uv}\right)^2+2\right)\cdot\left(\left(\frac{mz^2}{uv}+\frac{nx^2}{wu}\right)^2+2\right)\geq 432\cdot(m+n)^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3039** Let be $M$ an interior point in $\Delta ABC$ and $x=MA, y=MB, z=MC$ and $u,v,w$ the distances of point $M$ to the sides $BC, CA, AB$ then:

$$\left(\left(\frac{x}{v+w}\right)^2+2\right)\cdot\left(\left(\frac{y}{w+u}+\frac{z^2}{vu}\right)^2+2\right)\cdot\left(\left(\frac{z}{u+v}+\frac{x^2}{wv}\right)^2+2\right)\geq 675$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3040** If $t\in\mathbb{R}$, then in any $\Delta ABC$ with the area $F$ the following inequality holds:

$$((m_a^2\cdot\sin^2 t+m_b^2\cos^2 t)^2+2)\cdot((m_b^2\sin^2 t+m_c^2\cos^2 t)^2+2)\cdot((m_c^2\sin^2 t+m_a^2\cos^2 t)^2+2)\geq 81\cdot F^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3041** If $a,b,c$ are three positive real numbers, then prove that:

$$\sum_{cyclic}\frac{1}{2(2a+b+c)}\geq\sum_{cyclic}\frac{1}{3(a+b)+2c}$$

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**J.3042** If $a,b,c>0$, then prove that:

$$\sum\frac{a}{b}-\frac{\sum a^2}{\sum ab}\geq 2$$

***Proposed by Neculai Stanciu – Romania***

**J.3043** If $a_k>0 (k=1,2,\dots,n)$, then prove that:

$$16\sum\frac{a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)}+\sum\frac{a_1^2+a_2^2}{a_1+a_2}\leq 3\sum_{k=1}^{n}a_k$$

***Proposed by Mihaly Bencze – Romania***

**J.3044** Prove that any triangle $ABC$ with usual notations is true the following

$$\sum a^3\,(b^2+c^2)\geq(12Rr)^2s$$

***Proposed by Neculai Stanciu – Romania***

**J.3045** If $a,b,c>0$ then prove that:

$$\sum\frac{6a^3}{\sqrt{2a(a+b)^3}+\sqrt{2b^2(a^2+b^2)}}-\sum a\geq 0$$

***Proposed by Neculai Stanciu – Romania***

**J.3046** If $x_k>0(k=1,2,\dots,n)$, then prove that:

$$\sum\frac{x_1^2+x_1x_2+x_2^2}{(x_1+x_2)(x_1^2+x_2^2)}\leq\frac{3}{4}\sum_{k=1}^{n}\frac{1}{x_k}$$

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**J.3047** If $a,b,c>0$, then prove that $3\sqrt{6}(\sum a^2-\sum ab)\geq(\sum|a-b|)^2$

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**J.3048** Prove that in all triangle $ABC$ holds the inequality

$$7\sum a^4+64s^2Rr\geq 5(s^2+r^2+4Rr)^2$$

***Proposed by Neculai Stanciu – Romania***

**J.3049** If $a,b,c>0$, then prove that

$$\sum\frac{2a^2+c(b-c)}{(b+c)(a+b+c)}\geq 1$$

***Proposed by Neculai Stanciu – Romania***

**J.3050** If $a,b,c>0$ and $abc=1$ and $\lambda\geq 0$ then:

$$\frac{a}{bc+\lambda c}+\frac{b}{ca+\lambda a}+\frac{c}{ab+\lambda b}\geq\frac{3}{\lambda+1}$$

***Proposed by Marin Chirciu – Romania***

**J.3051** If $a,b,c>0$ and $\lambda\geq 0$ then:

$$\frac{a^2}{b\sqrt{ab+\lambda bc}}+\frac{b^2}{c\sqrt{bc+\lambda ca}}+\frac{c^2}{b\sqrt{ca+\lambda ab}}\geq\frac{3}{\sqrt{\lambda+1}}$$

***Proposed by Marin Chirciu – Romania***

**J.3052** In $\Delta ABC$ the following relationship holds:

$$\sum\sqrt{r_a^2+\lambda r_a r_b+r_b^2}\geq p\sqrt{3(\lambda+2)},\text{ where } -2\leq\lambda\leq 2$$

***Proposed by Marin Chirciu – Romania***

**J.3053** In $\Delta ABC$ the following relationship holds:

$$\sum\sqrt{\cot^2\frac{A}{2}+\lambda\cot\frac{A}{2}\cot\frac{B}{2}+\cot^2\frac{B}{2}}\geq 3\sqrt{3(\lambda+2)},\text{ where } -2\leq\lambda\leq 2$$

***Proposed by Marin Chirciu – Romania***

**J.3054** Let $x,y,z>0$. In $\Delta ABC$ the following relationship holds:

$$\sum\frac{y+z}{x}h_a\geq 18r$$

***Proposed by Marin Chirciu – Romania***

**J.3055** In $\Delta ABC$ the following relationship holds:

$$\sum\sqrt{h_a^2+\lambda h_a h_b+h_b^2}\geq 9r\sqrt{\lambda+2},\text{ where } -2\leq\lambda\leq 2$$

***Proposed by Marin Chirciu – Romania***

**J.3056** In $\Delta ABC$ the following relationship holds:

$$\frac{3}{2}\sqrt[6]{\frac{4r^5}{R^2}}\leq\sum\sqrt{w_a\cos\frac{B}{2}\cos\frac{C}{2}}\leq\frac{4R+r}{\sqrt{2R}}$$

***Proposed by Marin Chirciu – Romania***

**J.3057** If $a,b,c>0$ such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$ and $\lambda>0$ then:

$$\frac{1}{a(\lambda a+1)}+\frac{1}{b\big((\lambda+1)b+1\big)}+\frac{1}{c\big((\lambda+2)c+1\big)}\geq\frac{3}{\lambda+2}$$

***Proposed by Marin Chirciu – Romania***

**J.3058** In $\Delta ABC$ the following relationship holds:

$$216r^3\leq(h_a+h_b)(h_b+h_c)(h_c+h_a)\leq\frac{8r}{3}(4R+r)^2$$

***Proposed by Marin Chirciu – Romania***

**J.3059** In $\Delta ABC$ the following relationship holds:

$$108Rr^2 \leq (r_a + r_b)(r_b + r_c)(r_c + r_a) \leq \frac{4R}{3}(4R + r)^2$$

***Proposed by Marin Chirciu – Romania***

**J.3060** In $\Delta ABC$ the following relationship holds:

$$\sum \sqrt{\tan^2\frac{A}{2} + \lambda \tan\frac{A}{2}\tan\frac{B}{2} + \tan^2\frac{B}{2}} \geq \sqrt{3(\lambda + 2)}, \text{ where } -2 \leq \lambda \leq 2$$

***Proposed by Marin Chirciu – Romania***

**J.3061** Let be $m \geq 0$ and $n_a, n_b, n_c$ are the Nagel cevians lenghts of $\Delta ABC$, $M$ an interior point and $d_a, d_b, d_c$ the distances $M$ to the sides $BC, CA$ respectively $AB$. Prove that:

$$\left(\frac{n_a}{d_a}\right)^m + \left(\frac{n_b}{d_b}\right)^m + \left(\frac{n_c}{d_c}\right)^m \geq 3^{m+1}$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3062** If $a, b, s, t > 0$ then $(a^2 + s^2)(b^2 + t^2) \geq (at + bs)^2$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3063** In any triangle $ABC$ with the area $F$ the following inequality:

$$\left(\frac{1}{h_a} + 2\right) \cdot \left(\frac{1}{h_b} + 2\right) \cdot \left(\frac{1}{h_c} + 2\right) \geq \frac{9 \cdot \sqrt[4]{27}}{\sqrt{F}}$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3064** If $a, b, c, x, y > 0$ and $ab + bc + ca = 3$ then:

$$\frac{1}{a^3(bx + cy)} + \frac{1}{b^3(cx + ax)} + \frac{1}{c^3(ax + by)} \geq \frac{3}{x + y}$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3065** In triangle $ABC$ with the area $F$ the following inequality holds:

$$\frac{a^4}{m_b^2} + \frac{b^4}{m_c^2} + \frac{c^4}{m_a^2} \geq \frac{16}{\sqrt{3}}F$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3066** Let be $M$ an interior point in triangle $ABC$ with the area $F$ such that that the angles $\angle BAC, \angle CMA, \angle CMA, \angle AMB$ are congruents, then:

$$\frac{a^2b^2}{x_a x_b} + \frac{b^2c^2}{x_b x_c} + \frac{c^2a^2}{x_c x_a} \geq 12\sqrt{3} \cdot F$$

***Proposed by D.M. Bătinețu – Giurgiu,Mihaly Bencze– Romania***

**J.3067** In any $\Delta ABC$ with the area $F$, the following inequality holds:

$$(m_a^4+2)(m_b^4+2)(m_c^4+2)\geq 81\cdot F^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3068** If $a,b,c,x,y,z,s>0$, then:

$$(a^2x+s)(b^2y+s)(c^2z+s)\geq\frac{3}{4}s^2\left(a\sqrt{x}+b\sqrt{y}+c\sqrt{z}\right)^2$$

(A generalization of Arkady Alt inequality and Hojoo Lee inequality)

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3069** If $M$ is an interior point in $ABCD$ tetrahedron, and $d_A,d_B,d_C,d_D$ are the distances of point $M$ to the faces $BCD,CDA,DAB$ respectively $ABC$ then:

$$\frac{h_a}{d_A}+\frac{h_B}{d_B}+\frac{h_C}{d_C}+\frac{h_D}{d_D}\geq 16$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze – Romania***

**J.3070** In any triangle $ABC$ having the lenght sides $a,b,c$ we have inequality:

$$\left(\arcsin\left(\frac{a}{b+c}\right)\right)^2+\left(\arcsin\left(\frac{b}{c+a}\right)\right)^2+\left(\arcsin\left(\frac{c}{a+b}\right)\right)^2\geq\frac{\pi^2}{12}$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze – Romania***

**J.3071** Let be $n_a,n_b,n_c$ Nagel cevians lengths of triangle $ABC$, $M$ an interior point in triangle, and $d_a,d_b,d_c$ the distances of point $M$ to the sides $BC,CA$ respectively $AB$. Prove that:

$$\frac{n_a}{d_a}+\frac{n_b}{d_c}+\frac{n_c}{d_c}\geq 9$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**J.3072** Let be $m\geq 0$ and $M$ an interior point in triangle $ABC$ having the area $F$ and $F_a=$ area $MBC$, $F_b=$ area $MCA$, $F_c=$ area $MAB$, then

$$\frac{a^{2m+2}}{F_b^m}+\frac{b^{2m+2}}{F_c^m}+\frac{c^{2m+2}}{F_a^m}\geq 2^{2m+2}\left(\sqrt{3}\right)^{m+1}F$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3073** Let be $m\geq 0$ and $ABC$ a triangle where $F$ is the area $ABC$ then:

$$\frac{a^2}{(b+c)^m\cdot h_a^m}+\frac{b^2}{(c+a)^m h_b^m}+\frac{c^2}{(a+b)^m h_c^m}\geq\frac{\sqrt{3}}{4^{m-1}F^{m-1}}$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3074** Let be $M$ an interior point $\Delta ABC$ with the area $F$ and $d_a, d_b, d_c$ the distances of point $M$ to the sides $BC, CA$ respectively $AB$. Prove that:

$$\frac{m_a}{d_a}+\frac{m_b}{d_b}+\frac{m_c}{d_c}\geq 9$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3075** If $x, y, z > 0$ then in any $\Delta ABC$ with the area $F$ the following inequality holds:

$$(a^4+b^4+z^2)(c^4+x^2+y^2)\geq 16(xy+yz+zx)F^4$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3076** Let $M$ be an interior point in $\Delta ABC$ with the area $F$ and $F_a =$ area $MBC, F_b =$ area $MCA$,

