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2001. If $p_a, p_b, p_c \rightarrow$

Spieker cevians in Δ ABC, then the following relationship holds :

$$3\sum_{cyc}\left((h_b + h_c)(p_a - w_a)\right) \geq 4s(h_a + h_b + h_c - 9r)$$

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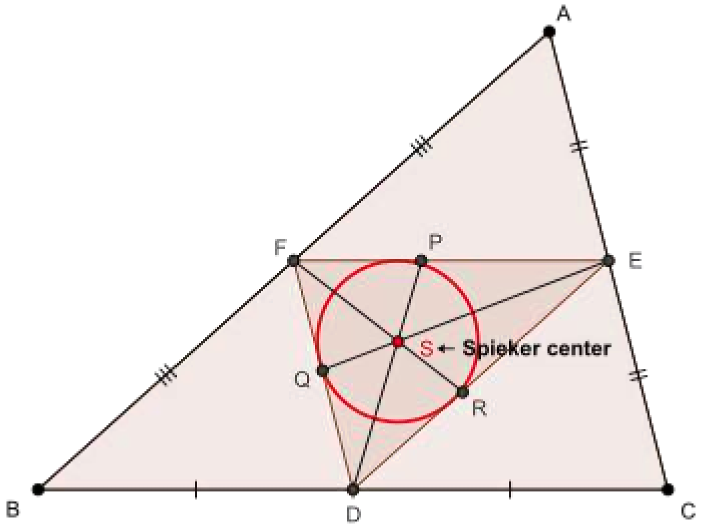


Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2}$$

$$= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\,AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow \mathrm{csin}\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, \mathrm{bsin}\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC] \Rightarrow \frac{1}{2}p_a\mathrm{csin}\alpha + \frac{1}{2}p_a\mathrm{bsin}\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \overset{?}{\geq} w_a+\frac{2}{3}\cdot\frac{(b-c)^2}{b+c} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\geq} s(s-a)$$

$$-\frac{s(s-a)(b-c)^2}{(2s-a)^2}+\frac{4}{9}\cdot\frac{(b-c)^4}{(b+c)^2}+\frac{4w_a}{3}\cdot\frac{(b-c)^2}{b+c}$$

$$\Leftrightarrow \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4}{9}\cdot\frac{(b-c)^2}{(b+c)^2} \boxed{\underset{(\blacksquare)}{\overset{?}{\geq}}} \frac{4w_a}{3(b+c)} \quad (\because (b-c)^2 \geq 0)$$

$$\text{We have : } \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4}{9}\cdot\frac{(b-c)^2}{(b+c)^2} \overset{\because a^2>(b-c)^2}{>} \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}$$

$$-\frac{4}{9}\cdot\frac{a^2}{(2s-a)^2}=\frac{9\left(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\right)-4a^2(2s+a)^2}{9(4s^2-a^2)^2}$$

$$=\frac{(s-a)\left(36s^3+18s^2a+5sa^2+a^3\right)}{9(4s^2-a^2)^2} \overset{s>a}{>} 0 \Rightarrow \text{LHS of } (\blacksquare)>0 \therefore (\blacksquare) \Leftrightarrow$$

$$\frac{\left(9T-4(b-c)^2(2s+a)^2\right)^2}{81(4s^2-a^2)^4} \overset{?}{\geq} \frac{16\left(s(s-a)(2s-a)^2-s(s-a)(b-c)^2\right)}{9(2s-a)^4}$$

$$\left(T=s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\right)$$

$$\Leftrightarrow \frac{81T^2+16(b-c)^4(2s+a)^4-72T(b-c)^2(2s+a)^2}{9(2s+a)^4} \overset{?}{\geq}$$

$$16s(s-a)(2s-a)^2-16s(s-a)(b-c)^2$$

$$\Leftrightarrow 16(2s+a)^4(b-c)^4-\left(72T(2s+a)^2-144s(s-a)(2s+a)^4\right)(b-c)^2+81T^2$$

$$-144s(s-a)(2s-a)^2(2s+a)^4 \boxed{\underset{(\blacksquare\blacksquare)}{\overset{?}{\geq}}} 0$$

$$\text{Now, LHS of } (\blacksquare\blacksquare) \text{ is a quadratic polynomial in } "(b-c)^2" \text{ with } \boxed{\text{discriminant}} =$$

$$\left(72T(2s+a)^2-144s(s-a)(2s+a)^4\right)^2$$

$$-64(2s+a)^4\left(81T^2-144s(s-a)(2s-a)^2(2s+a)^4\right)$$

$$=72^2(2s+a)^4Q^2-$$

$$64.9(2s+a)^4\left(9T^2-16s(s-a)(2s-a)^2(2s+a)^4\right)$$

$$\left(Q=s(3s+a)(2s-a)^2-s(s-a)(2s+a)^2\right)$$

$$=64.9(2s+a)^4\left(9Q^2-9T^2+16s(s-a)(2s-a)^2(2s+a)^4\right)$$

$$=-4s.64.9(2s+a)^4.a^7\left(176t^7-288t^6-40t^5+208t^4-13t^3-50t^2+3t+4\right)$$

$$\left(t=\frac{s}{a}\right)$$

$$=\boxed{-256s.9(2s+a)^4.a^7(t-1)^2\left(\begin{matrix}120t^5+\left(24t^5-24t^3\right)+\left(32t^5-32t^2\right)+\\ \left(64t^4-64t^3\right)+11t+4\end{matrix}\right)}<0$$

$$\because t>1 \Rightarrow LHS\ of\ (\blacksquare\blacksquare)>0 \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true}$$

$$\therefore \boxed{p_a \geq w_a + \frac{2}{3}\cdot\frac{(b-c)^2}{b+c}} \text{ and analogs}$$

$$\Rightarrow 3\sum_{cyc}\left((h_b+h_c)(p_a-w_a)\right) - 4s(h_a+h_b+h_c-9r)$$

$$= 3\sum_{cyc}\left(\left(\frac{ca+ab}{2R}\right)\left(\frac{2}{3}\cdot\frac{(b-c)^2}{b+c}\right)\right) - 4s\left(\frac{s^2+4Rr+r^2}{2R} - 9r\right)$$

$$= \frac{1}{R}\sum_{cyc}\left(a(b^2+c^2-2bc)\right) - 4s\left(\frac{s^2-14Rr+r^2}{2R}\right)$$

$$= \frac{1}{R}\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) - 9abc\right) - 2s\left(\frac{s^2-14Rr+r^2}{R}\right)$$

$$= \frac{1}{R}.2s(s^2+4Rr+r^2-18Rr) - 2s\left(\frac{s^2-14Rr+r^2}{R}\right) = 0$$

$$\therefore 3\sum_{cyc}\left((h_b+h_c)(p_a-w_a)\right) \geq 4s(h_a+h_b+h_c-9r)\ \forall\,\Delta\,ABC,$$

with equality iff Δ ABC is equilateral (QED)

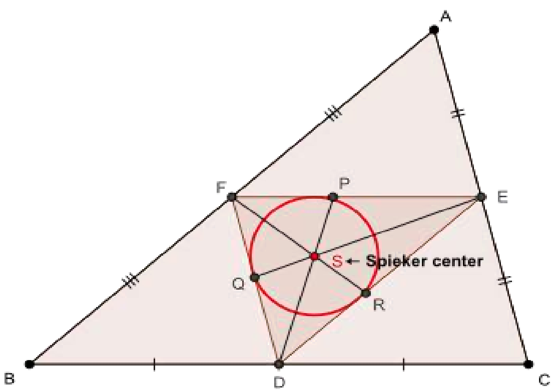
2002. If $p_a, p_b, p_c \rightarrow$

Spieker cevians in Δ ABC, then the following relationship holds :

$$p_a + p_b + p_c \leq w_a + w_b + w_c + \frac{4}{3}(max\{a,b,c\} - min\{a,b,c\})$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now}, 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now}, \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2 s}\big(ca(s-b) + ab(s-c)\big) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\big(b^2+c^2+ab+ca\big) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)}$$

$$\text{Now, } b^3+c^3-abc+a\big(4m_a^2\big) = b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)$$

$$= (b+c)\big(b^2-bc+c^2\big) + a\big(b^2-bc+c^2\big) + a\big(b^2+c^2-a^2\big)$$

$$= 2s\big(b^2-bc+c^2\big) + a\big(b^2-bc+c^2+bc-a^2\big)$$

$$= (2s+a)\big(b^2-bc+c^2\big) + a\left(\frac{(b+c)^2-(b-c)^2}{4} - a^2\right)$$

$$= (2s+a)\big(b^2-bc+c^2\big) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\big(b^2-bc+c^2\big) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now}, p_a \overset{?}{\le} w_a+\frac{2}{3}|b-c| \Leftrightarrow p_a^2 \overset{?}{\le} w_a^2+\frac{4}{9}(b-c)^2+\frac{4}{3}.w_a.|b-c|$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-s(s-a)+\frac{s(s-a)(b-c)^2}{(2s-a)^2}$$

$$-\frac{4}{9}(b-c)^2 \overset{?}{\le} \frac{4}{3}.w_a.|b-c|$$

$$\Leftrightarrow \left(\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4}{9}\right)|b-c| \overset{?}{\le} \frac{4}{3}.w_a \ (\because |b-c| \ge 0)$$

$$\Leftrightarrow \frac{(20s^4-18s^3a-s^2a^2-a^4)|b-c|}{9(4s^2-a^2)^2} \overset{?}{\le} \frac{w_a}{3}$$

$$\Leftrightarrow \frac{(s-a)(20s^3+2s^2a+sa^2+a^3)|b-c|}{3(4s^2-a^2)^2} \overset{?}{\le} w_a$$

$$\Leftrightarrow \frac{(s-a)^2(20s^3+2s^2a+sa^2+a^3)^2(b-c)^2}{9(4s^2-a^2)^4} \overset{?}{\le} s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}$$

$$\Leftrightarrow \frac{(s-a)(20s^3+2s^2a+sa^2+a^3)^2+9s(2s-a)^2(2s+a)^4}{9(4s^2-a^2)^4}.(b-c)^2 \overset{?}{\le} s \text{ and}$$

$$\because (b-c)^2 < a^2 \therefore \text{in order to prove this, it suffices to prove :}$$

$$9s(4s^2-a^2)^4 \overset{?}{>} a^2(s-a)(20s^3+2s^2a+sa^2+a^3)^2+9sa^2(2s-a)^2(2s+a)^4$$

$$\Leftrightarrow 2304t^9-3280t^7-256t^6+1044t^5+288t^4-69t^3-33t^2+t+1 \overset{?}{>} 0$$

$$\left(t=\frac{s}{a}\right) \Leftrightarrow (t-1)\begin{pmatrix} 2212t^8+92(t^8-t^6)+884(t^7-t^6)+1232(t^7-t^5)+ \\ 188(t^7-t^4)+100t^3+28t^2+2(t^2-t)+t^2-1 \end{pmatrix} \overset{?}{>} 0$$

$$\rightarrow \text{true} \because t = \frac{s}{a} > 1 \therefore p_a \le w_a + \frac{2}{3}|b-c| \text{ and analogs}$$

$$\Rightarrow \boxed{p_a + p_b + p_c \le w_a + w_b + w_c + \frac{2}{3}(|a-b| + |b-c| + |c-a|)} \rightarrow (m)$$

$$\text{Now, we shall prove that}: \frac{1}{2}(|b-c| + |c-a| + |a-b|)$$
$$= max\{a,b,c\} - \min\{a,b,c\}$$

$$\boxed{\text{Case }(1)}\ a \ge b \ge c \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(b-c+a-c+a-b)$$
$$= a - c = max\{a,b,c\} - \min\{a,b,c\}$$

$$\boxed{\text{Case }(2)}\ a \ge c \ge b \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(c-b+a-c+a-b)$$
$$= a - b = max\{a,b,c\} - \min\{a,b,c\}$$

$$\boxed{\text{Case }(3)}\ b \ge c \ge a \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(b-c+c-a+b-a)$$
$$= b - a = max\{a,b,c\} - \min\{a,b,c\}$$

$$\boxed{\text{Case }(4)}\ b \ge a \ge c \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(b-c+a-c+b-a)$$
$$= b - c = max\{a,b,c\} - \min\{a,b,c\}$$

$$\boxed{\text{Case }(5)}\ c \ge a \ge b \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(c-b+c-a+a-b)$$
$$= c - b = max\{a,b,c\} - \min\{a,b,c\}$$

$$\boxed{\text{Case }(6)}\ c \ge b \ge a \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(c-b+c-a+b-a)$$
$$= c - a = max\{a,b,c\} - \min\{a,b,c\} \therefore \text{combining all 6 cases, we conclude}:$$

$$\boxed{\frac{1}{2}(|b-c| + |c-a| + |a-b|) = max\{a,b,c\} - \min\{a,b,c\}} \rightarrow (n) \therefore (m) \text{ and } (n)$$

$$\Rightarrow p_a + p_b + p_c \le w_a + w_b + w_c + \frac{4}{3}(max\{a,b,c\} - \min\{a,b,c\})\ \forall\ \Delta\ ABC,$$
$$\text{with equality iff } \Delta\ ABC \text{ is equilateral (QED)}$$

2003. In ΔABC the following relationship holds:

$$\sum \sqrt{m_a^2 + rr_a} = \frac{1}{2}\sum (r_c + r)\sqrt{\frac{r_a + r_b}{r_c - r}} = \sum (a+b)cosC = 2s$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$1)\ \sum \sqrt{m_a^2 + rr_a}$$

$$m_a^2 + rr_a = \frac{1}{4}((2(b^2+c^2) - a^2) + (a^2 - (b-c)^2) =$$

$$= \frac{1}{4}(2b^2 + 2c^2 - a^2 + a^2 - b^2 - c^2 + 2bc) = \frac{1}{4}(b^2 + c^2 + 2bc) = \frac{1}{4}(b+c)^2$$

$$Here\ \boxed{a^2=(b-c)^2+4rr_a}\ (true)$$

$$\sum\sqrt{m_a^2+rr_a}=\sum\frac{1}{2}(b+c)=2s\quad(1)$$

$$2)\ \frac{1}{2}\sum(r_c+r)\sqrt{\frac{r_a+r_b}{r_c-r}}$$

$$(r_c+r)=s\tan\frac{C}{2}+(s-c)\tan\frac{C}{2}=(2s-c)\tan\frac{C}{2}=(a+b)\tan\frac{C}{2}$$

$$\frac{r_a+r_b}{r_c-r}=\frac{4R+r-r_c}{r_c-r}=\frac{4R}{r_c-r}-1=\frac{4R}{s\tan\frac{C}{2}-(s-c)\tan\frac{C}{2}}-1=\frac{4R}{c\tan\frac{C}{2}}-1=$$

$$\frac{4R}{4R\sin\frac{C}{2}\cos\frac{C}{2}\tan\frac{C}{2}}-1=\frac{1}{\sin^2\frac{C}{2}}-1$$

$$=ctg^2\frac{C}{2}\quad Here\ \boxed{r_a+r_b+r_c=4R+r}\ (true)$$

$$(r_c+r)\sqrt{\frac{r_a+r_b}{r_c-r}}=(a+b)\tan\frac{C}{2}.\cot\frac{C}{2}=(a+b)$$

$$\frac{1}{2}\sum(r_c+r)\sqrt{\frac{r_a+r_b}{r_c-r}}=2s\quad(2)$$

$$3)\ \sum(a+b)\cos C$$

$$\sum(a+b)\cos C=(a+b)\cos C+(b+c)\cos A+(a+c)\cos B=$$
$$=(b\cos C+c\cos B)+(c\cos A+a\cos C)+(a\cos B+b\cos A)=a+b+c=2s$$

$$Here\ \boxed{\begin{array}{l}a=b.\cos C+c.\cos B\\ b=c.\cos A+a.\cos C\\ c=a.\cos B+b.\cos A\end{array}}\ (true)$$

$$\sum(a+b)\cos C=2s\quad(3)$$

The expression from (1), (2) *and* (3) *has been proved.*

2004. In ΔABC the following relationship holds:

$$\prod_{cyc}\frac{\frac{r_a}{r_b}+1}{\frac{h_a}{h_b}+1}=\prod_{cyc}\frac{r_a-r}{h_a-r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\frac{r_a-r}{h_a-r}=\frac{\frac{F}{s-a}-r}{\frac{2F}{a}-r}=\frac{\frac{r(s-s+a)}{s-a}}{\frac{r(2s-a)}{a}}=\frac{a^2}{(s-a)(2s-a)}=\frac{a^2}{(s-a)(b+c)}$$

$$\prod_{cyc}\frac{r_a-r}{h_a-r}=\prod_{cyc}\frac{a^2}{(s-a)(b+c)}=$$

$$=\frac{a^2b^2c^2}{(s-a)(s-b)(s-c)(a+b)(b+c)(c+a)}\ (A)$$

$$\frac{\frac{r_a}{r_b}+1}{\frac{h_a}{h_b}+1}=\frac{\frac{s-b}{s-a}+1}{\frac{b}{a}+1}=\frac{a}{a+b}\frac{2s-b-a}{s-a}=\frac{ac}{(s-a)(a+b)}$$

$$\prod_{cyc}\frac{\frac{r_a}{r_b}+1}{\frac{h_a}{h_b}+1}=\prod_{cyc}\frac{ac}{(s-a)(a+b)}=$$

$$=\frac{a^2b^2c^2}{(s-a)(s-b)(s-c)(a+b)(b+c)(c+a)}\ (B)$$

$$From\ (A)\ and\ (B) we\ get\ \prod\frac{\left(\frac{r_a}{r_b}+1\right)}{\frac{h_a}{h_b}+1}=\prod\frac{r_a-r}{h_a-r}$$

2005. In any Δ ABC, following relationship holds:

$$\sum_{cyc}\frac{1}{a^5+c^5}\left(\frac{a^5}{b^2\sin B}+\frac{c^8}{b^5\sin C}\right)\geq\frac{2}{R^2\sqrt{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A', B', C' > 0, (A'+B'), (B'+C'), (C'+A')\ \text{form sides of a triangle}$$

$$(\because (A'+B')+(B'+C') > (C'+A')\ \text{and analogs})$$

$$\Rightarrow \sqrt{A'+B'}, \sqrt{B'+C'}, \sqrt{C'+A'}\ \text{form sides of a triangle with area F (say) and } 16F^2$$

$$=2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$

$$= 2\sum_{cyc}\left(\sum_{cyc} A'B' + B'^2\right) - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B'$$

$$= 6\sum_{cyc} A'B' + 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B' \Rightarrow 4F = 2\sqrt{\sum_{cyc} A'B'} \to (1)$$

$$\text{Now, } \forall\, x, y, z > 0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{1}{a^5+c^5}\left(\frac{a^5}{b^2\sin B} + \frac{c^8}{b^5\sin C}\right) + \frac{1}{a^5+b^5}\left(\frac{b^5}{c^2\sin C} + \frac{a^8}{c^5\sin A}\right)$$

$$+\frac{1}{b^5+c^5}\left(\frac{c^5}{a^2\sin A} + \frac{b^8}{a^5\sin B}\right)$$

$$= 2R.\frac{c^5a^5}{a^5b^5+b^5c^5}.\frac{b^5}{c^5a^5}.\left(\frac{a^5}{b^3} + \frac{c^7}{b^5}\right) + 2R.\frac{a^5b^5}{c^5a^5+b^5c^5}.\frac{c^5}{a^5b^5}.\left(\frac{b^5}{c^3} + \frac{a^7}{c^5}\right)$$

$$+2R.\frac{b^5c^5}{c^5a^5+a^5b^5}.\frac{a^5}{b^5c^5}.\left(\frac{c^5}{a^3} + \frac{b^7}{a^5}\right)$$

$$= 2R.\frac{b^5c^5}{c^5a^5+a^5b^5}.\left(\frac{a^2}{b^5} + \frac{b^2}{c^5}\right) + 2R.\frac{c^5a^5}{a^5b^5+b^5c^5}.\left(\frac{b^2}{c^5} + \frac{c^2}{a^5}\right)$$

$$+2R.\frac{a^5b^5}{c^5a^5+b^5c^5}.\left(\frac{c^2}{a^5} + \frac{a^2}{b^5}\right)$$

$$= 2R.\frac{x}{y+z}(B'+C') + 2R.\frac{y}{z+x}(C'+A') + 2R.\frac{z}{x+y}(A'+B')$$

$$\left(x = b^5c^5, y = c^5a^5, z = a^5b^5, A' = \frac{c^2}{a^5}, B' = \frac{a^2}{b^5}, C' = \frac{b^2}{c^5}\right)$$

$$= 2R.\frac{x}{y+z}.\sqrt{B'+C'}^2 + 2R.\frac{y}{z+x}.\sqrt{C'+A'}^2 + 2R.\frac{z}{x+y}.\sqrt{A'+B'}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2R.\,2\sqrt{\sum_{cyc} A'B'.\frac{\sqrt{3}}{2}}$$

$$= 2R.\sqrt{3\sum_{cyc}\left(\frac{c^2}{a^5}\cdot\frac{a^2}{b^5}\right)} \overset{A-G}{\geq} 6R.\sqrt[6]{\frac{c^2}{a^5}\cdot\frac{a^2}{b^5}\cdot\frac{a^2}{b^5}\cdot\frac{b^2}{c^5}\cdot\frac{b^2}{c^5}\cdot\frac{c^2}{a^5}} = 6R.\sqrt[6]{\frac{1}{a^6b^6c^6}} = \frac{6R}{4Rrs}$$

$$= \frac{3}{2rs} \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\geq} \frac{6}{R.\,3\sqrt{3}R} \therefore \frac{1}{a^5+c^5}\left(\frac{a^5}{b^2\sin B}+\frac{c^8}{b^5\sin C}\right)$$

$$+\frac{1}{a^5+b^5}\left(\frac{b^5}{c^2\sin C}+\frac{a^8}{c^5\sin A}\right)+\frac{1}{b^5+c^5}\left(\frac{c^5}{a^2\sin A}+\frac{b^8}{a^5\sin B}\right) \geq \frac{2}{R^2.\sqrt{3}}$$

$\forall\ \Delta\ ABC,'' ='' $ iff $\Delta\ ABC$ is equilateral (QED)

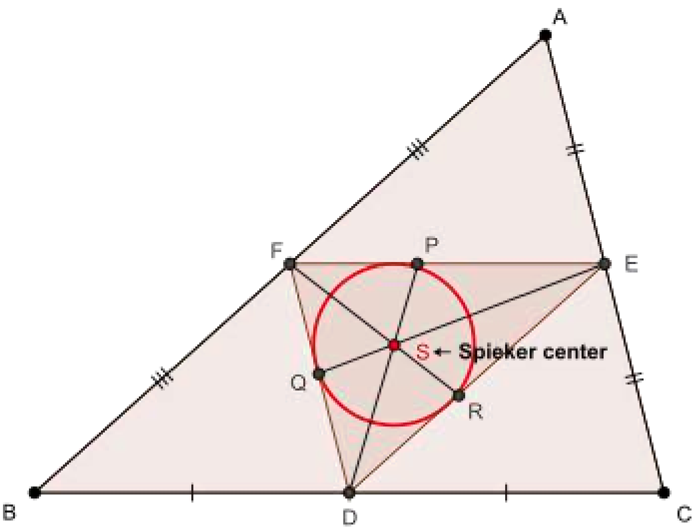
2006.

If $p_a, p_b, p_c \to$ Spieker cevians and $n_a, n_b, n_c \to$ Nagel cevians in $\Delta\ ABC$, then the following relationship holds :

$$\frac{p_a-h_a}{n_a+h_a}+\frac{p_b-h_b}{n_b+h_b}+\frac{p_c-h_c}{n_c+h_c} \geq \frac{4r}{3s}\left(\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}-3\right)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of $\Delta\ DEF = r'$ (say)

$$\text{Now}, 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\ \text{and}\ m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now}, \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\big(b^2+c^2+ab+ca\big)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)}$$

$$\text{Now, }b^3+c^3-abc+a\big(4m_a^2\big)=b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)$$

$$=(b+c)\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2\big)+a\big(b^2+c^2-a^2\big)$$

$$=2s\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2+bc-a^2\big)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$=s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a\overset{?}{\geq}h_a+\frac{2}{3}.\frac{(b-c)^2}{a}\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\overset{?}{\geq}s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}+\frac{4}{9}.\frac{(b-c)^4}{a^2}+\frac{4h_a}{3}.\frac{(b-c)^2}{a}$$

$$\Leftrightarrow\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2}\boxed{\underset{(\blacksquare)}{\overset{?}{\geq}}}\frac{4h_a}{3a}\ (\because (b-c)^2\geq 0)$$

$$\text{We have: }\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2}\overset{\because a^2>(b-c)^2}{>}\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4}{9}$$

$$=\frac{9s(3s+a)a^2+9s(s-a)(2s+a)^2-4a^2(2s+a)^2}{9a^2(2s+a)^2}$$

$$=\frac{4(s-a)(9s^3+9s^2a+5sa^2+a^3)}{9a^2(2s+a)^2}\overset{s>a}{>}0\Rightarrow \text{LHS of }(\blacksquare)>0\ \therefore(\blacksquare)\Leftrightarrow$$

$$\frac{T^2}{a^4(2s+a)^4}+\frac{16(b-c)^4}{81a^4}-\frac{8T(b-c)^2}{9a^4(2s+a)^2}\overset{?}{\geq}\frac{16}{9a^2}.\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$\left(T=s(3s+a)a^2+s(s-a)(2s+a)^2\right)$$

$$\Leftrightarrow\frac{16(b-c)^4}{81a^4}-\frac{8(b-c)^2}{9a^4}\left(\frac{T}{(2s+a)^2}-2s(s-a)\right)+\frac{T^2}{a^4(2s+a)^4}-\frac{16s(s-a)}{9a^2}$$

$$\Leftrightarrow\left(\frac{4(b-c)^2}{9}\right)^2+\frac{4(b-c)^2}{9}.\frac{4s(2s^3-3sa^2-a^3)}{(2s+a)^2}$$

$$+\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}\boxed{\underset{(\blacksquare\blacksquare)}{\overset{?}{\geq}}}0$$

Now, LHS of $(\blacksquare\blacksquare)$ is a quadratic polynomial in $''\frac{4(b-c)^2}{9}''$ whose $\boxed{\text{discriminant}}$

$$=\frac{16s^2\left(2s^3-3sa^2-a^3\right)^2}{(2s+a)^4}-4.\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}$$
$$=-\frac{16sa^7}{9(2s+a)^4}.\left(44t^5-28t^4-49t^3+10t^2+19t+4\right)\left(t=\frac{s}{a}\right)$$
$$=-\frac{16sa^7}{9(2s+a)^4}.(t-1)^2\left(44t^3+60t^2+27t+4\right)<0\Rightarrow \text{LHS of }(\blacksquare\blacksquare)>0$$
$$\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true}\therefore p_a\geq h_a+\frac{2}{3}.\frac{(b-c)^2}{a}\to(m)$$
Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$
$$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s\left(b^2+c^2\right)-2sbc$$
$$=an_a^2+a\left(as-s^2\right)\Rightarrow s\left(b^2+c^2-a^2-2bc\right)=an_a^2-as^2\Rightarrow an_a^2=$$
$$as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=$$
$$s\left(s-\frac{a^2-(b-c)^2}{a}\right)=s\left(s-a+\frac{(b-c)^2}{a}\right)\Rightarrow n_a^2=s(s-a)+\frac{s}{a}.(b-c)^2$$
$$\Rightarrow\frac{4r}{3s}.\left(\frac{n_a}{h_a}-1\right)=\frac{4ra}{3s.2rs}.\left(\frac{n_a^2-h_a^2}{n_a+h_a}\right)$$
$$=\frac{2a}{3s^2}.\left(\frac{s(s-a)+\frac{s}{a}.(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}}{n_a+h_a}\right)=\frac{2a}{3s^2}.\frac{s^2}{a^2}.\frac{(b-c)^2}{n_a+h_a}$$
$$=\frac{2}{3}.\frac{(b-c)^2}{a}.\frac{1}{n_a+h_a}\overset{\text{via }(m)}{\leq}\frac{p_a-h_a}{n_a+h_a}\therefore\frac{p_a-h_a}{n_a+h_a}\geq\frac{4r}{3s}.\left(\frac{n_a}{h_a}-1\right)\text{ and analogs}$$
$$\Rightarrow\frac{p_a-h_a}{n_a+h_a}+\frac{p_b-h_b}{n_b+h_b}+\frac{p_c-h_c}{n_c+h_c}\geq\frac{4r}{3s}\left(\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}-3\right)$$
$\forall\,\Delta ABC$, with equality iff ΔABC is equilateral (QED)

2007. In ΔABC the following relationship holds:

$$\sum\frac{a^2}{(\sin B+\sin C)^2}\geq 3R^2$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a^2}{(\sin B+\sin C)^2}=\frac{4R^2\sin^2A}{(\sin B+\sin C)^2}=\left(\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}.\cos\frac{B-C}{2}}\right)^2.4R^2=$$
$$4R^2\left(\frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}.\cos\frac{B-C}{2}}\right)^2\geq 4R^2\sin^2\frac{A}{2}\left(\text{Because }\cos\frac{B-C}{2}\leq 1\right)$$

$$\sum_{cyc}\frac{a^2}{(sinB+sinC)^2}\geq 4R^2\sum_{cyc}sin^2\frac{A}{2}$$

$$\sum_{cyc}sin^2\frac{A}{2}=\frac{3}{2}-\frac{1}{2}\sum_{cyc}cosA=\frac{3}{2}-\frac{1}{2}\left(1+\frac{r}{R}\right)=1-\frac{r}{2R}\overset{Euler}{\geq}1-\frac{1}{4}=\frac{3}{4}\quad(1)$$

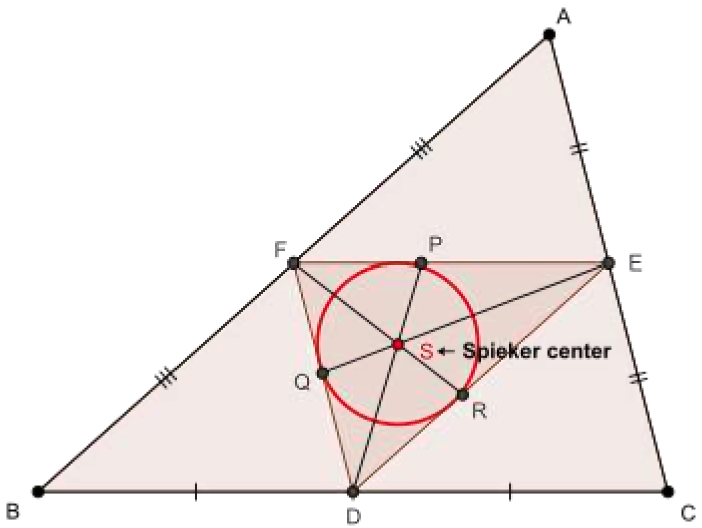
$$\sum_{cyc}\frac{a^2}{(sinB+sinC)^2}\geq 4R^2\cdot\frac{3}{4}=3R^2$$

2008. If $p_a, p_b, p_c \rightarrow$ Spieker cevians in Δ ABC, then the following relationship holds:

$$p_a+p_b+p_c+12r\geq\frac{7}{3}(h_a+h_b+h_c)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\rightarrow(1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$

$$=\frac{\pi}{2}-\frac{A-B}{2}\text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\rightarrow(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2} \stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2 + c^2 + ab + ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES}, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)((z+x)+(x+y)-2(y+z))}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore\ \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2}\right)$$

$$= s(s-a) + (b-c)^2\left(\left(\frac{s}{2s+a}\right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4}\right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2.\left(\frac{s}{2s+a} + \frac{1}{2}\right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now}, p_a \overset{?}{\geq} \frac{2b^2-bc+2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \overset{?}{\geq} \frac{\left(2b^2-bc+2c^2\right)^2}{36R^2.\frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \overset{?}{\geq} \left(\frac{2b^2-bc+2c^2}{3bc} - 1\right)\left(\frac{2b^2-bc+2c^2}{3bc} + 1\right)$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \overset{?}{\geq} \frac{4(b-c)^2\left(b^2+bc+c^2\right)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \overset{?}{\geq} \frac{4(b-c)^2\left(b^2+bc+c^2\right)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \overset{?}{\geq} \frac{b^2+bc+c^2}{9b^2c^2} \quad \left(\because (b-c)^2 \geq 0\right)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \overset{?}{\geq} \left(b^2+bc+c^2\right)(s-b)(s-c)$$

$$= \left(b^2+bc+c^2\right)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \overset{?}{\geq} \left(bc-s(s-a)\right)\left(b^2+c^2\right) + b^2c^2 - bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \overset{?}{\geq} \left(bc-s(s-a)\right)\left((2s-a)^2 - 2bc\right)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \overset{?}{\geq}$$
$$\left(bc-s(s-a)\right)(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc\left(s(s-a) + (2s-a)^2\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2}.b^2c^2 - \left(5s^2 - 5sa + a^2\right).bc + s(s-a)(2s-a)^2 \underset{(\blacksquare)}{\overset{?}{\geq}} 0}$$

Now, LHS of $(\blacksquare)$ is a quadratic polynomial with discriminant $=$

$$\left(5s^2 - 5sa + a^2\right)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2}.s(2s-a)^2$$

$$= \frac{-a^2\left(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4\right)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\left((s-a)(12s^2+6sa+5a^2)+6a^3\right)}{(2s+a)^2} < 0\ (\because s > a)$$

$$\therefore (\blacksquare) \text{ is true (strict inequality)} \therefore p_a \geq \frac{2b^2-bc+2c^2}{6R} \text{ and analogs}$$

$$\Rightarrow p_a+p_b+p_c+12r-\frac{7}{3}(h_a+h_b+h_c) \geq \sum_{cyc}\frac{2b^2-bc+2c^2}{6R}-\frac{7}{6R}\sum_{cyc}bc+12r$$

$$= \frac{4\sum_{cyc}a^2-8\sum_{cyc}bc+72Rr}{6R} = \frac{8(s^2-4Rr-r^2)-8(s^2+4Rr+r^2)+72Rr}{6R}$$

$$= \frac{8r(R-2r)}{6R} \overset{Euler}{\geq} 0 \therefore p_a+p_b+p_c+12r \geq \frac{7}{3}(h_a+h_b+h_c)$$

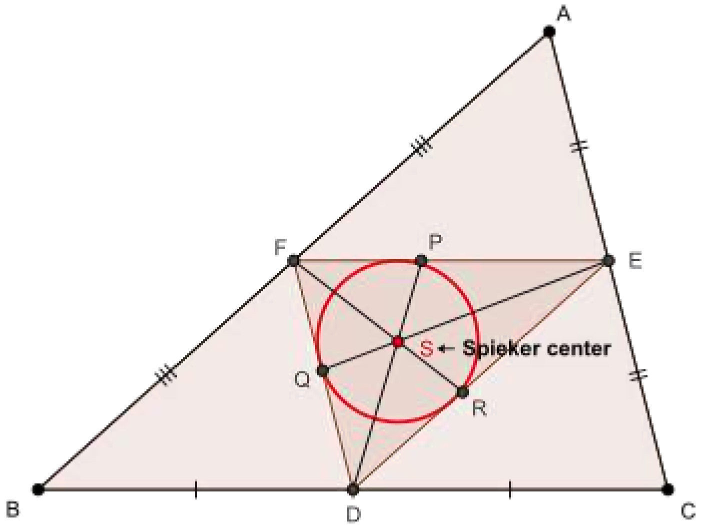
$\forall\ \Delta\, ABC$, with equality iff $\Delta\, ABC$ is equilateral (QED)