$F_c =$ area $MAB$. If $x, y, z > 0$ then:

$$\frac{x^2a^8}{(y+z)^2F_b}+\frac{y^2b^8}{(z+x)^2F}+\frac{zc^8}{(x+y)^2F_a}\geq 64F^3$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3077** If $a, b, c, x, y > 0$ then:

$$\frac{1}{c(ax+by)^2}+\frac{1}{a(bx+cy)^2}+\frac{1}{b(cx+ay)^2}\geq\frac{81}{(x+y)^2(a+b+c)^3}$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3078** If $m \geq 0$ and $M$ an interior point $\Delta ABC$ with the area $F$ then if $x, y, z$ are the distances point $M$ to the sides $BC, CA$ respectively $AB$, then:

$$xa^{m+1}+yb^{m+1}+zc^{m+1}\geq\frac{2^{m+1}}{(x+y+z)^m}\cdot F^{m+1}$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**J.3079** If $n_a, n_b, n_c -$ Nagel cevians, $g_a, g_b, g_c -$ Gergonne cevians in $\Delta ABC$ then:

$$n_ag_a+n_bg_b+n_cg_c\leq s^2\sqrt{2-\frac{2r}{R}}.$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**J.3080** If $n_a, n_b, n_c -$ Nagel cevians, $g_a, g_b, g_c -$ Gergonne cevians in $\Delta ABC$ then:

$$\frac{n_ag_a}{h_a}+\frac{n_bg_b}{h_b}+\frac{n_cg_c}{h_c}\leq\frac{20R-13r}{3}.$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.**

**PROBLEMS FOR SENIORS**

**S.2981** If $a, b, c > 0$ then prove that:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \frac{a^2+2b^2}{a^3+2b^3}+\frac{b^2+2c^2}{b^3+2c^3}+\frac{c^2+2a^2}{c^3+2a^3}$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**S.2982** In $\Delta ABC$ the following relationship holds:

$$\sum \frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2026}}{r^{2026}} \ge 2^{2026}+\sum \frac{a^{2025}}{b^{2025}+c^{2025}}$$

***Proposed by Nguyen Van Canh – Vietnam***

**S.2983** In any acute triangles $\Delta ABC$ and $\forall n \in \mathbb{N}$ the following relationship holds:

1. $\frac{a^n}{b^n+c^n}\cdot(\sec B+\sec C)+\frac{b^n}{c^n+a^n}\cdot(\sec C+\sec A)+\frac{c^n}{a^n+b^n}\cdot(\sec A+\sec B) \ge 6$

2. $\frac{h_a^n}{h_b^n+h_c^n}\cdot(\sec B+\sec C)+\frac{h_b^n}{h_c^n+h_a^n}\cdot(\sec C+\sec A)+\frac{h_c^n}{h_a^n+h_b^n}\cdot(\sec A+\sec B) \ge 6$

***Proposed by Zaza Mzhavanadze – Georgia***

**S.2984** Prove that:

$$\tan^2\left(\frac{2\pi}{21}\right)+\tan^2\left(\frac{8\pi}{21}\right)+\tan^2\left(\frac{10\pi}{21}\right)=93+20\sqrt{21}$$

***Proposed by Vasile Mircea Popa – Romania***

**S.2985** If $a, b, c > 0$ and $a+b+c \le 2$ then prove that:

$$\frac{a\sqrt{a}}{a+\sqrt{ab}+b}+\frac{b\sqrt{b}}{b+\sqrt{bc}+c}+\frac{c\sqrt{c}}{c+\sqrt{ca}+a}+\frac{1}{27\sqrt{abc}} \le \frac{13\sqrt{6}}{36}$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**S.2986** In any triangles $\Delta ABC$ the following relationship holds:

$$\frac{a}{b\cdot(\sin A+\sin B)+c\cdot\sin C}+\frac{b}{c\cdot(\sin A+\sin B)+a\cdot\sin C}+\frac{c}{a\cdot(\sin A+\sin B)+b\cdot\sin C}\geq\frac{2}{\sqrt{3}}$$

***Proposed by Zaza Mzhavanadze – Georgia***

**S.2987** If $x,y,z>0, x+y+z=2$ then:

$$\sum_{cyc}x^2(y^2+x^2)(z^2+x^2)\geq\frac{8}{9}$$

When equality holds?

***Proposed by Khaled Abd Almouti – Syria***

**S.2988** $p,q,r\in\mathbb{R}$ (prime), $n\in\mathbb{N}$

$$4n^2+p^4+24p^2q^2+24p^2qr+6p^2r^2+p^2+16q^4+32q^3r+24q^2r^2+2q^2+8qr^3+$$

$$+10q+r^4+25=m$$

$$4nq+8p^3q+4p^3r+32pq^3+48pq^2r+24pqr^2+2pq+4pr^3+10p=k$$

Solve it $\Rightarrow m=k$

***Proposed by Kerimov Elsen,Kenan Rustemov-Azerbaijan***

**S.2989** If $x\in\mathbb{R}$ then in $\Delta ABC$ holds:

$$\sum_{cyc}a\sqrt{(a\sin x)^2+(b\cos x)^2}\geq 2\sqrt{6}F$$

***Proposed by Daniel Sitaru – Romania***

**S.2990** $x;y\in\mathbb{R}^{\pm}$ solve the system equation

$$\begin{cases}\frac{3x}{y}+\frac{3y}{y-2x}+\frac{12y}{x-y}=-8\\ x^2+xy+y=3\end{cases}$$

***Proposed by Amin Hajiyev-Azerbaijan***

**S.2991** In $\Delta ABC$ the following relationship holds:

$$2022\min\left\{\sum\sqrt{\frac{a}{b}},\sum\frac{b}{a}\right\}+\frac{R^2}{r^2}\geq 4+2022\max\left\{\sum\sqrt{\frac{b}{a}},\sum\frac{a}{b}\right\}.$$

***Proposed by Nguyen Van Canh-Vietnam***

**S.2992** If $a=\min(a,b,c)$ in acute $\Delta ABC$ then:

$$\frac{2m_a}{r_a} \geq \frac{n_b}{h_c} + \frac{n_c}{h_b}$$

***Proposed by Bogdan Fuștei – Romania***

**S.2993** In $\Delta ABC$ the following relationship holds:

$$\sqrt{\frac{r_a + r_b + r_c}{h_a + h_b + h_c}}\left(\sqrt{\frac{2(a+c)-b}{2(a+b)-c}} + \sqrt{\frac{2(a+b)-c}{2(a+c)-b}}\right) \leq \sqrt{\frac{2R}{r}}$$

***Proposed by Bogdan Fuștei – Romania***

**S.2994** In $\Delta ABC$ the following relationship holds:

$$\sum \frac{\tan\frac{A}{2}}{\tan\frac{A}{2} + 2\tan\frac{B}{2}} \leq \frac{81R^3}{8r(4R+r)^2}$$

***Proposed by Marin Chirciu – Romania***

**S.2995** In $\Delta ABC$ the following relationship holds:

$$\prod \frac{\cos B + \cos C}{\sqrt{r_b^2 + r_c^2}} \leq \frac{\sqrt{2}}{216Rr^2}$$

***Proposed by Marin Chirciu – Romania***

**S.2996** If $a, b > 0$ such that $a^2 + b^2 = 2$ and $\lambda \geq 0, n \geq 0$ then find the maximum of

$$\frac{(a+b+n)(\lambda a + \lambda b(\lambda^2+1)ab)}{(a+\lambda)(b+\lambda)}$$

***Proposed by Marin Chirciu – Romania***

**S.2997** Let $\lambda > 0$ fixed. Solve in $\mathbb{R}$

$$(\lambda+1+x)\sqrt{\lambda - x} + (\lambda+1-x)\sqrt{\lambda+x} = 2\sqrt{\lambda x^2 + \lambda(\lambda+1)^2}$$

***Proposed by Marin Chirciu – Romania***

**S.2998** If $a, b, c > 0$ such that $\sum\frac{1}{a} = \sum\frac{1}{bc}$ and $\lambda \geq 0$ then:

$$\frac{a+\lambda}{b} + \frac{b+\lambda}{c} + \frac{c+\lambda}{a} \geq 3(\lambda+1)$$

***Proposed by Marin Chirciu – Romania***

**S.2999** If $a_1, a_2, \dots, a_n, a_{n+1} > 0$ such that $a_1 + a_2 + \cdots + a_n + a_{n+1} = n+1$ then

$$(a_1^n + n)(a_2^n + n)\dots(a_n^n + n)(a_{n+1}^n + n) \geq (n+1)^{n+1}$$

***Proposed by Marin Chirciu – Romania***

**S.3000** If $a, b, c > 0$ such that $a + b + c = 6$ and $\lambda \geq 0$ then:

$$\frac{a+\lambda}{\sqrt{b^3+1}} + \frac{b+\lambda}{\sqrt{c^3+1}} + \frac{c+\lambda}{\sqrt{a^3+1}} \geq \lambda + 2$$

***Proposed by Marin Chirciu – Romania***

**S.3001** If $a, b, c > 0$ such that $a + b + c = 6$ and $n \geq 0, \lambda \geq 0$ then:

$$\frac{na+\lambda}{\sqrt{b^3+1}} + \frac{nb+\lambda}{\sqrt{c^3+1}} + \frac{nc+\lambda}{\sqrt{a^3+1}} \geq 2n + \lambda$$

***Proposed by Marin Chirciu – Romania***

**S.3002** In $\Delta ABC$ the following relationship holds:

$$\frac{\sum m_a^2}{\sum m_b m_c} + \lambda \cdot \frac{2r}{R} \leq \lambda + 1, \text{ where } \lambda \geq \frac{1}{2}$$

***Proposed by Marin Chirciu – Romania***

**S.3003** In $\Delta ABC$ the following relationship holds:

$$36r \leq \sum \frac{a^2}{(p-a)\tan\frac{A}{2}} \leq \frac{9R^2}{r}$$

***Proposed by Marin Chirciu – Romania***

**S.3004** In $\Delta ABC$ the following relationship holds: $\sum \frac{a^2}{(p-a)\sin\frac{A}{2}} \geq 8p$

***Proposed by Marin Chirciu – Romania***

**S.3005** If $x, y > 0$ such that $x^2 + y^2 = 1$ and $n \in \mathbb{N}^*$, then:

$$x^{2n+1} + y^{2n+1} \geq \frac{\sqrt{2}}{2^{n-1}} xy$$

***Proposed by Marin Chirciu – Romania***

**S.3006** If $a, b, c > 0$ such that $abc = 1$, then:

$$\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} \leq \frac{1}{6}(a+b+c)^2$$

***Proposed by Marin Chirciu – Romania***

**S.3007** In $\Delta ABC$ the following relationship holds:

$$4 \leq \sum \frac{bc}{w_a^2} \leq \frac{2R}{r}$$

***Proposed by Marin Chirciu – Romania***

**S.3008** If $a,b,c \geq 0$ such that $a+b+c=n$ and $n,\lambda > 0$ then:

$$\sum \frac{a^2+ab+b^2}{(a+b+\lambda)^2} \leq \frac{n}{2\lambda}$$

***Proposed by Marin Chirciu – Romania***

**S.3009** Let $\lambda \geq 0$. Solve for $x,y,z \in \mathbb{N}$

$$\left(1+\frac{x}{y+\lambda z}\right)^2+\left(1+\frac{y}{z+\lambda x}\right)^2+\left(1+\frac{z}{x+\lambda y}\right)^2=3\left(\frac{\lambda+2}{\lambda+1}\right)^2$$

***Proposed by Marin Chirciu – Romania***

**S.3010** If $a,b,c > 0$ such that $abc=8^n, n \in \mathbb{N}$, then:

$$\frac{ab+4^n}{a+2^n}+\frac{bc+4^n}{b+2^n}+\frac{ca+4^n}{c+2^n} \geq 3 \cdot 2^n$$

***Proposed by Marin Chirciu – Romania***

**S.3011** Let $a \geq 1$ fixed. Solve in $\mathbb{R}$: $\frac{1}{\sqrt{x-1+a^2}}+\frac{1}{\sqrt{(a^2-1)x+1}}=\frac{2\sqrt{2}}{a\sqrt{x+1}}$