2009. In any $\Delta\, ABC$, the following relationship holds :

$$\left(\frac{m_a m_b m_c}{F}\right)^2 \geq \sum_{cyc}\left(r_a r_b\sqrt{\frac{p_c}{w_c}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of $\Delta\, DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$
$= \frac{\pi}{2} - \frac{A-B}{2}$ *and* $m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$
$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$
$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$
$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$
$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\big(b^3+c^3-abc+a(4m_a^2)\big)$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\big(b^3+c^3-abc+a(4m_a^2)\big)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\Big(\big(a^2+2a(b+c)+(b+c)^2\big)+\big((b+c)^2+2a(b+c)+a^2\big)-a^2\Big)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\big(2(a+b+c)^2-a^2\big)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, }m_a^4w_a\overset{\text{Lascu}+A-G}{\geq}m_a^3.\,s(s-a)\overset{?}{\geq}s^2(s-a)^2.\,p_a\Leftrightarrow m_a^6\overset{?}{\geq}s^2(s-a)^2.\,p_a^2$$

$$\Leftrightarrow\frac{m_a^4}{s^2(s-a)^2}-1\overset{?}{\geq}\frac{p_a^2}{m_a^2}-1$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow}\frac{\left(s(s-a)+\frac{(b-c)^2}{4}\right)^2-s^2(s-a)^2}{s^2(s-a)^2}\overset{?}{\geq}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2.m_a^2}$$

$$\Leftrightarrow m_a^2.\frac{\frac{(b-c)^4}{16}+s(s-a).\frac{(b-c)^2}{2}}{s^2(s-a)^2}\overset{?}{\geq}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\Leftrightarrow (b-c)^2\left(m_a^2.\frac{\frac{(b-c)^2}{16}+\frac{s(s-a)}{2}}{s^2(s-a)^2}-\frac{8s^2-a^2}{4(2s+a)^2}\right)\underset{(\blacksquare)}{\overset{?}{\geq}}0$$

$\because (b-c)^2\geq 0$ *and* $m_a^2\geq s(s-a)$ $\therefore$ in order to prove $(\blacksquare)$, it suffices to prove :

$$\frac{s(s-a).\frac{s(s-a)}{2}}{s^2(s-a)^2}-\frac{8s^2-a^2}{4(2s+a)^2}>0\Leftrightarrow 2(2s+a)^2>8s^2-a^2\Leftrightarrow 8sa+3a^2>0$$

$$\to \text{true}\Rightarrow(\blacksquare)\text{ is true}\therefore m_a^4w_a\geq s^2(s-a)^2.p_a\Rightarrow\boxed{\frac{m_a^4}{s^2(s-a)^2}\geq\frac{p_a}{w_a}}$$

$$\Rightarrow\frac{m_a^2}{r_br_c}\geq\sqrt{\frac{p_a}{w_a}}\Rightarrow m_a^2\geq r_br_c.\sqrt{\frac{p_a}{w_a}}\text{ and analogs}\Rightarrow\sum_{cyc}m_a^2\overset{(\bullet\bullet)}{\geq}\sum_{cyc}\left(r_br_c.\sqrt{\frac{p_a}{w_a}}\right)$$

$$\text{Now, }\prod_{cyc}\frac{a^2}{4}\overset{?}{\geq}\frac{F^2}{36}.\sum_{cyc}a^2\Leftrightarrow\frac{1}{64}.16R^2r^2s^2\overset{?}{\geq}\frac{r^2s^2}{36}.\sum_{cyc}a^2\Leftrightarrow 9R^2\overset{?}{\geq}\sum_{cyc}a^2$$

$\to$ true via Leibnitz $\therefore \prod_{cyc}\frac{a^2}{4}\geq\frac{F^2}{36}.\sum_{cyc}a^2$ and implementing this on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose area as a consequence of trivial calculations $=\frac{F}{3}$, we get :

$$\prod_{cyc}\frac{\left(\frac{4}{9}m_a^2\right)}{4}\geq\frac{\left(\frac{F^2}{9}\right)}{36}.\frac{4}{9}\sum_{cyc}m_a^2\Rightarrow\frac{1}{729}.m_a^2m_b^2m_c^2\geq\frac{F^2}{81.4}.\frac{4}{9}\sum_{cyc}m_a^2$$

$$\Rightarrow\frac{m_a^2m_b^2m_c^2}{F^2}\geq\sum_{cyc}m_a^2\overset{\text{via }(\bullet\bullet)}{\geq}\sum_{cyc}\left(r_br_c.\sqrt{\frac{p_a}{w_a}}\right)\therefore\left(\frac{m_am_bm_c}{F}\right)^2\geq\sum_{cyc}\left(r_ar_b.\sqrt{\frac{p_c}{w_c}}\right)$$

$\forall\,\Delta ABC$, " = " iff ΔABC is equilateral (QED)

2010. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{(2R+r-r_a)(3R+r-r_b)+R^2}\geq\frac{1}{R^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sum(2R+r-r_a)(3R+r-r_b)=$$

$$=\sum(6R^2+5Rr-2Rr_b-3Rr_a-r(r_a+r_b)+r_ar_b+r^2)=$$

$$=3.6R^2+3.5Rr-2R\sum r_b-3R\sum r_a-2r\sum r_a+\sum r_ar_b+3r^2=$$

$$=18R^2+15Rr-2R(4R+r)-3R(4R+r)-2r(4R+r)+s^2+3r^2=$$

$$=-2R^2+2Rr+r^2+s^2\overset{Gerretsen}{\leq}-2R^2+2Rr+r^2+4R^2+4Rr+3r^2=$$

$$=2R^2+6Rr+4r^2\overset{Euler}{\leq}2R^2+6R.\frac{R}{2}+4\left(\frac{R}{2}\right)^2=6R^2\ (1)$$

$$\sum_{cyc}\frac{1}{(2R+r-r_a)(3R+r-r_b)+R^2}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(1+1+1)^2}{\sum(2R+r-r_a)(3R+r-r_b)+3R^2}\overset{(1)}{\geq}\frac{9}{6R^2+3R^2}=\frac{1}{R^2}$$

$$Equality\ holds\ for\ a=b=c$$

2011. In ΔABC the following relationships holds:

$$1.\ \frac{a}{\sin\frac{A}{2}}+\frac{b}{\sin\frac{B}{2}}+\frac{c}{\sin\frac{C}{2}}\geq 12\sqrt{3}r,$$

$$2.\ \frac{a^2}{\sin\frac{A}{2}}+\frac{b^2}{\sin\frac{B}{2}}+\frac{c^2}{\sin\frac{C}{2}}\geq 72r^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$1.\frac{a}{\sin\frac{A}{2}}+\frac{b}{\sin\frac{B}{2}}+\frac{c}{\sin\frac{C}{2}}\geq 12\sqrt{3}r$

$$\frac{a}{\sin\frac{A}{2}}+\frac{b}{\sin\frac{B}{2}}+\frac{c}{\sin\frac{C}{2}}\overset{A-G}{\geq}3\sqrt[3]{\frac{abc}{\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}}\ (1)$$

$$\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\leq\frac{1}{8};\ Let's\ prove\ it\ ...$$

$$a^2=(b-c)^2+4bc\sin^2\frac{A}{2}\to\begin{cases}a\ge 2\sin\frac{A}{2}\sqrt{bc}\\ b\ge \sin\frac{B}{2}\sqrt{ac}\\ c\ge 2\sin\frac{C}{2}\sqrt{ab}\end{cases}\to\ \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\le\frac{1}{8}\ (2)$$

From (1) and (2) we have :

$$\frac{a}{\sin\frac{A}{2}}+\frac{b}{\sin\frac{B}{2}}+\frac{c}{\sin\frac{C}{2}}\ge$$

$$\ge\left(3\sqrt[3]{8abc}=6\sqrt[3]{4R.S}\overset{R\ge 2r}{\overbrace{\ge}}6\sqrt[3]{8r.S}\ge 12\sqrt[3]{r.3\sqrt{3}r^2}\right)=12r\sqrt{3}\ \ (True)$$

2. $\dfrac{a^2}{\sin\frac{A}{2}}+\dfrac{b^2}{\sin\frac{B}{2}}+\dfrac{c^2}{\sin\frac{C}{2}}\ge 72r^2$

$$\frac{a^2}{\sin\frac{A}{2}}+\frac{b^2}{\sin\frac{B}{2}}+\frac{c^2}{\sin\frac{C}{2}}\overset{A-G}{\overbrace{\ge}}3\sqrt{\frac{(abc)^2}{\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}}$$

If we use formulas ,we get :

$$\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\le\frac{1}{8}\ \ and\ \ abc=4RS\ ,R\ge 2r\ \ Euler\ \ ,S\ge 3\sqrt{3}r^2\ \ Mitrinovici$$

$$\frac{a^2}{\sin\frac{A}{2}}+\frac{b^2}{\sin\frac{B}{2}}+\frac{c^2}{\sin\frac{C}{2}}\overset{A-G}{\overbrace{\ge}}3\sqrt[3]{8(abc)^2}=$$

$$=\left(6\sqrt[3]{(4RS)^2}=6\sqrt[3]{\left(4.2r.3\sqrt{3}r^2\right)^2}\right)=72r^2\ (True)$$

2012.

$$\text{If in }\Delta\text{ ABC, the following relationship holds : }\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)},\text{ then}$$

$$\text{prove that : }1\le\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\le\frac{R}{r}-1$$

Proposed by Tapas Das-India

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}\Rightarrow 2\sin A\sin(B-C)=2\sin C\sin(A-B)$$

$$\Rightarrow\cos(A-B+C)-\cos(A+B-C)=\cos(C-A+B)-\cos(C+A-B)$$

$$\Rightarrow 2\cos(\pi-2B)=\cos(\pi-2A)+\cos(\pi-2C)\Rightarrow -2\cos 2B=-\cos 2A-\cos 2C$$

$$\Rightarrow 1-2\sin^2 A+1-2\sin^2 C=2-4\sin^2 B\Rightarrow \frac{2b^2}{4R^2}=\frac{a^2+c^2}{4R^2}$$
$$\Rightarrow 2b^2=a^2+c^2\to(1)$$
$$\text{Now,}\frac{R}{r}-1-\left(\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\right)\overset{Bandila}{\geq}$$
$$\frac{c}{a}+\frac{a}{c}-1-\left(\frac{a^2}{b^2+c^2}+\frac{c^2}{a^2+b^2}\right)\overset{\text{via }(1)}{=}$$
$$\frac{c^2+a^2-ca}{ca}-\left(\frac{a^2}{\frac{a^2+c^2}{2}+c^2}+\frac{c^2}{a^2+\frac{a^2+c^2}{2}}\right)\overset{?}{\geq}0$$
$$\Leftrightarrow \frac{c^2+a^2-ca}{ca}\overset{?}{\geq}\frac{2a^2(3a^2+c^2)+2c^2(3c^2+a^2)}{(3c^2+a^2)(3a^2+c^2)}$$
$$\Leftrightarrow 3t^6-9t^5+13t^4-14t^3+13t^2-9t+3\overset{?}{\geq}0\ \left(t=\frac{a}{c}\right)$$
$$\Leftrightarrow \frac{1}{16}(t-1)^2\left((12t^2+13)(2t-1)^2+4t+35\right)\overset{?}{\geq}0\to\text{true}$$
$$\therefore \frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\leq\frac{R}{r}-1$$
$$\text{Again,}\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}-1=\frac{a^2}{\frac{a^2+c^2}{2}+c^2}+\frac{c^2}{a^2+\frac{a^2+c^2}{2}}-1$$
$$=\frac{2a^2(3a^2+c^2)+2c^2(3c^2+a^2)-(3c^2+a^2)(3a^2+c^2)}{(3c^2+a^2)(3a^2+c^2)}$$
$$=\frac{3(a^4-2a^2c^2+c^4)}{(3c^2+a^2)(3a^2+c^2)}=\frac{3(c^2-a^2)^2}{(3c^2+a^2)(3a^2+c^2)}\geq 0$$
$$\therefore \frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\geq 1\text{ and so,}$$
$$1\leq\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\leq\frac{R}{r}-1\text{ whenever}$$
$$\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)},''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}\Leftrightarrow \sin A\sin(B-C)=\sin C\sin(A-B)$$
$$\Leftrightarrow \cos(A-B+C)-\cos(A+B-C)=\cos(C-A+B)-\cos(C+A-B)$$
$$\overset{A+B+C=\pi}{\Leftrightarrow}\ -\cos(2B)+\cos(2C)=-\cos(2A)+\cos(2B)$$
$$\overset{\cos 2x=1-2\sin^2 x}{\Leftrightarrow}\ (1-2\sin^2 A)+(1-2\sin^2 C)=2(1-2\sin^2 B)$$

$$\overset{a=2R\sin A}{\Leftrightarrow} \quad a^2+c^2=2b^2.$$

$$\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}=\frac{a^2}{b^2+c^2}+\frac{c^2}{a^2+b^2}=\frac{2a^2}{a^2+3c^2}+\frac{2c^2}{3a^2+c^2}$$

$$\overset{CBS}{\geq}\frac{2(a^2+c^2)^2}{a^2(a^2+3c^2)+c^2(3a^2+c^2)}=\frac{2(a^2+c^2)^2}{(a^2+c^2)^2+4a^2c^2}\overset{AM-GM}{\geq}$$

$$\geq\frac{2(a^2+c^2)^2}{(a^2+c^2)^2+(a^2+c^2)^2}=1.$$

$$\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}=\frac{2a^2}{a^2+3c^2}+\frac{2c^2}{3a^2+c^2}=$$

$$=\frac{2a^2}{(a^2+c^2)+2c^2}+\frac{2c^2}{2a^2+(a^2+c^2)}$$

$$\overset{AM-GM}{\leq}\frac{2a^2}{2ac+2c^2}+\frac{2c^2}{2a^2+2ac}=\frac{a^3+c^3}{ac(a+c)}=\frac{a^2+c^2-ac}{ac}=\frac{a}{c}+\frac{c}{a}-1\overset{Bandila}{\leq}\frac{R}{r}-1$$

Equality holds iff ΔABC is equilateral.

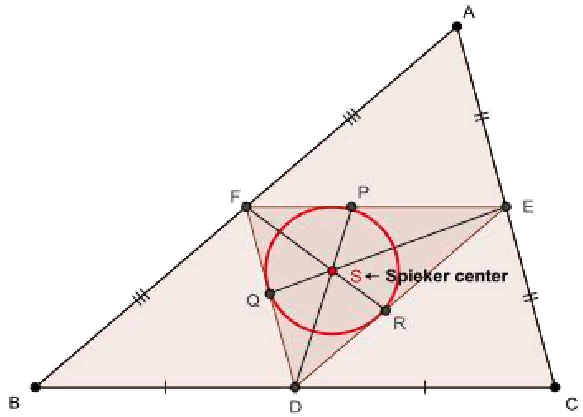
2013.If $p_a, p_b, p_c \to$

Spieker cevians in Δ ABC, then the following relationship holds :

$$p_a+p_b+p_c \geq h_a+h_b+h_c+\frac{4}{3}\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{s}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= 2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$= \frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s}-2\mathrm{Rr}$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s}+2\mathrm{Rr}$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right) = \frac{ab+ca}{4}-2\mathrm{Rr} \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\,AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\,AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, } b^3+c^3-abc+a\left(4m_a^2\right) = b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$

$$= (b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$

$$= 2s\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2+bc-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$
$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$
$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$
$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$
$$\text{Now, } p_a\overset{?}{\geq}h_a+\frac{2}{3}.\frac{(b-c)^2}{a}\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$
$$\overset{?}{\geq}s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}+\frac{4}{9}.\frac{(b-c)^4}{a^2}+\frac{4h_a}{3}.\frac{(b-c)^2}{a}$$
$$\Leftrightarrow\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2}\ \boxed{\overset{?}{\underset{(\blacksquare)}{\geq}}}\ \frac{4h_a}{3a}\ (\because (b-c)^2\geq 0)$$
$$\text{We have: }\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2}\overset{\because a^2>(b-c)^2}{>}\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4}{9}$$

$$=\frac{9s(3s+a)a^2+9s(s-a)(2s+a)^2-4a^2(2s+a)^2}{9a^2(2s+a)^2}$$

$$=\frac{4(s-a)(9s^3+9s^2a+5sa^2+a^3)}{9a^2(2s+a)^2}\overset{s>a}{>}0\Rightarrow \text{LHS of } (\blacksquare)>0\therefore(\blacksquare)\Leftrightarrow$$

$$\frac{T^2}{a^4(2s+a)^4}+\frac{16(b-c)^4}{81a^4}-\frac{8T(b-c)^2}{9a^4(2s+a)^2}\overset{?}{\geq}\frac{16}{9a^2}\cdot\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$\left(T=s(3s+a)a^2+s(s-a)(2s+a)^2\right)$$

$$\Leftrightarrow\frac{16(b-c)^4}{81a^4}-\frac{8(b-c)^2}{9a^4}\left(\frac{T}{(2s+a)^2}-2s(s-a)\right)+\frac{T^2}{a^4(2s+a)^4}-\frac{16s(s-a)}{9a^2}$$

$$\Leftrightarrow\left(\frac{4(b-c)^2}{9}\right)^2+\frac{4(b-c)^2}{9}\cdot\frac{4s(2s^3-3sa^2-a^3)}{(2s+a)^2}$$

$$+\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}\boxed{\underset{(\blacksquare\blacksquare)}{\overset{?}{\geq}}}0$$

Now, LHS of $(\blacksquare\blacksquare)$ is a quadratic polynomial in "$\frac{4(b-c)^2}{9}$" whose $\boxed{\text{discriminant}}$

$$=\frac{16s^2(2s^3-3sa^2-a^3)^2}{(2s+a)^4}-4.\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}$$

$$=-\frac{16sa^7}{9(2s+a)^4}\cdot(44t^5-28t^4-49t^3+10t^2+19t+4)\ \left(t=\frac{s}{a}\right)$$

$$=-\frac{16sa^7}{9(2s+a)^4}\cdot(t-1)^2(44t^3+60t^2+27t+4)<0\Rightarrow \text{LHS of } (\blacksquare\blacksquare)>0$$

$$\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true}\therefore p_a\geq h_a+\frac{2}{3}\cdot\frac{(b-c)^2}{a}\overset{s>a}{\geq}h_a+\frac{2}{3}\cdot\frac{(b-c)^2}{s}\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}p_a\geq\sum_{cyc}h_a+\frac{2}{3s}\cdot\sum_{cyc}(b-c)^2$$

$$\therefore p_a+p_b+p_c\geq h_a+h_b+h_c+\frac{4}{3}\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{s}$$

$\forall\,\Delta$ ABC, with equality iff Δ ABC is equilateral (QED)

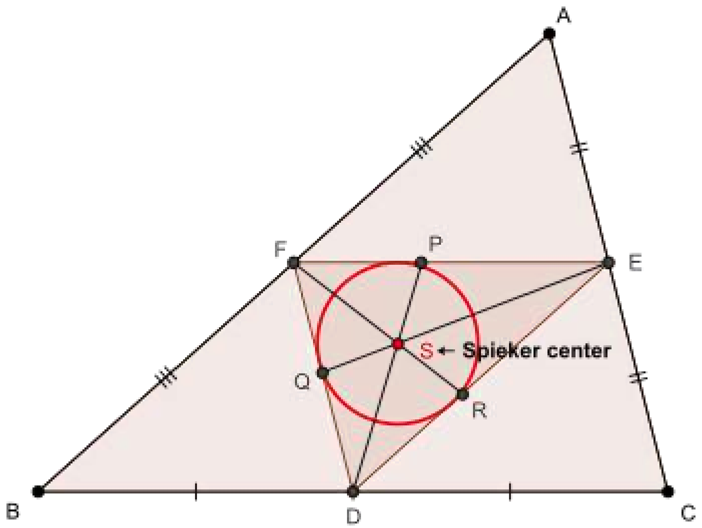
2014. In any Δ ABC with

$p_a\to$ Spieker cevian, the following relationship holds :

$$p_a\leq h_a+\frac{16R}{9}\left(\frac{b-c}{a}\right)^2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say)
and inradius of Δ DEF $= r'(say)$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2-bc+2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{(2b^2-bc+2c^2)^2}{36R^2.\frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2-h_a^2}{h_a^2} \geq \left(\frac{2b^2-bc+2c^2}{3bc}-1\right)\left(\frac{2b^2-bc+2c^2}{3bc}+1\right)$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}}{h_a^2}$$

$$\geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2} \; (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (b^2+bc+c^2)(s-b)(s-c)$$

$$= (b^2+bc+c^2)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq \big(bc-s(s-a)\big)(b^2+c^2)+b^2c^2-bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}-b^2c^2+bcs(s-a) \geq \big(bc-s(s-a)\big)((2s-a)^2-2bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}-b^2c^2+bcs(s-a) \geq$$

$$\big(bc-s(s-a)\big)(2s-a)^2-2b^2c^2+2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}+b^2c^2+s(s-a)(2s-a)^2-bc(s(s-a)+(2s-a)^2) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3-12sa^2-4a^3}{4(s-a)(2s+a)^2}.b^2c^2-(5s^2-5sa+a^2).bc+s(s-a)(2s-a)^2 \overset{(\square)}{\geq} 0}$$

Now, LHS of $(\square)$ is a quadratic polynomial with discriminant $=$

$$(5s^2-5sa+a^2)^2-\frac{25s^3-12sa^2-4a^3}{(2s+a)^2}.s(2s-a)^2$$

$$= \frac{-a^2(12s^4-18s^3a+5s^2a^2+2sa^3-a^4)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\big((s-a)(12s^2+6sa+5a^2)+6a^3\big)}{(2s+a)^2} < 0 \; (\because s > a)$$

$$\therefore (\blacksquare) \text{ is true (strict inequality)}$$

$$\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \ \forall \Delta ABC \to (m)$$

$$\text{Again}, p_a \leq h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2 \Leftrightarrow \frac{p_a^2 - h_a^2}{p_a + h_a} \leq \frac{16R}{9} \cdot \frac{(b-c)^2}{a^2}$$

$$\Leftrightarrow \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{p_a + h_a} \leq \frac{16R}{9} \cdot \frac{(b-c)^2}{a^2}$$

$$\Leftrightarrow \frac{s^4}{(2s+a)^2} \overset{(\blacksquare\blacksquare)}{\leq} \frac{4R}{9} \cdot (p_a + h_a) \ (\because (b-c)^2 \geq 0)$$

$$\text{Now, via } (m), \frac{4R}{9} \cdot (p_a + h_a) \geq \frac{4R}{9} \cdot \frac{2b^2 - bc + 2c^2 + 3bc}{6R} = \frac{4}{27} \cdot (b^2 + bc + c^2) \geq$$

$$\frac{4}{27} \cdot \frac{3}{4} \cdot (2s-a)^2 \overset{?}{>} \frac{s^4}{(2s+a)^2} \Leftrightarrow 4s^2 - a^2 \overset{?}{>} 3s^2 \Leftrightarrow s^2 \overset{?}{>} a^2 \to \text{true} \Rightarrow (\blacksquare\blacksquare)$$

$$\text{is true} \therefore p_a \leq h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2 \ \forall \Delta ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$

2015. In ΔABC the following relationship holds:

$$\sqrt{\frac{1}{2}(n_a{}^2 + g_a{}^2) + \frac{1}{2} \cdot \frac{\cos A}{\sin^2 \frac{A}{2}} \cdot (b-c)^2} \geq \frac{1}{2}(r_b + r_c) + \sqrt{h_a}\left(\sqrt{\frac{1}{2}(r_b + r_c)} - \sqrt{h_a}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$n_a{}^2 + g_a{}^2 + \frac{\cos A}{\sin^2 \frac{A}{2}} \cdot (b-c)^2 =$$

$$= s\left(s - a + \frac{(b-c)^2}{a}\right) + (s-a)\left(s - \frac{(b-c)^2}{a}\right) + \frac{\cos A}{\sin^2 \frac{A}{2}} \cdot (b-c)^2 =$$

$$= 2s(s-a) + \left(1 + \frac{\cos A}{\sin^2 \frac{A}{2}}\right)(b-c)^2 = 2s(s-a) + \frac{(b-c)^2}{\tan^2 \frac{A}{2}} =$$

$$= 2s(s-a) + \frac{(s-a)^2(b-c)^2}{r^2}$$

$$= 2s(s-a) + \frac{(s-a)^2[a^2 - 4(s-b)(s-c)]}{r^2} = \left(\frac{a(s-a)}{r}\right)^2 - 2s(s-a) =$$

$$= \left(F \cdot \frac{a}{(s-b)(s-c)}\right)^2 - 2\frac{F^2}{(s-b)(s-c)} = (r_b + r_c)^2 - 2r_b r_c = {r_b}^2 + {r_c}^2 =$$

And since $\frac{1}{r_b} + \frac{1}{r_c} = \frac{s-b}{F} + \frac{s-c}{F} = \frac{a}{F} = \frac{2}{h_a}$, then:

$$\frac{1}{2}h_a(r_b + r_c) = r_b r_c \text{ and } h_a = \frac{2r_b r_c}{r_b + r_c}.$$

So the desired inequality can be rewritten as follows

$$\sqrt{\frac{{r_b}^2 + {r_c}^2}{2}} \geq \frac{r_b + r_c}{2} + \sqrt{r_b r_c} - \frac{2r_b r_c}{r_b + r_c} \Leftrightarrow \sqrt{\frac{{r_b}^2 + {r_c}^2}{2}} - \sqrt{r_b r_c} \geq \frac{r_b + r_c}{2} - \frac{2r_b r_c}{r_b + r_c}$$

$$\Leftrightarrow \frac{(r_b - r_c)^2}{\sqrt{2({r_b}^2 + {r_c}^2)} + 2\sqrt{r_b r_c}} \geq \frac{(r_b - r_c)^2}{2(r_b + r_c)},$$

which is true because:

$$\sqrt{2({r_b}^2 + {r_c}^2)} + 2\sqrt{r_b r_c} \overset{CBS}{\leq} \sqrt{(2+2)\left(({r_b}^2 + {r_c}^2) + 2r_b r_c\right)} = 2(r_b + r_c).$$

So the proof is complete. Equality holds iff $r_b = r_c \Leftrightarrow b = c$.

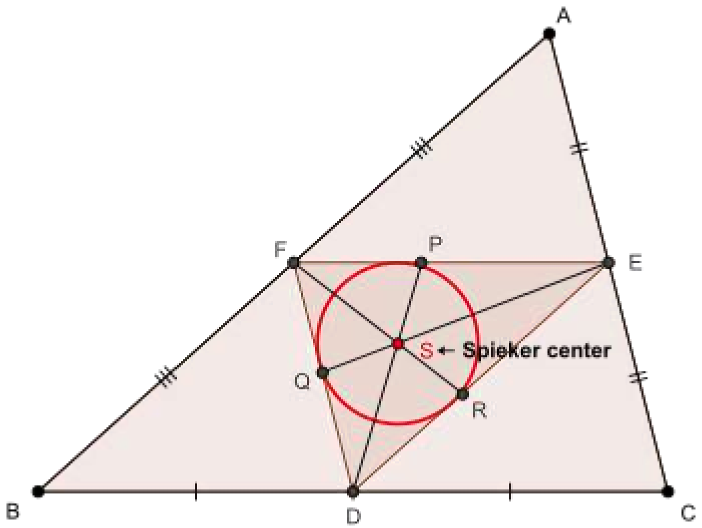
2016.

In any Δ ABC with $p_a \rightarrow$ Spieker cevian, the following relationship holds :

$$p_a \leq h_a + \frac{64}{27}(R - 2r)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'(say)$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we $arrive$ at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\text{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(\text{ii})}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$
$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$
$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$
$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$
$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore\boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore(\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{(2b^2 - bc + 2c^2)^2}{36R^2 . \frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \geq \left(\frac{2b^2 - bc + 2c^2}{3bc} - 1\right)\left(\frac{2b^2 - bc + 2c^2}{3bc} + 1\right)$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2}$$

$$\geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2} \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (b^2+bc+c^2)(s-b)(s-c)$$

$$= (b^2+bc+c^2)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq \big(bc - s(s-a)\big)(b^2+c^2) + b^2c^2 - bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq \big(bc - s(s-a)\big)\big((2s-a)^2 - 2bc\big)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq$$

$$\big(bc - s(s-a)\big)(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc\big(s(s-a) + (2s-a)^2\big) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2}.b^2c^2 - (5s^2 - 5sa + a^2).bc + s(s-a)(2s-a)^2 \overset{(\text{?})}{\geq} 0}$$

Now, LHS of (?) is a quadratic polynomial with discriminant =

$$(5s^2 - 5sa + a^2)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2}.s(2s-a)^2$$

$$= \frac{-a^2(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\big((s-a)(12s^2 + 6sa + 5a^2) + 6a^3\big)}{(2s+a)^2} < 0 \ (\because s > a)$$

$\therefore$ (?) is true (strict inequality)

$$\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R}\ \forall\, \Delta\, ABC \to (m)$$

$$\text{Again, } p_a \leq h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2 \Leftrightarrow \frac{p_a^2 - h_a^2}{p_a + h_a} \leq \frac{16R}{9}\cdot\frac{(b-c)^2}{a^2}$$

$$\Leftrightarrow \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{p_a + h_a} \leq \frac{16R}{9}\cdot\frac{(b-c)^2}{a^2}$$

$$\Leftrightarrow \frac{s^4}{(2s+a)^2} \overset{(■■)}{\leq} \frac{4R}{9}.(p_a + h_a)\ (\because (b-c)^2 \geq 0)$$

$$\text{Now, via } (m), \frac{4R}{9}.(p_a + h_a) \geq \frac{4R}{9}\cdot\frac{2b^2 - bc + 2c^2 + 3bc}{6R} = \frac{4}{27}.(b^2 + bc + c^2) \geq$$

$$\frac{4}{27}\cdot\frac{3}{4}.(2s-a)^2 \overset{?}{>} \frac{s^4}{(2s+a)^2} \Leftrightarrow 4s^2 - a^2 \overset{?}{>} 3s^2 \Leftrightarrow s^2 \overset{?}{>} a^2 \to \text{true} \Rightarrow (■■)$$

$$\text{is true} \therefore p_a \leq h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2 \overset{?}{\leq} h_a + \frac{64}{27}(R - 2r)$$

$$\Leftrightarrow \frac{4}{3}(R-2r).a^2 \overset{?}{\geq} R(b-c)^2 \Leftrightarrow$$

$$\frac{4}{3}.R\left(1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2}\right).16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}$$

$$\overset{?}{\geq} R.16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}$$

$$\Leftrightarrow \frac{4}{3}\left(1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2}\right)\left(1 - \sin^2\frac{A}{2}\right) \overset{?}{\geq} 1 - \cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 3\cos^2\frac{B-C}{2} - 16(x - x^3)\cos\frac{B-C}{2} + 1 + 12x^2 - 16x^4 \boxed{\underset{(■■■)}{\overset{?}{\geq}}} 0\ \left(x = \sin\frac{A}{2}\right)$$

Now, LHS of (■■■) is a quadratic polynomial in $\cos\frac{B-C}{2}$ with discriminant =

$$256(x - x^3)^2 - 12(1 + 12x^2 - 16x^4) = 256x^6 - 320x^4 + 112x^2 - 12$$

$= 4(4x^2 - 1)^2(4x^2 - 3) \leq 0$ iff $x \leq \frac{\sqrt{3}}{2}$ and so, when $x \leq \frac{\sqrt{3}}{2}$, discriminant ≤ 0

$\Rightarrow$ LHS of (■■■) $\geq 0 \Rightarrow$ (■■■) is true and we now focus on the scenario when :

when $x > \frac{\sqrt{3}}{2}$ and then, in order to prove (■■■), it suffices to prove :

$$\cos\frac{B-C}{2} > \frac{8(x - x^3) + (4x^2 - 1).\sqrt{4x^2 - 3}}{3}\ \left(\because x > \frac{\sqrt{3}}{2} > \frac{1}{2} \Rightarrow 4x^2 - 1 > 0\right)$$

$$\text{and} \because \cos\frac{B-C}{2} = \frac{b+c}{a}\sin\frac{A}{2} \overset{\frac{b+c}{a} > 1}{>} x \therefore \text{it suffices to prove :}$$

$$x > \frac{8(x - x^3) + (4x^2 - 1).\sqrt{4x^2 - 3}}{3} \Leftrightarrow 8x^3 - 5x > (4x^2 - 1).\sqrt{4x^2 - 3}$$

$$\Leftrightarrow (8x^3-5x)^2 > (4x^2-3)(4x^2-1)^2 \left(\because x > \frac{\sqrt{3}}{2} \Rightarrow 8x^2 > 6 > 5 \Rightarrow 8x^3 > 5x\right)$$

$$\Leftrightarrow 3-3x^2 > 0 \Rightarrow 1 > x^2 \to \text{true} \Rightarrow (\blacksquare\blacksquare\blacksquare) \text{ is true and combining both cases,}$$

$$(\blacksquare\blacksquare\blacksquare) \text{ is true } \forall \Delta\, ABC \therefore p_a \le h_a + \frac{64}{27}(R-2r),$$

$$'' = '' \text{ iff } \sin\frac{A}{2} = \frac{1}{2} \text{ and } B = C \Rightarrow '' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

2017. **In ΔABC the following relationship holds:**

$$\sum \frac{h_a\sqrt{h_a}}{w_a\sqrt{r_a}} \ge 3\sqrt{\frac{2r}{R}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$h_ah_bh_c = \frac{8F^3}{abc} = \frac{8r^3s^3}{4Rrs} = \frac{2r^2s^2}{R} \quad (1)$$

$$r_ar_br_c = s^2r \ \ (2),\ w_aw_bw_c \overset{w_a\le\sqrt{s(s-a)}}{\le} \sqrt{\prod s(s-a)} = \sqrt{s^3sr^2} = s^2r, \qquad (3)$$

$$\sum \frac{h_a\sqrt{h_a}}{w_a\sqrt{r_a}} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{h_ah_bh_c\sqrt{h_ah_bh_c}}{w_aw_bw_c\sqrt{r_ar_br_c}}} \overset{(1),(2),(3)}{\ge}$$

$$\ge 3\sqrt[3]{\frac{(h_ah_bh_c)^{\frac{3}{2}}}{(s^2r)^{\frac{3}{2}}}} = 3\sqrt{\frac{h_ah_bh_c}{s^2r}} = 3\sqrt{\frac{\frac{2r^2s^2}{R}}{s^2r}} = 3\sqrt{\frac{2r}{R}}$$

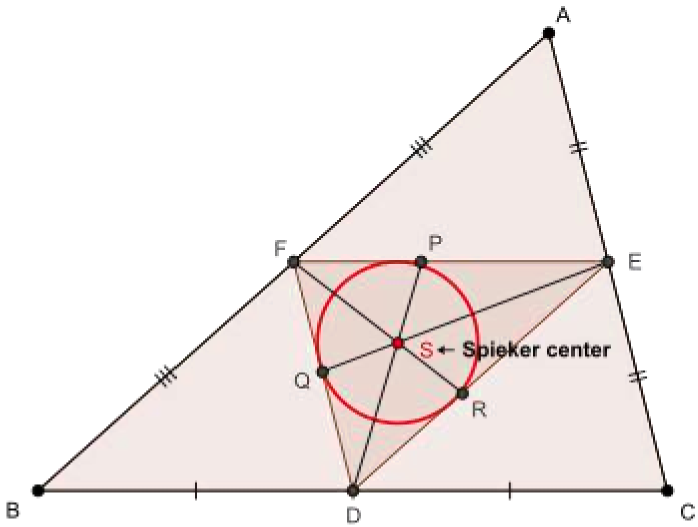
Equality holds for an equilateral triangle

2018.

In any $\Delta\, ABC$ with $p_a \to$ Spieker cevian, $n_a \to$ Nagel cevians, the following relationships hold : $\dfrac{s(b-c)^2}{a(2s+a)} \le n_a - p_a \le \dfrac{s|b-c|}{2s+a}$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and m(∡BAX) = α and m(∡CAX) = β (say)
and inradius of Δ DEF = r′(say)

$$\text{Now, } 16[\text{DEF}]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [\text{DEF}] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\,\text{DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we $arrive$ at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES},\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}\geq 0\Rightarrow p_a\overset{(\bullet\bullet)}{\geq} m_a$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$

$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-a+\frac{(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2\overset{(\bullet\bullet\bullet\bullet\bullet)}{=}s(s-a)+\frac{s}{a}.(b-c)^2$$

Now, $n_a-p_a\overset{?}{\leq}\frac{s|b-c|}{2s+a}\Leftrightarrow n_a^2\overset{?}{\leq}p_a^2+s^2.\frac{(b-c)^2}{(2s+a)^2}+2p_a.\frac{s|b-c|}{2s+a}$

$$\overset{\text{via }(\bullet\bullet\bullet\bullet)\text{ and }(\bullet\bullet\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s}{a}.(b-c)^2-s(s-a)-\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\overset{?}{\leq}s^2.\frac{(b-c)^2}{(2s+a)^2}+2p_a.\frac{s|b-c|}{2s+a}$$

$$\Leftrightarrow\left(\frac{s}{a}-\frac{s(3s+a)}{(2s+a)^2}-\frac{s^2}{(2s+a)^2}\right)(b-c)^2\overset{?}{\leq}2p_a.\frac{s|b-c|}{2s+a}\Leftrightarrow$$

$$\frac{4s^3|b-c|^2}{a(2s+a)}\leq 2s.p_a.|b-c|\Leftrightarrow ap_a\overset{?}{\geq}\frac{2s^2}{2s+a}.|b-c|\ (\because |b-c|\geq 0)$$

$$\overset{\text{via }(\bullet\bullet\bullet\bullet)}{\Leftrightarrow} a^2\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\overset{?}{\geq}\frac{4s^4}{(2s+a)^2}.(b-c)^2$$

$$\Leftrightarrow a^2s(s-a)-\frac{4s^4-a^2s(3s+a)}{(2s+a)^2}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow a^2s(s-a)-\frac{s(s-a)(2s+a)^2}{(2s+a)^2}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow s(s-a)(a^2-(b-c)^2)\overset{?}{\geq}0\to\text{true}\therefore n_a-p_a\leq\frac{s|b-c|}{2s+a}$$

Now, $b^3+c^3-abc+a(4m_a^2)=b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$

$$=\sum_{\text{cyc}}a^3-abc+2a.2bc\cos A=2s(s^2-6Rr-3r^2)-4Rrs+16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet\bullet\bullet\bullet\bullet\bullet)}{=}2s(s^2-8Rr-3r^2+8Rr\cos A)\therefore(\bullet),(\bullet\bullet\bullet\bullet\bullet\bullet\bullet)$$

$$\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}.2s(s^2-8Rr-3r^2+8Rr\cos A)$$

$$=\frac{4s^2}{(2s+a)^2}.\left(s^2-8Rr-3r^2+8Rr\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=\frac{4s^2}{(2s+a)^2}.\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)<\frac{4s^4}{(2s+a)^2}\Rightarrow p_a<\frac{2s^2}{2s+a}\therefore n_a-p_a$$

$$=\frac{n_a^2-p_a^2}{n_a+p_a}\geq\frac{\left(\frac{s}{a}-\frac{s(3s+a)}{(2s+a)^2}\right)(b-c)^2}{s+\frac{2s^2}{2s+a}}\quad(\because n_a^2=s^2-2h_ar_a<s^2\Rightarrow n_a<s)$$

$$=\frac{s(4s^2+sa)(b-c)^2}{a(2s+a)^2.\left(\frac{4s^2+sa}{2s+a}\right)}=\frac{s(b-c)^2}{a(2s+a)}\therefore n_a-p_a\geq\frac{s(b-c)^2}{a(2s+a)}\text{ and so,}$$

$$\frac{s(b-c)^2}{a(2s+a)}\leq n_a-p_a\leq\frac{s|b-c|}{2s+a}\ \forall\Delta ABC,$$

with equality iff Δ ABC is equilateral (QED)

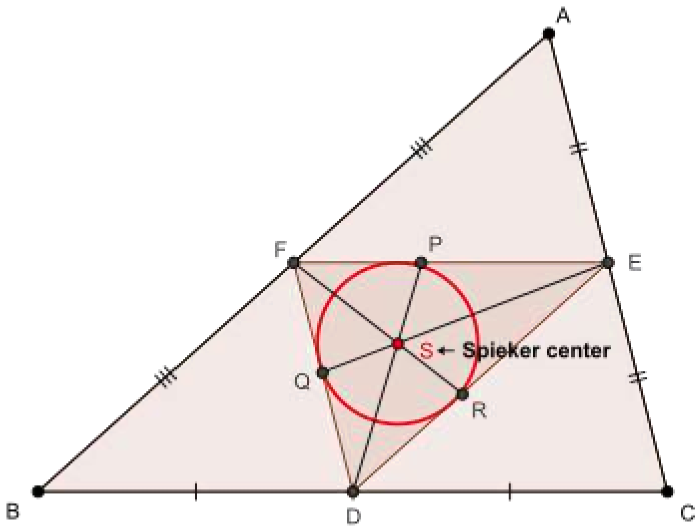
2019.

In any Δ ABC with $p_a\to$ Spieker cevian, the following relationship holds :

$$\frac{2s(b-c)^2}{4s^2-a^2}\leq p_a-w_a\leq\frac{2sa|b-c|}{4s^2-a^2}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and m(∡BAX) = α and m(∡CAX) = β (say) and inradius of Δ DEF = r′(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES},\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a - w_a \geq \frac{2s(b-c)^2}{4s^2-a^2} \Leftrightarrow p_a^2 - w_a^2 \geq$$

$$\frac{4s^2(b-c)^4}{(4s^2-a^2)^2}+\frac{4s.w_a.(b-c)^2}{4s^2-a^2} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} \overset{(\blacksquare)}{\geq} \frac{4s.w_a}{4s^2-a^2}\ (\because (b-c)^2 \geq 0)$$

$$\text{We have : } \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} >$$

$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2a^2}{(4s^2-a^2)^2}$$

$$= \frac{s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2a^2}{(4s^2-a^2)^2} = \frac{8s^2(2s+a)(s-a)}{(4s^2-a^2)^2} > 0$$

$$\therefore (\blacksquare) \Leftrightarrow \frac{(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2(b-c)^2)^2}{(4s^2-a^2)^4}$$

$$\geq \frac{16s^2}{(4s^2-a^2)^2}.\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow \frac{16s^4(b-c)^4-8s^2(b-c)^2(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)+(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)^2}{(4s^2-a^2)^2}$$

$$\geq \frac{16s^2(s(s-a)(2s-a)^2-s(s-a)(b-c)^2)}{(2s-a)^2}$$

$$\Leftrightarrow 16s^4(b-c)^4-8s^2(b-c)^2\left(\begin{matrix}s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\\ -2s(s-a)(2s+a)^2\end{matrix}\right)$$

$$+(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)^2-16s^3(s-a)(4s^2-a^2)^2 \geq 0$$

$$\Leftrightarrow 16s^4(b-c)^4-16s^3(b-c)^2(4s^3-4s^2a+sa^2+a^3)$$

$$+16a^2s^3(4s^3-4s^2a+a^3) \geq 0$$

$$\Leftrightarrow s(b-c)^4-(4s^3-4s^2a+sa^2+a^3)(b-c)^2+a^2(4s^3-4s^2a+a^3) \overset{(\blacksquare\blacksquare)}{\geq} 0$$

and in order to prove $(\blacksquare\blacksquare)$, it suffices to prove :

$$(b-c)^2 \leq \frac{(4s^3-4s^2a+sa^2+a^3)-\sqrt{\delta}}{2s}, \text{ where } \delta =$$

$$(4s^3-4s^2a+sa^2+a^3)^2-4sa^2(4s^3-4s^2a+a^3) \text{ and } \because (b-c)^2 < a^2$$

$$\therefore \text{ it suffices to prove : } 2sa^2 \leq (4s^3-4s^2a+sa^2+a^3)$$

$$-\sqrt{(4s^3-4s^2a+sa^2+a^3)^2-4sa^2(4s^3-4s^2a+a^3)}$$

$$\Leftrightarrow \sqrt{(s-a)^2(4s^2-a^2)^2} \leq 4s^3-4s^2a-sa^2+a^3 = (s-a)(4s^2-a^2) \rightarrow \text{true}$$

$$\Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true } \therefore p_a - w_a \ge \frac{2s(b-c)^2}{4s^2-a^2}$$

$$\text{Again, } p_a - w_a \le \frac{2sa|b-c|}{4s^2-a^2} \Leftrightarrow \frac{s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2}{(4s^2-a^2)^2}.(b-c)^2$$

$$\le \frac{4s^2a^2(b-c)^2}{(4s^2-a^2)^2} + \frac{4sa.w_a.|b-c|}{4s^2-a^2}$$

$$\Leftrightarrow \frac{s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2 - 4s^2a^2}{(4s^2-a^2)^2}.|b-c| \le \frac{4sa.w_a}{4s^2-a^2}$$

$$(\because |b-c| \ge 0) \Leftrightarrow \frac{8s^2(2s+a)(s-a)}{4s^2-a^2}.|b-c| \le 4sa.w_a$$

$$\Leftrightarrow \frac{4s^2(2s+a)^2(s-a)^2}{(4s^2-a^2)^2}.(b-c)^2 \le a^2\left(s(s-a) - \frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow \frac{4s^2(s-a)^2 + a^2s(s-a)}{(2s-a)^2}.(b-c)^2 \le a^2s(s-a)$$

$$\Leftrightarrow \frac{s(s-a)(4s^2-4sa+a^2)}{(2s-a)^2}.(b-c)^2 \le a^2s(s-a)$$

$$\Leftrightarrow s(s-a).(b-c)^2 \le a^2s(s-a) \Leftrightarrow s(s-a)(a^2-(b-c)^2) \overset{?}{\ge} 0 \to \text{true}$$

$$\therefore p_a - w_a \le \frac{2sa|b-c|}{4s^2-a^2} \text{ and so, } \frac{2s(b-c)^2}{4s^2-a^2} \le p_a - w_a \le \frac{2sa|b-c|}{4s^2-a^2}$$

$\forall\, \Delta$ ABC, with equality iff Δ ABC is equilateral (QED)

2020. In ΔABC the following relationship holds:

$$3r \le \frac{\sum r_a \sin A}{\sum \sin A} \le 2R - r$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum r_a \sin A = \sum \frac{F}{s-a}\frac{a}{2R} = \frac{F}{2R}\sum \frac{a}{s-a} = \frac{F}{2R}\cdot\frac{\sum a(s-b)(s-c)}{(s-a)(s-b)(s-c)} =$$

$$= \frac{F}{2R}\cdot\frac{(\sum(as^2 - s(ab+ac) + abc))}{sr^2} =$$

$$= \frac{F}{2R}\cdot\frac{s^2(a+b+c) - 2s(ab+bc+ca) + 3abc}{sr^2} =$$

$$=\frac{2s^3-2s(s^2+r^2+4Rr)+12Rrs}{sr^2}\cdot\frac{F}{2R}=\frac{2sr(2R-r)}{sr^2}\frac{rs}{2R}=(2R-r)\frac{s}{R}\ (1)$$

$$\frac{\sum r_a\sin A}{\sum\sin A}\overset{(1)}{=}(2R-r)\frac{s}{R}\frac{1}{\frac{s}{R}}=2R-r\overset{Euler}{\geq}4r-r=3r$$

Equality holds for $a=b=c$

2021. **In any acute Δ ABC, following relationship holds:**

$$\sin A.\,m_aw_a>F\geq 3\sqrt{3}r^2+\sqrt{2}r(R-2r)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$F\geq 3\sqrt{3}r^2+\sqrt{2}.r(R-2r)\Leftrightarrow s-\sqrt{2}(R-2r)\geq 3\sqrt{3}r$$

$$\Leftrightarrow\left(s-\sqrt{2}(R-2r)\right)^2\geq 27r^2$$

$$\left(\begin{array}{c}\because\Delta\text{ ABC is acute}\Rightarrow\prod_{cyc}\cos A>0\Rightarrow s>2R+r>2R\\ \Rightarrow s-\sqrt{2}(R-2r)>(2-\sqrt{2})R+2\sqrt{2}.r>0\end{array}\right)$$

$$\Leftrightarrow s^2+2(R-2r)^2-2\sqrt{2}.s(R-2r)\geq 27r^2$$

$$\Leftrightarrow s^2+2(R-2r)^2-27r^2\geq 2\sqrt{2}.s(R-2r)\Leftrightarrow(s^2+2(R-2r)^2-27r^2)^2$$

$$\geq 8s^2(R-2r)^2\left(\because s^2-27r^2+2(R-2r)^2\overset{Mitrinovic}{\geq}2(R-2r)^2\geq 0\right)$$

$$\Leftrightarrow s^4-(4R^2-16Rr+70r^2)s^2+4R^4-32R^3r-12R^2r^2+304Rr^3+361r^4\boxed{\overset{(*)}{\geq}}0$$

and $\because(s^2-2R^2-8Rr-3r^2)^2\geq 0\therefore$ **in order to prove** $(*)$, **it suffices to prove:**

LHS of $(*)\geq(s^2-2R^2-8Rr-3r^2)^2$

$$\Leftrightarrow(4R-8r)s^2\boxed{\overset{(**)}{\geq}}8R^3+11R^2r-32Rr^2-44r^3$$

Now, $\because\Delta$ **ABC is acute** $\therefore(4R-8r)s^2\overset{Walker}{\geq}(4R-8r)(2R^2+8Rr+3r^2)\overset{?}{\geq}$ $8R^3+11R^2r-32Rr^2-44r^3\Leftrightarrow 40r^2(R^2-4Rr+4r^2)\overset{?}{\geq}0\Leftrightarrow 40r^2(R-2r)^2\overset{?}{\geq}0$ $\rightarrow$ **true** $\Rightarrow(**)\Rightarrow(*)$ **is true** $\therefore F\geq 3\sqrt{3}r^2+\sqrt{2}.r(R-2r)\ \forall$ *acute* Δ ABC

Again, $\sin A.\,m_aw_a-F\overset{Lascu+A-G}{\geq}\frac{a}{2R}.s(s-a)-\frac{abc}{4R}=\frac{a}{2R}.(2s(s-a)-bc)$

$$=\frac{abc}{2R}\cdot\left(2\cos^2\frac{A}{2}-1\right)>0 \because \Delta\, ABC \text{ is acute} \Rightarrow 0<\frac{A}{2}<\frac{\pi}{4}\Rightarrow \cos\frac{A}{2}>\frac{1}{\sqrt{2}}$$

$$\Rightarrow 2\cos^2\frac{A}{2}-1>0 \therefore \sin A\,.\,m_a w_a > F\ \forall \text{ acute } \Delta\, ABC \text{ and so,}$$

$$\sin A\,.\,m_a w_a > F \geq 3\sqrt{3}r^2+\sqrt{2}.r(R-2r)\ \forall \text{ acute } \Delta\, ABC$$

$$,''=''\ \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

2022. In ΔABC the following relationship holds:

$$-\frac{3(R-2r)}{R}\leq(2\cos A-1)(2\cos B-1)(2\cos C-1)\leq\frac{(R-2r)(16R^2-Rr-8r^2)}{(16R-5r)R^2}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$(2\cos A-1)(2\cos B-1)(2\cos C-1)=$$

$$=-1+2\sum\cos A-4\sum\cos A\cos B+8\cos A\cos B\cos C=$$

$$=-1+2\left(1+\frac{r}{R}\right)-4\frac{s^2+r^2-4R^2}{4R^2}+8\,\frac{s^2-(2R+r)^2}{4R^2}=$$

$$=\frac{-R^2+2R^2+2Rr-s^2-r^2+4R^2+2s^2-2(4R^2+4Rr+r^2)}{R^2}=$$

$$=\frac{s^2-3R^2-6Rr-3r^2}{R^2}\ (1)$$

$$(2\cos A-1)(2\cos B-1)(2\cos C-1)=\frac{s^2-3R^2-6Rr-3r^2}{R^2}\overset{Gerretsen}{\geq}$$

$$\geq\frac{16Rr-5r^2-3R^2-6Rr-3r^2}{R^2}=\frac{-3R^2+10Rr-8r^2}{R^2}\overset{Euler}{\geq}$$

$$\geq\frac{-3R^2+10Rr-8r.\frac{R}{2}}{R^2}=\frac{-3R^2+6Rr}{R^2}=-\frac{3(R-2r)}{R}$$

$$(2\cos A-1)(2\cos B-1)(2\cos C-1)=$$

$$=\frac{s^2-3R^2-6Rr-3r^2}{R^2}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2-3R^2-6Rr-3r^2}{R^2}=$$

$$=\frac{R(R-2r)}{R^2}=\frac{(R-2r)R(16R-5r)}{(16R-5r)R^2}=\frac{(R-2r)(16R^2-5Rr)}{(16R-5r)R^2}=$$

$$=\frac{(R-2r)(16R^2-Rr-4Rr)}{(16R-5r)R^2}\overset{Euler}{\leq}\frac{(R-2r)(16R^2-Rr-4.2r.r)}{(16R-5r)R^2}=$$

$$=\frac{(R-2r)(16R^2-Rr-8r^2)}{(16R-5r)R^2}$$

$$Equality\ holds\ for\ A=B=C$$

2023. **In ΔABC the following relationship holds:**

$$\frac{R^2+6Rr-\sqrt{R(R-2r)^3}}{4R^2}\leq\prod\cos\frac{A-B}{2}\leq\frac{R^2+6Rr+\sqrt{R(R-2r)^3}}{4R^2}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$Blundon's\ inequality:$$

$$2R^2+10Rr-r^2-2\sqrt{R(R-2r)^3}\leq s^2\leq 2R^2+10Rr-r^2+2\sqrt{R(R-2r)^3}\ (1)$$

$$\prod\cos\frac{A-B}{2}=\frac{s^2+r^2+2Rr}{8R^2}\overset{(1)}{\geq}\frac{2R^2+10Rr-r^2-2\sqrt{R(R-2r)^3}+r^2+2Rr}{8R^2}=$$
$$=\frac{2R^2+12Rr-2\sqrt{R(R-2r)^3}}{8R^2}=\frac{R^2+6Rr-\sqrt{R(R-2r)^3}}{4R^2}$$

$$\prod\cos\frac{A-B}{2}=\frac{s^2+r^2+2Rr}{8R^2}\overset{(1)}{\leq}\frac{2R^2+10Rr-r^2+2\sqrt{R(R-2r)^3}+r^2+2Rr}{8R^2}=$$
$$=\frac{2R^2+12Rr+2\sqrt{R(R-2r)^3}}{8R^2}=\frac{R^2+6Rr+\sqrt{R(R-2r)^3}}{4R^2}$$

$$Equality\ holds\ for\ an\ equilateral\ triangle$$

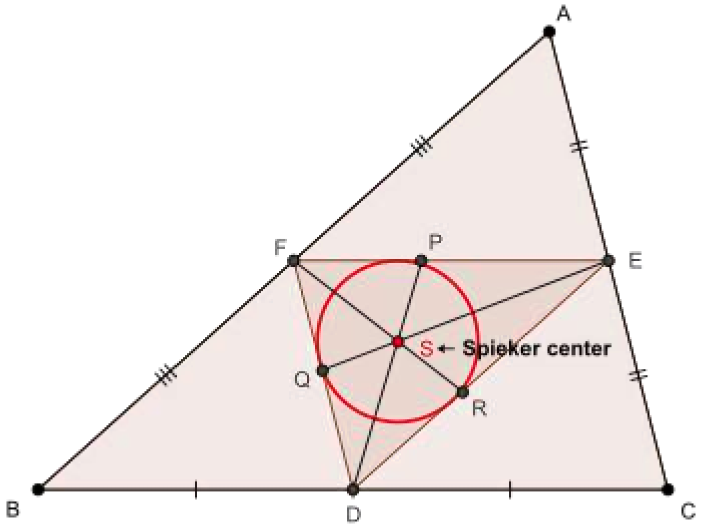
2024. **In any Δ ABC with $p_a, p_b, p_c \rightarrow$**

Spieker cevians, the following relationship holds:

$$\frac{p_a+w_a}{2m_a+\sqrt{p_aw_a}}+\frac{p_b+w_b}{2m_b+\sqrt{p_bw_b}}+\frac{p_c+w_c}{2m_c+\sqrt{p_cw_c}}\leq 2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Let AS produced meet BC } at \text{ X } and \text{ m}(\measuredangle BAX) = \alpha \; and \; m(\measuredangle CAX) = \beta \; (say)$$
$$and \text{ inradius of } \Delta \, DEF = r' (say)$$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$
$$= \frac{\pi}{2} - \frac{A-B}{2} \; and \; m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

$$\text{Via } (1), (2) \; and \text{ using cosine law on } \Delta \, AFS \; and \; \Delta \, AES, \text{ we arrive at}:$$

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$
$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, }\frac{p_a+w_a}{2m_a+\sqrt{p_aw_a}}\le\frac{2}{3}\Leftrightarrow 3(p_a+w_a)-2\sqrt{p_aw_a}\le 4m_a$$

$$\Leftrightarrow 9(p_a+w_a)^2+4p_aw_a-12(p_a+w_a).\sqrt{p_aw_a}\le 16m_a^2 \text{ and } \because$$

$$-12(p_a+w_a).\sqrt{p_aw_a}\overset{A-G}{\le}-24p_aw_a \therefore \text{ it suffices to prove :}$$

$$9(p_a^2+w_a^2)+18p_aw_a+4p_aw_a-24p_aw_a\le 16m_a^2 \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$9\left(2s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)-16\left(s(s-a)+\frac{(b-c)^2}{4}\right)$$

$$\le 2p_aw_a\Leftrightarrow\left(\frac{9s(3s+a)}{(2s+a)^2}-\frac{9s(s-a)}{(2s-a)^2}-4\right).(b-c)^2\le 2p_aw_a-2s(s-a)$$

$$\Leftrightarrow\frac{4s^4-36s^3a+25s^2a^2+9sa^3-2a^4}{(4s^2-a^2)^2}.(b-c)^2\ \boxed{\overset{(\blacksquare)}{\le}}\ p_aw_a-s(s-a)$$

$\boxed{\text{Case 1}}$ $4s^4-36s^3a+25s^2a^2+9sa^3-2a^4\le 0$ *and then* : LHS of $(\blacksquare)\le 0$

$$\le \text{RHS of }(\blacksquare)\left(\begin{array}{c}\because p_a^2=s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\ge m_a^2=s(s-a)+\frac{(b-c)^2}{4}\\ \Rightarrow p_aw_a-s(s-a)\ge m_aw_a-s(s-a)\overset{Lascu+A-G}{\ge}0\end{array}\right)$$

$\boxed{\text{Case 2}}$ $4s^4-36s^3a+25s^2a^2+9sa^3-2a^4>0$

$$\left(\Leftrightarrow t=\frac{s}{a}\ge 8.20584\ (\text{approximately})\right)\text{ and then : }(\blacksquare)\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{4s^4-36s^3a+25s^2a^2+9sa^3-2a^4}{(4s^2-a^2)^2}.(b-c)^2\right)^2\le$$

$$\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow\left(\frac{(4s^4-36s^3a+25s^2a^2+9sa^3-2a^4)^2}{(4s^2-a^2)^4}+\frac{s^2(3s+a)(s-a)}{(4s^2-a^2)^2}\right)(b-c)^4\le$$

$$\left(\begin{array}{c}\frac{s^2(3s+a)(s-a)}{(2s+a)^2}-\frac{s^2(s-a)^2}{(2s-a)^2}\\ -\frac{2s(s-a)(4s^4-36s^3a+25s^2a^2+9sa^3-2a^4)}{(4s^2-a^2)^2}\end{array}\right)(b-c)^2\Leftrightarrow$$

$$\frac{(4s^4-36s^3a+25s^2a^2+9sa^3-2a^4)^2+s^2(3s+a)(s-a)(4s^2-a^2)^2}{(4s^2-a^2)^4}.(b-c)^4$$

$$\le\frac{\left(\begin{array}{c}s^2(3s+a)(s-a)(2s-a)^2-s^2(s-a)^2(2s+a)^2\\ -2s(s-a)(4s^4-36s^3a+25s^2a^2+9sa^3-2a^4)\end{array}\right)}{(4s^2-a^2)^2}.(b-c)^2$$

$$\Leftrightarrow (b-c)^2\left(\frac{4(s-a)\left(\begin{matrix}16s^7-64s^6a+300s^5a^2-128s^4a^3\\-135s^3a^4+13s^2a^5+8sa^6-a^7\end{matrix}\right)}{(4s^2-a^2)^4}.(b-c)^2 - \frac{4sa(s-a)(16s^3-12s^2a-4sa^2+a^3)}{(4s^2-a^2)^2}\right)\le 0$$

$$\Leftrightarrow \left(\begin{matrix}16s^7-64s^6a+300s^5a^2-128s^4a^3-135s^3a^4\\+13s^2a^5+8sa^6-a^7\end{matrix}\right)(b-c)^2 \overset{(\blacksquare\blacksquare)}{\le}$$

$$sa(16s^3-12s^2a-4sa^2+a^3)(4s^2-a^2)^2\ (\because (b-c)^2\ge 0 \text{ and } (s-a)>0)$$

Now, $16s^7-64s^6a+300s^5a^2-128s^4a^3-135s^3a^4+13s^2a^5+8sa^6-a^7$

$$=(s-a)(16s^6-48s^5a+252s^4a^2+124s^3a^3-11s^2a^4+2sa^5+10a^6)+9a^7$$

$$=(s-a)\left(\begin{matrix}(16s^6+225s^4a^2-120s^5a)+27s^4a^2+113s^3a^3\\+11s^2a^3(s-a)+2sa^5+10a^6\end{matrix}\right)+9a^7$$

$$=(s-a)\left(\begin{matrix}(4s^3-15s^2a)^2+72s^5a+27s^4a^2+113s^3a^3\\+11s^2a^3(s-a)+2sa^5+10a^6\end{matrix}\right)+9a^7\overset{s>a}{>}0$$

$$\text{and}\because (b-c)^2<a^2\ \therefore \text{LHS of }(\blacksquare\blacksquare)<$$

$$\left(\begin{matrix}16s^7-64s^6a+300s^5a^2-128s^4a^3-135s^3a^4\\+13s^2a^5+8sa^6-a^7\end{matrix}\right)a^2\overset{?}{<}$$

$$sa(16s^3-12s^2a-4sa^2+a^3)(4s^2-a^2)^2\Leftrightarrow$$

$$256t^8-208t^7-128t^6-188t^5+176t^4+115t^3-17t^2-7t+1\overset{?}{>}0$$

$$\Leftrightarrow (t-1)\left((t-2)\left(\begin{matrix}256t^6+560t^5+1040t^4+1812t^3\\+3532t^2+7087t+14180\end{matrix}\right)+28359\right)\overset{?}{>}0$$

$$\to \text{true} \because t\ge 8.20584\ (\text{approximately})>2\Rightarrow (t-1),(t-2)>0\Rightarrow(\blacksquare\blacksquare)$$

$$\Rightarrow (\blacksquare)\text{ is true (strict inequality) and combining both cases, }(\blacksquare)\text{ is true }\forall\Delta ABC$$

$$\Rightarrow \frac{p_a+w_a}{2m_a+\sqrt{p_aw_a}}\le\frac{2}{3}\text{ and analogs }\forall\Delta ABC\Rightarrow$$

$$\frac{p_a+w_a}{2m_a+\sqrt{p_aw_a}}+\frac{p_b+w_b}{2m_b+\sqrt{p_bw_b}}+\frac{p_c+w_c}{2m_c+\sqrt{p_cw_c}}\le 2\ \forall\Delta ABC,$$

with equality iff ΔABC is equilateral (QED)

2025. In ΔABC the following relationship holds:

$$\frac{3}{4}\le\cos^2 A+\cos^2 B+\cos^2 C\le\frac{33}{8}-\frac{s^2}{2R^2}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$\cos^2 A+\cos^2 B+\cos^2 C=3-\sum\sin^2 A=3-\frac{\sum a^2}{4R^2}\ (1)$$

$$From\ (1)\cos^2 A+\cos^2 B+\cos^2 C=3-\frac{\sum a^2}{4R^2}\overset{Leibniz}{\geq}3-\frac{9R^2}{4R^2}=3-\frac{9}{4}=\frac{3}{4}$$

$$From\ (1)\cos^2 A+\cos^2 B+\cos^2 C=3-\frac{\sum a^2}{4R^2}=3-\frac{2(s^2-r^2-4Rr)}{4R^2}=$$

$$=3+\frac{r^2+4Rr}{2R^2}-\frac{s^2}{2R^2}=3+\frac{1}{2}\left(\frac{r}{R}\right)^2+2\frac{r}{R}-\frac{s^2}{2R^2}\overset{Euler}{\leq}3+\frac{1}{2}\left(\frac{1}{2}\right)^2+2.\frac{1}{2}-\frac{s^2}{2R^2}=$$

$$=4+\frac{1}{8}-\frac{s^2}{2R^2}=\frac{33}{8}-\frac{s^2}{2R^2}$$

Equality holds for an equilateral triangle

2026. In any Δ ABC, the following relationship holds :

$$\left|\sin\frac{A-B}{2}.\sin\frac{B-C}{2}.\sin\frac{C-A}{2}\right|\leq\frac{1}{24\sqrt{3}}.\frac{\sqrt{(s^2-12Rr-3r^2)^3}}{R^2r}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sin^2\frac{A-B}{2}.\sin^2\frac{B-C}{2}.\sin^2\frac{C-A}{2}=\prod_{cyc}\frac{\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}}{\sin^2\frac{A}{2}}=\prod_{cyc}\frac{\frac{(b-c)^2}{16R^2}}{\sin^2\frac{A}{2}}$$

$$=\frac{1}{256.16R^6}.\frac{16R^2}{r^2}.\prod_{cyc}(b-c)^2\therefore\prod_{cyc}\sin^2\frac{B-C}{2}=\frac{1}{256R^4r^2}.\prod_{cyc}(b-c)^2\to(1)$$

$$\text{Now,}\prod_{cyc}(b-c)^2=\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4-2abc\sum_{cyc}a^3-2\sum_{cyc}a^3b^3$$

$$+2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)-6a^2b^2c^2$$

$$=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)-16Rrs^2(s^2-6Rr-3r^2)$$

$$-2\left(\left(\sum_{cyc}ab\right)^3-3abc\prod_{cyc}(b+c)\right)+2abc\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)-6a^2b^2c^2$$

$$=2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-16Rrs^2(s^2-6Rr-3r^2)$$

$$-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)+16Rrs^2(s^2+4Rr+r^2)$$

$$-240R^2r^2s^2 = 4r^2(-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3)$$

$$\therefore \prod_{cyc} \sin^2 \frac{B - C}{2} \overset{via\ (1)}{=} \frac{1}{256R^4r^2} . 4r^2(-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3)$$

$$\overset{?}{\leq} \frac{(s^2 - 12Rr - 3r^2)^3}{3.576R^4r^2} \Leftrightarrow s^4 - (36Rr - 18r^2)s^2 + 81r^2(2R - r)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow s^4 - 2s^2.9r(2R - r) + \left(9r(2R - r)\right)^2 \overset{?}{\geq} 0 \Leftrightarrow \left(s^2 - 9r(2R - r)\right)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore \prod_{cyc} \sin^2 \frac{B - C}{2} \leq \frac{(s^2 - 12Rr - 3r^2)^3}{3.576R^4r^2}$$

$$\Rightarrow \left|\sin\frac{A - B}{2} . \sin\frac{B - C}{2} . \sin\frac{C - A}{2}\right| \leq \frac{1}{24\sqrt{3}} . \frac{\sqrt{(s^2 - 12Rr - 3r^2)^3}}{R^2r}$$

$\forall \Delta\ ABC,'' ='' $ iff $\Delta\ ABC$ is equilateral (QED)

2027. In acute triangle ABC :

$$r_a < \frac{5}{2}R$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$r_a < \frac{5}{2}R \Leftrightarrow s\tan\frac{A}{2} < \frac{5}{2}R \Leftrightarrow \frac{s}{4R}.\tan\frac{A}{2} < \frac{5}{8} \Leftrightarrow \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\tan\frac{A}{2} < \frac{5}{8} \Leftrightarrow$$

$$\Leftrightarrow \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} < \frac{5}{8}.$$

Let $x := \cos\frac{\pi - A}{2} \in \left(0, \frac{\sqrt{2}}{2}\right)$. By AM – GM and Jensen Inequalities, we have

$$\cos\frac{B}{2}\cos\frac{C}{2} \leq \left(\frac{\cos\frac{B}{2} + \cos\frac{C}{2}}{2}\right)^2 = \cos^2\frac{B + C}{4} = \frac{1}{2}\left(1 + \cos\frac{B + C}{2}\right)$$

$$\Rightarrow \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leq \cos\frac{\pi - A}{2}.\frac{1}{2}\left(1 + \cos\frac{\pi - A}{2}\right) = \frac{x(1 + x)}{2} \leq \frac{\sqrt{2}}{4}\left(1 + \frac{\sqrt{2}}{2}\right)$$

$$= \frac{2\sqrt{2} + 2}{8} < \frac{5}{8}.$$

2028.