***Proposed by Marin Chirciu – Romania***

**S.3012** If $a,b,c > 0$ and $n \in \mathbb{N}, n \geq 2$ then find the minimum of

$$(a+b+3n-3)\sum \sqrt[8]{\frac{(a^2+1)(b^2+1)}{ab(a+1)(b+1)}}$$

***Proposed by Marin Chirciu – Romania***

**S.3013** In $\Delta ABC$ the following relationship holds:

$$4\left(\frac{R}{r}-1\right) \leq \sum \frac{bc}{r_a^2} \leq 4\left(\frac{R}{r}-1\right)^2$$

***Proposed by Marin Chirciu – Romania***

**S.3014** If $ABC, A_1B_1C_1, A_2B_2C_2$ are triangles of areas respectively $F, F_1, F_2$ then:

$$a_1a_2a^2+b_1b_2b^2+c_1c_2c^2 \geq 16\sqrt{F_1 \cdot F_2} \cdot F$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**S.3015** Let be $x,y,z > 0$ then in $\Delta ABC$ the following inequality holds:

$$\frac{x}{y+z} \cdot \frac{a}{h_a}+\frac{y}{z+x} \cdot \frac{b}{h_b}+\frac{z}{x+y} \cdot \frac{c}{h_c} \geq \sqrt{3}$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**S.3016** Let $ABC, A_1B_1C_1$ be triangles of area $F$ respectively $F_1$ and sides of lengths $a, b, c$ respectively $a_1, b_1, c_1$. Prove that:

$$\frac{a \cdot a_1^2}{h_a} + \frac{b \cdot b_1^2}{h_b} + \frac{c \cdot c_1^2}{h_c} \geq 8 \cdot F_1$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**S.3017** In any $\Delta ABC$ the following inequality holds:

$$\frac{a^2}{h_b^2} + \frac{b^2}{h_c^2} + \frac{c^2}{h_a^2} \geq 4$$

***Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania***

**S.3018** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$\left(\left(\frac{1}{h_a} + \frac{1}{h_b}\right)^2 + 2\right) \cdot \left(\left(\frac{1}{h_b} + \frac{1}{h_c}\right)^2 + 2\right) \cdot \left(\left(\frac{1}{h_c} + \frac{1}{h_a}\right)^2 + 2\right) \geq \frac{12 \cdot \sqrt{3}}{F}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania***

**S.3019** If $a_k > 0$ $(k = 1,2, \dots, n)$, then prove that:

$$\frac{1}{3}\sum_{cyc}\left(\frac{a_1^3}{a_2^2 - a_2a_3 + a_3^2} + \frac{a_2^3}{a_3^2 - a_3a_1 + a_1^2} + \frac{a_3^3}{a_1^2 - a_1a_2 + a_2^2}\right) \geq \sum_{k=1}^{n} a_k$$

***Proposed by Neculai Stanciu – Romania***

**S.3020** If $x, y, z \in \mathbb{R}$ such that $3x + y + 2z \geq 3$ and $-x + 2y + 4z \geq 5$, then find:

$$\min(x + 2y + 4z).$$

***Proposed by Neculai Stanciu – Romania***

**S.3021** Find: $\sum_{\substack{y<x<-1 \\ x,y\in\mathbb{Z}}} 2^{x+1} \cdot 3^y \cdot 5$

***Proposed by Titu Zvonaru, Neculai Stanciu – Romania***

**S.3022** Prove that the numbers on form $3^{2^n} - 1$ it is written as the sum of two perfect squares for any natural number $n$.

***Proposed by Neculai Stanciu – Romania***

**S.3023** Prove that for all $n \in \mathbb{N}$ the expression:

$$\frac{4^{n+2} - 4}{3} - \frac{(n + 1)(3n + 8)}{2}$$

is divisible by 9.

***Proposed by Neculai Stanciu – Romania***

**S.3024** Prove that in all triangles $ABC$ holds:

$$\sum(1-\cos A-\cos 2A-\cos(B-C))^2=\left(\frac{s^2-4R-r^2}{R^2}\right)^2-\left(\frac{s^2+r^2+4Rr}{2R^2}\right)^2$$

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**S.3025** Let $a,b,c\in\mathbb{R}$. Statement 1) $a,b,c$ are in geometrical progression; Statement 2) $(\sum a^2)^2=(\sum ab)^2$; Statement 3) $(\sum ab)^3=abc(\sum a)^3$. Which of the following three statements from above are equivalent?

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**S.3026** Solve:

$$2\left(\sqrt{x}+\sqrt{x-1}+\sqrt{x-2}\right)=\sqrt{x+1}+\sqrt{x+5}+\sqrt{x+9}$$

***Proposed by Sakthi Vel-India***

**S.3027** If $a,b,c>0$ and $a^2+b^2+c^2=3$ then $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq 3$.

***Proposed by Nguyen Hung Cuong – Vietnam***

**S.3028** If $a,b,c>0$, then prove that:

$$\prod_{k=1}^{2n}\left(\sum\left(\frac{a}{b}\right)^{2^k}\right)\geq\left(\sum\frac{a}{b}\right)^n\left(\sum\frac{a}{c}\right)^n$$

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**S.3029** If $a,b,c>0$ and $abc=1,\forall n\in\mathbb{N}$ then prove that:

$$\frac{b(b+c)^{3n}+a(a+c)^{3n}}{c(a+b)}+\frac{b(a+b)^{3n}+c(a+c)^{3n}}{a(b+c)}+\frac{a(a+b)^{2n}+c(b+c)^{3}n}{b(a+c)}\geq 3\cdot 8^n$$

***Proposed by Zaza Mzhavanadze – Georgia***

**S.3030** Solve the equation:

$$\left[\frac{x^3+3x^2+2x+2}{3}\right]=\frac{1}{3}x^3+\frac{1}{3}$$

***Proposed by Ibrahim Abdullayev Masalli – Azerbaijan***

**S.3031** If $a_k>0\left(k=\overline{1,n}\right)$, then prove the inequality:

$$\sum_{cyclic}\frac{3a_1}{a_2+2a_3}\geq n$$

***Proposed by Neculai Stanciu – Romania***

**S.3032** If $a, b, c > 0$, then prove that:

$$abc\sum\frac{a^2+b^2}{c}\leq 2\sum a^4$$

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**S.3033** Prove that in all triangles $ABC$ holds:

$$\sum\frac{m_a^{k+1}}{m_b+m_c-m_a}\geq\sum m_a^k \text{ for all } k\in\mathbb{N}.$$

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**S.3034** If $a+b+c+abc=4$ then $a^2+b^2+c^2\geq 3$.

***Proposed by Nguyen Hung Cuong – Vietnam***

**S.3035** If $a+b=3$ and $a^2+b^2\geq 5$ then: $a^4+b^4+6a^2b^2\geq 41$

***Proposed by Nguyen Hung Cuong – Vietnam***

**S.3036** Let $0\leq a\leq 1\leq b\leq 2\leq c\leq 2023$. Find the maximum and minimum value of the expression:

$$P=\sqrt{\frac{b}{a+1}}+\sqrt[3]{\frac{a+1}{a+2}}$$

***Proposed by Nguyen Van Canh – Vietnam***

**S.3037** In any acute triangle $ABC$ holds:

$$\frac{1}{h_a}\sqrt{\tan A}+\frac{1}{h_b}\sqrt{\tan B}+\frac{1}{h_c}\sqrt{\tan C}\geq\frac{2}{R}\sqrt[4]{3}$$

***Proposed by Vasile Mircea Popa – Romania***

**S.3038** If $a>0, b>0, c>0$ then prove:

$$\frac{a^8(a^2+bc)}{(b+c)^{10}}+\frac{b^8(b^2+ca)}{(c+a)^{10}}+\frac{c^8(c^2+ab)}{(a+b)^{10}}\geq\frac{3}{512}$$

***Proposed by Zaza Mzhavanadze – Georgia***

**S.3039** If $a, b, c, d > 0$ and $\forall n\in\mathbb{N}$ prove that:

$$\frac{a}{nb+(n+1)c+(n+2)d}+\frac{b}{nc+(n+1)d+(n+2)a}+\frac{c}{nd+(n+1)a+(n+2)b}+$$
$$+\frac{d}{na+(n+1)b+(n+2)c}\geq\frac{4}{3(n+1)}$$

***Proposed by Zaza Mzhavanadze – Georgia***

**S.3040** If $a, b, c \in \mathbb{R}^+$ then:

$$\frac{a+b}{a+b+2c}+\frac{b+c}{b+c+2a}+\frac{a+c}{a+c+2b}\geq\frac{3}{2}$$

***Proposed by Samed Ahmedov – Azerbaijan***

**S.3041** Let $a, b, c \geq 0$ and $a^2+b^2+c^2 \neq 0$. Prove that:

$$\sum\frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc}\geq\frac{a+b+c}{2}$$

***Proposed by Nguyen Van Canh – Vietnam***

**S.3042** If $x, y > 0, x+y=1$ and $\lambda \geq 0, n \in \mathbb{N}$ then:

$$\left(\lambda+\frac{1}{x^n}\right)\left(\lambda+\frac{1}{y^n}\right)\geq(\lambda+2^n)^2$$

***Proposed by Marin Chirciu – Romania***

**S.3043** If $ABCDEFGHJKLM$ is a regular 13-gon, then prove that: $AE(AC-AB)=AF(AD-AC)$

***Proposed by Neculai Stanciu – Romania***

**S.3044** If $a_k > 0\ \forall k=\overline{1,n}$, then prove that

$$\frac{\prod_{k=1}^{n}(a_k^n+n-1)}{(\sum_{k=1}^{n}a_k)^n}$$

***Proposed by Neculai Stanciu – Romania***

**S.3045** Let $ABC$ be a triangle, the altitude $h_a$ intersects the circumcircle in point $E$ the bisector of angle $\angle EAC$ intersects the circumcircle in point $D$. The tangent line to circle in point $D$ intersects the lines $AE$ and $BC$ in points $M$ and $N$. 1) Determine all triangles $ABC$ for which $BHCE$ is rhombus.

2) Determine all trinagles $ABC$ for which $ABMN$ is cyclic

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**S.3046** If $x_i > 0\ (i=1,2,\dots,n)$ and $k \in \mathbb{N}$, then prove that:

$$2\sum_{i=1}^{n}x_i^{2k+2}\geq\sum_{cyclic}x_1x_2\left(x_1^{2k}+x_2^{2k}\right)\geq\sum_{cyclic}x_1^2x_2^2\left(x_1^{2k-2}+x_2^{2k-2}\right)\geq\dots\geq$$

$$\geq\sum_{cyclic}x_1^kx_2^k(x_1^2+x_2^2)\geq2\sum_{cyclic}x_1^{k+1}x_2^{k+1}$$

***Proposed by Mihaly Bencze, Neculai Stanciu – Romania***

**S.3047** If $x_i > 0(i=1,2,\dots,n)$ and $k \in \mathbb{N}^*$, then prove that:

$$2\sum_{i=1}^{n} x_i^{2k+1} \geq \sum_{cyclic} x_1x_2(x_1^{2k-1}+x_2^{2k-1}) \geq \sum_{cyclic} x_1^2x_2^2(x_1^{2k-3}+x_2^{2k-3}) \geq \cdots \geq \sum_{cyclic} x_1^kx_2^k(x_1+x_2)$$

***Proposed by Neculai Stanciu – Romania***

**S.3048** Prove that the following equations

(a) $x_1^5 + x{+}_2^5 + x + 3^5 + x_4^5 + x_5^5 = 0$ and (b) $x_1^7 + x_2^7 + x_3^7 + x_4^7 + x_5^7 + x_6^7 + x_7^7 + x_8^7 = 0$

have an infinitely solutions in $\mathbb{Z}$

***Proposed by Neculai Stanciu – Romania***

**S.3049** If $ABCDEFGIJK$ is a regular $11$ – gon, then prove that:

$$AE^2 - AD^2 = AB \cdot AE \cdot \left(\frac{AD}{AB} - \frac{AE}{AC}\right)$$

***Proposed by Neculai Stanciu – Romania***

**S.3050** Solve for positive real numbers the following system:

$$\begin{cases} \dfrac{x^3}{(1+y)(1+z)} = \dfrac{6y-x-z-2}{8} \\ \dfrac{y^3}{(1+z)(1+x)} = \dfrac{6z-y-x-2}{8} \\ \dfrac{z^3}{(1+x)(1+y)} = \dfrac{6x-z-y-2}{8} \end{cases}$$