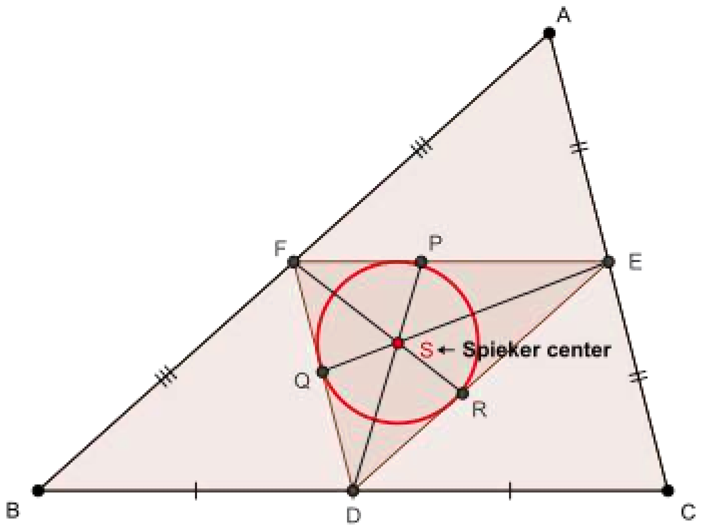
If p_a, p_b, p_c

→ Spieker cevians in Δ ABC, then the following relationship holds :

$$\frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \geq 3 + \frac{2}{3} . \frac{a^2 + b^2 + c^2 - ab - bc - ca}{F}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$
$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$
$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$
$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$
$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$
$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$
$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$
$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$
$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$
$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$
$$\therefore\ \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now}, b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$
$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$
$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$
$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$
$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$
$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$=s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$
$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$
$$\text{Now}, p_a\overset{?}{\geq}h_a+\frac{2}{3}.\frac{(b-c)^2}{a}\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$
$$\overset{?}{\geq}s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}+\frac{4}{9}.\frac{(b-c)^4}{a^2}+\frac{4h_a}{3}.\frac{(b-c)^2}{a}$$
$$\Leftrightarrow\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2}\boxed{\overset{?}{\underset{(\blacksquare)}{\geq}}}\frac{4h_a}{3a}\ (\because (b-c)^2\geq 0)$$

We have : $\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2} \overset{\because a^2>(b-c)^2}{>} \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4}{9}$

$$=\frac{9s(3s+a)a^2+9s(s-a)(2s+a)^2-4a^2(2s+a)^2}{9a^2(2s+a)^2}$$

$$=\frac{4(s-a)(9s^3+9s^2a+5sa^2+a^3)}{9a^2(2s+a)^2} \overset{s>a}{>} 0 \Rightarrow \text{LHS of } (\blacksquare)>0 \therefore (\blacksquare) \Leftrightarrow$$

$$\frac{T^2}{a^4(2s+a)^4}+\frac{16(b-c)^4}{81a^4}-\frac{8T(b-c)^2}{9a^4(2s+a)^2} \overset{?}{\geq} \frac{16}{9a^2}.\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$\left(T=s(3s+a)a^2+s(s-a)(2s+a)^2\right)$$

$$\Leftrightarrow \frac{16(b-c)^4}{81a^4}-\frac{8(b-c)^2}{9a^4}\left(\frac{T}{(2s+a)^2}-2s(s-a)\right)+\frac{T^2}{a^4(2s+a)^4}-\frac{16s(s-a)}{9a^2}$$

$$\Leftrightarrow \left(\frac{4(b-c)^2}{9}\right)^2+\frac{4(b-c)^2}{9}.\frac{4s(2s^3-3sa^2-a^3)}{(2s+a)^2}$$

$$+\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4} \boxed{\overset{?}{\underset{(\blacksquare\blacksquare)}{\geq}}} 0$$

Now, LHS of $(\blacksquare\blacksquare)$ is a quadratic polynomial in $''\frac{4(b-c)^2}{9}''$ whose $\boxed{\text{discriminant}}$

$$=\frac{16s^2(2s^3-3sa^2-a^3)^2}{(2s+a)^4}-4.\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}$$

$$=-\frac{16sa^7}{9(2s+a)^4}.(44t^5-28t^4-49t^3+10t^2+19t+4)\ \left(t=\frac{s}{a}\right)$$

$$=-\frac{16sa^7}{9(2s+a)^4}.(t-1)^2(44t^3+60t^2+27t+4)<0 \Rightarrow \text{LHS of } (\blacksquare\blacksquare)>0$$

$$\Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true} \therefore p_a \geq h_a+\frac{2}{3}.\frac{(b-c)^2}{a} \Rightarrow \frac{p_a}{h_a} \geq 1+\frac{2}{3}.\frac{(b-c)^2}{ah_a}$$

$$=1+\frac{2}{3}.\frac{(b-c)^2}{2F} \Rightarrow \frac{p_a}{h_a} \geq 1+\frac{(b-c)^2}{3F} \text{ and analogs} \Rightarrow$$

$$\sum_{cyc}\frac{p_a}{h_a} \geq 3+\frac{1}{3F}.\sum_{cyc}(b-c)^2 \therefore \frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c} \geq 3+\frac{2}{3}.\frac{a^2+b^2+c^2-ab-bc-ca}{F}$$

$\forall\ \Delta$ ABC, with equality iff Δ ABC is equilateral (QED)

2029. In any Δ ABC, the following relationship holds :

$$\frac{m_a w_a}{r_a h_a} \geq \frac{r}{R-r+\sqrt{R(R-2r)}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

If p $\rightarrow$ semi $-$ perimeter, $\frac{m_a w_a}{r_a h_a} \overset{\text{Lascu}+\text{A-G}}{\geq} \frac{pa(p-a)^2}{2r^2p^2}$

$$=\frac{a(p-a)^2}{2(p-a)(p-b)(p-c)}=\frac{a}{2p}\cdot\frac{p(p-a)}{(p-b)(p-c)}=\frac{a}{2p}\cdot\cot^2\frac{A}{2}$$

$$\overset{?}{\geq}\frac{r}{R-r+\sqrt{R(R-2r)}}\Leftrightarrow R-r+\sqrt{R(R-2r)}\overset{?}{\geq}\frac{2pr}{a}\cdot tan^2\frac{A}{2}$$

$$\Leftrightarrow R-4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+R.\sqrt{1-4sc+c^2}\overset{?}{\geq}$$

$$\frac{2.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}\cdot\frac{s^2}{1-s^2}\left(s=\sin\frac{A}{2}\ and\ c=\cos\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-2s(c-s)+\sqrt{1-4sc+c^2}\overset{?}{\geq}2(s+c)(c-s).\frac{s^2}{1-s^2}$$

$$\Leftrightarrow\sqrt{1-4sc+c^2}\overset{?}{\geq}2(s+c)(c-s).\frac{s^2}{1-s^2}-1+2s(c-s)$$

$$\Leftrightarrow\boxed{\sqrt{1-4sc+c^2}\underset{(*)}{\underbrace{\overset{?}{\geq}}}\frac{2s^2c^2+2c(s-s^3)-(s^2+1)}{1-s^2}}$$

We note that for an equilateral triangle, $c=1$ and $s=\frac{1}{2}$ and then : LHS of $(*)=$ RHS of $(*)=0$ and moreover, if LHS of $(*)\leq 0$, then $(*)$ is evidently true and so, we now consider the case when : $2s^2c^2+2c(s-s^3)-(s^2+1)>0$ and then :

$$(*)\Leftrightarrow 1-4sc+c^2\geq\frac{\left(2s^2c^2+2c(s-s^3)-(s^2+1)\right)^2}{(1-s^2)^2}$$

$$\Leftrightarrow -c^4s-2c^3(1-s^2)+c^2(3s-s^3)+2c(1-s^2)+s^3-2s\geq 0$$

$$\Leftrightarrow -c^4s+(2c-2c^3)(1-s^2)+s^3(1-c^2)+s(3c^2-2)\geq 0$$

$$\Leftrightarrow s(3c^2-2-c^4)+(1-c^2)\left(s^3+2c(1-s^2)\right)\geq 0$$

$$\Leftrightarrow s(1-c^2)(c^2-2)+(1-c^2)\left(s^3+2c(1-s^2)\right)\geq 0$$

$$\Leftrightarrow\boxed{(1-c^2)\left(sc^2-2s+s^3+2c(1-s^2)\right)\overset{(**)}{\geq}0}$$

We have : $c=\cos\frac{B-C}{2}=\frac{b+c}{a}.\sin\frac{A}{2}>\sin\frac{A}{2}=s\Rightarrow c>s$ and so, LHS of $(**)>$

$$(1-c^2)\left(s^3-2s+s^3+2s(1-s^2)\right)\left(\because 1-c^2=1-\cos^2\frac{B-C}{2}>0\right)$$

$=(1-c^2).0\Rightarrow$ LHS of $(**)>0\Rightarrow(**)\Rightarrow(*)$ is true (strict inequality) and hence, combining both scenarios, $(*)$ is true $\forall\,\Delta\,ABC$

$$\therefore\frac{m_aw_a}{r_ah_a}\geq\frac{r}{R-r+\sqrt{R(R-2r)}}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

2030. **In ΔABC the following relationship holds:**

$$\frac{h_a}{r_a+r_b+r_c}>\frac{r}{2R}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$2\sum ab-\sum a^2=$$
$$=2(s^2+r^2+4Rr)-2(s^2-r^2-4Rr)=$$
$$=4r(4R+r)\ (1) a^2+(b+c)^2\overset{AM-GM}{\geq}2a(b+c)\ (2)$$
we need to show
$$\frac{h_a}{r_a+r_b+r_c}>\frac{r}{2R}\ \ or,$$
$$\frac{bc}{(4R+r)2R}>\frac{r}{2R}\ \ or, bc>r(4R+r)\ or,$$
$$4\ bc>4r(4R+r)\ or,$$
$$4bc\overset{(1)}{>}2\sum ab-\sum a^2$$
$$or, \sum a^2+4bc>2\sum ab\ \ or,$$
$$\sum a^2+2bc>2ab+2ac\ or,$$
$$(b^2+2bc+c^2)+a^2>2a(b+c)\ \ or,$$

$$a^2+(b+c)^2>2a(b+c)\ true\ \big(using\ (2)\big)$$

2031. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{w_a}{h_a\cos A}\leq\frac{3R^2}{10r^2-2R^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ \ then\ h_a\leq h_b\leq h_c\ and$$
$$\cos A\leq\cos B\leq\cos C\ \ (acute)\ and\ h_a\cos A\leq h_b\cos B\leq h_c\cos C$$

$$\sum w_a\leq\sum\sqrt{s(s-a)}\overset{CBS}{\leq}\sqrt{3s(3s-a-b-c)}=s\sqrt{3}\overset{Mitrinovic}{\leq}$$
$$\leq\ 3\sqrt{3}\ \frac{R}{2}\sqrt{3}=\frac{9R}{2}\ (1)$$
$$.\sum\tan A=\frac{2sr}{s^2-(2R+r)^2}=\frac{2F}{s^2-(2R+r)^2}\leq Walker's$$
$$\leq\frac{2F}{2R^2+8Rr+3r^2-(2R+r)^2}=$$

$$=\frac{2F}{4Rr+2r^2-2R^2}\overset{Euler}{\le}\frac{2F}{4.2r.r+2r^2-2R^2}\le\frac{2F}{10r^2-2R^2}\ (2)$$

$$\sum_{cyc}\frac{w_a}{h_a\cos A}\le\sum\frac{\sqrt{s(s-a)}}{h_a\cos A}\overset{Chebyshev}{\le}\frac{1}{3}\left(\sum w_a\right)\left(\sum\frac{1}{h_a\cos A}\right)=$$

$$=\frac{1}{3}\left(\sum w_a\right)\left(\sum\frac{a}{2F\cos A}\right)=\frac{1}{3}\left(\sum w_a\right)\left(\sum\frac{2R\sin A}{2F\cos A}\right)=$$

$$=\frac{R}{3F}\left(\sum w_a\right)\left(\sum\tan A\right)\overset{(1)\&(2)}{\le}\frac{R}{3F}\frac{9R}{2}\frac{2F}{10r^2-2R^2}=\frac{3R^2}{10r^2-2R^2}$$

Equality holds for an equilateral triangle

2032. In ΔABC the following relationship holds:

$$\frac{2F}{R}\le(b+c-a)\sin\frac{A}{2}+(c+a-b)\sin\frac{B}{2}+(a+b-c)\sin\frac{C}{2}\le\frac{F}{r}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\sin\frac{A}{2}\overset{Jensen}{\le}3\sin\frac{A+B+C}{6}=3\sin\frac{\pi}{6}=\frac{3}{2}\ (1)$$

$$(b+c-a)\sin\frac{A}{2}+(c+a-b)\sin\frac{B}{2}+(a+b-c)\sin\frac{C}{2}\overset{Chebyshev}{\le}$$

$$\le\frac{1}{3}\left(\sum(b+c-a)\right)\left(\sum\sin\frac{A}{2}\right)\le 2s.\frac{3}{2}.\frac{1}{3}=s=\frac{sr}{r}=\frac{F}{r}$$

$$(b+c-a)\sin\frac{A}{2}+(c+a-b)\sin\frac{B}{2}+(a+b-c)\sin\frac{C}{2}=$$

$$=\sum(b+c-a)\sin\frac{A}{2}=\sum 2(s-a)\sin\frac{A}{2}\overset{AM-GM}{\ge}6\sqrt[3]{\prod(s-a)\sin\frac{A}{2}}=$$

$$=6\left(\frac{sr^2r}{4R}\right)^{\frac{1}{3}}=6\left(\frac{s^3r^3}{4Rs^2}\right)^{\frac{1}{3}}\overset{Mitrinovic}{\ge}6\left(\frac{s^3r^3}{4R\frac{27}{4}R^2}\right)^{\frac{1}{3}}=\frac{6rs}{3R}=\frac{2F}{R}$$

Equality holds for $a=b=c$

2033. In ΔABC the following relationship holds:

$$\frac{\csc A+\csc B+\csc C}{\sqrt[3]{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}}\ge 2\sqrt[3]{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\csc A=\sum\frac{1^2}{\sin A}\overset{Bergstom}{\ge}\frac{(1+1+1)^2}{\sum\sin A}=\frac{9}{\frac{s}{R}}=\frac{9R}{s}=\sqrt[3]{\frac{9^3R^3}{s^3}}\ (1) and$$

$$\sum\tan\frac{A}{2}=\frac{4R+r}{s}\ (2)$$

$$\frac{\csc A+\csc B+\csc C}{\sqrt[3]{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}}=\frac{\sum\csc A}{\sqrt[3]{\sum\tan\frac{A}{2}}}\overset{(1)\&(2)}{\ge}\frac{\sqrt[3]{\frac{9^3R^3}{s^3}}}{\sqrt[3]{\frac{4R+r}{s}}}=$$

$$=\sqrt[3]{\frac{9^3R^3}{s^2(4R+r)}}\overset{Mitrinovic\,\&\,Euler}{\ge}\sqrt[3]{\frac{9^3R^3}{\frac{27R^2}{4}\left(4R+\frac{R}{2}\right)}}=\sqrt[3]{\frac{9^3R^3}{\frac{27R^2}{4}\left(\frac{9R}{2}\right)}}=\sqrt[3]{8\times 3}=2\sqrt[3]{3}$$

$$Equality\ holds\ for\ A=B=C=\frac{\pi}{3}$$

2034. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{\sqrt{r_a+r_b}}{c}\right)^n\ge\frac{3}{R^{\frac{n}{2}}}\qquad n\in\mathbb{N}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$(r_a+r_b)(r_b+r_c)(r_c+r_a)=\sum r_a\sum r_ar_b-r_ar_br_c=(4R+r)s^2-s^2r=4Rs^2\ (1)$$

$$a^2b^2c^2=16R^2r^2s^2\overset{Euler}{\le}16R^2\left(\frac{R}{2}\right)^2s^2=4R^4s^2\quad(2)$$

$$\sum_{cyc}\left(\frac{\sqrt{r_a+r_b}}{c}\right)^n=\sum_{cyc}\left(\sqrt{\frac{r_a+r_b}{c^2}}\right)^n\overset{AM-GM}{\geq}$$

$$\geq 3\left(\sqrt[3]{\sqrt{\frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{a^2b^2c^2}}}\right)^n\overset{(1)\&(2)}{\geq}3\left(\sqrt[3]{\frac{4Rs^2}{4R^4s^2}}\right)^{\frac{n}{2}}=\frac{3}{R^{\frac{n}{2}}}$$

Equality holds for an equilateral triangle

2035. In ΔABC the following relationship holds:

$$\sum\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}}\leq\frac{s\sqrt{2}}{r}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$Let\ \tan\frac{A}{2}=x,\tan\frac{B}{2}=y,\tan\frac{C}{2}=z.$$

We know that in $\Delta\ ABC$:

$$\sum\tan\frac{A}{2}\tan\frac{B}{2}=1\ or,\sum xy=1\ (1)$$

$$xyz=\tan\frac{A}{2},\tan\frac{B}{2}\tan\frac{C}{2}=\frac{r}{s}\ (2)\ and$$

$$\sum x=\sum\tan\frac{A}{2}=\frac{(4R+r)}{s}\overset{s^2\geq 3r(4R+r)}{\leq}\frac{s^2}{3rs}=\frac{s}{3r}\quad(3)$$

$$\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}=xy-x^2y^2=xy(1-xy)\overset{(1)}{=}$$

$$=xy(xy+yz+zx-xy)=xyz(x+y)\ (4)$$

$$\sum\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}}=\sum\frac{x+y}{\sqrt{xy-x^2y^2}}\overset{(4)}{=}$$

$$=\sum\frac{x+y}{\sqrt{xyz(x+y)}}=\frac{1}{\sqrt{xyz}}\sum\sqrt{x+y}\overset{CBS}{\leq}$$

$$\leq\frac{1}{\sqrt{xyz}}\sqrt{3(2x+2y+2z)}=\frac{1}{\sqrt{xyz}}\sqrt{6(x+y+z)}\overset{(2)\&(3)}{\leq}\sqrt{\frac{s}{r}}\sqrt{6\frac{s}{3r}}=\frac{s\sqrt{2}}{r}$$

Equality holds for an equilateral triangle

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{tan\frac{A}{2}+tan\frac{B}{2}}{\sqrt{tan\frac{A}{2}tan\frac{B}{2}-tan^2\frac{A}{2}tan^2\frac{B}{2}}}=\sum_{cyc}\frac{\frac{r_a+r_b}{p}}{\sqrt{\frac{r_ar_b}{p^2}-\frac{r_a^2r_b^2}{p^4}}}=\sum_{cyc}\frac{\frac{4R}{p}\cdot\frac{p(p-c)}{ab}}{\sqrt{\frac{p(p-c)}{p^2}-\frac{p^2(p-c)^2}{p^4}}}=$$

$$=\sum_{cyc}\frac{\frac{4R}{p}\cdot\frac{pc(p-c)}{4Rrp}}{\sqrt{\frac{p(p-c)}{p^2}\cdot\left(1-\frac{p(p-c)}{p^2}\right)}}=\sum_{cyc}\frac{\frac{c(p-c)}{rp}}{\sqrt{\frac{p(p-c)}{p^2}\cdot\frac{pc}{p^2}}}=\sum_{cyc}\frac{\frac{c(p-c)}{rp}\cdot p}{\sqrt{c(p-c)}}=\frac{1}{r}\cdot\sum_{cyc}\sqrt{c(p-c)}\le$$

$$\stackrel{CBS}{\le}\frac{1}{r}\cdot\sqrt{\sum_{cyc}c}\cdot\sqrt{\sum_{cyc}(p-c)}=\frac{1}{r}\cdot\sqrt{2p.p}\therefore\sum_{cyc}\frac{tan\frac{A}{2}+tan\frac{B}{2}}{\sqrt{tan\frac{A}{2}tan\frac{B}{2}-tan^2\frac{A}{2}tan^2\frac{B}{2}}}\le\frac{p.\sqrt{2}}{r}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

2036. In any $\Delta\,ABC$, the following relationship holds :

$$3\le\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}\le 3\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}=\sum_{cyc}\frac{bc}{(2R\sin A)a}=\sum_{cyc}\frac{bc}{a^2}=\frac{1}{16R^2r^2s^2}\cdot\sum_{cyc}b^3c^3$$

$$=\frac{(s^2+4Rr+r^2)^3-3(4Rrs)\left(2s(s^2+2Rr+r^2)\right)}{16R^2r^2s^2}\stackrel{?}{\le}3\left(\frac{R}{2r}\right)^3$$

$$\Leftrightarrow rs^6-(12Rr^2-3r^3)s^4-\left(6R^5-3r^5\right)s^2+r^4(4R+r)^3\boxed{\underset{(*)}{\stackrel{?}{\le}}}0$$

$$\text{Now, LHS of }(*)\stackrel{Gerretsen}{\le}\left(r(4R^2+4Rr+3r^2)-(12Rr^2-3r^3)\right)s^4$$
$$-\left(6R^5-3r^5\right)s^2+r^4(4R+r)^3$$
$$=r(4R^2-8Rr+6r^2)s^4-\left(6R^5-3r^5\right)s^2+r^4(4R+r)^3$$
$$\stackrel{Gerretsen}{\le}\left(r(4R^2-8Rr+6r^2)(4R^2+4Rr+3r^2)-\left(6R^5-3r^5\right)\right)s^2+r^4(4R+r)^3$$
$$\stackrel{?}{\le}0\Leftrightarrow\left(6R^5-16R^4r+16R^3r^2-4R^2r^3-21r^5\right)s^2\boxed{\underset{(**)}{\stackrel{?}{\ge}}}r^4(4R+r)^3$$

$$\text{Now, } 6R^5-16R^4r+16R^3r^2-4R^2r^3-21r^5=$$
$$(R-2r)(4R^4+2R^3(R-2r)+8R^2r^2+12Rr^3+24r^4)+27r^5\overset{\text{Euler}}{\geq}27r^5>0$$
$$\therefore \text{LHS of } (**)\overset{\text{Gerretsen}}{\geq}(6R^5-16R^4r+16R^3r^2-4R^2r^3-21r^5)(16Rr-5r^2)\overset{?}{\geq}$$
$$r^4(4R+r)^3\Leftrightarrow 48t^6-143t^5+168t^4-104t^3-14t^2-174t+52\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left(24t^5+24t^4(t-2)+t^4+74t^3+44t^2+61t+13(t-2)\right)\overset{?}{\geq}0\to\text{true}$$
$$\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\Rightarrow(*)\text{ is true }\therefore\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}\leq 3\left(\frac{R}{2r}\right)^3$$
$$\text{Again, }\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}=$$
$$\frac{(s^2+4Rr+r^2)^3-3(4Rrs)\left(2s(s^2+2Rr+r^2)\right)}{16R^2r^2s^2}\overset{?}{\geq}3$$
$$\Leftrightarrow s^6-(12Rr-3r^2)s^4-r^2s^2(48R^2-3r^2)+r^3(4R+r)^3\ \boxed{\underset{(***)}{\overset{?}{\geq}}}\ 0$$
$$\text{Also, LHS of } (***)\overset{\text{Gerretsen}}{\geq}(4Rr-2r^2)s^4-r^2s^2(48R^2-3r^2)+r^3(4R+r)^3$$
$$\overset{\text{Gerretsen}}{\geq}\left((4Rr-2r^2)(16Rr-5r^2)-r^2(48R^2-3r^2)\right)s^2+r^3(4R+r)^3\overset{?}{\geq}0$$
$$\Leftrightarrow(16R^2-52Rr+13r^2)s^2+r(4R+r)^3\ \boxed{\underset{(****)}{\overset{?}{\geq}}}\ 0$$

$\boxed{\text{Case 1}}$ $16R^2-52Rr+13r^2\geq 0$ and then : LHS of $(****)\geq r(4R+r)^3>0$
$\Rightarrow(****)$ is true (strict inequality)

$\boxed{\text{Case 2}}$ $16R^2-52Rr+13r^2<0$ and then : LHS of $(****)\overset{\text{Gerretsen}}{\geq}$
$$(16R^2-52Rr+13r^2)(4R^2+4Rr+3r^2)+r(4R+r)^3\overset{?}{\geq}0$$
$$\Leftrightarrow 16t^4-20t^3-15t^2-23t+10\overset{?}{\geq}0$$
$$\Leftrightarrow(t-2)(16t^3+12t^2+3(t-2)+6t+1)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$
$\Rightarrow(****)$ is true and so, combining both cases, $(****)\Rightarrow(***)$ is true $\forall\,\Delta ABC$

$$\therefore\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}\geq 3\text{ and so, }3\leq\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}\leq 3\left(\frac{R}{2r}\right)^3$$

$\forall\,\Delta ABC$, " = " iff ΔABC is equilateral (QED)

2037. In any ΔABC, the following relationship holds :

$$-1+19\left(\frac{r}{R}\right)^4 \le \sum_{cyc}\cos^2 B\cos^2 C \le \frac{137}{32}-\frac{131}{2}\left(\frac{r}{R}\right)^4$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\cos^2 B\cos^2 C = \sum_{cyc}\left(\left(1-\frac{b^2}{4R^2}\right)\left(1-\frac{c^2}{4R^2}\right)\right) =$$

$$= 3-\sum_{cyc}\frac{b^2+c^2}{4R^2}+\sum_{cyc}\frac{b^2c^2}{16R^4} \overset{\text{Goldstone}}{\le} 3-\frac{s^2-4Rr-r^2}{R^2}+\frac{4R^2s^2}{16R^4} =$$

$$= \frac{12R^2-4s^2+16Rr+4r^2+s^2}{4R^2} \le$$

$$\overset{\text{Gerretsen}}{\le} \frac{12R^2+16Rr+4r^2-3(16Rr-5r^2)}{4R^2} \overset{?}{\le} \frac{137R^4-16.131r^4}{32R^4}$$

$$\Leftrightarrow 41t^4+256t^3-152t^2-2096 \overset{?}{\ge} 0 \quad \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(41t^3+338t^2+524t+1048) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \sum_{cyc}\cos^2 B\cos^2 C \le \frac{137}{32}-\frac{131}{2}\left(\frac{r}{R}\right)^4$$

$$\text{Now, }\left(\sum_{cyc} ab\right)^2 - 24Rrs^2 =$$

$$= s^4+(4Rr+r^2)^2+2(4Rr+r^2)s^2-24Rrs^2 \overset{\text{Gerretsen}}{\ge}$$

$$\ge (16Rr-5r^2)s^2+2(4Rr+r^2)s^2-24Rrs^2+(4Rr+r^2)^2 = r^2((4R+r)^2-3s^2)$$

$$\overset{\text{Doucet or Trucht}}{\ge} 0 \Rightarrow \sum_{cyc} b^2c^2 \ge \frac{1}{3}\left(\sum_{cyc} ab\right)^2 \ge \frac{24Rrs^2}{3} \Rightarrow \sum_{cyc} b^2c^2 \ge 8Rrs^2 \to (1)$$

$$\therefore \sum_{cyc}\cos^2 B\cos^2 C = 3-\frac{s^2-4Rr-r^2}{R^2}+\sum_{cyc}\frac{b^2c^2}{16R^4} \overset{\text{via }(1)}{\ge}$$

$$\ge 3-\frac{s^2-4Rr-r^2}{R^2}+\frac{8Rrs^2}{16R^4} = \frac{6R^4-2R^2(s^2-4Rr-r^2)+Rrs^2}{2R^4} =$$

$$= \frac{6R^4+2R^2(4Rr+r^2)-(2R^2-Rr)s^2}{2R^4} =$$

$$\overset{\text{Gerretsen}}{\ge} \frac{6R^4+2R^2(4Rr+r^2)-(2R^2-Rr)(4R^2+4Rr+3r^2)}{2R^4} \overset{?}{\ge} -1+19\left(\frac{r}{R}\right)^4$$

$$=\frac{19r^4-R^4}{R^4}\Leftrightarrow 4t^3+3t-38\overset{?}{\geq}0\Leftrightarrow(t-2)(4t^2+8t+19)\overset{?}{\geq}0\to\text{true}$$

$$\because t\overset{\text{Euler}}{\geq}2\therefore\sum_{cyc}\cos^2B\cos^2C\geq-1+19\left(\frac{r}{R}\right)^4\ \text{and so},-1+19\left(\frac{r}{R}\right)^4\leq$$

$$\sum_{cyc}\cos^2B\cos^2C\leq\frac{137}{32}-\frac{131}{2}\left(\frac{r}{R}\right)^4\ \forall\,\Delta\,ABC,''=''\ \text{iff }\Delta\text{ ABC is equilateral (QED)}$$

2038. In ΔABC holds:

$$\sum_{cyc}\frac{r_a(2r_a-3r)}{{r_a}^2+18r^2}\geq 1$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{r_a(2r_a-3r)}{{r_a}^2+18r^2}=\sum_{cyc}\left(2-\frac{3r(r_a+12r)}{({r_a}^2+9r^2)+9r^2}\right)\overset{AM-GM}{\geq}\sum_{cyc}\left(2-\frac{3r(r_a+12r)}{6rr_a+9r^2}\right)=$$

$$=\sum_{cyc}\left(2-\frac{r_a+12r}{2r_a+3r}\right)=\sum_{cyc}\left(\frac{3}{2}-\frac{21r}{2(2r_a+3r)}\right)\overset{CBS}{\geq}\sum_{cyc}\left[\frac{3}{2}-\frac{21r}{2.9}\left(\frac{2}{r_a}+\frac{1}{3r}\right)\right]=$$

$$=\sum_{cyc}\left(\frac{10}{9}-\frac{7r}{3r_a}\right)=\frac{10}{3}-\frac{7r}{3}.\sum_{cyc}\frac{1}{r_a}=\frac{10}{3}-\frac{7r}{3}.\frac{1}{r}=1.$$

Equality holds iff ΔABC is equilateral.

2039. In ΔABC holds:

$$\sum_{cyc}\frac{h_a(2h_a-3r)}{{h_a}^2+18r^2}\geq 1$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{h_a(2h_a-3r)}{{h_a}^2+18r^2}=\sum_{cyc}\left(2-\frac{3r(h_a+12r)}{({h_a}^2+9r^2)+9r^2}\right)\overset{AM-GM}{\geq}\sum_{cyc}\left(2-\frac{3r(h_a+12r)}{6rh_a+9r^2}\right)=$$

$$=\sum_{cyc}\left(2-\frac{h_a+12r}{2h_a+3r}\right)=\sum_{cyc}\left(\frac{3}{2}-\frac{21r}{2(2h_a+3r)}\right)\overset{CBS}{\underset{\approx}{\geq}}\sum_{cyc}\left[\frac{3}{2}-\frac{21r}{2.9}\left(\frac{2}{h_a}+\frac{1}{3r}\right)\right]=$$

$$=\sum_{cyc}\left(\frac{10}{9}-\frac{7r}{3h_a}\right)=\frac{10}{3}-\frac{7r}{3}.\sum_{cyc}\frac{1}{h_a}=\frac{10}{3}-\frac{7r}{3}.\frac{1}{r}=1.$$

Equality holds iff ΔABC is equilateral.

2040. In ΔABC the following relationship holds:

$$\sum a\sqrt{\frac{b^2+c^2}{2}}\leq\frac{9R^4}{4r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum a\sqrt{\frac{b^2+c^2}{2}}\overset{CBS}{\leq}\sqrt{\left(\sum a^2\right)\left(\sum\frac{b^2+c^2}{2}\right)}=\left(\sum a^2\right)\overset{Leibniz}{\leq}$$

$$\leq 9R^2=\frac{9R^4}{R^2}\overset{Euler}{\leq}\frac{9R^4}{(2r)^2}=\frac{9R^4}{4r^2}$$

$$Equality\ holds\ for\ a=b=c$$

2041. In ΔABC the following relationship holds:

$$\sum\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}}\leq\frac{p\sqrt{2}}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}}=\left(\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}.\tan\frac{B}{2}}\right)^{\frac{1}{2}}.\left(\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}.\tan\frac{B}{2}}\right)^{\frac{1}{2}}=$$

$$=\left(\frac{\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}.\cos\frac{B}{2}}}{\tan\frac{A}{2}.\tan\frac{B}{2}}\right)^{\frac{1}{2}}.(\tan\frac{A}{2}+\tan\frac{B}{2})^{\frac{1}{2}}=\left(\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}.\sin\frac{B}{2}}\right)^{\frac{1}{2}}.\left(ctg\frac{C}{2}\right)^{\frac{1}{2}}=$$

$$=\frac{\cos\frac{C}{2}}{(\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2})^{\frac{1}{2}}}=\frac{\cos\frac{C}{2}}{\left(\frac{r}{4R}\right)^{\frac{1}{2}}}=\left(\frac{4R}{r}\right)^{\frac{1}{2}}.\cos\frac{C}{2}=\frac{\sqrt{4Rr}}{r}.\cos\frac{C}{2}=$$

$$=\frac{1}{r}.\sqrt{4Rr}.\cos\frac{C}{2}=\frac{1}{r}\sqrt{\frac{abc}{p}}.\sqrt{\frac{p(p-c)}{ab}}=\frac{1}{r}\sqrt{c(p-c)}$$

$$\sum\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}}=\frac{1}{r}.\sum\sqrt{c(p-c)}\overset{CBS}{\leq}$$

$$\leq\frac{1}{r}.\sqrt{\sum c}.\sqrt{\sum(p-c)}=\frac{1}{r}.\sqrt{2p}.\sqrt{p}=\frac{p\sqrt{2}}{r}$$

Equality holds for $a=b=c$.

2042. **In ΔABC the following relationship holds:**

$$\sum\left(\frac{AI}{b+c}\right)^2\geq\frac{r^2}{R^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$AI=\frac{r}{\sin\frac{A}{2}},BI=\frac{r}{\sin\frac{B}{2}},CI=\frac{r}{\sin\frac{C}{2}}\ \ and$$

$$AI\cdot BI\cdot CI=\frac{r^3}{\prod\sin\frac{A}{2}}=r^3\frac{4R}{r}\overset{Euler}{\geq}r^2.4.2r=8r^3=(2r)^3\ \ (1)$$

$$\sqrt[3]{(a+b)(b+c)(c+a)}\overset{AM-GM}{\leq}\frac{2(a+b+c)}{3}=\frac{4s}{3}\overset{Mitrinovic}{\leq}\frac{4}{3}\cdot\frac{3\sqrt{3}R}{2}=2\sqrt{3}R\ \ (2)$$

$$\sum\left(\frac{AI}{b+c}\right)^2\overset{AM-GM}{\geq}3\left(\sqrt[3]{\frac{AI\cdot BI\cdot CI}{(a+b)(b+c)(c+a)}}\right)^2\overset{(1)\&(2)}{\geq}3.\frac{((2r)^3)^{\frac{2}{3}}}{\left(2\sqrt{3}R\right)^2}=\frac{12r^2}{12R^2}=\frac{r^2}{R^2}$$

Equality holds for an equilateral triangle

2043. **In ΔABC the following relationship holds:**

$$\frac{\tan A+\tan B+\tan C}{\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}+\cot\frac{B}{2}\cot\frac{C}{2}+\cot\frac{C}{2}\cot\frac{A}{2}}}\geq\sqrt[6]{243}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\textit{For acute triangle } \cos(A-B)\leq 1\ (1)$$

$$(\tan A+\tan B)=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B}=$$

$$=\frac{2\sin(A+B)}{2\cos A\cos B}\overset{A+B+C=\pi}{=}\frac{2\sin C}{\cos(A+B)+\cos(A-B)}\overset{(1)}{\geq}\frac{2\sin C}{\cos(A+B)+1}=$$

$$\overset{A+B+C=\pi}{=}\frac{2\sin C}{1-\cos C}=\frac{4\sin\frac{C}{2}\cos\frac{C}{2}}{2\sin^2\frac{C}{2}}=2\cot\frac{C}{2},$$

$$\textit{similarly}, (\tan C+\tan B)\geq 2\cot\frac{A}{2},\quad (\tan C+\tan A)\geq 2\cot\frac{B}{2}$$

$$\textit{Using above result we get}\sum\tan A\geq\sum\cot\frac{A}{2}\ \ (3)$$

$$\sum\cot\frac{A}{2}\cot\frac{B}{2}\overset{\forall x\,y,z>0\ 3\sum xy\leq(\sum x)^2}{\leq}\frac{\left(\sum\cot\frac{A}{2}\right)^2}{3}\ (4)$$

$$:\frac{\tan A+\tan B+\tan C}{\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}+\cot\frac{B}{2}\cot\frac{C}{2}+\cot\frac{C}{2}\cot\frac{A}{2}}}\overset{(4)\&(3)}{\geq}\frac{\sum\cot\frac{A}{2}}{\sqrt[3]{\frac{\left(\sum\cot\frac{A}{2}\right)^2}{3}}}=\sqrt[3]{3}\left(\sum\cot\frac{A}{2}\right)^{\frac{1}{3}}=$$

$$=\sqrt[3]{3}\left(\frac{s}{r}\right)^{\frac{1}{3}}\overset{Mitrinovic}{\geq}\sqrt[3]{3}(3\sqrt{3})^{\frac{1}{3}}=3^{\frac{1}{2}}.\sqrt[3]{3}=\sqrt[6]{3^5}=\sqrt[6]{243}$$

$$Equality\ holds\ for\ A=B=C=\frac{\pi}{3}$$

2044. In ΔABC the following relationship holds:

$$\frac{1}{R}+\frac{1}{r}\geq\frac{9\sqrt{3}}{2s}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$We\ need\ to\ show\ \frac{1}{R}+\frac{1}{r}\geq\frac{9\sqrt{3}}{2s}\ \ or\ 2s(R+r)\geq 9\sqrt{3}Rr\ or$$

$$4s^2(R+r)^2\geq 243\ R^2r^2$$

$$4(16Rr-5r^2)(R+r)^2\geq 243R^2r^2(Gerretsen)\ or$$

$$4\left(\frac{16R}{r}-5\right)\left(\frac{R}{r}+1\right)^2\geq 243\ \left(\frac{R}{r}\right)^2\ \ or$$

$$4(16x-5)(x+1)^2\geq 243x^2\left(\frac{R}{r}=x\geq 2\ Euler\right)\ or$$

$$4(16x-5)(x^2+2x+1)\geq 243x^2$$

$$4(16x^3+27x^2+6x-5)\geq 243x^2\ or$$

$$64x^3-135x^2+24x-20\geq 0\ or$$

$$(x-2)(64x^2-7x+10)\geq 0\ true\ as\ x\geq 2$$

$$Equality\ holds\ \ for\ an\ \ equilateral\ triangle$$

2045. In ΔABC the following relationship holds:

$$\cot A+\cot B+\cot C+\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\geq 2(\csc A+\csc B+\csc C)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$We\ know\ that\ in\ \Delta ABC:\sum\tan A\tan B\ =1+\frac{4R^2}{s^2-(2R+r)^2}$$

$$\prod \tan A = \frac{2sr}{s^2 - (2R+r)^2}$$

$$\sum \sin A \sin B = \frac{s^2 + r^2 + 4Rr}{4R^2}, \quad \prod \sin A = \frac{sr}{2R^2}$$

Using the above result we get:

$$\sum \csc A = \sum \frac{1}{\sin A} = \frac{\sum \sin A \sin B}{\prod \sin A} = \frac{s^2 + r^2 + 4Rr}{2sr} \text{ and}$$

$$\sum \cot A = \sum \frac{1}{\tan A} = \frac{\sum \tan A \tan B}{\prod \tan A} = \frac{s^2 - (2R+r)^2 + 4R^2}{2sr},$$

$$\prod \cot \frac{A}{2} = \frac{s}{r}$$

We need to show:

$$\cot A + \cot B + \cot C + \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \geq 2(\csc A + \csc B + \csc C)$$

$$\frac{s^2 - (2R+r)^2 + 4R^2}{2sr} + \frac{s}{r} \geq \frac{s^2 + r^2 + 4Rr}{2sr} \text{ or}$$

$$s^2 - (2R+r)^2 + 4R^2 + 2s^2 - 2s^2 - 2r^2 - 8Rr \geq 0 \text{ or}$$

$$16Rr - 5r^2 - 4R^2 - 4Rr - r^2 + 4R^2 - 2r^2 - 8Rr \geq 0 \ (\text{Gerretsen}) \text{ or}$$

$$4Rr - 8r^2 \geq 0 \text{ or } R \geq 2r \text{ true Euler}$$

Equality holds for $A = B = C$

2046. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{4m_b^2 - b^2}{4m_a^2 - a^2} = \sum_{cyc} \frac{tanA}{tanB}$$

Proposed by Ertan Yildirim-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{4m_b^2-b^2}{4m_a^2-a^2}=\sum_{cyc}\frac{2(a^2+c^2)-b^2-b^2}{2(b^2+c^2)-a^2-a^2}=$$

$$=\sum_{cyc}\frac{a^2+c^2-b^2}{b^2+c^2-a^2}=\sum_{cyc}\frac{2accosB}{2bccosA}=\sum_{cyc}\frac{acosB}{bcosA}=$$

$$=\sum_{cyc}\frac{2RsinAcosB}{2RsinbcosA}=\sum_{cyc}\frac{\frac{sinA}{cosA}}{\frac{sinB}{cosB}}=\sum_{cyc}\frac{tanA}{tanB}$$

2047. In ΔABC the following relationship holds:

$$\frac{m_a}{a}+\frac{w_b}{b}+\frac{h_c}{c}\ge\frac{3\sqrt{3}r}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{m_a}{a}+\frac{w_b}{b}+\frac{h_c}{c}\ge\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}=\frac{ah_a}{a^2}+\frac{bh_b}{b^2}+\frac{ch_c}{c^2}=$$

$$=\frac{2F}{a^2}+\frac{2F}{b^2}+\frac{2F}{c^2}=2F\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=2F\left(\frac{1^3}{a^2}+\frac{1^3}{b^2}+\frac{1^3}{c^2}\right)\ge$$

$$\overset{RADON}{\geq}2F\cdot\frac{(1+1+1)^3}{(a+b+c)^2}=\frac{54F}{4s^2}=\frac{27rs}{2s^2}=\frac{27r}{2s}\ge$$

$$\overset{MITRINOVIC}{\geq}\frac{27r}{2\cdot\frac{3\sqrt{3}R}{2}}=\frac{9r}{\sqrt{3}R}=\frac{3\sqrt{3}r}{R}$$

Equality holds for $a=b=c$.