***Proposed by Neculai Stanciu – Romania***

**S.3051** In $\Delta ABC$ the following relationship holds:

$$\sum \sqrt{\cot^2\frac{A}{2} + \cot\frac{A}{2}\cot\frac{B}{2} + \cot^2\frac{B}{2}} \geq 9$$

***Proposed by Marin Chirciu – Romania***

**S.3052** In $\Delta ABC$ the following relationship holds:

$$\frac{4(4R+r)}{3\sqrt{3}r} \leq \sum \frac{\cot\frac{A}{2}}{\sin B \sin C} \leq \frac{2R(4R+r)}{3r^2\sqrt{3}}$$

***Proposed by Marin Chirciu – Romania***

**S.3053** If $a, b, c > 0$ such that $abc = 1$ and $n \in \mathbb{N}, n \geq 2, \lambda \geq 0$ then:

$$\frac{a^nb}{b+\lambda} + \frac{b^nc}{c+\lambda} + \frac{c^na}{a+\lambda} \geq \frac{3^{3-n}}{\lambda+1}$$

***Proposed by Marin Chirciu – Romania***

**S.3054** Let $x, y, z > 0$. In $\Delta ABC$ the following relationship holds: $\sum \frac{y+z}{x} r_a \geq 18r$

***Proposed by Marin Chirciu – Romania***

**S.3055** Let $x, y, z > 0$. In $\Delta ABC$ the following relationship holds:

$$\sum \frac{y+z}{x} m_a \geq 18r$$

***Proposed by Marin Chirciu – Romania***

**S.3056** In $\Delta ABC$ the following relationship holds:

$$729 \sum \tan^3 \frac{B}{2} \tan^3 \frac{C}{2} \leq \sum \cot^3 \frac{B}{2} \cot^3 \frac{C}{2}$$

***Proposed by Marin Chirciu – Romania***

**S.3057** If $a, b, c > 0$ such that $a + b + c = 3$ and $n \geq 0, \lambda \geq 0$ then:

$$\frac{n+a^2}{b+\lambda c} + \frac{n+b^2}{c+\lambda a} + \frac{n+c^2}{a+\lambda b} \geq \frac{3(n+1)}{\lambda+1}$$

***Proposed by Marin Chirciu – Romania***

**S.3058** If $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 3$ and $1 < \lambda \leq 2$ then:

$$\lambda(a+b+c) + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3(\lambda+1)$$

***Proposed by Marin Chirciu – Romania***

**S.3059** Let $\Delta ABC$ be a equilateral triangle. Let $D \in [BC], E \in [CA], F \in [AB]$ such that $|BD| = |CE| = |AF|$. If $[AD] \cap [CF] = \{K\}, [AD] \cap [BE] = \{L\}, [CF] \cap [BE] = \{M\}$,

$[AD] \cap [FE] = \{P\}, [BE] \cap [FD] = \{Q\}$ and $[DE] \cap [CF] = \{R\}$. Prove that the triangles ΔABC,ΔDEF,ΔKLM and ΔPQR have the same circumcenter.

***Proposed by Mehmet Șahin-Turkiye***

**S.3060** Let $\Delta ABC$ be a triangle . Its side lenghts are $a, b, c$, the inradius is $r$ and its the exradii are $r_a, r_b, r_c$ . Prove that:

$$\frac{r_a}{b+c-r} + \frac{r_b}{c+a-r} + \frac{r_c}{a+b-r} \geq \frac{27r}{4s-3r}$$

***Proposed by Mehmet Șahin-Turkiye***

**S.3061** If $n_a, n_b, n_c -$ Nagel cevians, $g_a, g_b, g_c -$ Gergonne cevians in $\Delta ABC$ then:

$$\frac{n_a g_a}{{h_a}^2} + \frac{n_b g_b}{{h_b}^2} + \frac{n_c g_c}{{h_c}^2} \leq \sqrt{5}\left(\frac{R}{r} - 2\right) + 3.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**S.3062** If $a, b, c > 0$, then:

$$(a^2 + 1936 \cdot 4) \cdot (b^2 + 1936 \cdot 4) \cdot (c^2 + 1936 \cdot 4) \geq 988^2(ab + bc + ca)$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**S.3063** If $a, b, c > 0$ and $b^2 + c^2 = a^2$ then:

$$(3 + a^4 + b^4 + c^4) \cdot \left(3 + \frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}\right) \geq 20$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**S.3064** Let be $x, y, z > 0$ and $M$ an interior point in triangle $ABC$ with the area $F$ and $d_a, d_b, d_c$ the distances of point $M$ to the sides $BC, CA$ respectively $AB$, then:

$$\frac{x^2 a^3}{(y+z)^2 \cdot d_a} + \frac{y^2 b^3}{(z+x)^2 d_b} + \frac{z^2 c^3}{(x+y)^2 d_c} \geq 6F$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**S.3065** In any $\Delta ABC$ with the area $F$ the following inequality holds:

$$\frac{\sin A}{h_a^2} + \frac{\sin B}{h_b^2} + \frac{\sin C}{h_c^2} \geq \frac{3}{2F}$$

***Proposed by D.M. Bătinețu – Giurgiu– Romania***

**S.3066** In $\Delta ABC$ with the area $F$ the following inequality holds:

$$\left(\frac{a^4}{(b+c)^2 h_a^2} + 2\right) \cdot \left(\frac{b^4}{(c+a)^2 h_b^2} + 2\right) \cdot \left(\frac{c^4}{(a+c)^2 h_c^2}\right) \geq 9$$

***Proposed by D.M. Bătinețu – Giurgiu– Romania***

**S.3067** Let be $M$ an interior point in triangle $ABC$ with the area $F$, and $d_a, d_b, d_c$ the distances of point $M$ to the sides $BC, CA$ respectively $AB$, then:

$$\frac{a^3 \cdot b^4}{\sqrt{h_a \cdot d_a}} + \frac{b^3 \cdot c^4}{\sqrt{h_b \cdot d_b}} + \frac{c^3 \cdot a^4}{\sqrt{h_c \cdot d_c}} \geq 64 \cdot F^3$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**S.3068** Let be $x, y, z > 0$ and $M$ an interior point in triangle $ABC$ with the area $F$ and $d_c, d_b, d_c$ the distances of point $M$ to the sides $BC, CA$ respectively $AB$, then:

$$\frac{x^3 \cdot a^4}{(y+z)^3 \cdot d_a^2} + \frac{y^3 \cdot b^4}{(z+x)^3 \cdot d_b^2} + \frac{z^3 \cdot c^4}{(x+y)^3 \cdot d_c^2} \geq 6\sqrt{3}F$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**S.3069** Let be $m \geq 0$ and $ABC$ a triangle with the area $F$, then:

$$\frac{a \cdot \cot^{m+1} A}{h_a^{2m+1}} + \frac{b \cdot \cot^{m+1} B}{h_b^{2m+1}} + \frac{c \cdot \cot^{m+1} C}{h_c^{2m+1}} \geq \frac{2}{3^m F^m}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**S.3070**. If $M$ is an interior point in triangle $ABC$ and $d_a, d_b, d_c$ are the distances of point $M$ to the sides $BC, CA$ respectively $AB$, then:

$$MA^2 + MB^2 + MC^2 \geq \frac{4}{3}(d_a + d_b + d_c)^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania***

**S.3071** Let be $A_1B_1C_1, A_2B_2C_2$ triangles with the areas $F_1$, respectively $F_2$, then:

$$(a_1^2a_2^2 + 2)(b_1^2b_2^2 + 2)(c_1^2c_2^2 + 2) \geq 144 \cdot F_1F_2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**S.3072** Let be $M$ an interior point in triangle $ABC$ with the area $F$ and $d_a, d_b, d_c$ the distances of point $M$ to the sides $BC, CA$ respectively $AB$, then:

$$\left(\frac{a^2b^4}{h_ad_a} + 2\right) \cdot \left(\frac{b^2c^4}{h_bd_b} + 2\right) \cdot \left(\frac{c^2a^4}{h_cd_c} + 2\right) \geq 432 \cdot F^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**S.3073** If $x \in \left(0, \frac{\pi}{2}\right)$ then in any $\Delta ABC$ with the area $F$ the following inequality holds:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}F \sin 2x + \sum_{cyc} (a \sin x - b \cos x)^2$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania***

**S.3074** If $p_a, p_b, p_c -$ Spieker cevians, $n_a, n_b, n_c -$ Nagel cevians in $\Delta ABC$ then:

$$n_a + n_b + n_c \leq p_a + p_b + p_c + \frac{4}{5}(max\{a,b,c\} - min\{a,b,c\})$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**S.3075** If $p_a, p_b, p_c -$ Spieker cevians, $n_a, n_b, n_c -$ Nagel cevians in $\Delta ABC$ then:

$$n_a + n_b + n_c \geq p_a + p_b + p_c + \frac{2}{3} . \frac{a^2 + b^2 + c^2 - ab - bc - ca}{s}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**S.3076** If $p_a, p_b, p_c -$ Spieker cevians, $n_a, n_b, n_c -$ Nagel cevians in $\Delta ABC$ then:

$$\frac{n_a - p_a}{h_a} + \frac{n_b - p_b}{h_b} + \frac{n_c - p_c}{h_c} \geq \frac{a^2 + b^2 + c^2 - ab - bc - ca}{3F}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**S.3077** If $p_a, p_b, p_c -$ Spieker cevians, $n_a, n_b, n_c -$ Nagel cevians in $\Delta ABC$ then:

$$n_a + n_b + n_c \geq p_a + p_b + p_c + \frac{2s(R - 2r)}{5R}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**S.3078** If $p_a, p_b, p_c -$ Spieker cevians in $\Delta ABC$ then:

$$p_a + p_b + p_c \leq w_a + w_b + w_c + \frac{4}{3}[max\{a, b, c\} - min\{a, b, c\}]$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**S.3079** If $p_a, p_b, p_c -$ Spieker cevians in $\Delta ABC$ then:

$$0 \leq \frac{p_a - w_a}{a} + \frac{p_b - w_b}{b} + \frac{p_c - w_c}{c} < \frac{4}{3}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**S.3080** If $p_a, p_b, p_c -$ Spieker cevians in $\Delta ABC$ then:

$$p_a + p_b + p_c \geq w_a + w_b + w_c + \frac{4s(R - 2r)}{15R}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.**

## UNDERGRADUATE PROBLEMS

**U.2981** Compute:

$$K=\int_0^1\sqrt{x+2021\sqrt{x+2020\sqrt{x}}}\,dx$$

***Proposed by Nguyen Van Canh –Vietnam***

**U.2982** Let:

$$\Omega_1=\sum_{k=0}^{\infty}e^{-e^{2\pi k}},\Omega_2=\sum_{k=1}^{\infty}\frac{\sin\left(4\ln k-k+\frac{\pi}{4}-\frac{B_2}{1.2k}+\frac{B_4}{3.4k^3}-\cdots\right)}{\sqrt{k\sin h\,(k)}},\qquad \Omega_3=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!\,(e^{2\pi k}-1)}$$

Prove that:

$$2\sqrt{\pi}\left(\sqrt{\pi}(\Omega_1-\Omega_3)-\Omega_2\right)=\pi-\gamma$$

where: $B_n$: denote the nth Bernouli number

***Proposed by Toubal Fethi -Algeria***

**U.2983**

$$\int_0^{\infty}\frac{\tan^{-1}(x^5)}{x^6+2}dx=\frac{2^{-\frac{17}{6}}}{45}\Omega'\left(\frac{1}{6}\right)+\pi 3^{-\frac{5}{2}}2^{-\frac{11}{6}}\left(\Omega\left(\frac{1}{2}\right)-\Omega\left(\frac{5}{6}\right)\right)+\pi\csc\left(\frac{\pi}{5}\right)\frac{2^{-\frac{17}{6}}}{15}\cdot$$

$$\cdot\left(\Omega\left(\frac{23}{30}\right)-\Omega\left(\frac{17}{30}\right)\right)+\pi\csc\left(\frac{2\pi}{5}\right)\frac{2^{-\frac{17}{6}}}{15}\left(\pi\left(\frac{29}{30}\right)-\Omega\left(\frac{11}{30}\right)\right)$$