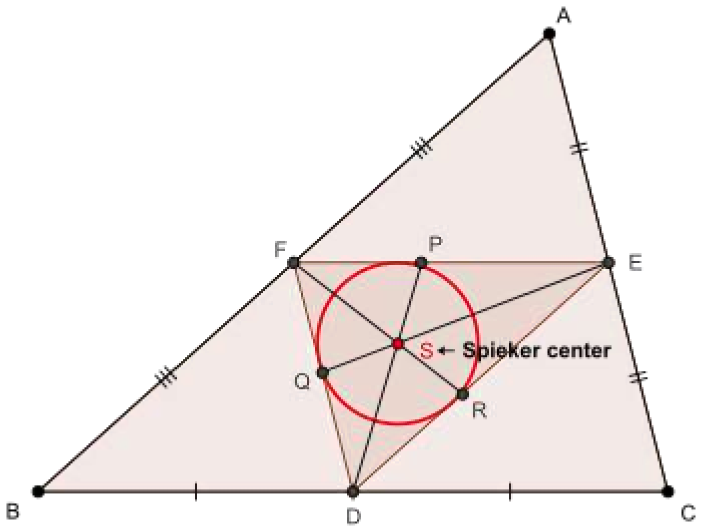
2048. In any Δ ABC with $p_a, p_b, p_c \to$

Spieker cevians, the following relationship holds :

$$p_a+p_b+p_c\ge m_a+m_b+m_c+\frac{s(R-2r)}{10R}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say)
and inradius of Δ DEF $= r'(say)$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we $arrive$ at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\ \frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, } p_a-m_a\ge\frac{(b-c)^2}{2(2s+a)}\Leftrightarrow p_a^2\ge m_a^2+\frac{(b-c)^4}{4(2s+a)^2}+\frac{m_a.(b-c)^2}{2s+a}$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow}\ \frac{(b-c)^2(8s^2-a^2-(b-c)^2)}{4(2s+a)^2}\overset{(\blacksquare)}{\ge}\frac{m_a.(b-c)^2}{2s+a}$$

$$\text{We note that}: 8s^2-a^2-(b-c)^2>8s^2-2a^2>0\ \therefore(\blacksquare)\Leftrightarrow$$

$$\frac{(8s^2-a^2)^2+(b-c)^4-2(8s^2-a^2)(b-c)^2}{16(2s+a)^2}$$

$$\ge\left(s(s-a)+\frac{(b-c)^2}{4}\right)\ (\because (b-c)^2\ge 0)$$

$$\Leftrightarrow (8s^2-a^2)^2-16s(s-a)(2s+a)^2+(b-c)^4$$
$$-2(b-c)^2(8s^2-a^2+2(2s+a)^2)\ge 0$$

$$\Leftrightarrow (b-c)^4-2(4s+a)^2(b-c)^2+a^2(32s^2+16sa+a^2)\overset{(\blacksquare\blacksquare)}{\ge}0$$

Now, $(\blacksquare\blacksquare)$ is a quadrilateral in $(b-c)^2$ with discriminant $=$
$$4(4s+a)^4-4a^2(32s^2+16sa+a^2)=256s^2(2s+a)^2$$

$\therefore$ in order to prove $(\blacksquare\blacksquare)$, it suffices to prove :

$$(b-c)^2\le\frac{2(4s+a)^2-16s(2s+a)}{2}=a^2\to\text{true}\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true}$$

$$\therefore p_a - m_a \geq \frac{(b-c)^2}{2(2s+a)} \text{ and analogs}$$

$$\therefore p_a + p_b + p_c \overset{(\blacksquare\blacksquare\blacksquare)}{\geq} m_a + m_b + m_c + \sum_{cyc} \frac{(b-c)^2}{2(2s+a)}$$

$$\text{We have}: \sum_{cyc} \frac{(b-c)^2}{2(2s+a)} \overset{(m)}{=} \frac{1}{4s(9s^2+6Rr+r^2)} \cdot \sum_{cyc} \left((b-c)^2(8s^2-2sa+bc)\right)$$

$$\text{and}, \sum_{cyc} \left((b-c)^2(8s^2-2sa+bc)\right)$$

$$= 8s^2 \sum_{cyc} (b-c)^2 - 2s. \sum_{cyc} a(b^2+c^2-2bc) + \sum_{cyc} \left(bc\left(\sum_{cyc} a^2 - a^2 - 2bc\right)\right)$$

$$= 16s^2(s^2-12Rr-3r^2) - 2s.\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) +$$

$$2(s^2-4Rr-r^2)(s^2+4Rr+r^2) - 2((s^2+4Rr+r^2)^2 - 16Rrs^2) - 8Rrs^2$$

$$= 16s^2(s^2-12Rr-3r^2) - 4s^2(s^2-14Rr+r^2) +$$

$$2(s^2-4Rr-r^2)(s^2+4Rr+r^2) - 2((s^2+4Rr+r^2)^2 - 16Rrs^2) - 8Rrs^2$$

$$\overset{\text{via }(m)}{\Leftrightarrow} \sum_{cyc} \frac{(b-c)^2}{2(2s+a)} =$$

$$\frac{\begin{pmatrix} 16s^2(s^2-12Rr-3r^2) - 4s^2(s^2-14Rr+r^2) + 2(s^2-4Rr-r^2)(s^2+4Rr+r^2) - \\ 2((s^2+4Rr+r^2)^2 - 16Rrs^2) - 8Rrs^2 \end{pmatrix}}{4s(9s^2+6Rr+r^2)}$$

$$\overset{?}{\geq} \frac{s(R-2r)}{10R} \Leftrightarrow$$

$$(21R+18r)s^4 - rs^2(326R^2+129Rr-2r^2) - 10Rr^2(4R+r)^2 \underset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{\overset{?}{\geq}} 0 \text{ and } \because$$

$$(21R+18r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (\blacksquare\blacksquare\blacksquare\blacksquare),$$

$$\text{it suffices to prove : LHS of } (\blacksquare\blacksquare\blacksquare\blacksquare) \geq (21R+18r)(s^2-16Rr+5r^2)^2 \Leftrightarrow$$

$$(346R^2+237Rr-178r^2)s^2 \overset{(\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare)}{\geq} r(5536R^3+1328R^2r-2345Rr^2+450r^3)$$

$$\text{Finally}, (346R^2+237Rr-178r^2)s^2 \overset{\text{Gerretsen}}{\geq} \begin{pmatrix} 346R^2+237Rr \\ -178r^2 \end{pmatrix}(16Rr-5r^2)$$

$$\overset{?}{\geq} r(5536R^3+1328R^2r-2345Rr^2+450r^3)$$

$$\Leftrightarrow 2r^3(367R^2-844Rr+220r^2) \overset{?}{\geq} 0 \Leftrightarrow 2r^3(R-2r)(367R-110r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (■■■■■) \Rightarrow (■■■■) \text{ is true} \therefore \sum_{cyc} \frac{(b-c)^2}{2(2s+a)} \geq \frac{s(R-2r)}{10R}$$

$$\overset{\text{via } (■■■)}{\Rightarrow} p_a + p_b + p_c \geq m_a + m_b + m_c + \frac{s(R-2r)}{10R} \forall \Delta ABC,$$

with equality iff ΔABC is equilateral (QED)

2049.In any ΔABC, the following relationship holds :

$$\frac{n_a}{m_b} + \frac{p_b}{w_c} + \frac{g_c}{h_a} \geq \frac{6r}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{n_a}{m_b} + \frac{p_b}{w_c} + \frac{g_c}{h_a} \geq \frac{h_a}{m_b} + \frac{h_b}{m_c} + \frac{h_c}{m_a} \overset{A-G}{\geq} 3 \cdot \sqrt[3]{\frac{h_a h_b h_c}{m_a m_b m_c}} \overset{m_a m_b m_c \leq \frac{Rs^2}{2}}{\geq} 3 \cdot \sqrt[3]{\frac{2r^2 s^2}{R \cdot \frac{Rs^2}{2}}}$$

$$= 3 \cdot \sqrt[3]{\frac{4r^2}{R^2}} \overset{?}{\geq} \frac{6r}{R} \Leftrightarrow \frac{4r^2}{R^2} \overset{?}{\geq} \frac{8r^3}{R^3} \Leftrightarrow R \overset{?}{\geq} 2r \rightarrow \text{true via Euler} \therefore \frac{n_a}{m_b} + \frac{p_b}{w_c} + \frac{g_c}{h_a} \geq \frac{6r}{R}$$

$\forall \Delta ABC$,'' ='' iff ΔABC is equilateral (QED)

Proof of $m_a m_b m_c \leq \frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)$$

$$= \frac{1}{64}\left(-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4 b^2 + \sum_{cyc} a^2 b^4\right) + 3a^2 b^2 c^2\right) \rightarrow (1)$$

$$\text{Now, } \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2 b^2 c^2 + \sum_{cyc}\left(a^2 b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2 b^2 c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \to (2)$$

$$\text{Also, } \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) =$$

$$\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \to (3) \therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2$$

$$= \frac{1}{64}\left(\begin{matrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{matrix}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(\begin{matrix} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{matrix}\right)$$

$$= \frac{1}{16}(s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3)$$

$$\leq \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\leq} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\leq} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$$

$$-r^3(4R+r)^3 \overset{?}{\leq} 0$$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \underbrace{\overset{?}{\geq}}_{(\bullet\bullet)} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(*)} s^2(16Rr - 5r^2)(8R - 16r)$$

$$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \text{ and}$$

$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\leq}}_{(**)} 20rs^2(4R^2+4Rr+3r^2)$$

$$(*),(**) \Rightarrow \text{in order to prove } (\bullet\bullet), \text{it suffices to prove :}$$

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$
$$\geq 20rs^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3 \geq 0$$
$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$$

$$\text{Now, LHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(***)} (108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$$

$$\text{and RHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\leq}}_{(****)} 27r^2(4R^2+4Rr+3r^2)$$

$$(***),(****) \Rightarrow \text{in order to prove } (\bullet\bullet\bullet), \text{it suffices to prove :}$$

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3 \geq 27r^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow 224t^3-587t^2+308t-60 \geq 0 \left(\text{where } t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$
$$\Rightarrow (\bullet) \text{ is true} \Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2} \text{ (QED)}$$

2050. In ΔABC the following relationship holds:

$$\frac{2}{3}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right) \geq \sqrt[5]{\frac{12sr}{R^2}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$\frac{2}{3}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right) \overset{AM-GM}{\geq} \frac{2}{3}.3\sqrt[3]{\left(\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}\right)} = 2\sqrt[3]{\frac{s}{4R}}$$

$$\textit{We need to show } 2\sqrt[3]{\frac{s}{4R}} \geq \sqrt[5]{\frac{12sr}{R^2}} \textit{ or } \left(2\sqrt[3]{\frac{s}{4R}}\right)^{15} \geq \left(\sqrt[5]{\frac{12sr}{R^2}}\right)^{15}$$

$$2^{15}\frac{s^5}{2^{10}R^5} \geq \frac{12^3s^3r^3}{R^6}$$

$$2^{15}\frac{s^5}{2^{10}R^5} \geq \frac{3^32^6s^3r^3}{R^6}$$

$$Rs^2 \ge 54r^3$$

$$R \cdot 27r^2 \overset{Mitrinovic}{\ge} 54r^3$$

$$R \ge 2r \ (Euler)$$

Equality holds for an equilateral triangle.

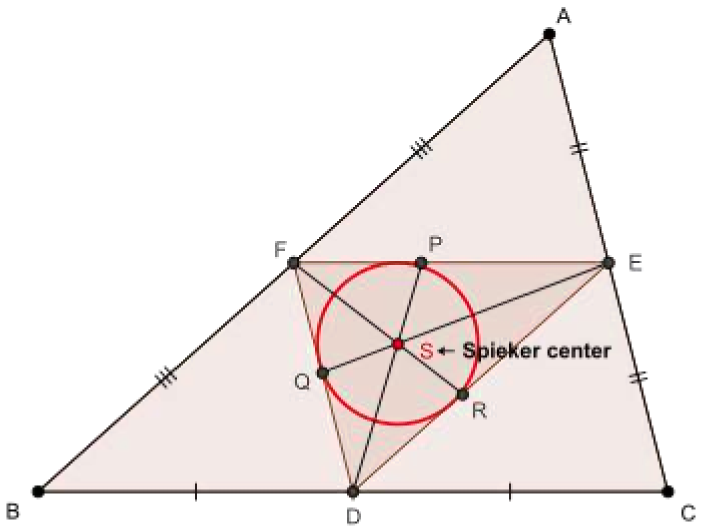
2051.

In any Δ ABC with $p_a \to$ Spieker cevian, the following relationship holds :

$$\frac{(b-c)^2}{2(2s+a)} \le p_a - m_a \le \frac{a|b-c|}{2(2s+a)}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\ \textit{and}\ m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\rightarrow(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on } \Delta\, AES, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, } p_a-m_a\le\frac{a|b-c|}{2(2s+a)}\Leftrightarrow p_a^2-m_a^2\le\frac{a^2(b-c)^2}{4(2s+a)^2}+\frac{am_a.|b-c|}{2s+a}$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-2a^2)}{4(2s+a)^2} \overset{(\blacksquare)}{\leq} \frac{am_a.|b-c|}{2s+a}$$

$$\text{Now, } a^2(4m_a^2) = a^2(2b^2+2c^2-a^2)$$

$$= 2a^2b^2 + 2c^2a^2 + 2b^2c^2 - a^4 - b^4 - c^4 + (b^4+c^4-2b^2c^2)$$

$$= 16F^2 + (b^2-c^2)^2 > (b^2-c^2)^2 \Rightarrow 2am_a > |b^2-c^2|$$

$$\therefore \frac{am_a.|b-c|}{2s+a} \geq \frac{(b-c)^2(2s-a)}{2(2s+a)} = \frac{2(b-c)^2(2s-a)(2s+a)}{4(2s+a)^2}$$

$$= \frac{(b-c)^2(8s^2-2a^2)}{4(2s+a)^2} \Rightarrow (\blacksquare) \text{ is true } \therefore p_a - m_a \leq \frac{a|b-c|}{2(2s+a)}$$

$$\text{Again, } p_a - m_a \geq \frac{(b-c)^2}{2(2s+a)} \Leftrightarrow p_a^2 \geq m_a^2 + \frac{(b-c)^4}{4(2s+a)^2} + \frac{m_a.(b-c)^2}{2s+a}$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-a^2-(b-c)^2)}{4(2s+a)^2} \overset{(\blacksquare\blacksquare)}{\geq} \frac{m_a.(b-c)^2}{2s+a}$$

$$\text{We note that}: 8s^2 - a^2 - (b-c)^2 > 8s^2 - 2a^2 > 0 \therefore (\blacksquare\blacksquare) \Leftrightarrow$$

$$\frac{(8s^2-a^2)^2 + (b-c)^4 - 2(8s^2-a^2)(b-c)^2}{16(2s+a)^2}$$

$$\geq \left(s(s-a) + \frac{(b-c)^2}{4}\right) \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow (8s^2-a^2)^2 - 16s(s-a)(2s+a)^2 + (b-c)^4$$
$$-2(b-c)^2(8s^2-a^2+2(2s+a)^2) \geq 0$$

$$\Leftrightarrow (b-c)^4 - 2(4s+a)^2(b-c)^2 + a^2(32s^2+16sa+a^2) \overset{(\blacksquare\blacksquare\blacksquare)}{\geq} 0$$

$$\text{Now, } (\blacksquare\blacksquare\blacksquare) \text{ is a quadrilateral in } (b-c)^2 \text{ with discriminant} =$$
$$4(4s+a)^4 - 4a^2(32s^2+16sa+a^2) = 256s^2(2s+a)^2$$

$$\therefore \text{ in order to prove } (\blacksquare\blacksquare\blacksquare), \text{it suffices to prove}:$$

$$(b-c)^2 \leq \frac{2(4s+a)^2 - 16s(2s+a)}{2} = a^2 \rightarrow \text{true} \Rightarrow (\blacksquare\blacksquare\blacksquare) \Rightarrow (\blacksquare\blacksquare) \text{ is true}$$

$$\therefore p_a - m_a \geq \frac{(b-c)^2}{2(2s+a)} \text{ and so, } \frac{(b-c)^2}{2(2s+a)} \leq p_a - m_a \leq \frac{a|b-c|}{2(2s+a)} \ \forall \Delta \text{ ABC},$$
$$\text{with equality iff } \Delta \text{ ABC is equilateral (QED)}$$

2052. In ΔABC the following relationship holds:

$$m_a w_a + m_b w_b + m_c w_c \geq s^2$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

Using the known inequalities:

$$m_a \geq \frac{b+c}{2}\cos\frac{A}{2} \text{ and } w_a = \frac{2bc}{b+c}\cos\frac{A}{2}$$

$$m_a w_a \geq \frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{2bc}{b+c}\cos\frac{A}{2} = bc\cos^2\frac{A}{2} = abc\frac{\cos^2\frac{A}{2}}{a} =$$

$$= abc\frac{\cos^2\frac{A}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{4Rrs}{4R}\cot\frac{A}{2} = rs\cot\frac{A}{2} \ (1)$$

$$m_a w_a + m_b w_b + m_c w_c = \sum m_a w_a \overset{(1)}{\geq} rs\sum\cot\frac{A}{2} = rs.\frac{s}{r} = s^2$$

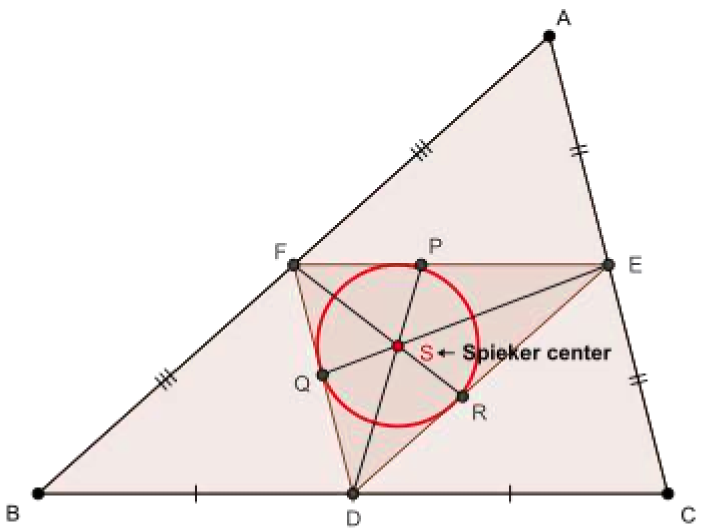
Equality holds for an equilateral triangle

2053.

In any Δ ABC with $p_a, p_b, p_c \to$ Spieker cevians, $n_a, n_b, n_c \to$ Nagel cevians, the following relationship holds : $n_a + n_b + n_c \geq p_a + p_b + p_c + \frac{2s(R-2r)}{5R}$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2 s}\big(ca(s-b) + ab(s-c)\big) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\Rightarrow p_a^2 - m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right) - m_a^2$$

$$= \frac{2s}{(2s+a)^2}(b^3+c^3-abc) - \left(1 - \frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3+c^3-abc) - (2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2 + 4a(b+c)(b-c)^2 + 2(b^2-c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right) + \left((b+c)^2+2a(b+c)+a^2\right) - a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2 - a^2\right) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \geq 0 \Rightarrow p_a \overset{(\bullet\bullet)}{\geq} m_a$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a) = an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s - a + \frac{(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} s(s-a) + \frac{s}{a}.(b-c)^2$$

Now, $b^3 + c^3 - abc + a(4m_a^2) = b^3 + c^3 + a^3 - abc + a(2b^2 + 2c^2 - a^2) - a^3$

$$= \sum_{cyc} a^3 - abc + 2a.2bc\cos A = 2s(s^2 - 6Rr - 3r^2) - 4Rrs + 16Rrs\cos A$$

$$\Rightarrow b^3 + c^3 - abc + a(4m_a^2) \overset{(\bullet\bullet\bullet\bullet\bullet\bullet\bullet)}{=} 2s(s^2 - 8Rr - 3r^2 + 8Rr\cos A) \therefore (\bullet), (\bullet\bullet\bullet\bullet\bullet\bullet\bullet)$$

$$\Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}.2s(s^2 - 8Rr - 3r^2 + 8Rr\cos A)$$

$$= \frac{4s^2}{(2s+a)^2}.\left(s^2 - 8Rr - 3r^2 + 8Rr\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= \frac{4s^2}{(2s+a)^2}.\left(s^2 - 3r^2 - 16Rr\sin^2\frac{A}{2}\right) < \frac{4s^4}{(2s+a)^2} \Rightarrow p_a < \frac{2s^2}{2s+a} \therefore n_a - p_a$$

$$= \frac{n_a^2 - p_a^2}{n_a + p_a} \geq \frac{\left(\frac{s}{a} - \frac{s(3s+a)}{(2s+a)^2}\right)(b-c)^2}{s + \frac{2s^2}{2s+a}} \quad (\because n_a^2 = s^2 - 2h_a r_a < s^2 \Rightarrow n_a < s)$$

$$= \frac{s(4s^2 + sa)(b-c)^2}{a(2s+a)^2.\left(\frac{4s^2+sa}{2s+a}\right)} = \frac{s(b-c)^2}{a(2s+a)} \therefore n_a - p_a \geq \frac{s(b-c)^2}{a(2s+a)} \textit{ and analogs}$$

$$\Rightarrow n_a + n_b + n_c \overset{(\blacksquare)}{\geq} p_a + p_b + p_c + \sum_{cyc}\frac{s(b-c)^2}{a(2s+a)}$$

$$\text{Now, } \sum_{cyc}\frac{(b-c)^2}{a} = \sum_{cyc}\frac{b^2+c^2+a^2}{a} - \sum_{cyc} a - \frac{2}{4Rrs}.\sum_{yc} b^2c^2$$

$$= \frac{1}{4Rrs}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) - \frac{8Rrs^2}{4Rrs} - \frac{2}{4Rrs}\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)$$

$$= \frac{1}{4Rrs}\left(\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2 - 2\sum_{cyc} ab\right) + 24Rrs^2\right)$$

$$= \frac{2(s^2 + 4Rr + r^2)\left(s^2 - 4Rr - r^2 - (s^2 + 4Rr + r^2)\right) + 24Rrs^2}{4Rrs}$$

$$= \frac{(2R-r)s^2 - r(4R+r)^2}{Rs} \overset{(m)}{=} \sum_{cyc} \frac{(b-c)^2}{a}$$

We have : $\sum_{cyc} \frac{(b-c)^2}{(2s+a)} \overset{(l)}{=} \frac{1}{2s(9s^2+6Rr+r^2)} \cdot \sum_{cyc} \left((b-c)^2(8s^2-2sa+bc)\right)$

and, $\sum_{cyc} \left((b-c)^2(8s^2-2sa+bc)\right) =$

$$8s^2 \sum_{cyc} (b-c)^2 - 2s. \sum_{cyc} a(b^2+c^2-2bc) + \sum_{cyc} \left(bc \left(\sum_{cyc} a^2 - a^2 - 2bc \right) \right)$$

$$= 16s^2(s^2-12Rr-3r^2) - 2s. \left(\left(\sum_{cyc} a \right) \left(\sum_{cyc} ab \right) - 9abc \right) +$$

$$2(s^2-4Rr-r^2)(s^2+4Rr+r^2) - 2((s^2+4Rr+r^2)^2 - 16Rrs^2) - 8Rrs^2$$

$$= 16s^2(s^2-12Rr-3r^2) - 4s^2(s^2-14Rr+r^2) +$$

$$2(s^2-4Rr-r^2)(s^2+4Rr+r^2) - 2((s^2+4Rr+r^2)^2 - 16Rrs^2) - 8Rrs^2 \overset{via\ (l)}{\Leftrightarrow}$$

$$\sum_{cyc} \frac{(b-c)^2}{(2s+a)} \overset{(n)}{=} \frac{\begin{pmatrix} 16s^2(s^2-12Rr-3r^2) - 4s^2(s^2-14Rr+r^2) + \\ 4r\left((2R-r)s^2 - r(4R+r)^2\right) \end{pmatrix}}{2s(9s^2+6Rr+r^2)}$$

We have : $\sum_{cyc} \frac{2s(b-c)^2}{a(2s+a)} = \sum_{cyc} \frac{(2s+a-a)(b-c)^2}{a(2s+a)}$

$$= \sum_{cyc} \frac{(b-c)^2}{a} - \sum_{cyc} \frac{(b-c)^2}{2s+a} \overset{via\ (m)\ and\ (n)}{=} \frac{(2R-r)s^2 - r(4R+r)^2}{Rs}$$

$$- \frac{16s^2(s^2-12Rr-3r^2) - 4s^2(s^2-14Rr+r^2) + 4r\left((2R-r)s^2 - r(4R+r)^2\right)}{2s(9s^2+6Rr+r^2)}$$

$$= \frac{2(9s^2+6Rr+r^2)\left((2R-r)s^2 - r(4R+r)^2\right) - R\sigma}{2Rs(9s^2+6Rr+r^2)}$$

$$\begin{pmatrix} \sigma = 16s^2(s^2-12Rr-3r^2) - 4s^2(s^2-14Rr+r^2) + \\ 4r\left((2R-r)s^2 - r(4R+r)^2\right) \end{pmatrix} \overset{?}{\geq} \frac{4s(R-2r)}{5R}$$

$$\Leftrightarrow (24R+27r)s^4 - r(364R^2+196Rr+42r^2)s^2 - 5r^2(4R+r)^3 \overset{?}{\underset{①}{\geq}} 0 \text{ and } \because$$

$(24R+27r)(s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore$ in order to prove ①,
it suffices to prove : LHS of ① $\geq (24R+27r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (101R^2+107Rr-78r^2)s^2 \overset{②}{\geq} r(1616R^3+828R^2r-915Rr^2+170r^3)$$

$$\text{Finally, LHS of } ② \overset{\text{Gerretsen}}{\geq} (101R^2+107Rr-78r^2)(16Rr-5r^2) \overset{?}{\geq}$$
$$r(1616R^3+828R^2r-915Rr^2+170r^3) \Leftrightarrow 379R^2-868Rr+220r^2 \overset{?}{\geq} 0$$
$$\Leftrightarrow (379R-110r)(R-2r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow ② \Rightarrow ① \text{ is true}$$
$$\therefore \sum_{cyc}\frac{s(b-c)^2}{a(2s+a)} \geq \frac{2s(R-2r)}{5R} \overset{\text{via }(\blacksquare)}{\Rightarrow} n_a+n_b+n_c \geq$$
$$p_a+p_b+p_c+\frac{2s(R-2r)}{5R}\ \forall\,\Delta\,ABC, \text{with equality iff } \Delta\,ABC \text{ is equilateral (QED)}$$

2054. **In any ΔABC, prove that:**

$$\left|\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}\right| \leq \frac{1}{24\sqrt{3}}\cdot\frac{\sqrt{(p^2-12Rr-3r^2)^3}}{R^2r}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Mollweide's formula, we have $\sin\frac{B-C}{2}=\frac{b-c}{a}.\cos\frac{A}{2}$ (and analogs), then:

$$\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}=\frac{(a-b)(b-c)(c-a)}{4pRr}.\frac{p}{4R}=\frac{(a-b)(b-c)(c-a)}{16R^2r},$$

and since $(a-b)^2+(b-c)^2+(c-a)^2=2(p^2-12Rr-3r^2)$,

then the desired inequality is equivalent to

$$3\sqrt{6}.|(a-b)(b-c)(c-a)| \leq \sqrt{[(a-b)^2+(b-c)^2+(c-a)^2]^3}.$$

WLOG, we assume that $a \geq b \geq c$.

Let $x := a-b, y := b-c$. The desired inequality becomes

$$3\sqrt{3}xy(x+y) \leq 2\sqrt{(x^2+y^2+xy)^3}$$

By AM − GM inequality, we have

$$3\sqrt{3}xy(x+y)=2\sqrt{27.xy.xy.\frac{(x+y)^2}{4}} \leq 2\sqrt{\left(xy+xy+\frac{(x+y)^2}{4}\right)^3}$$
$$\leq 2\sqrt{(x^2+y^2+xy)^3}$$

So the proof is complete. Equality holds iff $x=y \Leftrightarrow a+c=2b$ and permutation.

2055. **In any Δ ABC, the following relationship holds:**

$$\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)(r_a^2+r_b^2+r_c^2) \leq \frac{\sqrt{r+4R}\left(r+4R-\sqrt{2r(r+4R)}\right)\left(\sqrt{r+4R}-\sqrt{2r}\right)}{2r^2}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)(r_a^2+r_b^2+r_c^2)\le\frac{\sqrt{r+4R}\left(r+4R-\sqrt{2r(r+4R)}\right)\left(\sqrt{r+4R}-\sqrt{2r}\right)}{2r^2}$$

$$\Leftrightarrow\frac{2(s^2-4Rr-r^2)}{4r^2s^2}.\left((4R+r)^2-2s^2\right)\le$$

$$\le\frac{\left(4R+r-\sqrt{2r(4R+r)}\right)\left(4R+r-\sqrt{2r(4R+r)}\right)}{2r^2}$$

$$\Leftrightarrow\frac{s^2-4Rr-r^2}{s^2}.\left((4R+r)^2-2s^2\right)\le(4R+r)^2+2r(4R+r)-2(4R+r).\sqrt{2r(4R+r)}$$

$$\Leftrightarrow(4R+r)^2-2s^2-\frac{(4Rr+r^2)(4R+r)^2}{s^2}+2(4Rr+r^2)\le$$

$$\le(4R+r)^2+2(4Rr+r^2)-2(4R+r).\sqrt{2(4Rr+r^2)}$$

$$\Leftrightarrow 2s^4+(4Rr+r^2)(4R+r)^2-2s^2(4R+r).\sqrt{2(4Rr+r^2)}\ge 0$$

$$\Leftrightarrow\left(\sqrt{2}s^2\right)^2+\left(\sqrt{4Rr+r^2}.(4R+r)\right)^2-2\left(\sqrt{2}s^2\right).\left(\sqrt{4Rr+r^2}.(4R+r)\right)\ge 0$$

$$\Leftrightarrow\left(\sqrt{2}s^2-\sqrt{4Rr+r^2}.(4R+r)\right)^2\ge 0\to\text{true}$$

$$\therefore\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)(r_a^2+r_b^2+r_c^2)\le$$

$$\le\frac{\sqrt{r+4R}\left(r+4R-\sqrt{2r(r+4R)}\right)\left(\sqrt{r+4R}-\sqrt{2r}\right)}{2r^2}\ \forall\,\Delta\,ABC\ (QED)$$

2056.

In any Δ ABC, $\forall\, x,y,z>0$, the following relationship holds :

$$\frac{x}{y+z}(1+\cos A)+\frac{y}{z+x}(1+\cos B)+\frac{z}{x+y}(1+\cos C)\ge\frac{\sqrt{3}}{2}(\sin A+\sin B+\sin C)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=\frac{s(s-a)}{bc}+\frac{s(s-b)}{ca}-\frac{s(s-c)}{ab}=$$

$$=\frac{s}{abc}\left(a(s-a)+b(s-b)-c(s-c)\right)=\frac{s}{abc}\left(x(y+z)+y(z+x)-z(x+y)\right)=$$

$$=\frac{s}{abc}(2xy)>0(x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y)$$

$$\therefore \cos^2\frac{C}{2}<\cos^2\frac{A}{2}+\cos^2\frac{B}{2}<\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)^2\Rightarrow \cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}>0$$

$and\ analogs \Rightarrow \cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}$ form sides of a triangle

$$\Rightarrow \sqrt{\cos\frac{A}{2}},\sqrt{\cos\frac{B}{2}},\sqrt{\cos\frac{C}{2}} \text{ form sides of } a \text{ triangle with } area$$

$$F_1=\frac{1}{4}.\sqrt{2\sum_{cyc}\cos\frac{A}{2}\cos\frac{B}{2}-\sum_{cyc}\cos^2\frac{A}{2}}$$

$$\text{Now,}\frac{x}{y+z}(1+\cos A)+\frac{y}{z+x}(1+\cos B)+\frac{z}{x+y}(1+\cos C)$$

$$=2\left(\frac{x}{y+z}.\left(\sqrt{\cos\frac{A}{2}}\right)^4+\frac{y}{z+x}.\left(\sqrt{\cos\frac{B}{2}}\right)^4+\frac{z}{x+y}.\left(\sqrt{\cos\frac{C}{2}}\right)^4\right)\geq$$

$$\overset{Tsintsifas}{\geq} 16F_1^2=2\sum_{cyc}\cos\frac{A}{2}\cos\frac{B}{2}-\sum_{cyc}\cos^2\frac{A}{2}\overset{?}{\geq}\frac{\sqrt{3}}{2}(\sin A+\sin B+\sin C)$$

$$\Leftrightarrow 2\sum_{cyc}\sin\frac{B+C}{2}\sin\frac{C+A}{2}-\sum_{cyc}\sin^2\frac{B+C}{2}\underset{①}{\overset{?}{\geq}}\frac{\sqrt{3}}{2}\left(\sum_{cyc}\sin(B+C)\right)$$

$$\text{Now, since}: \frac{B+C}{2}+\frac{C+A}{2}+\frac{A+B}{2}=\pi,\therefore X=\frac{B+C}{2},Y=\frac{C+A}{2},Z=\frac{A+B}{2}$$

form $angles$ of a triangle XYZ with sides x,y,z and semiperimeter, circumradius, inradius $=s',R'$ and r' (say) and then

$$:①\Leftrightarrow 2\sum_{cyc}\sin X\sin Y-\sum_{cyc}\sin^2 X\overset{?}{\geq}\frac{\sqrt{3}}{2}.\sum_{cyc}\sin 2X$$

$$\Leftrightarrow \frac{1}{4R'^2}\left(2\sum_{cyc} xy - \sum_{cyc} x^2\right) \overset{?}{\geq} \frac{\sqrt{3}}{2}.4\sin X \sin Y \sin Z = 2\sqrt{3}.\frac{xyz}{8R'^3} = 2\sqrt{3}.\frac{4R'r's'}{8R'^3}$$

$$\Leftrightarrow 2\sum_{cyc} xy - \sum_{cyc} x^2 \overset{?}{\geq} 4\sqrt{3}.r's' \rightarrow \text{which is true via Hadwiger} - \text{Finsler}$$

$$\Rightarrow ① \text{ is true} \therefore \frac{x}{y+z}(1+\cos A) + \frac{y}{z+x}(1+\cos B) + \frac{z}{x+y}(1+\cos C)$$

$$\geq \frac{\sqrt{3}}{2}(\sin A + \sin B + \sin C)\ \forall\ \Delta\ ABC,'' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

2057. In ΔABC the following relationship holds:

$$\frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} \geq \frac{9}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum a.m_a^2 = \frac{1}{4}\sum a(2b^2 + 2c^2 - a^2) = \frac{1}{4}\left(2\sum a(b^2+c^2) - \sum a^3\right)\ (1)$$

$$2\sum a(b^2+c^2) - \sum a^3 = 2\sum a^2(b+c) - \sum a^3 =$$

$$= 2\sum a^2(2s-a) - \sum a^3 = 4s\sum a^2 - 2\sum a^3 - \sum a^3 =$$

$$= 8s(s^2 - r^2 - 4Rr) - 6s(s^2 - 3r^2 - 6Rr) =$$

$$= 8s^3 - 8s(4Rr + r^2) - 6s^3 + 6s(6Rr + 3r^2) =$$

$$= 2s^3 - 2s(16Rr + 4r^2 - 9r^2 - 18Rr) =$$

$$= 2s(s^2 + 2Rr + 5r^2) \overset{Gerretsen}{\geq} 2s(16Rr - 5r^2 + 2Rr + 5r^2) = 36Rrs\ (2)$$

$$\sum am_a^2 = \frac{1}{4}\sum a(2b^2 + 2c^2 - a^2) = \frac{1}{4}\left(2\sum a(b^2+c^2) - \sum a^3\right) \overset{(2)}{\geq}$$

$$\geq \frac{1}{4}.36Rrs = 9Rrs$$

$$\frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} = \sum \frac{m_a^2}{bc} = \frac{1}{abc}\sum am_a^2 \geq \frac{1}{abc}.9Rrs = \frac{9Rrs}{4Rrs} = \frac{9}{4}$$

Equality holds for an equilateral triangle

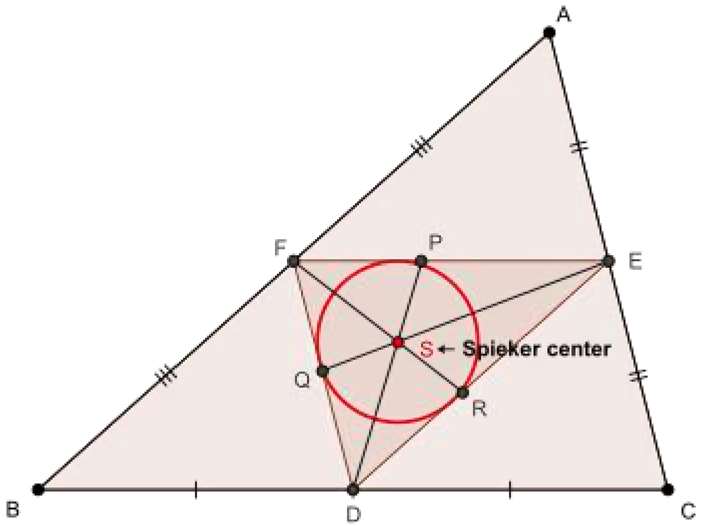
2058. In any $\Delta\,ABC$ with $p_a, p_b, p_c \to$

Spieker cevians, the following relationship holds:

$$p_a + p_b + p_c \geq w_a + w_b + w_c + \frac{4s(R-2r)}{15R}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of $\Delta\,DEF = r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\,DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on $\Delta\,AFS$ and $\Delta\,AES$, we arrive at:

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

Via sine law on Δ AFS, $\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$

$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}$ *and* via sine law on Δ AES, $b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$

Now, $[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

Now, $b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a-w_a \ge \frac{2s(b-c)^2}{4s^2-a^2} \Leftrightarrow p_a^2-w_a^2 \ge$$

$$\frac{4s^2(b-c)^4}{(4s^2-a^2)^2}+\frac{4s.w_a.(b-c)^2}{4s^2-a^2} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} \boxed{\overset{(\blacksquare)}{\ge}} \frac{4s.w_a}{4s^2-a^2} \quad (\because (b-c)^2 \ge 0)$$

$$\text{We have : } \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} >$$

$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2a^2}{(4s^2-a^2)^2}$$

$$= \frac{s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2a^2}{(4s^2-a^2)^2}=\frac{8s^2(2s+a)(s-a)}{(4s^2-a^2)^2}>0$$

$$\therefore (\blacksquare) \Leftrightarrow \frac{(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2(b-c)^2)^2}{(4s^2-a^2)^4}$$

$$\ge \frac{16s^2}{(4s^2-a^2)^2}.\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow \frac{\begin{array}{c}16s^4(b-c)^4-8s^2(b-c)^2(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)+\\(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)^2\end{array}}{(4s^2-a^2)^2}$$

$$\ge \frac{16s^2(s(s-a)(2s-a)^2-s(s-a)(b-c)^2)}{(2s-a)^2}$$

$$\Leftrightarrow 16s^4(b-c)^4-8s^2(b-c)^2\begin{pmatrix}s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\\-2s(s-a)(2s+a)^2\end{pmatrix}$$

$$+(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)^2-16s^3(s-a)(4s^2-a^2)^2\geq 0$$
$$\Leftrightarrow 16s^4(b-c)^4-16s^3(b-c)^2(4s^3-4s^2a+sa^2+a^3)$$
$$+16a^2s^3(4s^3-4s^2a+a^3)\geq 0$$
$$\Leftrightarrow s(b-c)^4-(4s^3-4s^2a+sa^2+a^3)(b-c)^2+a^2(4s^3-4s^2a+a^3)\overset{(\blacksquare\blacksquare)}{\boxed{\geq}}0$$
and in order to prove $(\blacksquare\blacksquare)$, **it suffices to prove :**
$$(b-c)^2\leq\frac{(4s^3-4s^2a+sa^2+a^3)-\sqrt{\delta}}{2s},\text{ where }\delta=$$
$$(4s^3-4s^2a+sa^2+a^3)^2-4sa^2(4s^3-4s^2a+a^3)\text{ and }\because(b-c)^2<a^2$$
$$\therefore\text{ it suffices to prove : }2sa^2\leq(4s^3-4s^2a+sa^2+a^3)$$
$$-\sqrt{(4s^3-4s^2a+sa^2+a^3)^2-4sa^2(4s^3-4s^2a+a^3)}$$
$$\Leftrightarrow\sqrt{(s-a)^2(4s^2-a^2)^2}\leq 4s^3-4s^2a-sa^2+a^3=(s-a)(4s^2-a^2)\to\text{true}$$
$$\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true }\therefore p_a-w_a\geq\frac{2s(b-c)^2}{4s^2-a^2}\text{ and analogs}$$
$$\therefore p_a+p_b+p_c\overset{(\blacksquare\blacksquare\blacksquare)}{\geq}w_a+w_b+w_c+\sum_{cyc}\frac{2s(b-c)^2}{4s^2-a^2}$$
$$\text{Now, }\sum_{cyc}\left((b-c)^2(4s^2-b^2)(4s^2-c^2)\right)=$$
$$16s^4\sum_{cyc}(b-c)^2-4s^2\sum_{cyc}\left((b-c)^2\left(\sum_{cyc}a^2-a^2\right)\right)$$
$$+\sum_{cyc}\left(b^2c^2\left(\sum_{cyc}a^2-a^2-2bc\right)\right)$$
$$=32s^4(s^2-12Rr-3r^2)-16s^2(s^2-4Rr-r^2)(s^2-12Rr-3r^2)+$$
$$4s^2\sum_{cyc}\left(a^2(b^2+c^2-2bc)\right)+2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)$$
$$-48R^2r^2s^2-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)$$
$$=32s^4(s^2-12Rr-3r^2)-16s^2(s^2-4Rr-r^2)(s^2-12Rr-3r^2)+$$
$$(10s^2-8Rr-2r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-64Rrs^4-48R^2r^2s^2$$
$$-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)$$
$$=4(6s^6-(64Rr+5r^2)s^4-r^2s^2(148R^2+76Rr+12r^2)-(4Rr+r^2)^3)$$
$$\therefore\sum_{cyc}\frac{2s(b-c)^2}{4s^2-a^2}=$$
$$\frac{8s(6s^6-(64Rr+5r^2)s^4-r^2s^2(148R^2+76Rr+12r^2)-(4Rr+r^2)^3)}{4s^2(9s^2+6Rr+r^2)(s^2+2Rr+r^2)}\overset{?}{\geq}\frac{4s(R-2r)}{15R}$$
$$\Leftrightarrow 15R(6s^6-(64Rr+5r^2)s^4-r^2s^2(148R^2+76Rr+12r^2)-(4Rr+r^2)^3)$$

$$\overset{?}{\geq} 2s^2(9s^2+6Rr+r^2)(s^2+2Rr+r^2)(R-2r)$$

$$\Leftrightarrow (72R+36r)s^6 - r(1008R^2 - Rr - 40r^2)s^4 -$$

$$r^2(2244R^3+1108R^2r+150Rr^2-4r^3)s^2 - 15Rr^3(4R+r)^3 \boxed{\overset{?}{\underset{①}{\geq}}} 0 \text{ and } \because$$

$$(72R+36r)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove ①, it suffices}$$

$$\text{to prove : LHS of ①} \geq (72R+36r)(s^2-16Rr+5r^2)^3 \Leftrightarrow$$

$$(2448R^2+649Rr-500r^2)s^4$$

$$-r(57540R^3-5804R^2r-11730Rr^2+2696r^3)s^2+$$

$$r^2(293952R^4-129744R^3r-52020R^2r^2+34185Rr^3-4500r^4) \boxed{\overset{②}{\geq}} 0 \text{ and}$$

$$\because (2448R^2+649Rr-500r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$

$$\text{to prove ②, it suffices to prove : LHS of ②} \geq$$

$$(2448R^2+649Rr-500r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (5199R^3+523R^2r-2690Rr^2+576r^3)s^2 \boxed{\overset{③}{\geq}}$$

$$r(293952R^4-129744R^3r-52020R^2r^2+34185Rr^3-4500r^4)$$

$$\text{Finally, LHS of ③} \overset{\text{Gerretsen}}{\geq} (5199R^3+523R^2r-2690Rr^2+576r^3)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(293952R^4-129744R^3r-52020R^2r^2+34185Rr^3-4500r^4)$$

$$\Leftrightarrow 6321t^3-16000t^2+7156t-880 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4642t^2+1679t(t-2)+440) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ③ \Rightarrow ②$$

$$\Rightarrow ① \text{ is true} \therefore \sum_{\text{cyc}} \frac{2s(b-c)^2}{4s^2-a^2} \geq \frac{4s(R-2r)}{15R} \overset{\text{via } (\blacksquare\blacksquare\blacksquare)}{\Rightarrow}$$

$$p_a+p_b+p_c \geq w_a+w_b+w_c+\frac{4s(R-2r)}{15R} \ \forall \ \Delta \ ABC,$$

with equality iff Δ ABC is equilateral (QED)

2059. In ΔABC the following relationship holds:

$$h_a+h_b+h_c-9r \leq 2(R-2r)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$h_a+h_b+h_c=\frac{ab+bc+ca}{2R}=\frac{s^2+r^2+4Rr}{2R} \overset{\text{Gerretsen}}{\leq}$$

$$\le \frac{4R^2+4Rr+3r^2+r^2+4Rr}{2R}=\frac{4R^2+8Rr+4r^2}{2R}$$

We need to show:

$$h_a+h_b+h_c-9r\le 2(R-2r),\ \frac{4R^2+8Rr+4r^2}{2R}-9r\le 2(R-2r)$$

$$4R^2-10Rr+4r^2\le 4R^2-8Rr$$

$$2Rr\ge 4r^2 \text{ or } R\ge 2r\ (\text{True Euler})$$

Equality holds for an equilateral triangle

2060. In ΔABC the following relationship holds:

$$\frac{m_a}{\sin\frac{A}{2}}+\frac{m_b}{\sin\frac{B}{2}}+\frac{m_c}{\sin\frac{C}{2}}\ge\frac{a^2+b^2+c^2}{2r}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Using the known inequalities $m_a\ge\frac{b+c}{2}\cos\frac{A}{2}$ *and*

$$\frac{m_a}{\sin\frac{A}{2}}\ge\frac{\frac{b+c}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}}=\frac{b+c}{2}\cot\frac{A}{2}=\frac{b+c}{2}\cdot\frac{s}{r_a}=\frac{s(2s-a)}{2r_a}=\frac{2s^2}{2r_a}-\frac{as}{2r_a}=$$

$$=\frac{2s^2}{2r_a}-\frac{a(s-a)}{2r}\ (1)$$

$$\frac{m_a}{\sin\frac{A}{2}}+\frac{m_b}{\sin\frac{B}{2}}+\frac{m_c}{\sin\frac{C}{2}}=\sum\frac{m_a}{\sin\frac{A}{2}}\overset{(1)}{\ge}\sum\frac{2s^2}{2r_a}-\sum\frac{a(s-a)}{2r}=$$

$$=\frac{2s^2}{2r}-\frac{2s^2-(a^2+b^2+c^2)}{2r}=\frac{a^2+b^2+c^2}{2r}$$

Equality holds for $A=B=C$

2061.

In any acute Δ ABC, the following relationship holds :

$$\frac{\cos A}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos B}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos C}{\cos\frac{A}{2}\cos\frac{B}{2}}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\cos A}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos B}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos C}{\cos\frac{A}{2}\cos\frac{B}{2}}\overset{A-G}{\geq}\sum_{cyc}\frac{4\cos A}{2\cos^2\frac{B}{2}+2\cos^2\frac{C}{2}}=$$

$$=\sum_{cyc}\frac{4\cos A}{1+\cos B+1+\cos C}=4\sum_{cyc}\frac{\cos^2 A}{2\cos A+\cos A\cos B+\cos A\cos C}$$

$$\overset{Bergstrom}{\geq}\frac{4\left(\sum_{cyc}\cos A\right)^2}{2\sum_{cyc}\cos A+2\sum_{cyc}\cos A\cos B}\overset{?}{\geq}2$$

$$\Leftrightarrow 2\left(\sum_{cyc}\cos A\right)^2\overset{?}{\geq}2\sum_{cyc}\cos A+\left(\sum_{cyc}\cos A\right)^2-\left(3-\sum_{cyc}\sin^2 A\right)$$

$$\Leftrightarrow\frac{(R+r)^2}{R^2}-\frac{2(R+r)}{R}+3-\frac{s^2-4Rr-r^2}{2R^2}\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{2R^2+4Rr+2r^2-4R^2-4Rr+6R^2-s^2+4Rr+r^2}{2R^2}\overset{?}{\geq}0$$

$$\Leftrightarrow 4R^2+4Rr+3r^2\overset{?}{\geq}s^2\to\text{true via Gerretsen}$$

$$\therefore\frac{\cos A}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos B}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos C}{\cos\frac{A}{2}\cos\frac{B}{2}}\geq 2\ \forall\ acute\ \Delta\ ABC,$$

$$''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

2062. In ΔABC the following relationship holds:

$$\sum r_a^2\geq\sum m_a^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum r_a^2=\left(\sum r_a\right)^2-2\sum r_ar_b=(4R+r)^2-2s^2\ and$$

$$\sum m_a^2=\frac{3}{4}\left(\sum a^2\right)=\frac{3}{4}.2(s^2-r^2-4Rr)=\frac{3s^2-3r^2-12Rr}{2}$$

$$We\ need\ to\ show,\ (4R+r)^2-2s^2\geq\frac{3s^2-3r^2-12Rr}{2}\ or$$

$$2(4R+r)^2-4s^2\geq 3s^2-3r^2-12Rr\ or$$

$$2(16R^2+8Rr+r^2)+12Rr+3r^2\geq 7s^2\ or$$

$$32R^2+28Rr+5r^2\overset{Gerretsen}{\geq}7(4R^2+4Rr+3r^2)or$$

$$4R^2\geq 16r^2\ or\ \ R\geq 2r\ \ true\ Euler$$

$$Equality\ holds\ for\ a=b=c$$

2063.

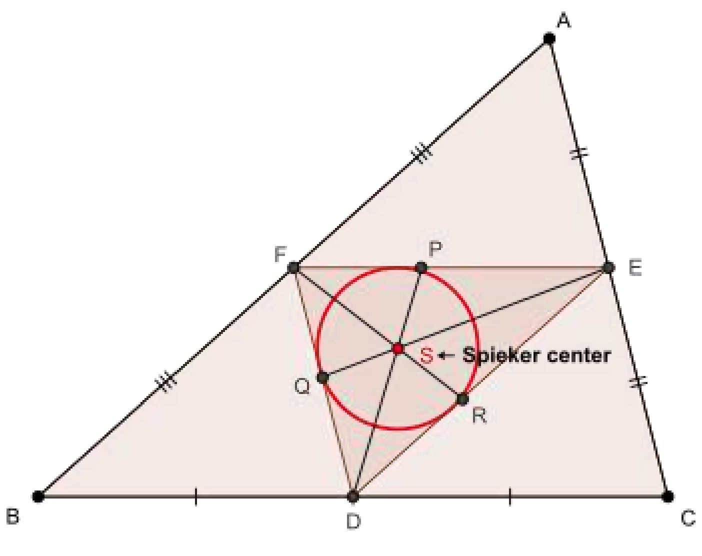
In any Δ ABC with p_a, p_b, p_c

$\rightarrow$ Spieker cevians, the following relationship holds :

$$(p_a+p_b+p_c)^2\geq 4s^2-\frac{9r(64R-53r)}{25}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Let AS produced meet BC } at \text{ X } and \text{ m}(\measuredangle BAX)=\alpha\ and\ m(\measuredangle CAX)=\beta\ (say)$$
$$and\ inradius\ of\ \Delta\ DEF=r'(say)$$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via $(1), (2)$ and using cosine law on Δ AFS and Δ AES, we arrive at:

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again}, \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(\mathrm{ii})}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now}, [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2=\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a(4m_a^2)\Big)\Rightarrow p_a^2-m_a^2=$$

$$\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a(4m_a^2)\Big)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\Big(\big(a^2+2a(b+c)+(b+c)^2\big)+\big((b+c)^2+2a(b+c)+a^2\big)-a^2\Big)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\big(2(a+b+c)^2-a^2\big)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\bullet)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, } p_a - m_a \geq \frac{(b-c)^2}{2(2s+a)} \Leftrightarrow p_a^2 \geq m_a^2 + \frac{(b-c)^4}{4(2s+a)^2} + \frac{m_a.(b-c)^2}{2s+a}$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-a^2-(b-c)^2)}{4(2s+a)^2} \overset{(\blacksquare)}{\geq} \frac{m_a.(b-c)^2}{2s+a}$$

$$\text{We note that : } 8s^2 - a^2 - (b-c)^2 > 8s^2 - 2a^2 > 0 \therefore (\blacksquare) \Leftrightarrow$$

$$\frac{(8s^2-a^2)^2 + (b-c)^4 - 2(8s^2-a^2)(b-c)^2}{16(2s+a)^2}$$

$$\geq \left(s(s-a) + \frac{(b-c)^2}{4}\right) \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow (8s^2-a^2)^2 - 16s(s-a)(2s+a)^2 + (b-c)^4 -$$
$$2(b-c)^2(8s^2-a^2+2(2s+a)^2) \geq 0$$

$$\Leftrightarrow (b-c)^4 - 2(4s+a)^2(b-c)^2 + a^2(32s^2+16sa+a^2) \overset{(\blacksquare\blacksquare)}{\geq} 0$$

Now, $(\blacksquare\blacksquare)$ is a quadrilateral in $(b-c)^2$ with discriminant $=$
$4(4s+a)^4 - 4a^2(32s^2+16sa+a^2) = 256s^2(2s+a)^2 \therefore$ in order to

$$\text{prove } (\blacksquare\blacksquare)\text{, it suffices to prove : } (b-c)^2 \leq \frac{2(4s+a)^2 - 16s(2s+a)}{2}$$

$$= a^2 \to \text{true} \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true} \therefore p_a - m_a \overset{(\bullet\bullet)}{\geq} \frac{(b-c)^2}{2(2s+a)} \text{ and analogs}$$

$$\text{Via Chu and Yang, } \left(\sum_{cyc} m_a\right)^2 \geq \frac{9s^2}{4} + \frac{7s^2}{4} - 28Rr + 29r^2 \overset{\text{Gerretsen}}{\geq}$$

$$\frac{9s^2}{4} + \frac{7(16Rr-5r^2)}{4} - 28Rr + 29r^2 = \frac{9s^2}{4} + \left(29 - \frac{35}{4}\right)r^2 > \frac{9s^2}{4} \Rightarrow \sum_{cyc} m_a \overset{(\bullet\bullet\bullet)}{>} \frac{3s}{2}$$

$$\text{We have : } \sum_{cyc} \frac{(b-c)^2}{(2s+a)} \overset{(l)}{=} \frac{1}{2s(9s^2+6Rr+r^2)} . \sum_{cyc}\left((b-c)^2(8s^2-2sa+bc)\right)$$

$$\text{and, } \sum_{cyc}\left((b-c)^2(8s^2-2sa+bc)\right) = 8s^2\sum_{cyc}(b-c)^2 - 2s.\sum_{cyc} a(b^2+c^2-2bc)$$

$$+ \sum_{cyc}\left(bc\left(\sum_{cyc} a^2 - a^2 - 2bc\right)\right)$$

$$= 16s^2(s^2-12Rr-3r^2) - 2s.\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) +$$

$$2(s^2-4Rr-r^2)(s^2+4Rr+r^2) - 2((s^2+4Rr+r^2)^2 - 16Rrs^2) - 8Rrs^2$$
$$= 16s^2(s^2-12Rr-3r^2) - 4s^2(s^2-14Rr+r^2) +$$

$$2(s^2-4Rr-r^2)(s^2+4Rr+r^2)-2((s^2+4Rr+r^2)^2-16Rrs^2)-8Rrs^2 \overset{via\ (l)}{\Rightarrow}$$

$$\sum_{cyc}\frac{(b-c)^2}{2(2s+a)} \overset{(m)}{=} \frac{4s^2(s^2-12Rr-3r^2)-s^2(s^2-14Rr+r^2)+r\left((2R-r)s^2-r(4R+r)^2\right)}{s(9s^2+6Rr+r^2)}$$

$$= \frac{\sigma}{s(9s^2+6Rr+r^2)}\ (say) \therefore (\bullet\bullet)\ and\ (m) \Rightarrow$$

$$p_a+p_b+p_c \geq \sum_{cyc} m_a + \frac{\sigma}{s(9s^2+6Rr+r^2)} \Rightarrow (p_a+p_b+p_c)^2 \geq$$

$$\left(\sum_{cyc} m_a\right)^2 + \frac{\sigma^2}{s^2(9s^2+6Rr+r^2)^2} + \frac{2\sigma}{s(9s^2+6Rr+r^2)}.\left(\sum_{cyc} m_a\right)$$

$$\overset{via\ Chu\ and\ Yang, and\ (\bullet\bullet\bullet)}{\geq} 4s^2-28Rr+29r^2+\frac{\sigma^2}{s^2(9s^2+6Rr+r^2)^2}+\frac{3\sigma}{9s^2+6Rr+r^2}$$

$$\overset{?}{\geq} 4s^2 - \frac{9r(64R-53r)}{25} \Leftrightarrow \frac{\sigma^2+3s^2(9s^2+6Rr+r^2)\sigma}{s^2(9s^2+6Rr+r^2)^2} \overset{?}{\geq} \frac{124r(R-2r)}{25}$$

$$\Leftrightarrow 2250s^8-(35094Rr-8763r^2)s^6-r^2(15392R^2-31652Rr-7489r^2)s^4+$$

$$r^3(13936R^3+26640R^2r+9002Rr^2+873r^3)s^2+25r^4(4R+r)^4 \underset{(1)}{\overset{?}{\geq}} 0$$

$$\text{Now, } T = 2250(s^2-16Rr+5r^2)^4+(108906Rr-36237r^2)(s^2-16Rr+5r^2)^3$$

$$+2r^2(878048R^2-590657Rr+106772r^2)(s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\geq} 0$$

$\therefore$ in order to prove (1), it suffices to prove : LHS of (1) $\geq T$

$$\Leftrightarrow (589575R^3-611967R^2r+243365Rr^2-33862r^3)s^2 \underset{(2)}{\overset{?}{\geq}}$$

$$r(9433200R^4-12568272R^3r+6508764R^2r^2-1537665Rr^3+138450r^4)$$

$$\text{Now, LHS of (2)} \overset{Rouche}{\geq} \begin{pmatrix}589575R^3-611967R^2r\\+243365Rr^2-33862r^3\end{pmatrix}\begin{pmatrix}2R^2+10Rr-r^2\\-2(R-2r).\sqrt{R^2-2Rr}\end{pmatrix}$$

$$\overset{?}{\geq} \text{RHS of (2)}$$

$$\Leftrightarrow (R-2r)\begin{pmatrix}1179150R^4-2403084R^3r+1539589R^2r^2\\-451693Rr^3+52294r^4\end{pmatrix}$$

$$\overset{?}{\geq} 2(R-2r).\sqrt{R^2-2Rr}.(589575R^3-611967R^2r+243365Rr^2-33862r^3)$$

$$and \because R-2r \overset{Euler}{\geq} 2 \therefore \text{it suffices to prove :}$$

$$(1179150R^4-2403084R^3r+1539589R^2r^2-451693Rr^3+52294r^4)^2 >$$

$$4(R^2-2Rr)(589575R^3-611967R^2r+243365Rr^2-33862r^3)^2$$

$$\left(\begin{matrix}\because 1179150R^4-2403084R^3r+1539589R^2r^2-451693Rr^3+52294r^4\\ = (R-2r)(1179150R^3-44784R^2r+1450021Rr^2+2448349r^3)+4948992r^4 \overset{Euler}{>} 0\end{matrix}\right)$$

$$\Leftrightarrow 986{,}948{,}550{,}000t^6-1{,}821{,}847{,}095{,}300t^5+$$

$$(1{,}559{,}557{,}564{,}533t^4-770{,}879{,}282{,}042t^3)+$$

$$(228,609,350,325t^2 - 38,068,587,132t) + 2,734,662,436 > 0 \left(t = \frac{R}{r}\right)$$

which is clearly true $\because t \overset{\text{Euler}}{\geq} 2$ and hence, $986,948,550,000t^6 \geq$
$2 * 986,948,550,000t^5 = 1,973,897,100,000t^5 > 1,821,847,095,300t^5$
(here, comma is the thousands − seperator) $\Rightarrow$ ② $\Rightarrow$ ① is true

$$\therefore (p_a + p_b + p_c)^2 \geq 4s^2 - \frac{9r(64R - 53r)}{25} \ \forall \Delta ABC,$$

with equality iff Δ ABC is equilateral (QED)

2064. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{h_a h_b}{4rr_c} = \sum_{cyc} \frac{h_a + h_b}{2(r + r_c)} = \frac{3}{2} + \sum_{cyc} \frac{h_a + h_b}{4(r_a + r)} = 3 + \sum_{cyc} \frac{h_a - h_b}{2(r_a - r_b)} = \sum_{cyc} \cos^2 \frac{A}{2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

Observation: $p = s$ −*semiperimeter*, $F = S$ − *area*

1) $\sum_{cyc} \frac{h_a h_b}{4rr_c}$

$$\frac{h_a h_b}{4rr_c} = \frac{\frac{2S}{a} \cdot \frac{2S}{b}}{4r.p\tan\frac{C}{2}} = \frac{4S^2}{4abrp.\tan\frac{C}{2}} = \frac{abc.pr}{4Rabrp.\tan\frac{C}{2}} = \frac{1}{4R} \cdot \frac{c}{\tan\frac{C}{2}} =$$

$$= \frac{1}{4R} \cdot \frac{2R\sin C}{\tan\frac{C}{2}} = \frac{\sin\frac{C}{2}.\cos\frac{C}{2}}{\tan\frac{C}{2}} = \cos^2\frac{C}{2} \qquad \boxed{\sum_{cyc} \frac{h_a h_b}{4rr_c} = \sum_{cyc} \cos^2\frac{C}{2}}$$

2) $\sum_{cyc} \frac{h_a + h_b}{2(r + r_c)}$

$$\frac{h_a + h_b}{2(r + r_c)} = \frac{\frac{2S}{a} + \frac{2S}{b}}{2\left(p\tan\frac{C}{2} + (p - c)\tan\frac{C}{2}\right)} = \frac{2S(a + b)}{2ab(a + b)\tan\frac{C}{2}} = \frac{abc}{4Rab.\tan\frac{C}{2}} =$$

$$= \frac{c}{4R\tan\frac{C}{2}} = \frac{2R\sin C}{4R\tan\frac{C}{2}} = \frac{\sin\frac{C}{2}.\cos\frac{C}{2}}{\tan\frac{C}{2}} = \cos^2\frac{C}{2} \qquad \boxed{\sum_{cyc} \frac{h_a + h_b}{2(r + r_c)} = \sum_{cyc} \cos^2\frac{C}{2}}$$

3) $\frac{3}{2} + \sum_{cyc} \frac{h_a + h_b}{4(r_a + r)}$

$$\frac{h_a + h_b}{4(r_a + r)} = \frac{\frac{2S}{a} + \frac{2S}{b}}{4(4R + r - r_c)} = \frac{2S(a + b)}{4ab\left(4R + (p - c)\tan\frac{C}{2} - p\tan\frac{C}{2}\right)} =$$

$$=\frac{2S(a+b)}{4ab\left(4R-ctan\frac{C}{2}\right)}=\frac{abc(a+b)}{2ab.4R.2R\left(2-sinCtan\frac{C}{2}\right)}=\frac{c(a+b)}{16R^2\left(2-2sin^2\frac{C}{2}\right)}=$$

$$=\frac{4R^2sinC(sinA+sinB)}{32R^2cos^2\frac{C}{2}}=\frac{16R^2.sin\frac{C}{2}.cos\frac{C}{2}}{32R^2cos^2\frac{C}{2}}.sin\frac{A+B}{2}.cos\frac{A-B}{2}=$$

$$=\frac{sin\frac{C}{2}.cos\frac{C}{2}.cos\frac{C}{2}}{2cos^2\frac{C}{2}}.cos\frac{A-B}{2}=\frac{1}{2}sin\frac{C}{2}\left(sin\frac{C+A-B}{2}+sin\frac{C+B-A}{2}\right)$$

$$=\frac{1}{4}(cosB+cosA)$$

$$\frac{3}{2}+\sum_{cyc}\frac{h_a+h_b}{4(r_a+r)}=\frac{3}{2}+\sum_{cyc}\frac{1}{4}(cosB+cosA)=\frac{3}{2}+\frac{1}{2}(cosA+cosB+cosC)=$$

$$=\frac{3}{2}+\frac{1}{2}\left(-3+2\left(cos^2\frac{A}{2}+cos^2\frac{B}{2}+cos^2\frac{C}{2}\right)\right)=\frac{3}{2}+\sum_{cyc}cos^2\frac{C}{2}-\frac{3}{2}=\sum_{cyc}cos^2\frac{C}{2}$$

$$\boxed{\frac{3}{2}+\sum_{cyc}\frac{h_a+h_b}{4(r_a+r)}=\frac{3}{2}+\sum_{cyc}cos^2\frac{C}{2}-\frac{3}{2}=\sum_{cyc}cos^2\frac{C}{2}}$$