Where

$$\Omega(n)=\int_0^{32}\frac{x^{n-1}-1}{x-1}dx$$

$$\Omega'\left(\frac{1}{6}\right)=\int_0^{32}\frac{x^{\frac{1}{6}-1}}{x-1}\ln x\,dx,\qquad \Omega\left(\frac{1}{2}\right)-\Omega\left(\frac{5}{6}\right)=\int_0^{32}\frac{x^{\frac{1}{2}-1}-x^{\frac{5}{6}-1}}{x-1}dx$$

$$\Omega\left(\frac{23}{30}\right)-\Omega\left(\frac{17}{30}\right)=\int_0^{32}\frac{x^{\frac{23}{30}-1}-x^{\frac{17}{30}-1}}{x-1}dx,\qquad \Omega\left(\frac{29}{30}\right)-\Omega\left(\frac{11}{30}\right)=\int_0^{32}\frac{x^{\frac{29}{30}-1}-x^{\frac{11}{30}-1}}{x-1}dx$$

***Proposed by Fao Ler-Iraq***

**U.2984**

$$\int_0^1 \frac{x^{-\frac{1}{6}}}{32-x}dx = 2^{\frac{11}{6}}\left(\ln\left(129\times x^{\frac{31}{6}}+315\times 2^{\frac{13}{3}}+297\times 2^{\frac{9}{2}}+873\times 2^{\frac{8}{3}}+1059\times 2^{\frac{11}{6}}+3265\right)\right)-$$

$$-2\ln 31-2\sqrt{3}\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{2^{\frac{11}{3}}+2^{\frac{1}{3}}-8}}\right)$$

***Proposed by Fao Ler-Iraq***

**U.2985** Find:

$$\Omega=\int_0^\infty \frac{\ln(1+x)}{(x^2+1)^2}dx$$

***Proposed by Vasile Mircea Popa – Romania***

**U.2986** Prove the below closed form

$$\int\int\int_{[0,1]^3}\left(\sum_{x,y,z}\sqrt{\frac{x}{\sqrt{y+\sqrt{z}}}}\right)dx\,dy\,dz=\frac{64}{77}\left(2\sqrt[4]{8}-1\right)$$

***Proposed by Ankush Kumar Parcha -India***

**U.2987** Find:

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(4\tan x)}{\sin 2x\ln(2\tan x)}dx$$

***Proposed by Nguyen Hung Cuong – Vietnam***

**U.2988** Prove the below closed form

$$\int\int_{[0,1]^2}\ln\left(\frac{1}{\sinh^2(x)+\cosh^2(y)}\right)dxdy=\frac{1}{4}Li_3\left(-\frac{1}{e^4}\right)+\frac{3\zeta(3)}{16}-\frac{\pi^2}{12}+2\ln 2-\frac{4}{3}$$

***Proposed by Ankush Kumar Parcha-India***

**U.2989** Find the general form of the integral and if $a_1a_2a_3\dots a_k=1\quad 2k>2n+1$

Prove that:

$$\int_0^\infty \frac{x^{2n}}{(x^2+a_1^2)(x^2+a_2^2)(x^2+a_3^2)\dots(x^2+a_k^2)}dx\le\frac{\pi}{2^k}\sum_{m=1}^{k}\frac{(-1)^n a_m^{\frac{4n-k}{2}}}{\prod_{t\in[1;k]\setminus m}(a_t-a_m)}$$

***Proposed by Abbaszade Yusif – Azerbaijan***

**U.2990** Prove the below closed form

$$\int_0^{2\pi} \frac{1+\sin(x)+\sin^2(x)}{1+\cos(x)+\cos^2(x)}dx = \frac{2}{3}\left(\sqrt{24+14\sqrt{3}}-3\right)\pi$$

***Proposed by Ankush Kumar Parcha-India***

**U.2991** Let be the sequence:

$$(u_n)_{n\geq 1}: \begin{matrix} u_1 = 1 \\ u_{n+1} = \frac{3s_n^2}{2s_n - 2}, n \geq 2 \end{matrix} \text{ such that: } s_n = u_1 + u_2 + \cdots + u_n$$

Find: $\lim_{n\to\infty}(u_n)$

***Proposed by Khaled Abd Imouti-Syria***

**U.2992** Let be $t \geq 0$ and $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that $\lim_{n\to\infty} \frac{n^{t+1}}{a_n} = a > 0$, $\lim_{n\to\infty} \frac{b_{n+1}}{a_n \cdot b_n} = b > 0$. Calculate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)^t} - \frac{\sqrt[n]{b_n}}{n^t}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania***

**U.2993** Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be sequences of real strictly positive numbers such that

$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a > 0$ and $\lim_{n\to\infty} \frac{b_{n+1}}{n \cdot b_n} = b > 0$. Calculate:

$$\lim_{n\to\infty} \sqrt[n]{a_n} \cdot \left(\frac{\sqrt[n+1]{b_{n+1}}}{n+1} - \frac{\sqrt[n]{b_n}}{n}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania***

**U.2994** If $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ are sequences of real strictly positive numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$, $\lim_{n\to\infty} \frac{b_{n+1}}{n^2 \cdot b_n} = b > 0$ and $c_{n+1}^n = a_n \cdot b_n \cdot c_n^n$; $\forall n \in \mathbb{N}^*$, calculate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{c_{n+1}}}{(n+1)^2} - \frac{\sqrt[n]{c_n}}{n^2}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania***

**U.2995** Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be sequences of real strictly positive numbers such that

$\lim_{n\to\infty}(b_{n+1} - b_n) = b > 0$ and $a_{n+1} = a_n \cdot b_n, \forall n \in \mathbb{N}^*$. Calculate:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania***

**U.2996** Let be $(a_n)_{n\geq 1}, (H_n)_{n\geq 1}$ where $a_{n+1} = a_n \cdot e^{H_n}, \forall n \in \mathbb{N}^*$ where $a_n > 0, \forall n \in \mathbb{N}^*$ and where $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Calculate:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania***

**U.2997** If $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ are sequences of real strictly positive numbers such that

$\lim_{n\to\infty} b_n = b > 0$ and $a_{n+1} = a_n \left(nb_n + \sqrt[n]{(2n-1)!!}\right), \forall n \geq 2$. Calculate:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania***

**U.2998** Let be $(H_n)_{n\geq 1}, (a_n)_{n\geq 1}$ be sequences of real strictly positive numbers such that

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, a_{n+1} = a_n \cdot e^{2H_n}, \forall n \in \mathbb{N}^*$$

Calculate:

$$\lim_{n\to\infty} \left(\frac{(n+1)^3}{\sqrt[n+1]{a_{n+1}}} - \frac{n^3}{\sqrt[n]{a_n}}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania***

**U.2999** Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ be sequences of real strictly positive numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{n\cdot a_n} = a > 0$, $\lim_{n\to\infty} \frac{b_{n+1}}{n^2\cdot b_n} = b > 0$ and $c_{n+1}^n \cdot a_n = c_n^n b_n, \forall n \geq 1$. Calculate:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{c_{n+1}} - \sqrt[n]{c_n}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania***

**U.3000** Let $(a_n)_{n\geq 1}$ be a sequence of real strictly positive numbers such that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$$

Calculate:

$$\lim_{n\to\infty} \sqrt[n]{a_n} \cdot \left(\frac{\sqrt[n+1]{(2n+1)!!}}{n+1} - \frac{\sqrt[n]{(2n-1)!!}}{n}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania***

**U.3001** Let be $(a_n)_{n\geq 2}, (b_n)_{n\geq 2}$ sequences of real strictly positive numbers such that

$a_{n+1} + \sqrt[n]{(2n-1)!!} = a_n + \sqrt[n+1]{(2n+1)!!}, \forall n \geq 2$ and $\lim_{n\to\infty} \frac{b_{n+1}}{a_n b_n} = b > 0$. Calculate:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania***

**U.3002** Let be $k \in \mathbb{N}^*$ and $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that

$\lim_{n\to\infty}\frac{\sqrt[k]{a_{n+1}}}{n\cdot\sqrt[k]{a_n}}=a>0$. Calculate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{k-1}}-\frac{\sqrt[n]{a_n}}{n^{k-1}}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania***

**U.3003** Prove the integral

$$\int_{-\infty}^{\infty}\big((1+i\,\mathrm{erf}(x))(1+i\,\mathrm{erfi}(x))\big)\,e^{-\pi x^2}dx=1-\frac{2}{\pi}\coth^{-1}(\pi)$$

where $\mathrm{erf}(x)$ , $\mathrm{erfi}(x)$ are Error functions

***Proposed by Srinivasa Raghava-AIRMC-India***

**U.3004** Prove the below closed – form:

$$S=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^n {}_2F_1\left(1,n+\frac{3}{2};n+\frac{5}{2};\frac{1}{2}\right)}{(n+1)(2n+3)}=\frac{12Li_2\left(\frac{1}{4}\left(\sqrt{2}+2\right)\right)-\pi^2+15\ln^2(2)-6\ln^2\left(\sqrt{2}+2\right)}{3\sqrt{2}}$$

***Proposed by Pham Duc Nam-Vietnam***

**U.3003** Prove the summation:

$$\sum_{n=0}^{\infty}\frac{2n-(-1)^n}{2n+(-1)^n}\frac{\binom{2n}{n}}{(2n+1)4^n}=\frac{5\pi}{8}-\frac{5}{6\sqrt{2}}-\frac{3}{4}\log\left(1+\sqrt{2}\right)$$

***Proposed by Srinivasa Raghava-AIRMC-India***

**U.3004** Prove that:

$$\int_0^{\pi}\arcsin\left(\frac{\sin x}{\sqrt{\frac{5}{4}-\cos x}}\right)dx=\frac{\pi^2}{4}$$

***Proposed by Khalaf Jssam – Iraq***

**U.3005** Find a closed form: $\Omega=\sum_{n=0}^{\infty}\ \left((-\varphi)^n-(\varphi-1)^n\right)\cdot\left(\frac{-1}{4}\right)^n$

***Proposed by Khaled Abd Imouti-Syria***

**U.3006** Show that for all $\{z \in \mathbb{C}: Re(z) > 0 \wedge z \neq 2\}$

$$\int_0^{\frac{\pi}{4}} \sin^{z-2}(2\theta)\tan(\theta)\,d\theta = \frac{1}{4-2z} + \frac{1}{4}B\left(\frac{z}{2}-1, \frac{1}{2}\right)$$

where $B(a,b)$ is the Beta function.