4) $3+\sum_{cyc}\frac{h_a-h_b}{2(r_a-r_b)}$

$$\frac{h_a-h_b}{2(r_a-r_b)}=\frac{\frac{2S}{a}-\frac{2S}{b}}{2\left(ptan\frac{A}{2}-ptan\frac{B}{2}\right)}=\frac{2S(b-a)}{2abp\left(tan\frac{A}{2}-tan\frac{B}{2}\right)}=\frac{abc.2R(sinB-sinA)}{4Rabp\frac{sin\frac{A-B}{2}}{cos\frac{A}{2}.cos\frac{B}{2}}}$$

$$=\frac{c}{p}.\frac{sin\frac{B-A}{2}.cos\frac{A+B}{2}.cos\frac{A}{2}.cos\frac{B}{2}}{sin\frac{A-B}{2}}=-\frac{2RsinC.sin\frac{C}{2}.cos\frac{A}{2}.cos\frac{B}{2}}{p}$$

$$=-\frac{4Rsin\frac{C}{2}.cos\frac{C}{2}.sin\frac{C}{2}.cos\frac{A}{2}.cos\frac{B}{2}}{p}=-\frac{4Rsin^2\frac{C}{2}\left(cos\frac{A}{2}.cos\frac{B}{2}.cos\frac{C}{2}\right)}{p}$$

$$=-4R.\frac{sin^2\frac{C}{2}.\frac{P}{4R}}{p}=-sin^2\frac{C}{2}=-\left(1-cos^2\frac{C}{2}\right)=cos^2\frac{C}{2}-1$$

$$\boxed{3+\sum_{cyc}\frac{h_a-h_b}{2(r_a-r_b)}=\sum_{cyc}cos^2\frac{C}{2}}$$

2065. In ΔABC the following relationship holds:

$$\left(\sum\csc A\right)\left(\sum\cot\frac{A}{2}\right)\geq 2\sum\cot\frac{A}{2}\cot\frac{B}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\left(\sum \csc A\right) = \sum \frac{1^2}{\sin A} \overset{Bergstrom}{\geq} \frac{(1+1+1)^2}{\sum \sin A} = \frac{9}{\frac{s}{R}} = \frac{9R}{s} \ (1)$$

$$\left(\sum \cot\frac{A}{2}\right) = \sum \frac{s}{r_a} = \frac{s}{r} \ (2)$$

$$2\sum \cot\frac{A}{2}\cot\frac{B}{2} = 2\sum \frac{s}{r_a}\cdot\frac{s}{r_b} = 2s^2\frac{\sum r_a}{r_a r_b r_c} =$$

$$= 2s^2\frac{4R+r}{s^2 r} = \frac{8R+2r}{r} \overset{Euler}{\leq} \frac{(8R+R)}{r} = \frac{9R}{r} \ (A)$$

$$\left(\sum \csc A\right)\left(\sum \cot\frac{A}{2}\right) \overset{(1)\&(2)}{\geq} \frac{9R}{s}\cdot\frac{s}{r} = \frac{9R}{r} \ (B)$$

$$\textit{From } (A) \textit{ and } (B) \textit{ we get } \left(\sum \csc A\right)\left(\sum \cot\frac{A}{2}\right) \geq 2\sum \cot\frac{A}{2}\cot\frac{B}{2}$$

$$\textit{Equality holds for } A = B = C$$

2066.In any Δ ABC with $n_a, n_b, n_c \to$

Nagel's cevians, the following relationship holds :

$$n_a + n_b + n_c + 18r \geq 3(h_a + h_b + h_c)$$

Proposed by Mohamed Amine Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - s(a^2 - (b-c)^2) = as(s-a) + s(b-c)^2 \Rightarrow n_a^2 \overset{(1)}{=} s(s-a) + \frac{s}{a}(b-c)^2$$

$$\text{Now, } n_a \overset{?}{\geq} \frac{b^2 - bc + c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \overset{?}{\geq} \frac{b^2 - bc + c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2} - 1 \overset{?}{\geq} \left(\frac{b^2 - bc + c^2}{bc}\right)^2 - 1$$

$$=\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\Leftrightarrow\frac{n_a^2-h_a^2}{h_a^2}\overset{?}{\geq}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}$$

$$\overset{\text{via }(1)}{\Leftrightarrow}s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}\overset{?}{\geq}$$

$$\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\cdot\frac{b^2c^2}{4R^2}\Leftrightarrow\left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2\overset{?}{\geq}\frac{(b-c)^2(b^2+c^2)}{4R^2}$$

$$\Leftrightarrow\frac{s^2}{a^2}\overset{?}{\geq}\frac{b^2+c^2}{4R^2}\ (\because (b-c)^2\geq 0)\Leftrightarrow 4R^2s^2\overset{?}{\geq}a^2b^2+c^2a^2\to\text{true}$$

$$(\text{strict inequality})\because 4R^2s^2\overset{\text{Goldstone}}{\geq}\sum_{\text{cyc}}a^2b^2>a^2b^2+c^2a^2$$

$$\therefore n_a\geq\frac{b^2-bc+c^2}{2R}\text{ and analogs}\Rightarrow n_a+n_b+n_c+18r-3(h_a+h_b+h_c)$$

$$\geq\frac{2\sum_{\text{cyc}}a^2-\sum_{\text{cyc}}ab-3\sum_{\text{cyc}}ab}{2R}+18r$$

$$=\frac{2(s^2-4Rr-r^2)-2(s^2+4Rr+r^2)+18Rr}{R}=\frac{2Rr-4r^2}{R}=\frac{2r(R-2r)}{R}\overset{\text{Euler}}{\geq}0$$

$$\therefore n_a+n_b+n_c+18r\geq 3(h_a+h_b+h_c)\ \forall\,\Delta\,ABC,$$
with equality iff Δ ABC is equilateral (QED)

2067. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{(b^2+c^2)\frac{\sin C}{\sin A}+(c^2+a^2)\frac{\sin B}{\sin A}}{\sqrt{\frac{b^2+c^2}{a^2+b^2}}.\sin C+\sqrt{\frac{c^2+a^2}{a^2+b^2}}.\sin B}\geq 48\sqrt{3}r^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, }\sqrt[3]{\left(\sum_{\text{cyc}}a^2+a^2\right)\left(\sum_{\text{cyc}}a^2+b^2\right)\left(\sum_{\text{cyc}}a^2+c^2\right)}$$

$$\overset{\text{Leibnitz}}{\leq}\sqrt[3]{(9R^2+a^2)(9R^2+b^2)(9R^2+c^2)}\overset{A-G}{\leq}\sum_{\text{cyc}}\frac{9R^2+a^2}{3}=\frac{27R^2+\sum_{\text{cyc}}a^2}{3}$$

$$\overset{\text{Leibnitz}}{\leq}\frac{27R^2+9R^2}{3}=12R^2\Rightarrow\left(\sum_{\text{cyc}}a^2+a^2\right)\left(\sum_{\text{cyc}}a^2+b^2\right)\left(\sum_{\text{cyc}}a^2+c^2\right)\leq$$

$$144 * 12R^6 \Rightarrow \sqrt{\left(\sum_{cyc} a^2 + a^2\right)\left(\sum_{cyc} a^2 + b^2\right)\left(\sum_{cyc} a^2 + c^2\right)} \le 24\sqrt{3}R^3 \to (1)$$

$$\text{Now, } \frac{(b^2+c^2)\frac{\sin C}{\sin A} + (c^2+a^2)\frac{\sin B}{\sin A}}{\sqrt{\frac{b^2+c^2}{a^2+b^2}}.\sin C + \sqrt{\frac{c^2+a^2}{a^2+b^2}}.\sin B} + \frac{(a^2+b^2)\frac{\sin C}{\sin B} + (c^2+a^2)\frac{\sin A}{\sin B}}{\sqrt{\frac{a^2+b^2}{b^2+c^2}}.\sin C + \sqrt{\frac{c^2+a^2}{b^2+c^2}}.\sin A}$$

$$+ \frac{(a^2+b^2)\frac{\sin B}{\sin C} + (b^2+c^2)\frac{\sin A}{\sin C}}{\sqrt{\frac{a^2+b^2}{c^2+a^2}}.\sin B + \sqrt{\frac{b^2+c^2}{c^2+a^2}}.\sin A}$$

$$\overset{CBS}{\ge} 2R.\frac{(b^2+c^2)\frac{c}{a} + (c^2+a^2)\frac{b}{a}}{\sqrt{\frac{b^2+c^2}{a^2+b^2} + \frac{c^2+a^2}{a^2+b^2}}.\sqrt{b^2+c^2}} + 2R.\frac{(a^2+b^2)\frac{c}{b} + (c^2+a^2)\frac{a}{b}}{\sqrt{\frac{a^2+b^2}{b^2+c^2} + \frac{c^2+a^2}{b^2+c^2}}.\sqrt{c^2+a^2}}$$

$$+2R.\frac{(a^2+b^2)\frac{b}{c} + (b^2+c^2)\frac{a}{c}}{\sqrt{\frac{a^2+b^2}{c^2+a^2} + \frac{b^2+c^2}{c^2+a^2}}.\sqrt{a^2+b^2}} \overset{A-G}{\ge} 4R.\frac{\sqrt{(b^2+c^2)(c^2+a^2)}.\sqrt{\frac{bc}{a^2}}.\sqrt{a^2+b^2}}{\sqrt{b^2+c^2}.\sqrt{\sum_{cyc} a^2 + c^2}} +$$

$$4R.\frac{\sqrt{(a^2+b^2)(c^2+a^2)}.\sqrt{\frac{ca}{b^2}}.\sqrt{b^2+c^2}}{\sqrt{c^2+a^2}.\sqrt{\sum_{cyc} a^2 + a^2}} + 4R.\frac{\sqrt{(a^2+b^2)(b^2+c^2)}.\sqrt{\frac{ab}{c^2}}.\sqrt{c^2+a^2}}{\sqrt{a^2+b^2}.\sqrt{\sum_{cyc} a^2 + b^2}}$$

$$\overset{A-G}{\ge} 12R.\sqrt[3]{\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{\sqrt{(\sum_{cyc} a^2 + a^2)(\sum_{cyc} a^2 + b^2)(\sum_{cyc} a^2 + c^2)}}} \overset{\substack{Cesaro\\ and\\ via\ (1)}}{\ge} 12R.\sqrt[3]{\frac{8a^2b^2c^2}{24\sqrt{3}R^3}}$$

$$= \frac{12R}{\sqrt{3}R}.\sqrt[3]{16R^2r^2s^2} \overset{\substack{Euler\\ and\\ Mitrinovic}}{\ge} \frac{12}{\sqrt{3}}.\sqrt[3]{64 * 27r^6} = \frac{48}{\sqrt{3}}.3r^2$$

$$\therefore \frac{(b^2+c^2)\frac{\sin C}{\sin A} + (c^2+a^2)\frac{\sin B}{\sin A}}{\sqrt{\frac{b^2+c^2}{a^2+b^2}}.\sin C + \sqrt{\frac{c^2+a^2}{a^2+b^2}}.\sin B} + \frac{(a^2+b^2)\frac{\sin C}{\sin B} + (c^2+a^2)\frac{\sin A}{\sin B}}{\sqrt{\frac{a^2+b^2}{b^2+c^2}}.\sin C + \sqrt{\frac{c^2+a^2}{b^2+c^2}}.\sin A}$$

$$+ \frac{(a^2+b^2)\frac{\sin B}{\sin C} + (b^2+c^2)\frac{\sin A}{\sin C}}{\sqrt{\frac{a^2+b^2}{c^2+a^2}}.\sin B + \sqrt{\frac{b^2+c^2}{c^2+a^2}}.\sin A} \ge 48\sqrt{3}r^2\ \forall\ \Delta\ ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)

2068.

In any non − obtuse Δ ABC, the following relationship holds :

$$r_a \le 2R + r \le \frac{R}{r} \cdot h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_a \le 2R + r \Leftrightarrow \frac{rs}{s-a} - \frac{rs}{s} \le 2R \Leftrightarrow \frac{rs.(s-s+a)}{s(s-a)} \le 2R$$

$$\Leftrightarrow \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\sin\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \le 2R \Leftrightarrow \sin^2\frac{A}{2} \le \frac{1}{2} \Leftrightarrow \sin\frac{A}{2} \le \frac{1}{\sqrt{2}}$$

$$\Leftrightarrow \frac{A}{2} \le \frac{\pi}{4}\left(\because 0 < \frac{A}{2} < \frac{\pi}{2}\right) \Leftrightarrow A \le \frac{\pi}{2} \to \text{true} \because \Delta\,ABC \text{ is non} - \text{obtuse} \therefore r_a \le 2R + r$$

$$Again, 2R + r \le \frac{R}{r}.h_a \Leftrightarrow R\left(\frac{2rs}{ra} - 2\right) \ge r \Leftrightarrow \frac{2(s-a)}{a} \ge \frac{r}{R}$$

$$\Leftrightarrow \frac{8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}} \ge 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Leftrightarrow \sin^2\frac{A}{2} \le \frac{1}{2} \Leftrightarrow \sin\frac{A}{2} \le \frac{1}{\sqrt{2}}$$

$$\Leftrightarrow \frac{A}{2} \le \frac{\pi}{4}\left(\because 0 < \frac{A}{2} < \frac{\pi}{2}\right) \Leftrightarrow A \le \frac{\pi}{2} \to \text{true} \because \Delta\,ABC \text{ is non} - \text{obtuse}$$

$$\therefore \frac{R}{r}.h_a \ge 2R + r \text{ and so}, r_a \le 2R + r \le \frac{R}{r}.h_a \ \forall \text{ non} - \text{obtuse } \Delta\,ABC$$

$$'' ='' \text{ iff } \Delta\,ABC \text{ is right} - \text{angled at } \hat{A} \text{ (QED)}$$

2069.

In any Δ ABC, the following relationship holds :

$$\frac{r}{s} + \frac{s}{R} + \frac{R}{r} \ge \frac{36 + 29\sqrt{3}}{18}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{r}{s} + \frac{s}{R} \overset{\substack{\text{Mitrinovic} \\ \text{and} \\ \text{Gerretsen + Euler}}}{\ge} \frac{2r}{3\sqrt{3}R} + \frac{\sqrt{3(4Rr + r^2)}}{R} = \frac{2\sqrt{3}r}{9R} + \frac{\sqrt{3(4Rr + r^2)}}{R}$$

$$\therefore \text{it suffices to prove : } \frac{2\sqrt{3}r}{9R} + \frac{\sqrt{3(4Rr + r^2)}}{R} + \frac{R}{r} \ge \frac{36 + 29\sqrt{3}}{18}$$

$$\Leftrightarrow \frac{R}{r} - 2 \ge \sqrt{3}\left(\frac{29}{18} - \frac{2r}{9R} - \sqrt{\frac{4r}{R} + \frac{r^2}{R^2}}\right)$$

$$\Leftrightarrow t - 2 \overset{(1)}{\ge} \sqrt{3}\left(\frac{29}{18} - \frac{2}{9t} - \sqrt{\frac{4}{t} + \frac{1}{t^2}}\right) \left(t = \frac{R}{r}\right)$$

$$\text{Let } f(t)=t-2-\sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right)\ \forall\, t\ge 2 \text{ and then :}$$

$$f'(t)=1-\sqrt{3}\left(\frac{2}{9t^2}+\frac{\frac{2}{t^2}+\frac{1}{t^3}}{\sqrt{\frac{4}{t}+\frac{1}{t^2}}}\right)\overset{\frac{1}{t^2}>0}{>}1-\sqrt{3}\left(\frac{2}{9t^2}+\frac{\frac{2}{t^2}+\frac{1}{t^3}}{2\sqrt{\frac{1}{t}}}\right)$$

$$=1-\sqrt{3}\left(\frac{2}{9m^4}+\frac{\frac{2}{m^4}+\frac{1}{m^6}}{\frac{2}{m}}\right)\ (m=\sqrt{t})$$

$$=1-\sqrt{3}\left(\frac{2}{9m^4}+\frac{1}{m^3}+\frac{1}{2m^5}\right)\overset{m=\sqrt{\frac{R}{r}}\ge\sqrt{2}\text{ via Euler}}{\ge}1-\sqrt{3}\left(\frac{2}{9.4}+\frac{1}{2\sqrt{2}}+\frac{1}{8\sqrt{2}}\right)$$

$$\approx 0.1383>0\Rightarrow f'(t)>0\ \forall\, t\ge 2\Rightarrow f(t)\text{ is }\uparrow\text{ on }[2,\infty)\Rightarrow f(t)\ge f(2)=0\ \forall\, t\ge 2$$

$$\therefore t-2\ge\sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right)\Rightarrow ① \text{ is true}\ \therefore \frac{r}{s}+\frac{s}{R}+\frac{R}{r}\ge\frac{36+29\sqrt{3}}{18}$$

$\forall\, \Delta ABC$, " = " iff ΔABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\frac{r}{s}+\frac{s}{R}+\frac{R}{r}=\left(\frac{r}{s}+\frac{2s}{27R}+\frac{\sqrt{3}R}{18r}\right)+\frac{25}{18}\left(\frac{s}{3R}+\frac{s}{3R}+\frac{\sqrt{3}R}{4r}\right)+\frac{(72-29\sqrt{3})R}{72r}$$

$$\overset{AM-GM}{\ge}\frac{3}{\sqrt[3]{81\sqrt{3}}}+\frac{25}{18}.3\sqrt[3]{\frac{s^2}{12\sqrt{3}Rr}}+\frac{(72-29\sqrt{3})R}{72r}$$

$$\overset{\substack{Cosnita\ Turtoiu\\ Euler}}{\ge}\frac{\sqrt{3}}{3}+\frac{25}{6}.\sqrt[3]{\frac{27}{12\sqrt{3}.2}}+\frac{(72-29\sqrt{3}).2}{72}=\frac{36+29\sqrt{3}}{18}.$$

Equality holds iff ΔABC is equilateral.

2070.In any Δ ABC, the following relationship holds :

$$\frac{R}{s}+\frac{s}{r}+\frac{r}{R}\ge\frac{9+58\sqrt{3}}{18}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{R}{s}+\frac{s}{r}-\frac{58\sqrt{3}}{18}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen + Euler}}}{\geq}\frac{2}{3\sqrt{3}}+\frac{\sqrt{3(4Rr+r^2)}}{r}-\frac{29\sqrt{3}}{9}=$$

$$=\frac{2}{3\sqrt{3}}+\frac{3.\sqrt{4Rr+r^2}}{\sqrt{3}r}-\frac{29}{3\sqrt{3}}=\frac{9.\sqrt{4Rr+r^2}-27r}{3\sqrt{3}r}=\frac{3.\sqrt{4Rr+r^2}-9r}{\sqrt{3}r}\overset{?}{\geq}\frac{1}{2}-\frac{r}{R}$$

$$\Leftrightarrow\frac{9(4Rr+r^2)+81r^2-54r.\sqrt{4Rr+r^2}}{3r^2}\overset{?}{\geq}\frac{(R-2r)^2}{4R^2}$$

$$\Leftrightarrow\frac{3(4R+r)+27r}{r}-\frac{(R-2r)^2}{4R^2}\overset{?}{\geq}\frac{18.\sqrt{4Rr+r^2}}{r}$$

$$\Leftrightarrow\frac{(4R^2(12R+30r)-r(R-2r)^2)^2}{16R^4r^2}\overset{?}{\geq}\frac{324(4Rr+r^2)}{r^2}$$

$$\left(\because\frac{3(4R+r)+27r}{r}\overset{\text{Euler}}{\geq}54>1\overset{\text{Euler}}{\geq}\frac{(R-2r)^2}{4R^2}\Rightarrow 4R^2(12R+30r)-r(R-2r)^2>0\right)$$

$$\Leftrightarrow 2304t^6-9312t^5+9361t^4+568t^3-936t^2-32t+16\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)^2\left((t-2)(2304t^3+4512t^2+8785t+17566)+35136\right)\overset{?}{\geq}0\to\text{true}$$

$$\because t\overset{\text{Euler}}{\geq}2\ \therefore\frac{R}{s}+\frac{s}{r}-\frac{58\sqrt{3}}{18}\geq\frac{1}{2}-\frac{r}{R}$$

$$\Rightarrow\frac{R}{s}+\frac{s}{r}+\frac{r}{R}\geq\frac{9+58\sqrt{3}}{18}\ \forall\Delta ABC,''=''\ \text{iff}\ \Delta ABC\ \text{is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\frac{R}{s}+\frac{s}{r}+\frac{r}{R}=\left(\frac{R}{s}+\frac{2s}{27r}\right)+\left(\frac{r}{R}+\frac{\sqrt{3}s}{18r}+\frac{\sqrt{3}s}{18r}\right)+\frac{(25-3\sqrt{3})s}{27r}$$

$$\overset{AM-GM}{\geq}2\sqrt{\frac{2R}{27r}}+3\sqrt[3]{\frac{s^2}{108Rr}}+\frac{(25-3\sqrt{3})s}{27r}$$

$$\overset{\substack{\text{Cosnita-Turtoiu}\\ \text{Euler and Mitrinovic}}}{\geq}2\sqrt{\frac{4}{27}}+3\sqrt[3]{\frac{27}{108.2}}+\frac{(25-3\sqrt{3}).3\sqrt{3}}{27}=\frac{9+58\sqrt{3}}{18}.$$

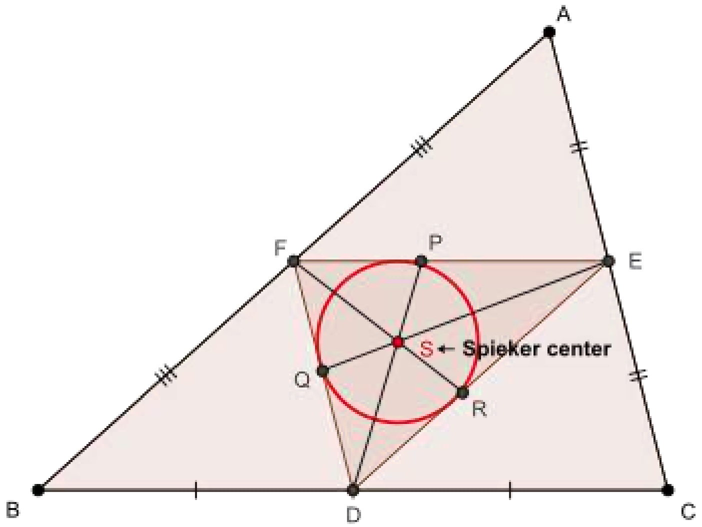
Equality holds iff ΔABC is equilateral.

2071. In any Δ ABC with $p_a, p_b, p_c \to$ Spieker cevians, the following relationship holds:

$$p_a + p_b + p_c \geq \frac{9}{2}\cdot\sqrt{2Rr}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at:

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, } b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)\Rightarrow p_a^2-m_a^2=$$

$$\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, }p_a-m_a\ge\frac{(b-c)^2}{2(2s+a)}\Leftrightarrow p_a^2\ge m_a^2+\frac{(b-c)^4}{4(2s+a)^2}+\frac{m_a.(b-c)^2}{2s+a}$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow}\frac{(b-c)^2(8s^2-a^2-(b-c)^2)}{4(2s+a)^2}\overset{(\blacksquare)}{\ge}\frac{m_a.(b-c)^2}{2s+a}$$

$$\text{We note that : }8s^2-a^2-(b-c)^2>8s^2-2a^2>0\therefore(\blacksquare)\Leftrightarrow$$

$$\frac{(8s^2-a^2)^2+(b-c)^4-2(8s^2-a^2)(b-c)^2}{16(2s+a)^2}$$

$$\ge\left(s(s-a)+\frac{(b-c)^2}{4}\right)\ (\because(b-c)^2\ge 0)$$

$$\Leftrightarrow(8s^2-a^2)^2-16s(s-a)(2s+a)^2+(b-c)^4-$$

$$2(b-c)^2(8s^2-a^2+2(2s+a)^2)\geq 0$$

$$\Leftrightarrow (b-c)^4-2(4s+a)^2(b-c)^2+a^2(32s^2+16sa+a^2)\overset{(\blacksquare\blacksquare)}{\geq} 0$$

Now, $(\blacksquare\blacksquare)$ is a quadrilateral in $(b-c)^2$ with discriminant $=$ $4(4s+a)^4-4a^2(32s^2+16sa+a^2)=256s^2(2s+a)^2$ $\therefore$ in order to prove $(\blacksquare\blacksquare)$, it suffices to prove : $(b-c)^2\leq\frac{2(4s+a)^2-16s(2s+a)}{2}$ $=a^2\to$ true $\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)$ is true $\therefore p_a-m_a\overset{(\bullet\bullet)}{\geq}\frac{(b-c)^2}{2(2s+a)}$ and analogs

Via Chu and Yang, $\left(\sum_{cyc}m_a\right)^2\geq\frac{9s^2}{4}+\frac{7s^2}{4}-28Rr+29r^2\overset{\text{Gerretsen}}{\geq}$

$$\frac{9s^2}{4}+\frac{7(16Rr-5r^2)}{4}-28Rr+29r^2=\frac{9s^2}{4}+\left(29-\frac{35}{4}\right)r^2>\frac{9s^2}{4}\Rightarrow\sum_{cyc}m_a\overset{(\bullet\bullet\bullet)}{>}\frac{3s}{2}$$

We have : $\sum_{cyc}\frac{(b-c)^2}{(2s+a)}\overset{(l)}{=}\frac{1}{2s(9s^2+6Rr+r^2)}.\sum_{cyc}\left((b-c)^2(8s^2-2sa+bc)\right)$

and, $\sum_{cyc}\left((b-c)^2(8s^2-2sa+bc)\right)=8s^2\sum_{cyc}(b-c)^2-2s.\sum_{cyc}a(b^2+c^2-2bc)$

$$+\sum_{cyc}\left(bc\left(\sum_{cyc}a^2-a^2-2bc\right)\right)$$

$$=16s^2(s^2-12Rr-3r^2)-2s.\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-9abc\right)+$$

$$2(s^2-4Rr-r^2)(s^2+4Rr+r^2)-2((s^2+4Rr+r^2)^2-16Rrs^2)-8Rrs^2$$

$$=16s^2(s^2-12Rr-3r^2)-4s^2(s^2-14Rr+r^2)+$$

$$2(s^2-4Rr-r^2)(s^2+4Rr+r^2)-2((s^2+4Rr+r^2)^2-16Rrs^2)-8Rrs^2\overset{\text{via }(l)}{\Rightarrow}$$

$$\sum_{cyc}\frac{(b-c)^2}{2(2s+a)}\boxed{\overset{(m)}{=}}\frac{4s^2(s^2-12Rr-3r^2)-s^2(s^2-14Rr+r^2)+r\left((2R-r)s^2-r(4R+r)^2\right)}{s(9s^2+6Rr+r^2)}$$

$$=\frac{\sigma}{s(9s^2+6Rr+r^2)}\text{ (say) }\therefore(\bullet\bullet)\text{ and }(m)\Rightarrow$$

$$p_a+p_b+p_c\geq\sum_{cyc}m_a+\frac{\sigma}{s(9s^2+6Rr+r^2)}\Rightarrow(p_a+p_b+p_c)^2\geq$$

$$\left(\sum_{cyc}m_a\right)^2+\frac{\sigma^2}{s^2(9s^2+6Rr+r^2)^2}+\frac{2\sigma}{s(9s^2+6Rr+r^2)}.\left(\sum_{cyc}m_a\right)$$

$$\overset{\text{via Chu and Yang,and }(\bullet\bullet\bullet)}{\geq} 4s^2-28Rr+29r^2+\frac{\sigma^2}{s^2(9s^2+6Rr+r^2)^2}+\frac{3\sigma}{9s^2+6Rr+r^2}$$

$$=\frac{s^2(4s^2-28Rr+29r^2)(9s^2+6Rr+r^2)^2+\sigma^2+3\sigma.s^2(9s^2+6Rr+r^2)}{s^2(9s^2+6Rr+r^2)^2}\overset{?}{\geq}\frac{81Rr}{2}$$

$$\Leftrightarrow 828s^8-(12237Rr-3936r^2)s^6-r^2(14668R^2-4462Rr-1294r^2)s^4$$

$$-r^3(3460R^3-1980R^2r-1051Rr^2-108r^3)s^2+2r^4(4R+r)^4 \boxed{\overset{?}{\underset{(1)}{\geq}}} 0$$

$$\text{Now, }T=828(s^2-16Rr+5r^2)^4+(40755Rr-12624r^2)(s^2-16Rr+5r^2)^3$$
$$+r^2(669764R^2-417935Rr+66454r^2)(s^2-16Rr+5r^2)^2+$$
$$4r^3(923775R^3-882507R^2r+291296Rr^2-32908r^3)(s^2-16Rr+5r^2)$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove (1), it suffices to prove : LHS of (1)} \geq T$$

$$\Leftrightarrow 9200t^4-32755t^3+34842t^2-13060t+1592\geq 0\ \left(t=\frac{R}{r}\right)\Leftrightarrow$$

$$(t-2)\left((t-2)(9200t^2+4045t+14222)+27648\right)\geq 0\to \text{true}\Rightarrow (1) \text{ is true}$$

$$\Rightarrow (p_a+p_b+p_c)^2\geq\frac{81Rr}{2}\therefore p_a+p_b+p_c\geq\frac{9}{2}.\sqrt{2Rr}$$

$\forall\,\Delta\,ABC$, with equality iff $\Delta\,ABC$ is equilateral (QED)

2072. In any $\Delta\,ABC$ the following relationship holds :

$$(R+s+r)\left(\frac{1}{R}+\frac{1}{s}+\frac{1}{r}\right)\geq\frac{33+29\sqrt{3}}{6}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{R}{s}+\frac{s}{r}-\frac{58\sqrt{3}}{18}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen + Euler}}}{\geq}\frac{2}{3\sqrt{3}}+\frac{\sqrt{3(4Rr+r^2)}}{r}-\frac{29\sqrt{3}}{9}$$

$$=\frac{2}{3\sqrt{3}}+\frac{3.\sqrt{4Rr+r^2}}{\sqrt{3}r}-\frac{29}{3\sqrt{3}}=\frac{9.\sqrt{4Rr+r^2}-27r}{3\sqrt{3}r}=\frac{3.\sqrt{4Rr+r^2}-9r}{\sqrt{3}r}\overset{?}{\geq}\frac{1}{2}-\frac{r}{R}$$

$$\Leftrightarrow\frac{9(4Rr+r^2)+81r^2-54r.\sqrt{4Rr+r^2}}{3r^2}\overset{?}{\geq}\frac{(R-2r)^2}{4R^2}$$

$$\Leftrightarrow\frac{3(4R+r)+27r}{r}-\frac{(R-2r)^2}{4R^2}\overset{?}{\geq}\frac{18.\sqrt{4Rr+r^2}}{r}$$

$$\Leftrightarrow\frac{(4R^2(12R+30r)-r(R-2r)^2)^2}{16R^4r^2}\overset{?}{\geq}\frac{324(4Rr+r^2)}{r^2}$$

$$\left(\because\frac{3(4R+r)+27r}{r}\overset{\text{Euler}}{\geq}54>1\overset{\text{Euler}}{\geq}\frac{(R-2r)^2}{4R^2}\atop\Rightarrow 4R^2(12R+30r)-r(R-2r)^2>0\right)$$

$$\Leftrightarrow 2304t^6-9312t^5+9361t^4+568t^3-936t^2-32t+16\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)^2\left((t-2)(2304t^3+4512t^2+8785t+17566)+35136\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{R}{s}+\frac{s}{r}-\frac{58\sqrt{3}}{18} \underset{\text{Mitrinovic}}{\geq} \frac{1}{2}-\frac{r}{R} \Rightarrow \frac{R}{s}+\frac{s}{r}+\frac{r}{R} \geq \frac{9+58\sqrt{3}}{18} \to \text{(i)}$$

$$\text{Again}, \frac{r}{s}+\frac{s}{R} \overset{\substack{\text{and}\\ \text{Gerretsen + Euler}}}{\geq} \frac{2r}{3\sqrt{3}R}+\frac{\sqrt{3(4Rr+r^2)}}{R} = \frac{2\sqrt{3}r}{9R}+\frac{\sqrt{3(4Rr+r^2)}}{R}$$

$$\therefore \text{in order to prove}: \frac{r}{s}+\frac{s}{R}+\frac{R}{r} \geq \frac{36+29\sqrt{3}}{18}, \text{it suffices to prove}:$$

$$\frac{2\sqrt{3}r}{9R}+\frac{\sqrt{3(4Rr+r^2)}}{R}+\frac{R}{r} \geq \frac{36+29\sqrt{3}}{18} \Leftrightarrow \frac{R}{r}-2 \geq \sqrt{3}\left(\frac{29}{18}-\frac{2r}{9R}-\sqrt{\frac{4r}{R}+\frac{r^2}{R^2}}\right)$$

$$\Leftrightarrow t-2 \overset{①}{\geq} \sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right) \left(t=\frac{R}{r}\right)$$

$$\text{Let } f(t) = t-2-\sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right) \forall t \geq 2 \text{ and then}:$$

$$f'(t) = 1-\sqrt{3}\left(\frac{2}{9t^2}+\frac{\frac{2}{t^2}+\frac{1}{t^3}}{\sqrt{\frac{4}{t}+\frac{1}{t^2}}}\right) \overset{\frac{1}{t^2}>0}{>} 1-\sqrt{3}\left(\frac{2}{9t^2}+\frac{\frac{2}{t^2}+\frac{1}{t^3}}{2\sqrt{\frac{1}{t}}}\right)$$

$$= 1-\sqrt{3}\left(\frac{2}{9m^4}+\frac{\frac{2}{m^4}+\frac{1}{m^6}}{\frac{2}{m}}\right) (m=\sqrt{t})$$

$$= 1-\sqrt{3}\left(\frac{2}{9m^4}+\frac{1}{m^3}+\frac{1}{2m^5}\right) \overset{m=\sqrt{\frac{R}{r}}\geq\sqrt{2} \text{ via Euler}}{\geq} 1-\sqrt{3}\left(\frac{2}{9.4}+\frac{1}{2\sqrt{2}}+\frac{1}{8\sqrt{2}}\right)$$

$$\approx 0.1383 > 0 \Rightarrow f'(t) > 0 \ \forall t \geq 2 \Rightarrow f(t) \text{ is } \uparrow \text{ on } [2,\infty) \Rightarrow f(t) \geq f(2) = 0 \ \forall t \geq 2$$

$$\therefore t-2 \geq \sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right) \Rightarrow ① \text{ is true} \therefore \frac{r}{s}+\frac{s}{R}+\frac{R}{r} \geq \frac{36+29\sqrt{3}}{18} \to \text{(ii)}$$

$$\text{We have}: (R+s+r)\left(\frac{1}{R}+\frac{1}{s}+\frac{1}{r}\right) = 3+\frac{R}{s}+\frac{R}{r}+\frac{s}{R}+\frac{s}{r}+\frac{r}{R}+\frac{r}{s}$$

$$= 3+\frac{R}{s}+\frac{s}{r}+\frac{r}{R}+\frac{r}{s}+\frac{s}{R}+\frac{R}{r} \overset{\text{via (i)+(ii)}}{\geq} 3+\frac{9+58\sqrt{3}}{18}+\frac{36+29\sqrt{3}}{18}$$

$$= \frac{33+29\sqrt{3}}{6} \ \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$(R+s+r)\left(\frac{1}{R}+\frac{1}{s}+\frac{1}{r}\right)=\frac{R}{r}+\frac{r}{R}+(R+r)\left(\frac{s}{Rr}+\frac{1}{s}\right)+3=$$
$$=\frac{3R}{4r}+\left(\frac{R}{4r}+\frac{r}{R}\right)+(R+r)\left(\frac{25s}{27Rr}+\left(\frac{2s}{27Rr}+\frac{1}{s}\right)\right)+3\geq$$
$$\overset{\substack{Cosnita-Turtoiu\\ Euler\ and\ AM-GM}}{\geq}\frac{3}{2}+1+3\sqrt{\frac{Rr}{2}}.\left(\frac{25}{27Rr}.\sqrt{\frac{27Rr}{2}}+2\sqrt{\frac{2}{27Rr}}\right)+3=$$
$$=\frac{33+29\sqrt{3}}{6}$$

Equality holds iff ΔABC is equilateral.

2073. In any acute ΔABC the following relationship holds :

$$\prod_{cyc}\left(\frac{1}{r_br_c-rr_a}+\frac{1}{r_cr_a-rr_b}-\frac{1}{r_ar_b+rr_c}\right)\geq\frac{27}{(abc)^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$r_br_c-rr_a=\frac{F^2}{(s-b)(s-c)}-\frac{r.sr}{s-a}=s(s-a)-(s-b)(s-c)=$$
$$=\frac{(b+c)^2-a^2}{4}-\frac{a^2-(b-c)^2}{4}=\frac{b^2+c^2-a^2}{2}$$
$$r_br_c+rr_a=\frac{F^2}{(s-b)(s-c)}+\frac{r.sr}{s-a}=s(s-a)+(s-b)(s-c)=$$
$$=2s^2-s(a+b+c)+bc=bc$$

Let $2x\coloneqq b^2+c^2-a^2\geq0,\ \ 2y\coloneqq c^2+a^2-b^2\geq0,\ \ 2z\coloneqq a^2+b^2-c^2\geq0$

We have

$$\frac{1}{r_br_c-rr_a}+\frac{1}{r_cr_a-rr_b}-\frac{1}{r_ar_b+rr_c}=\frac{2}{b^2+c^2-a^2}+\frac{2}{c^2+a^2-b^2}-\frac{1}{ab}=$$
$$=\frac{1}{x}+\frac{1}{y}-\frac{1}{\sqrt{(z+x)(z+y)}}=$$
$$=\frac{(x+y)\sqrt{(z+x)(z+y)}-xy}{xy\sqrt{(z+x)(z+y)}}\overset{AM-GM\,\&\,CBS}{\geq}\frac{2\sqrt{xy}.\left(z+\sqrt{xy}\right)-xy}{xy\sqrt{(z+x)(z+y)}}=$$
$$=\frac{2z+\sqrt{xy}}{\sqrt{xy(z+x)(z+y)}}\overset{AM-GM}{\geq}\frac{3\sqrt[3]{z^2\sqrt{xy}}}{\sqrt{xy(z+x)(z+y)}}$$

Therefore

$$\prod_{cyc}\left(\frac{1}{r_b r_c - r r_a}+\frac{1}{r_c r_a - r r_b}-\frac{1}{r_a r_b + r r_c}\right)\geq \prod_{cyc}\frac{3\sqrt[3]{z^2\sqrt{xy}}}{\sqrt{xy(z+x)(z+y)}}=$$

$$=\frac{27}{(x+y)(y+z)(z+x)}=\frac{27}{(abc)^2}$$

Equality holds iff ΔABC is equilateral.