***Proposed Vincent Nguyen-USA***

**U.3007** Find:

$$\Omega = \frac{\ln\left(-\sum_{k=1}^{\infty}\left(\frac{(-1)^k}{(2k+1)^2}\right)\cdot\frac{2}{\sqrt{2}}\cdot\frac{2}{\sqrt{2+\sqrt{2}}}\cdot\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdots\right)}{\ln\left(-\frac{e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}}}{2i}\right)}$$

***Proposed by Basir Ahmad Alizada-Afghanistan***

**U.3008** Prove that:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sqrt[3]{\frac{\sin 3x}{\sin 2y}}\,dx\,dy = \frac{\pi}{2\sqrt{3}}$$

***Proposed by Ghulam Shah Naseri-Afghanistan***

**U.3009** Evaluate:

$$\Omega = \sum_{n=1}^{\infty}\left(\int_0^1 \frac{x!}{(x-n)!}dx\right)\frac{1}{3^n n!}$$

***Proposed by Toubal Fethi -Algeria***

**U.3010** If $\{a;b;c\} \in \mathbb{Z}^+ \ \ a \neq b \neq c$

$$\Omega = \int_0^1\int_0^x \frac{(1-t)(1+t)}{bx-\frac{ax-xt}{c}}dt\,dx = 600\,Li_2\left(-\frac{1}{11}\right) + \frac{215}{4}$$

Prove that: $c^2 = a^2 + b^2 \ \ a > b$

***Proposed by Amin Hajiyev-Azerbaijan***

**U.3011** Find $\alpha =?$ such that:

$$\sum_{x=2}^{\infty}(-1)^x\frac{\sin^2(x-1)\cos^2(x-1)}{(x-1)^2} = \frac{(\pi-\alpha)^\alpha}{\alpha^3}$$

***Proposed by Tahirov Shirvan –Azerbaijan***

**U.3012** Prove the below closed form.

$$\int\int_{[0,1]^2} \frac{\ln(xy)}{xy} \ln\left(\frac{x^2y^2 - xy + 1}{x^2y^2 + xy + 1}\right) dx\,dy = \frac{13\pi^4}{324}$$

***Proposed by Ankush Kumar Parcha-India***

**U.3013** Prove that:

$$\int_1^\infty \frac{\ln(\sqrt{x})\ln^2(x+1)}{(x+1)^2} dx = \frac{1}{24}\left(21\zeta(3) + 2\pi^2 + 4\ln(2)\left(6 + \ln^2(2) + \ln(8)\right)\right)$$

Note: $\zeta(3) \to$ Apery's constant

***Proposed by Cosqun Memmedov and Shirvan Tahirov-Azerbaijan***

**U.3014** Find:

$$\Omega = \lim_{n\to\infty}\left(\log n + \lim_{x\to 0} \frac{1 - (1 + x^2)^{H_n - 1}}{x^2}\right)$$

***Proposed by Khaled Abd Imouti -Syria***

**U.3015** If $x > e$ then:

$$\ln\left(\sqrt[6]{x \cdot \ln x \cdot \ln(\ln x)}\right) + \sqrt{\frac{x + \ln x + \ln(\ln x)}{3}} < 1$$

***Proposed by Khaled Abd Imouti-Syria***

**U.3016**$x_1, x_2, x_3, x_4$ are the real roots of equation $ax^4 + bx^3 - cx - d = 0$

$(b{:}\,c{:}\,d = a{:}\,2a{:}\,a, a > 0, b > 0, c < 0, d < 0)$ prove that: $0 < \frac{a}{b} \le 0{,}3125$

***Proposed by Samir Cabiyev – Azerbaijan***

**U.3017** Determine all real functions derivable such that:

$$f(x) = \sqrt{2024 + \int_0^x \left[\left(f(t)\right)^2 + \left(f'(t)\right)^2\right] dt}, \forall x \in \mathbb{R}^*$$

***Proposed by Neculai Stanciu – Romania***

**U.3018** Prove that $\left[\left(45 + \sqrt{2010}\right)^n\right] + \left[\left(45 + \sqrt{2011}\right)^n\right]$ is even for any natural number $n$, where $[*]$ is GIF.

***Proposed by Neculai Stanciu – Romania***

**U.3019** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$p_a + p_b + p_c \le m_a + m_b + m_c + \frac{1}{3}(max\{a,b,c\} - min\{a,b,c\})$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**U.3020** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$0 \le \frac{p_a - m_a}{a} + \frac{p_b - m_b}{b} + \frac{p_c - m_c}{c} < \frac{1}{3}$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**U.3021** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$p_a + p_b + p_c \le m_a + m_b + m_c + R - 2r$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**U.3022** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$p_a + p_b + p_c \ge m_a + m_b + m_c + \frac{a^2 + b^2 + c^2 - ab - bc - ca}{3s}$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**U.3023** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$p_a + p_b + p_c \le \frac{7}{3}s - (7\sqrt{3} - 9)r$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3024** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$h_a(p_a + w_a) + h_b(p_b + w_b) + h_c(p_c + w_c) \le 2s^2.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3025** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$\frac{p_a - m_a + w_a}{h_a} + \frac{p_b - m_b + w_b}{h_b} + \frac{p_c - m_c + w_c}{h_c} \le \frac{R}{r} + 1.$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**U.3026** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$\frac{r_a}{p_a} + \frac{r_b}{p_b} + \frac{r_c}{p_c} \ge 3$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**U.3027** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$r_a(p_a + w_a) + r_b(p_b + w_b) + r_c(p_c + w_c) \le 2s^2$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**U.3028** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$\frac{p_a}{m_a} + \frac{p_b}{m_b} + \frac{p_c}{m_c} \le \frac{R}{2r} + 2.$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**U.3029** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$\frac{p_a + w_a}{2m_a + \sqrt{p_a w_a}} + \frac{p_b + w_b}{2m_b + \sqrt{p_b w_b}} + \frac{p_c + w_c}{2m_c + \sqrt{p_c w_c}} \le 2.$$

***Proposed by Mohamed Amine Ben Ajiba-Morocco***

**U.3030** Prove the below closed form

$$\int_0^1 \frac{x}{\sinh(x)} dx = Li_2\left(-\frac{1}{e}\right) - Li_2\left(\frac{1}{e}\right) + \frac{\pi^2}{4} - \ln\left(\frac{e+1}{e-1}\right)$$

Where, $Li_2(z)$ is the dilogarithm or Spence's function.

***Proposed by Ankush Kumar Parcha-India***

**U.3031**

$$k(x) := \sum_{0 \le n} \frac{\binom{2n}{n}^2}{2^{4n}} x^n,$$

$$\int_0^{\frac{1}{2}} \frac{4x(1-x)k(1-x)k(x)(k(1-x)^2 - k(x)^2)}{1-2x} \cdot (6k(1-x)^2 k(x)^2 - k(1-x)^4 - k(x)^4) dx = \frac{\pi^5}{2\Gamma\left(\frac{3}{4}\right)^{16}}$$

***Proposed by Hikmat Mammdov-Azerbaijan***

**U.3032** Find:

$$\sum_{n=1}^{\infty} \frac{H_n - H_{\frac{n}{2}}}{n} \binom{2n}{n} \left(-\frac{1}{4}\right)^n$$

***Proposed by Hikmat Mammadov-Azerbaijan***

**U.3033** If

$$\Omega := \frac{1}{4}\int_0^{\infty} \frac{dx}{(1+e^x)^2} + \frac{4}{9}\int_0^{\infty} \frac{dy}{(2+e^y)^2} + \frac{9}{16}\int_0^{\infty} \frac{dz}{(3+e^z)^2} + \cdots$$

Then, show that

$$\Omega = 2\pi^2 \ln A - \frac{\pi^2}{6}\gamma + \zeta(3) - \frac{\pi^2}{6}\ln(2\pi e)$$

Where: $A$: Glaisher – Kinkelin constant, $\gamma$: Euler – Mascheroni constant, $\zeta(3)$: Apery's constant

***Proposed by Ankush Kumar Parcha-India***

**U.3034** Prove that:

$$\int_0^\infty e^{-x}\ {}_3\widetilde{F_2}\left(\frac{1}{2},\frac{3}{2},\frac{5}{2};2,2;e^{-x}\right)dx = \frac{8}{3}\left(1 - \frac{4\Gamma\left(\frac{3}{4}\right)^4}{\pi^3}\right)$$

${}_3\widetilde{F_2}(a,b,c;p,q,x)$ is Generalized hypergeometric function

***Proposed by Srinivasa Raghava-AIRMC-India***

**U.3035** Find:

$$\Omega = \int_0^1 \frac{\sqrt{x}\ln x}{x^3 - x\sqrt{x} + 1}dx$$

***Proposed by Vasile Mircea Popa – Romania***

**U.3036** Prove that:

$$\int_0^\infty \left(\frac{\tanh\left(\sqrt{\frac{\pi}{8}}x\right)}{\cosh\left(\sqrt{\frac{\pi}{2}}x\right)}\right)^{\frac{3}{2}} dx = \frac{\Gamma\left(\frac{9}{8}\right) - \frac{3\Gamma\left(\frac{5}{8}\right)^2}{\Gamma\left(\frac{1}{8}\right)}}{\Gamma\left(\frac{5}{8}\right)}$$

***Proposed by Srinivasa Raghava-AIRMC-India***

**U.3037** If we have the integral

$$\alpha = \int_{-\infty}^{\infty} \frac{\sqrt[5]{x} - \frac{1}{\sqrt[5]{x}}}{\sqrt[5]{x} + \frac{1}{\sqrt[5]{x}}}\frac{dx}{1 + x + x^2}$$

then find the value of $\sqrt[8]{-(81\alpha^8 + 9936\alpha^6 + 19296\alpha^4 + 5376\alpha^2)}$.

***Proposed by Srinivasa Raghava-AIRMC-India***

**U.3038** If:

$$\Omega := \int_0^1\int_0^1 \frac{\ln\left((x^0 + y^0) + (x^1 + y^1) + (x^2 + y^2) + (x^3 + y^3) + \cdots\right)}{(1 - x) + (1 - y)}dx\,dy$$

Then, show that: $\Omega = \frac{\pi^2}{6} + \ln^2(2) + 2\ln(2)$

***Proposed by Ankush Kumar Parcha-India***

**U.3039** Find:

$$\Omega = \int_1^{\infty} \frac{\sqrt{x}\ln x}{x^4 + x^2 + 1}dx$$

***Proposed by Vasile Mircea Popa – Romania***

**U.3040** If $n \geq 2, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ then:

$$n + \left(\sum_{k=2}^{n} \frac{1}{(H_k)^p}\right)^{\frac{1}{p}} \left(\sum_{k=2}^{n} (H_k)^{2q}\right)^{\frac{1}{q}} > 1 + n \cdot \log n$$

***Proposed by Khaled Abd Imouti-Syria***

**U.3041** Prove that:

$$\prod_{n,m=1}^{\infty} \frac{n + \zeta(m+1)}{n - \frac{2}{\log 2}\beta(2m)} = \prod_{m=1}^{\infty} \frac{\Gamma\left(1 - \frac{2}{\log 2}\beta(2m)\right)}{\zeta^2(m+1)\Gamma\big(\zeta(m+1)\big)}$$

$\zeta(x)$: Zeta function, $\Gamma(x)$: Gamma function, $\beta(x)$: Dirichlet Beta function

***Proposed by Hikmat Mammadov-Azerbaijan***

**U.3042** In $\Delta ABC$ the following relationship holds:

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \geq \frac{3}{2}\sqrt{\frac{R}{r} + 2}.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3043** If $n_a, n_b, n_c$ – Nagel cevians in $\Delta ABC$ then:

$$\frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \geq \sqrt{\frac{4R}{r} + 1}.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3044** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$\frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \geq \frac{1}{5}\sqrt{\frac{64R}{r} + 97}.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3045** In $\Delta ABC$ the following relationship holds:

$$\sqrt{\frac{R}{r}}+3-\sqrt{2}\le\frac{\sqrt{r_br_c}}{h_a}+\frac{\sqrt{r_cr_a}}{h_b}+\frac{\sqrt{r_ar_b}}{h_c}\le\sqrt{\frac{2R}{r}}+1.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3046** If $p_a$ – Spieker cevian in $\Delta ABC$ then:

$$p_a\le\frac{(b+c)^2}{16r}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3047** If $p_a, p_b, p_c$ – Spieker cevians, $n_a, n_b, n_c$ – Nagel cevians in $\Delta ABC$ then:

$$\frac{(3p_a-2n_a)^2}{h_a}+\frac{(3p_b-2n_b)^2}{h_b}+\frac{(3p_c-2n_c)^2}{h_c}\le 2R+5r$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3048** Find the largest positive constant $k$ such that the following inequality

$$(a+b+c)\left(a\sqrt{a^2+bc}+b\sqrt{b^2+ca}+c\sqrt{c^2+ab}\right)$$
$$\ge k\big(ab(a+b)+bc(b+c)+ca(c+a)\big),$$

holds for all nonnegative real numbers $a, b, c$.

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3049** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$\frac{p_a}{b+c}+\frac{p_b}{c+a}+\frac{p_c}{a+b}\le\frac{s}{4r}.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3050** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$\frac{3p_a-2m_a}{h_a}+\frac{3p_b-2m_b}{h_b}+\frac{3p_c-2m_c}{h_c}\le\frac{2R}{r}-1$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3051** If $p_a, p_b, p_c$ – Spieker cevians and $I$ – incenter in $\Delta ABC$ then:

$$(3p_a+w_a).AI+(3p_b+w_b).BI+(3p_c+w_c).CI\ge 2(a^2+b^2+c^2).$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3052** In $\Delta ABC$ the following relationship holds:

$$\sin^8 A \,.\, \cos A + \sin^8 B \,.\, \cos B + \sin^8 C \,.\, \cos C \le \frac{243}{512}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3053** Let $a, b, c > 0$ such that $abc = 1$. Prove that:

$$\frac{1}{a^3 + a^2 + a} + \frac{1}{b^3 + b^2 + b} + \frac{1}{c^3 + c^2 + c} + \frac{8}{a + b + c} \ge \frac{11}{3}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3054** In $\Delta ABC$ the following relationship holds:

$$1 + \frac{4R}{r} \ge \sqrt{\cot^2\frac{A}{2} + \cot^2\frac{B}{2} + 3} + \sqrt{\cot^2\frac{B}{2} + \cot^2\frac{C}{2} + 3} + \sqrt{\cot^2\frac{C}{2} + \cot^2\frac{A}{2} + 3} \ge \frac{5\sqrt{3}s}{6r} + \frac{3}{2}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3055** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$(3p_a + w_a)\sin\frac{A}{2} + (3p_b + w_b)\sin\frac{B}{2} + (3p_c + w_c)\sin\frac{C}{2} \ge 2(h_a + h_b + h_c)$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3056** If $p_a, n_a, g_a$ – Spieker cevian, Nagel cevian, Gergonne cevian

in $\Delta ABC$ then: $w_a + 2p_a \le 4m_a - w_a \le 3p_a \le 2m_a + n_a \le g_a + 2n_a$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3057** If $p_a$ – Spieker cevian in $\Delta ABC$ then:

$$p_a \ge \frac{2b^2 - bc + 2c^2}{6R}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3058** If $n_a$ – Nagel cevian in $\Delta ABC$ then:

$$n_a \ge \frac{b^2 - bc + c^2}{2R}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3059** If $n_a, n_b, n_c$ – Nagel cevians in $\Delta ABC$ then:

$$\frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \ge \frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} - 3$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3060** If $p_a -$ Spieker cevian in $\Delta ABC$ then:

$$\frac{p_a}{h_a} \le \frac{2R}{3r} - \frac{1}{3}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3061** If $p_a, p_b, p_c -$ Spieker cevians in $\Delta ABC$ then:

$$\frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \ge \frac{2}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) - 1$$

***Proposed by** Mohamed Amine Ben Ajiba – **Morocco***

**U.3062** If $n_a, n_b, n_c -$ Nagel cevians in $\Delta ABC$ then:

$$\sqrt{n_a n_b} + \sqrt{n_b n_c} + \sqrt{n_c n_a} \ge \frac{a^2 + b^2 + c^2}{2R}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3063** If $p_a -$ Spieker cevian in acute $\Delta ABC$ then:

$$p_a \ge \sqrt{\frac{b^2 + c^2}{2}} . \cos\frac{A}{2}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3064** If $p_a, p_b, p_c -$ Spieker cevians in $\Delta ABC$ then:

$$\sqrt{p_a p_b} + \sqrt{p_b p_c} + \sqrt{p_c p_a} \ge \frac{2(a^2 + b^2 + c^2) + ab + bc + ca}{6R}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3065** If $p_a, n_a, g_a -$ Spieker cevian, Nagel cevian, Gergonne cevian

in $\Delta ABC$ then: $4n_a + 5g_a \le 9p_a \le 16m_a - 7g_a$.

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3066** If $n_a, n_b, n_c -$ Nagel cevians in $\Delta ABC$ then:

$$\frac{\sqrt{n_b n_c}}{h_a} + \frac{\sqrt{n_c n_a}}{h_b} + \frac{\sqrt{n_a n_b}}{h_c} \ge \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3067** If $p_a, p_b, p_c -$ Spieker cevians in $\Delta ABC$ then:

$$\frac{\sqrt{p_b p_c}}{h_a}+\frac{\sqrt{p_c p_a}}{h_b}+\frac{\sqrt{p_a p_b}}{h_c}\ge\frac{\sqrt{6}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-6\right)+3$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3068** If $p_a, p_b, p_c$ – Spieker cevians, $n_a, n_b, n_c$ – Nagel cevians in $\Delta ABC$ then:

$$\frac{\sqrt{p_a n_a}}{h_a}+\frac{\sqrt{p_b n_b}}{h_b}+\frac{\sqrt{p_c n_c}}{h_c}\ge\frac{\sqrt{6}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+3-2\sqrt{6}.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3069** If $p_a, p_b, p_c$ – Spieker cevians in $\Delta ABC$ then:

$$\frac{\sqrt{m_a p_a}}{h_a}+\frac{\sqrt{m_b p_b}}{h_b}+\frac{\sqrt{m_c p_c}}{h_c}\ge\frac{\sqrt{3}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+3-2\sqrt{3}.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3070** If $n_a, n_b, n_c$ – Nagel cevians in $\Delta ABC$ then:

$$\frac{\sqrt{m_a n_a}}{h_a}+\frac{\sqrt{m_b n_b}}{h_b}+\frac{\sqrt{m_c n_c}}{h_c}\ge\frac{\sqrt{2}}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+3-3\sqrt{2}.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3071** If $(H_n)_{n\ge1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$ and $(a_n)_{n\ge1}$ is a sequence of real strictly positive numbers such that $a_{n+1}=a_n+ne^{H_n}\cdot\sin\frac{\pi}{n^2}$. Find $\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{n!}}$

***Proposed by D.M. Bătinețu – Giurgiu-Romania***

**U.3072** Let be $(a_n)_{n\ge1}, a_n>0, \forall n\in\mathbb{N}^*$ such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$. Find:

$$\lim_{n\to\infty}\left(\sin\frac{x\sqrt[n]{n!}+1}{\sqrt[n]{n!}}-\sin x\right)\sqrt[n]{a_n}$$

where $x\in\mathbb{R}$.

***Proposed by D.M. Bătinețu – Giurgiu-Romania***

**U.3073** Let be $t>0$ and $(a_n)_{n\ge1}$ a sequence of real striclty positive numbers such that

$$\lim_{n\to\infty}\left(\frac{(n+1)^{t+2}}{\sqrt[n+1]{a_{n+1}}}-\frac{n^{t+2}}{\sqrt[n]{a_n}}\right)\cdot\sin\frac{\pi}{n}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania***

**U.3074** Let be $(H_n)_{n\ge1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$. Find:

$$\lim_{n\to\infty} e^{2H_n}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sin\frac{\pi}{n^2}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania***

**U.3075**. If $(H_n)_{n\geq 1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$ find:

$$\lim_{n\to\infty}\frac{e^{3H_n}}{\sqrt[n]{n!}}\cdot\sin\frac{\pi^2}{n^2}$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania***

**U.3076** If $s>0$, find:

$$\lim_{n\to\infty}\left(\frac{(n+1)^{s+1}}{\sqrt[n+1]{\left((n+1)!\right)^s}}-\frac{n^{s+1}}{\sqrt[n]{(n!)^s}}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania***

**U.3077** Let be $(a_n)_{n\geq 1},(H_n)_{n\geq 1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$ and $a_{n+1}=a_n+e^{H_n}, \forall n\geq 1$. Find:

$$\lim_{n\to\infty}\left(a_n\cdot\sqrt[n]{n!}\cdot\sin\frac{\pi}{n^3}\right)$$

***Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania***

**U.3078** If $n_a,n_b,n_c-$ Nagel cevians, $g_a,g_b,g_c-$ Gergonne cevians in $\Delta ABC$ then:

$$n_ag_a+n_bg_b+n_cg_c\geq s^2\sqrt{1+\frac{16r^2(R-2r)}{s^2R}}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3079** If $n_a,n_b,n_c-$ Nagel cevians, $g_a,g_b,g_c-$ Gergonne cevians in $\Delta ABC$ then:

$$\frac{n_ag_a}{h_a}+\frac{n_bg_b}{h_b}+\frac{n_cg_c}{h_c}\geq\sqrt{16R^2+24Rr-31r^2}.$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

**U.3080** If $n_a,n_b,n_c-$ Nagel cevians, $g_a,g_b,g_c-$ Gergonne cevians in $\Delta ABC$ then:

$$\frac{n_ag_a}{h_a^{\,2}}+\frac{n_bg_b}{h_b^{\,2}}+\frac{n_cg_c}{h_c^{\,2}}\geq\frac{\sqrt{R^2+3Rr-r^2}}{r}$$

***Proposed by Mohamed Amine Ben Ajiba – Morocco***

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# ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-AUTUMN 2025

## PROBLEMS FOR JUNIORS

**JP.556 In acute $\Delta ABC$ the following relationship holds:**

$$\sum_{cyc}\sin^2 A\,(\cos B+\cos C)\le 2+\frac{1}{8}\left(13\frac{r}{R}-15\left(\frac{r}{R}\right)^2-6\left(\frac{r}{R}\right)^3\right)$$

*Proposed by Marian Ursărescu – Romania*

**JP.557 In $\Delta ABC$ the following relationship holds:**

$$\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}+\left(\frac{a}{c}+\frac{c}{a}\right)\cos^2\frac{B}{2}+\left(\frac{a}{b}+\frac{b}{a}\right)\cos^2\frac{C}{2}\le\frac{3}{2}\left(\frac{R}{r}+1\right)$$

*Proposed by Marian Ursărescu – Romania*

**JP.558 If $a,b,c>0;x\ge 0$ then:**

$$\sum_{cyc}\frac{(a^3+x)(b^3+x)}{ac^2+x}\ge\sum_{cyc}\frac{(ba^2+x)(cb^2+x)}{c^3+x}$$

*Proposed by Daniel Sitaru – Romania*

**JP.559 If $x,y,z>0$ then:**

$$9\sum_{cyc}\left(\frac{2x+y+z}{x^2+2}\right)^2\le 2\sum_{cyc}(x^2+2)(y^2+2)$$

*Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania*

**JP.560 If $x,y,z>0;xyz\le 1$ then:**

$$\sum_{cyc}\left(\frac{1}{x^7}+\frac{1}{y^5}+\frac{1}{z^3}\right)\geq\frac{3(x+y+z)}{xyz}$$

*Proposed by Daniel Sitaru – Romania*

**JP.561 Solve for real numbers:**

$$\log_{2\sqrt{8+2\sqrt{15}}}(x^2+x+2)=\log_{\sqrt{4+\sqrt{15}}}(x^2+x+1)$$

*Proposed by Marian Ursărescu and Florică Anastase – Romania*

**JP.562 If $a,b,c\in(0,1)$ and $x,y,z>0$ such that $a=(bc)^x, b=(ca)^y, c=(ab)^z$ and $xyz=1$ then holds:**

$$\sqrt[n]{\sum_{cyc}a^n(a^n+y+2)^{2n-1}}\geq 6\cdot\sqrt[3]{abc}, n\in\mathbb{N}^*, n\geq 2$$

*Proposed by Marian Ursărescu and Florică Anastase – Romania*

**JP.563 In acute triangle $ABC$, $A', B', C'$ are symmetric points of the points $A,B,C$ to the sides $BC, AC$ and $AB$ respectively. Prove that:**

$$\frac{\sigma[A'B'C']}{\sigma[ABC]}=4\left(\frac{r}{R}\right)^2+8\cdot\frac{r}{R}-1$$

**where $\sigma[ABC]$ is area of $\Delta ABC$.**

*Proposed by Marian Ursărescu and Florică Anastase – Romania*

**JP.564 If $x,y,z>0; x^2+y^2+z^2=\frac{3}{4}$ then:**

$$4(x+y+z)+2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 18$$

*Proposed by Daniel Sitaru – Romania*

**JP.565 If $a,b>0$ then:**

$$3\sqrt[3]{a}+5\sqrt[5]{a}+3\sqrt[3]{b}+5\sqrt[5]{b}\leq 4\sqrt{a}+4\sqrt{b}+8$$

*Proposed by Daniel Sitaru – Romania*

**JP.566 In $\Delta ABC$ the following relationship holds:**

$$8\leq\sum\left(\frac{b}{c}+\frac{c}{b}\right)\frac{a^2}{m_a^2}\leq\frac{4R}{r}\left(\frac{R}{r}-1\right)$$

*Proposed by Marin Chirciu – Romania*

**JP.567 Let $ABC$ be a triangle with inradius $r$ and circumradius $R$. Prove that:**

$$\frac{9\sqrt{3}}{2}\frac{r^2}{R^2} \le \sin^3 A + \sin^3 B + \sin^3 C \le \frac{9\sqrt{6}}{8}\sqrt{1-\frac{r}{R}}$$

***Proposed by George Apostolopoulos – Greece***

**JP.568 Let $ABC$ be a triangle, with inradius $r$, and circumradius $R$. Prove that:**

$$4 \le \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \le 2\left(\frac{R}{2r}\right)^4 + 2$$

***Proposed by George Apostolopoulos – Greece***

**JP.569 In $\Delta ABC$ prove that:**

$$\left|\frac{a^2-b^2}{ab}\right| + \left|\frac{b^2-c^2}{bc}\right| + \left|\frac{c^2-a^2}{ca}\right| < \frac{3R}{r}$$

***Proposed by Ertan Yildirim-Turkiye***

**JP.570 In $\Delta ABC$ the following relationship holds:**

$$\frac{12R^2p}{R} \le \sum w_a\sqrt{\frac{b^2+c^2}{2}} \le \frac{9R^2}{2}\sqrt{3}$$

***Proposed by Marin Chirciu – Romania***

## PROBLEMS FOR SENIORS

**SP.556 We consider the function $f: (0,\infty) \to \mathbb{R}, f(x) = \frac{\sqrt{3x+\sin x}}{x}$. Prove that it is integrable and prove the inequality:**

$$\int_{\varepsilon}^{8} f(x)\,dx < 10 + \ln 2$$

**where $1 > \varepsilon > 0$ and $\varepsilon \to 0$.**

***Proposed by Adalbert Kovacs – Romania***

**SP.557 Prove the following inequality:**

$$\int_{\frac{1}{2024}}^{3} \frac{\sqrt{\sin x}}{x}\,dx < 2 + \ln 3$$

***Proposed by Adalbert Kovacs – Romania***

**SP.558 In $\Delta ABC, O-$ circumcenter. If the bisector from angle $A$, altitude from angle $B$ and $CO$ circumcevian are in concurrence, then holds:**

$$\sqrt[3]{-1+3\frac{r}{R}-\frac{3}{2}\left(\frac{r}{R}\right)^2}\le \cos A\le \sqrt[3]{\frac{1}{2}\left(\frac{r}{R}\right)^2}$$

***Proposed by Marian Ursărescu – Romania***

**SP.559 In acute $\Delta ABC, AD, BE, CF$ – symmedians. Prove that:**

$$\frac{r^3}{R}\le \frac{AF\cdot BD\cdot CE}{AB+BC+CA}\le \frac{Rr}{4}$$

***Proposed by Marian Ursărescu – Romania***

**SP.560 For $k\in\mathbb{N}$ fixed and $\alpha>0$ evaluate:**

$$L=\lim_{n\to\infty}\frac{1}{\sqrt{n^\alpha}}\cdot\left(\prod_{i=1}^{k}\frac{n+k+i}{n+i}\right)^{n^\alpha}$$

***Proposed by Marian Ursărescu and Florică Anastase – Romania***

**SP.561 Let be the function $f:[0,1]\to\mathbb{R}$ integrable such that $f(1)=1$ and**

$$\int_x^y f(t)\,dt=\frac{1}{2}\big(yf(y)-xf(x)\big),\forall x,y\in[0,1]$$

**Find:**

$$I=\int_0^{\frac{\pi}{4}} f(x)\cdot\tan^2 x\,dx.$$

***Proposed by Marian Ursărescu and Florică Anastase – Romania***

**SP.562 If $0<a\le b<\frac{\pi}{2}$ then:**

$$\int_a^b\frac{2+\sqrt{2}(\sin x+\cos x)}{\sin\left(\frac{\pi}{4}+x\right)}\ge 4(b-a)$$

***Proposed by Daniel Sitaru – Romania***

**SP.563 If $a,b,c>0, a+b+c=3$, then:**

$$\sum\frac{2a^4}{3a^2-2a+5}\ge 1$$

***Proposed by Marin Chirciu – Romania***

**SP.564** If $a, b, c > 0, (a+b-1)^2 = ab$ and $\lambda \ge 0$ then:

$$\frac{1}{ab} + \frac{1}{a^2+b^2} + \frac{\lambda\sqrt{ab}}{a+b} \ge 1 + \sqrt{\lambda}$$

***Proposed by Marin Chirciu – Romania***

**SP.565** If $a_1, a_2, \dots, a_n > 0, a_1 + a_2 + \dots + a_n = n$ then:

$$\left(1+\frac{1}{a_1}\right)^{a_2^2}\left(1+\frac{1}{a_2}\right)^{a_3^2} \cdot \dots \cdot \left(1+\frac{1}{a_n}\right)^{a_1^2} \ge 2^n$$

***Proposed by Marin Chirciu – Romania***

**SP.566** If $\lambda > 0$ then find:

$$\int_0^1 \frac{(x^2e^x + (\lambda+1)x + 1)e^x}{\lambda + xe^x} dx$$

***Proposed by Marin Chirciu – Romania***

**SP.567** Prove that $\frac{3}{2}$ is the largest positive value of the constant $k$ such that the inequality

$$(a+k)^2 + (b+k)^2 + (c+k)^2 + (d+k)^2 + (e+k)^2 + (f+k)^2 \ge 6(1+k)^2$$

holds whenever $a \ge b \ge 1 \ge c \ge d \ge e \ge f \ge 0$ satisfying

$$ab + bc + cd + de + ef + fa = 6$$

***Proposed by Vasile Cîrtoaje – Romania***

**SP.568** Let $a_1 \ge 1 \ge a_2 \ge \dots \ge a_n \ge 0$ such that $a_1 + a_2 + \dots + a_n = n$. Prove that

$$a_1a_2 + a_2a_3 + \dots + a_na_1 \le n$$

***Proposed by Vasile Cîrtoaje – Romania***

**SP.569** Let $a, b, c$ be positive real numbers such that at most one of them is greater that $1$ and $abc = 1$. Prove that

$$\frac{11(b^2+c^2)-10a^2}{b+c} + \frac{11(c^2+a^2)-10b^2}{c+a} + \frac{11(a^2+b^2)-10c^2}{a+b} \le 18$$

***Proposed by Vasile Cîrtoaje – Romania***

**SP.570** Let $a, b > 1$ fixed. Solve the equation:

$$\left(\frac{x}{ab^2}\right)^{\log_{\sqrt{a}} x} = \left(\frac{x}{a^2b}\right)^{\log_{\sqrt{b}} x}$$

***Proposed by Marin Chirciu – Romania***

## UNDERGRADUATE PROBLEMS

**UP.556 Let be $f:[a,b]\to\mathbb{R}; 0<a\le b, f$ continuous, convex and**

$$\int_0^a f(x)\,dx = a;\ \int_0^b f(x)\,dx = 2b$$

**Prove that:**

$$\int_0^{\frac{a+b}{2}} f(x)\,dx \le a+b$$

***Proposed by Daniel Sitaru – Romania***

**UP.557 Prove:**

$$\int_0^{\frac{\pi}{2}} \frac{\cos\theta \arccos\left(\frac{\cos^2\theta}{1+\sin^2\theta}\right)}{1+\sin^2\theta}\,d\theta = \frac{3}{8}\zeta(2).$$

***Proposed by Said Attaoui – Algeria***

**UP.558 Prove that $3$ is the greatest positive value of $k$ such that:**

$$\frac{1}{a+k}+\frac{1}{b+k}+\frac{1}{c+k}+\frac{1}{d+k}+\frac{1}{e+k}\ge\frac{5}{1+k}$$

**for any $a\ge b\ge c\ge d\ge 1\ge e\ge 0$ satisfying $ab+bc+de+ea=5$.**

***Proposed by Vasile Cîrtoaje – Romania***

**UP.559 For given $n\ge 2$, prove that $n-1$ is the smaller positive value of the constant $k$ such that**

$$a_1+a_2+\cdots+a_n\ge\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$$

**for any positive numbers $a_i$ with $a_1\le a_2\le\cdots\le a_n$ and $a_1^k a_n\ge 1$.**

***Proposed by Vasile Cîrtoaje – Romania***

**UP.560 Prove:**

$$\int_0^\infty\int_0^\infty \frac{(x+y)e^{-(x+y)}}{1-e^{-(x+y)}}\,dx\,dy = 2\zeta(3)$$

***Proposed by Said Attaoui – Algeria***

**UP.561 Prove:**

$$\frac{2}{3}\int_0^{\pi}\frac{x\sin(x)}{1+\cos^2(x)}dx=\zeta(2)$$

*Proposed by Said Attaoui – Algeria*

**UP.562 Prove:**

$$\int_0^{\infty}\frac{\sin(2x)}{x}\log(x)\,dx=-\frac{\pi}{2}(\log 2+\gamma)$$

**Is there a way to prove that?**

$$\int_0^1\frac{\log\left(\frac{1-x}{x}\right)}{x(1-x)}\cos\left(\frac{1-2x+2x^2}{2x(1-x)}\right)\sin\left(\frac{1-2x}{2x(1-x)}\right)dx=-\frac{\pi\gamma}{2}$$

**where $\gamma$ design the Euler – Mascheroni constant given by $\Gamma'(1)=\gamma=0.557215..$ and $\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-t}\,dt$ is the gamma function.**

*Proposed by Said Attaoui –Algeria*

**UP.563 Calculate the integral:**

$$\int_1^{\infty}\frac{\sqrt{x}\ln^2 x}{(x+1)(x^2+1)}dx$$

*Proposed by Vasile Mircea Popa – Romania*

**UP.564 Calculate the integral:**

$$\int_0^{8}\frac{\arctan(x)}{\sqrt{x^4-x^2+1}}dx$$

*Proposed by Vasile Mircea Popa – Romania*

**UP.565 Evaluate:**

$$\sum_{q=0}^{\infty}(-1)^q\left(\frac{q}{(2q+1)(q+1)}\right)^2$$

*Proposed by Said Attaoui-Algeria*

**UP.566 If $0<a\le b<1$ then:**

$$\int_a^b\int_a^b\int_a^b\sqrt{\frac{xyz}{(1-x)(1-y)(1-z)}}\,dx\,dy\,dz\ge(b^2-a^2)^3$$

*Proposed by Daniel Sitaru – Romania*

**UP.567 If $-2 < a \leq b < 2$ then:**

$$\int_a^b \int_a^b \frac{x}{x^2 - xy + 1} dx\, dy \leq \left(\frac{2-a}{2-b}\right)^{ab}$$

***Proposed by Daniel Sitaru – Romania***

**UP.568 Let $a \neq 0, a + b \neq 0$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:**

$$f(2) = 1 \text{ and } af\big(f(x)\big) + bf(x) = a + b, \forall x \in \mathbb{R}.$$

***Proposed by Marin Chirciu – Romania***

**UP.569 Let $\lambda \in \mathbb{N}^*$ fixed. Evaluate:**

$$\lim_{n\to\infty} \left\{\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1}\right\}.$$

***Proposed by Marin Chirciu – Romania***

**UP.570 Solve for real numbers the equation:**

$$\sqrt{18 + x} + \sqrt{7 - x} = x^2 - 11x + 25$$

***Proposed by Marin Chirciu – Romania***

**All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.**

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