2074.In any Δ ABC with $n_a, n_b, n_c \to$

Nagel cevians, the following relationship holds :

$$n_a+n_b+n_c \geq m_a+m_b+m_c+\frac{n_a^2+n_b^2+n_c^2-s^2}{2s}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=$$

$$as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-a+\frac{(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2\overset{(\bullet)}{=}s(s-a)+\frac{s}{a}.(b-c)^2$$

$$\text{Now, } n_a-m_a\overset{?}{\geq}\frac{(b-c)^2}{2a}\overset{\text{via }(\bullet)}{\Leftrightarrow}s(s-a)+\frac{s}{a}.(b-c)^2-s(s-a)-\frac{(b-c)^2}{4}$$

$$\overset{?}{\geq}\frac{(b-c)^4}{4a^2}+m_a.\frac{(b-c)^2}{a}\Leftrightarrow\frac{4s-a}{4}-\frac{(b-c)^2}{4a}\underset{(1)}{\overset{?}{\geq}}m_a\ (\because (b-c)^2\geq 0)$$

$$\text{Now, }\frac{4s-a}{4}-\frac{(b-c)^2}{4a}>\frac{4s-a}{4}-\frac{a^2}{4a}=\frac{4s-2a}{4}>0\ \therefore (1)\Leftrightarrow$$

$$\frac{(a(4s-a)-(b-c)^2)^2}{16a^2}\overset{?}{\geq}s(s-a)+\frac{(b-c)^2}{4}$$

$$\Leftrightarrow a^2(4s-a)^2+(b-c)^4-2a(4s-a)(b-c)^2\overset{?}{\geq}16a^2s(s-a)+4a^2(b-c)^2$$

$$\Leftrightarrow (b-c)^4-(8sa+2a^2)(b-c)^2+a^2(16s^2-8sa+a^2-16s^2+16sa)\overset{?}{\geq}0$$

$$\Leftrightarrow (b-c)^4 - (8sa+2a^2)(b-c)^2 + a^2(8sa+a^2) \overset{?}{\underset{②}{\geq}} 0$$

Now, LHS of ② is a quadratic polynomial in $(b-c)^2$ with discriminant $=$ $(8sa+2a^2)^2 - 4a^2(8sa+a^2) = 64a^2s^2$ ∴ in order to prove ②,

it suffices to prove : $(b-c)^2 \overset{?}{\leq} \frac{8sa+2a^2-\sqrt{64a^2s^2}}{2} \Leftrightarrow a^2 \overset{?}{\geq} (b-c)^2 \to$ true

$$\Rightarrow ② \Rightarrow ① \text{ is true } \therefore n_a - m_a \geq \frac{(b-c)^2}{2a} \text{ and analogs} \to (l)$$

$$\text{Now, } \sum_{cyc}\frac{(b-c)^2}{a} = \sum_{cyc}\frac{b^2+c^2+a^2}{a} - \sum_{cyc}a - \frac{2}{4Rrs}.\sum_{yc}b^2c^2$$

$$= \frac{1}{4Rrs}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right) - \frac{8Rrs^2}{4Rrs} - \frac{2}{4Rrs}\left(\left(\sum_{cyc}ab\right)^2 - 16Rrs^2\right)$$

$$= \frac{1}{4Rrs}\left(\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2 - 2\sum_{cyc}ab\right) + 24Rrs^2\right)$$

$$= \frac{2(s^2+4Rr+r^2)\left(s^2-4Rr-r^2-(s^2+4Rr+r^2)\right)+24Rrs^2}{4Rrs}$$

$$= \frac{(2R-r)s^2 - r(4R+r)^2}{Rs} \overset{(m)}{=} \sum_{cyc}\frac{(b-c)^2}{a}$$

$$\text{As deduced earlier, } an_a^2 = as^2 - \frac{4s(s-b)(s-c)(s-a)}{(s-a)} = as^2 - a.\frac{4F^2}{a(s-a)}$$

$$= as^2 - 2a.\frac{2F}{a}.\frac{F}{s-a} \Rightarrow n_a^2 = s^2 - 2h_ar_a = s^2 - 2.\frac{2rs^2\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}$$

$$\Rightarrow n_a^2 = s^2 - \frac{rs^2}{R}.\sec^2\frac{A}{2} \text{ and analogs} \Rightarrow \frac{n_a^2+n_b^2+n_c^2-s^2}{2s}$$

$$= \frac{3s^2 - \frac{rs^2}{R}.\frac{s^2+(4R+r)^2}{s^2} - s^2}{2s} = \frac{(2R-r)s^2 - r(4R+r)^2}{2Rs} \overset{\text{via (m)}}{=}$$

$$\sum_{cyc}\frac{(b-c)^2}{2a} \overset{\text{via }(l)}{\leq} \sum_{cyc}(n_a - m_a) \therefore n_a + n_b + n_c \geq m_a + m_b + m_c$$

$$+\frac{n_a^2+n_b^2+n_c^2-s^2}{2s} \; \forall \Delta ABC, \text{with equality iff } \Delta ABC \text{ is equilateral (QED)}$$

2075. In any ΔABC, the following relationship holds:

$$\frac{m_aw_a}{r_ah_a} \geq \frac{r}{R-r+\sqrt{R(R-2r)}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Using the known inequality $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$, we have:

$$\frac{m_a w_a}{r_a h_a} \geq \frac{bc\cos^2\frac{A}{2}}{r_a h_a} = \frac{s(s-a).(s-a)a}{2F^2} = \frac{(s-a)^2 a}{2sr^2}.$$

So it suffices to prove that :

$$\frac{(s-a)^2 a}{2sr^2} \geq \frac{r}{R-r+\sqrt{R(R-2r)}} = \frac{2r}{\left(\sqrt{R}+\sqrt{R-2r}\right)^2}$$

$$\Leftrightarrow 2\sqrt{sr}\left(\sqrt{R}+\sqrt{R-2r}\right) \geq \frac{2\sqrt{sr}.2\sqrt{sr^3}}{(s-a)\sqrt{a}} \Leftrightarrow \sqrt{abc}+\sqrt{abc-8sr^2} \geq \frac{4sr^2}{(s-a)\sqrt{a}}$$

$$\Leftrightarrow \sqrt{abc-8sr^2} \geq \frac{4sr^2}{(s-a)\sqrt{a}} - \sqrt{abc}. \quad (1)$$

If $RHS_{(1)} \leq 0$, the inequality (1) is true. Assume now that $RHS_{(1)} \geq 0$. We have

$$(1) \Leftrightarrow -8sr^2 \geq \frac{16(sr^2)^2}{a(s-a)^2} - \frac{8sr^2\sqrt{bc}}{(s-a)} \Leftrightarrow \sqrt{bc} \geq \frac{2sr^2}{a(s-a)} + s - a$$
$$= \frac{2(s-b)(s-c)}{a} + s - a.$$

By CBS and HM − GM inequalities, we have

$$\sqrt{bc} = \sqrt{[(s-c)+(s-a)][(s-b)+(s-a)]} \geq \sqrt{(s-b)(s-c)} + s - a =$$

$$\geq \frac{2(s-b)(s-c)}{(s-b)+(s-c)} + s - a = \frac{2(s-b)(s-c)}{a} + s - a$$

which completes the proof. Equality holds iff $b = c \leq a$.

2076. In any Δ ABC with $n_a, n_b, n_c \to$

Nagel cevians, the following relationship holds :

$$\frac{n_a - m_a}{h_a} + \frac{n_b - m_b}{h_b} + \frac{n_c - m_c}{h_c} \geq \frac{a^2+b^2+c^2-ab-bc-ca}{2F}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=$$
$$as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-a+\frac{(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2\overset{(\bullet)}{=}s(s-a)+\frac{s}{a}.(b-c)^2$$
$$\text{Now, } n_a-m_a\overset{?}{\geq}\frac{(b-c)^2}{2a}\overset{\text{via }(\bullet)}{\Leftrightarrow} s(s-a)+\frac{s}{a}.(b-c)^2-s(s-a)-\frac{(b-c)^2}{4}$$
$$\overset{?}{\geq}\frac{(b-c)^4}{4a^2}+m_a.\frac{(b-c)^2}{a}\Leftrightarrow\frac{4s-a}{4}-\frac{(b-c)^2}{4a}\underset{①}{\overset{?}{\geq}}m_a\ (\because (b-c)^2\geq 0)$$
$$\text{Now, }\frac{4s-a}{4}-\frac{(b-c)^2}{4a}>\frac{4s-a}{4}-\frac{a^2}{4a}=\frac{4s-2a}{4}>0\ \therefore ①\Leftrightarrow$$
$$\frac{(a(4s-a)-(b-c)^2)^2}{16a^2}\overset{?}{\geq}s(s-a)+\frac{(b-c)^2}{4}$$
$$\Leftrightarrow a^2(4s-a)^2+(b-c)^4-2a(4s-a)(b-c)^2\overset{?}{\geq}16a^2s(s-a)+4a^2(b-c)^2$$
$$\Leftrightarrow (b-c)^4-(8sa+2a^2)(b-c)^2+a^2(16s^2-8sa+a^2-16s^2+16sa)\overset{?}{\geq}0$$
$$\Leftrightarrow (b-c)^4-(8sa+2a^2)(b-c)^2+a^2(8sa+a^2)\underset{②}{\overset{?}{\geq}}0$$

Now, LHS of ② is a quadratic polynomial in $(b-c)^2$ with discriminant $=(8sa+2a^2)^2-4a^2(8sa+a^2)=64a^2s^2$ ∴ in order to prove ②,

$$\text{it suffices to prove : }(b-c)^2\overset{?}{\leq}\frac{8sa+2a^2-\sqrt{64a^2s^2}}{2}\Leftrightarrow a^2\overset{?}{\geq}(b-c)^2\rightarrow\text{true}$$
$$\Rightarrow ②\Rightarrow ①\text{ is true }\therefore n_a-m_a\geq\frac{(b-c)^2}{2a}\Rightarrow\frac{n_a-m_a}{h_a}\geq\frac{a(b-c)^2}{2a.2F}$$
$$\Rightarrow\frac{n_a-m_a}{h_a}\geq\frac{(b-c)^2}{4F}\text{ and analogs}$$
$$\Rightarrow\frac{n_a-m_a}{h_a}+\frac{n_b-m_b}{h_b}+\frac{n_c-m_c}{h_c}\geq\sum_{cyc}\frac{(b-c)^2}{4F}$$
$$=\frac{a^2+b^2+c^2-ab-bc-ca}{2F}\ \forall\ \Delta ABC,\text{ with equality iff }\Delta ABC\text{ is equilateral (QED)}$$

2077. In any ΔABC, for $\alpha \le \frac{17+15\sqrt{3}}{5}$ and $\beta = (2+3\sqrt{3}+\alpha).3^{\frac{-3}{4}}$. Prove that:

$$R+s+\alpha r \ge \beta\sqrt{F}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality is equivalent to

$$R+s \ge (2+3\sqrt{3})\sqrt{\frac{F}{3\sqrt{3}}}+\alpha\left(\sqrt{\frac{F}{3\sqrt{3}}}-r\right)$$

Since $s \ge 3\sqrt{3}r$, then $\sqrt{\frac{F}{3\sqrt{3}}}-r \ge 0$, so it suffices to prove that

$$R+s \ge (2+3\sqrt{3})\sqrt{\frac{F}{3\sqrt{3}}}+\frac{17+15\sqrt{3}}{5}.\left(\sqrt{\frac{F}{3\sqrt{3}}}-r\right) \text{ or}$$

$$R+\left(\sqrt{s}-\sqrt{3\sqrt{3}r}\right)^2+\frac{17}{5}r \ge \frac{27}{5}\sqrt{\frac{F}{3\sqrt{3}}}.$$

$$\text{or } R+\frac{17}{5}r \ge \frac{27}{5}\sqrt{\frac{F}{3\sqrt{3}}} \text{ or } (5R+17r)^4 \ge 27^3s^2r^2.$$

By Gerretsen's inequality, we have

$$27^3s^2r^2 \le$$

$$\le 27^3(4R^2+4Rr+3r^2)r^2$$
$$=(5R+17r)^4-(R-2r)^2(625R^2+11000Rr+6118r^2) \le$$

$$\le (5R+17r)^4.$$

Equality holds iff ΔABC is equilateral.

2078. In ΔABC the following relationship holds:

$$m_a \cot\frac{A}{2} + m_b \cot\frac{B}{2} + m_c \cot\frac{C}{2} \geq \frac{2s^3}{9Rr}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$m_a \cot\frac{A}{2} + m_b \cot\frac{B}{2} + m_c \cot\frac{C}{2} = \sum m_a \cot\frac{A}{2} \overset{m_a \geq \sqrt{s(s-a)}}{\geq}$$

$$\geq \sum \sqrt{s(s-a)}\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = \sum \frac{s(s-a)}{\sqrt{(s-b)(s-c)}} =$$

$$= s\sum \frac{(s-a)^{\frac{3}{2}}}{\sqrt{(s-b)(s-a)(s-c)}} = \frac{s}{\sqrt{sr^2}}\sum (s-a)^{\frac{3}{2}} =$$

$$= \frac{\sqrt{s}}{r}\sum (s-a)^{\frac{3}{2}} \overset{CBS}{\geq} \frac{\sqrt{s}}{r}\frac{\left(\sum(s-a)\right)^{\frac{3}{2}}}{3^{\frac{1}{2}}} = \frac{\sqrt{s}}{r}\frac{(s)^{\frac{3}{2}}}{3^{\frac{1}{2}}} = \frac{s^2}{r\sqrt{3}}$$

$$= \frac{s^3}{rs\sqrt{3}} \overset{Mitrinovic}{\geq} \frac{s^3}{\frac{r3\sqrt{3}R}{2}\sqrt{3}} = \frac{2s^3}{9Rr}$$

Equality holds for an equilateral triangle.

2079. In $\Delta ABC, b^2 + bc + c^2 \leq a^2$. Prove that :

$$\frac{9r}{4R} < \frac{h_a + h_b + h_c}{r_a + r_b + r_c} < \frac{18}{25}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{9r}{4R} < \frac{h_a + h_b + h_c}{r_a + r_b + r_c} \Leftrightarrow \frac{9r}{4R} < \frac{ab+bc+ca}{2R(4R+r)} \Leftrightarrow 9.4r(4R+r) < 8(ab+bc+ca)$$

$$\Leftrightarrow 9[2(ab+bc+ca) - (a^2+b^2+c^2)] < 8(ab+bc+ca)$$

$$\Leftrightarrow 0 < 5(b+c-a)^2 + 4(a^2 - b^2 - bc - c^2) + 8(b-c)^2,$$

which is true. $(\because b+c-a>0)$

$$\frac{h_a+h_b+h_c}{r_a+r_b+r_c}<\frac{18}{25}\Leftrightarrow 25\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)<18\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)$$

$$\overset{CBS}{\Leftarrow} 25\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)<18\left(\frac{1}{b+c-a}+\frac{4}{2a}\right)\Leftrightarrow 25\left(\frac{1}{b}+\frac{1}{c}\right)<\frac{18}{b+c-a}+\frac{11}{a}.$$

Since $bc\geq(b+c)^2-a^2>0$, it suffices to prove that

$$\frac{25(b+c)}{(b+c)^2-a^2}<\frac{18}{b+c-a}+\frac{11}{a}\overset{x:=\frac{b+c}{a}>1}{\Leftrightarrow}\frac{25x}{x^2-1}<\frac{18}{x-1}+11\Leftrightarrow 0<11x^2-7x+7,$$

which is true for $x>1$. So the proof is complete.

2080. In any Δ ABC with $n_a, n_b, n_c \rightarrow$

Nagel cevians, the following relationship holds :

$$n_a+n_b+n_c\geq m_a+m_b+m_c+\frac{1}{2}\left(s-3\sqrt{3}r\right)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=$$
$$as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-a+\frac{(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2\overset{(\bullet)}{=}s(s-a)+\frac{s}{a}.(b-c)^2$$

$$\text{Now, } n_a-m_a\overset{?}{\geq}\frac{(b-c)^2}{2a}\overset{\text{via }(\bullet)}{\Leftrightarrow}s(s-a)+\frac{s}{a}.(b-c)^2-s(s-a)-\frac{(b-c)^2}{4}$$
$$\overset{?}{\geq}\frac{(b-c)^4}{4a^2}+m_a.\frac{(b-c)^2}{a}\Leftrightarrow\frac{4s-a}{4}-\frac{(b-c)^2}{4a}\overset{?}{\underset{①}{\geq}}m_a\ (\because (b-c)^2\geq 0)$$

$$\text{Now, }\frac{4s-a}{4}-\frac{(b-c)^2}{4a}>\frac{4s-a}{4}-\frac{a^2}{4a}=\frac{4s-2a}{4}>0\therefore ①\Leftrightarrow$$

$$\frac{(a(4s-a)-(b-c)^2)^2}{16a^2}\overset{?}{\geq}s(s-a)+\frac{(b-c)^2}{4}$$

$$\Leftrightarrow a^2(4s-a)^2+(b-c)^4-2a(4s-a)(b-c)^2\overset{?}{\geq}16a^2s(s-a)+4a^2(b-c)^2$$

$$\Leftrightarrow (b-c)^4-(8sa+2a^2)(b-c)^2+a^2(16s^2-8sa+a^2-16s^2+16sa)\overset{?}{\geq}0$$

$$\Leftrightarrow (b-c)^4-(8sa+2a^2)(b-c)^2+a^2(8sa+a^2)\underset{②}{\overset{?}{\geq}}0$$

Now, LHS of ② is a quadratic polynomial in $(b-c)^2$ with discriminant $=(8sa+2a^2)^2-4a^2(8sa+a^2)=64a^2s^2$ ∴ in order to prove ②,

it suffices to prove : $(b-c)^2\overset{?}{\leq}\frac{8sa+2a^2-\sqrt{64a^2s^2}}{2}\Leftrightarrow a^2\overset{?}{\geq}(b-c)^2\to$ true

$$\Rightarrow ② \Rightarrow ① \text{ is true } \therefore n_a-m_a\geq\frac{(b-c)^2}{2a} \text{ and analogs}\to(m)$$

$$\text{Now, }\sum_{cyc}\frac{(b-c)^2}{a}=\sum_{cyc}\frac{b^2+c^2+a^2}{a}-\sum_{cyc}a-\frac{2}{4Rrs}\cdot\sum_{yc}b^2c^2$$

$$=\frac{1}{4Rrs}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)-\frac{8Rrs^2}{4Rrs}-\frac{2}{4Rrs}\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)$$

$$=\frac{1}{4Rrs}\left(\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2-2\sum_{cyc}ab\right)+24Rrs^2\right)$$

$$=\frac{2(s^2+4Rr+r^2)\left(s^2-4Rr-r^2-(s^2+4Rr+r^2)\right)+24Rrs^2}{4Rrs}$$

$$=\frac{(2R-r)s^2-r(4R+r)^2}{Rs}\overset{(n)}{=}\sum_{cyc}\frac{(b-c)^2}{a}$$

$$\therefore (m)\text{ and }(n)\Rightarrow\sum_{cyc}(n_a-m_a)\geq\frac{(2R-r)s^2-r(4R+r)^2}{2Rs}\overset{?}{\geq}\frac{1}{2}(s-3\sqrt{3}r)$$

$$\Leftrightarrow (R-r)s^2+3\sqrt{3}Rrs-r(4R+r)^2\underset{(*)}{\overset{?}{\geq}}0$$

We have, via Mitrinovic : LHS of $(*)\geq(R-r)s^2+2rs^2-r(4R+r)^2$

$$=(R+r)s^2-r(4R+r)^2\overset{\text{Gerretsen}}{\geq}(R+r)(16Rr-5r^2)-r(4R+r)^2$$

$$=3r^2(R-2r)\overset{\text{Euler}}{\geq}0\Rightarrow(*)\text{ is true }\therefore\sum_{cyc}(n_a-m_a)\geq\frac{1}{2}(s-3\sqrt{3}r)$$

$$\therefore n_a+n_b+n_c\geq m_a+m_b+m_c+\frac{1}{2}(s-3\sqrt{3}r)\ \forall\Delta ABC,$$

with equality iff Δ ABC is equilateral (QED)

2081.

In any ΔABC, for $k=\frac{\sqrt[4]{3}(373\sqrt{3}+1323)}{1794}$, prove that:

$$\frac{1}{R+s}+\frac{1}{s+r}+\frac{1}{r+R}\le\frac{k}{\sqrt{F}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$R+s=\sqrt{R.R}+\sqrt{s.s}\overset{Mitrinovic\ \&\ Euler}{\geq}\sqrt{\frac{2s}{3\sqrt{3}}.2r}+\sqrt{s.3\sqrt{3}r}=\frac{(2+3\sqrt{3})\sqrt{F}}{\sqrt[4]{27}}.$$

$$s+r=\frac{s}{3\sqrt{3}}+r+\frac{3\sqrt{3}-1}{3\sqrt{3}}\sqrt{s.s}\overset{AM-GM\ \&\ Mitrinovic}{\geq}2\sqrt{\frac{sr}{3\sqrt{3}}}+\frac{3\sqrt{3}-1}{3\sqrt{3}}\sqrt{s.3\sqrt{3}rs}$$
$$=\frac{(1+3\sqrt{3})\sqrt{F}}{\sqrt[4]{27}}.$$

$$r+R\overset{Mitrinovic}{\geq}r+\frac{2s}{3\sqrt{3}}=r+\frac{s}{3\sqrt{3}}+\sqrt{\frac{s.s}{27}}\overset{AM-GM\ \&\ Mitrinovic}{\geq}2\sqrt{\frac{sr}{3\sqrt{3}}}+\sqrt{\frac{sr}{3\sqrt{3}}}=\sqrt{\sqrt{3}F}.$$

Then

$$\frac{1}{R+s}+\frac{1}{s+r}+\frac{1}{r+R}\le\frac{\sqrt[4]{27}}{(2+3\sqrt{3})\sqrt{F}}+\frac{\sqrt[4]{27}}{(1+3\sqrt{3})\sqrt{F}}+\frac{1}{\sqrt{\sqrt{3}F}}=\frac{k}{\sqrt{F}}.$$

Equality holds iff ΔABC is equilateral.

2082. In any Δ ABC, the following relationship holds :

$$\frac{R-r-\sqrt{R(R-2r)}}{r}\le\frac{r_a}{h_a}\le\frac{R-r+\sqrt{R(R-2r)}}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{r_a}{h_a}+1=\frac{rpa}{2rp(p-a)}+1\ (p\to\text{semi}-\text{perimeter})=\frac{a}{b+c-a}+1$$

$$=\frac{b+c}{2(p-a)}=\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\Rightarrow\frac{r_a}{h_a}+1\overset{①}{=}\frac{c}{c-s}$$

$$\left(\text{where } c=\cos\frac{B-C}{2}, s=\sin\frac{A}{2}\right)$$

$$\text{Again, }\frac{R\pm\sqrt{R(R-2r)}}{r}=\frac{1}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{R.\sqrt{1-4sc+4s^2}}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\therefore\frac{R\pm\sqrt{R(R-2r)}}{r}\overset{②}{=}\frac{1}{2s(c-s)}\pm\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}$$

$$\text{Now, ① and ②}\Rightarrow\frac{r_a}{h_a}\le\frac{R-r+\sqrt{R(R-2r)}}{r}\Leftrightarrow$$

$$\frac{c}{c-s}\le\frac{1}{2s(c-s)}+\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}\Leftrightarrow\frac{2sc-1}{2s(c-s)}\le\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}\Leftrightarrow$$

$$2sc-1\overset{(*)}{\le}\sqrt{1-4sc+4s^2}\left(\because c-s=\frac{b+c}{a}.\sin\frac{A}{2}-\sin\frac{A}{2}=\frac{b+c-a}{a}.\sin\frac{A}{2}>0\right)$$

and if $2sc-1<0$, *then* : $(*)$ is trivi*ally* true *and* so, we now focus on :
$2sc-1\ge 0$ *and* then : $(*)\Leftrightarrow 4s^2c^2-4sc+1\le 1-4sc+4s^2$

$$\Leftrightarrow 4s^2(c^2-1)\le 0\to\text{true}\because c^2=\cos^2\frac{B-C}{2}\le 1\Rightarrow(*)\text{ is true }\forall\,\Delta\,ABC$$

$$\therefore\frac{r_a}{h_a}\le\frac{R-r+\sqrt{R(R-2r)}}{r}$$

$$\text{Also, ① and ②}\Rightarrow\frac{r_a}{h_a}\ge\frac{R-r-\sqrt{R(R-2r)}}{r}\Leftrightarrow$$

$$\frac{c}{c-s}\ge\frac{1}{2s(c-s)}-\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}\Leftrightarrow\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}\ge\frac{1-2sc}{2s(c-s)}$$

$\Leftrightarrow\sqrt{1-4sc+4s^2}\overset{(**)}{\ge}1-2sc$ *and* if $1-2sc<0$, *then* : $(**)$ is trivi*ally* true
and so, we now focus on : $1-2sc\ge 0$ *and* then : $(**)\Leftrightarrow$
$1-4sc+4s^2\ge 4s^2c^2-4sc+1\Leftrightarrow 4s^2(1-c^2)\ge 0$

$$\to\text{true}\because c^2=\cos^2\frac{B-C}{2}\le 1\Rightarrow(**)\text{ is true }\forall\,\Delta\,ABC$$

$$\therefore\frac{r_a}{h_a}\ge\frac{R-r-\sqrt{R(R-2r)}}{r}\text{ and hence,}$$

$$\frac{R-r-\sqrt{R(R-2r)}}{r}\le\frac{r_a}{h_a}\le\frac{R-r+\sqrt{R(R-2r)}}{r}\ \forall\,\Delta\,ABC\ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $\frac{r_a}{h_a}=\frac{a}{2(s-a)}=\frac{1}{2}\left(\frac{s}{s-a}-1\right)$

$=\frac{1}{2}\left(\frac{r_a}{r}-1\right)$, so the desired inequality is equivalent to

$$2R-r-2\sqrt{R(R-2r)}\le r_a\le 2R-r+2\sqrt{R(R-2r)}.$$

We have

$$4\le(r_b+r_c)\left(\frac{1}{r_b}+\frac{1}{r_c}\right)=(4R+r-r_a)\left(\frac{1}{r}-\frac{1}{r_a}\right)\Leftrightarrow 4rr_a\le(4R+r-r_a)(r_a-r)$$

$$\Leftrightarrow {r_a}^2-2(2R-r)r_a+r(4R+r)\le 0$$

$$\Leftrightarrow\left(r_a-2R+r+2\sqrt{R(R-2r)}\right)\left(r_a-2R+r-2\sqrt{R(R-2r)}\right)\le 0$$

$$\Leftrightarrow 2R-r-2\sqrt{R(R-2r)}\le r_a\le 2R-r+2\sqrt{R(R-2r)}.$$

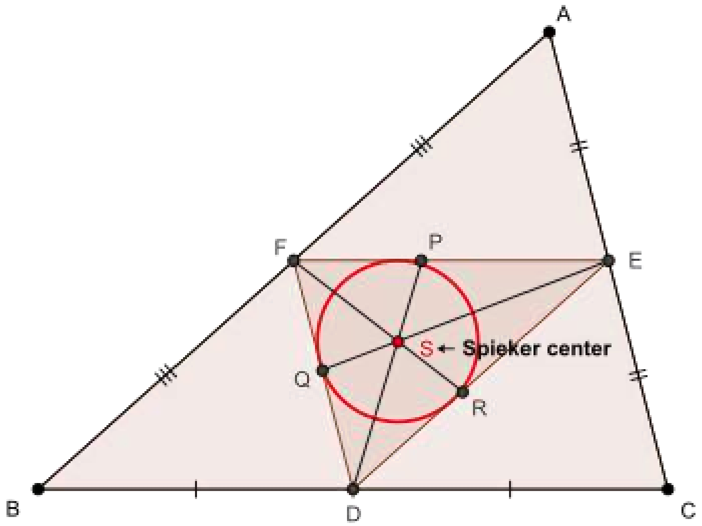
which completes the proof. Equality holds iff $b=c$.

2083.In any Δ ABC with $p_a, p_b, p_c\to$ Spieker cevians, the following relationship holds :

$$0\le\frac{p_a-w_a}{a}+\frac{p_b-w_b}{b}+\frac{p_c-w_c}{c}<\frac{4}{3}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say)

$$and\ inradius\ of\ \Delta\ DEF = r'(say)$$

$$\text{Now}, 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\ DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2}\ and\ m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

$$Via\ (1),(2)\ and\ using\ cosine\ law\ on\ \Delta\ AFS\ and\ \Delta\ AES, we\ arrive\ at:$$

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now}, \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\stackrel{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2\stackrel{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Again, $p_a - w_a \le \frac{2sa|b-c|}{4s^2-a^2} \Leftrightarrow \frac{s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2}{(4s^2-a^2)^2}.(b-c)^2$

$$\le \frac{4s^2a^2(b-c)^2}{(4s^2-a^2)^2}+\frac{4sa.w_a.|b-c|}{4s^2-a^2}$$

$$\Leftrightarrow \frac{s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2a^2}{(4s^2-a^2)^2}.|b-c| \le \frac{4sa.w_a}{4s^2-a^2}$$

$$(\because |b-c| \ge 0) \Leftrightarrow \frac{8s^2(2s+a)(s-a)}{4s^2-a^2}.|b-c| \le 4sa.w_a$$

$$\Leftrightarrow \frac{4s^2(2s+a)^2(s-a)^2}{(4s^2-a^2)^2}.(b-c)^2 \le a^2\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow \frac{4s^2(s-a)^2+a^2s(s-a)}{(2s-a)^2}.(b-c)^2 \le a^2s(s-a)$$

$$\Leftrightarrow \frac{s(s-a)(4s^2-4sa+a^2)}{(2s-a)^2}.(b-c)^2 \le a^2s(s-a)$$

$$\Leftrightarrow s(s-a).(b-c)^2 \le a^2s(s-a) \Leftrightarrow s(s-a)(a^2-(b-c)^2) \overset{?}{\ge} 0 \to \text{true}$$

$$\therefore p_a - w_a \le \frac{2sa|b-c|}{4s^2-a^2} \text{ and analogs}$$

$$\Rightarrow \frac{p_a-w_a}{a}+\frac{p_b-w_b}{b}+\frac{p_c-w_c}{c} \le \sum_{cyc}\frac{2s.|b-c|}{4s^2-a^2} \overset{|b-c|<a \text{ and analogs}}{<}$$

$$\frac{2s}{(\prod_{cyc}(2s-a)).(\prod_{cyc}(2s+a))}.\sum_{cyc}\left(a(4s^2-b^2)(4s^2-c^2)\right)$$

$$= \frac{2s\left(2s.16s^4-4s^2(\sum_{cyc}ab^2+\sum_{cyc}a^2b)+4Rrs(s^2+4Rr+r^2)\right)}{\left(2s(s^2+2Rr+r^2)\right).\left(2s(9s^2+6Rr+r^2)\right)}$$

$$= \frac{4s^2\left(16s^4-4s^2(s^2-2Rr+r^2)+2Rr(s^2+4Rr+r^2)\right)}{4s^2(s^2+2Rr+r^2)(9s^2+6Rr+r^2)}$$

$$= \frac{4s^2\left(16s^4-4s^2(s^2-2Rr+r^2)+2Rr(s^2+4Rr+r^2)\right)}{4s^2(s^2+2Rr+r^2)(9s^2+6Rr+r^2)} \overset{?}{<} \frac{4}{3}$$

$$\Leftrightarrow 2r\left((33R+26r)s^2+r(12R^2+13Rr+2r^2)\right) \overset{?}{>} 0 \to \text{true}$$

$$\therefore \frac{p_a-w_a}{a}+\frac{p_b-w_b}{b}+\frac{p_c-w_c}{c} < \frac{4}{3} \text{ and } p_a^2-w_a^2 =$$

$$\frac{s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2}{(4s^2-a^2)^2}.(b-c)^2 \geq 0 \Rightarrow \frac{p_a-w_a}{a} \geq 0$$

$$\text{and analogs} \therefore \frac{p_a-w_a}{a}+\frac{p_b-w_b}{b}+\frac{p_c-w_c}{c} \geq 0 \text{ and so,}$$

$$0 \leq \frac{p_a-w_a}{a}+\frac{p_b-w_b}{b}+\frac{p_c-w_c}{c} < \frac{4}{3}\ \forall \Delta\ ABC,$$

with equality iff Δ ABC is equilateral (QED)

2084. In any Δ ABC with $\hat{A} \geq 90°$, the following relationship holds :

$$m_a h_a \geq w_a^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_a h_a \geq w_a^2 \Leftrightarrow \left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$\geq \left(s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)^2$$

$$\Leftrightarrow -\frac{s^2(s-a)^2(b-c)^2}{a^2}+s(s-a).\frac{(b-c)^2}{4}-\frac{s(s-a)(b-c)^4}{4a^2}$$

$$\geq -\frac{2s^2(s-a)^2(b-c)^2}{(b+c)^2}+\frac{s^2(s-a)^2(b-c)^4}{(b+c)^4} \text{ and } \because (b-c)^2 \geq 0 \therefore \text{it suffices}$$

$$\text{to prove : } \frac{1}{4}-\frac{s(s-a)}{a^2}-\frac{(b-c)^2}{4a^2}+\frac{s(s-a)}{(b+c)^2}\left(2-\frac{(b-c)^2}{(b+c)^2}\right) > 0$$

$$\Leftrightarrow \frac{a^2-(b+c)^2+a^2-(b-c)^2}{4a^2}+\frac{s(s-a)}{(b+c)^4}.(b^2+c^2+6bc) > 0$$

$$\text{and} \because a^2 \geq b^2+c^2 \therefore \text{it suffices to prove :}$$

$$\frac{2(b^2+c^2)-(b+c)^2-(b-c)^2}{4a^2}+\frac{s(s-a)}{(b+c)^4}.(b^2+c^2+6bc) > 0$$

$$\Leftrightarrow \frac{s(s-a)}{(b+c)^4}.(b^2+c^2+6bc) > 0 \to \text{true} \therefore m_a h_a \geq w_a^2\ \forall \Delta\ ABC \text{ with } \hat{A} \geq 90°,$$

$$'' = '' \text{ iff } \hat{B} = \hat{C} < 45° \text{ (QED)}$$

2085. In any Δ ABC, the following relationship holds :

$$2R-r-2\sqrt{R(R-2r)} \leq r_a \leq 2R-r+2\sqrt{R(R-2r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\mathbf{r}_a+\mathbf{r}=\frac{\mathbf{rp}}{\mathbf{p}-a}+\frac{\mathbf{rp}}{\mathbf{p}}=\frac{\mathbf{r(b+c)}}{\mathbf{p}-a}\ (\mathbf{p}\to\text{semi}-\text{perimeter})$$

$$=\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\therefore r_a+r\overset{①}{=}4Rsc\left(\text{where } c=\cos\frac{B-C}{2},s=\sin\frac{A}{2}\right)$$

$$\text{Again},2R\pm 2\sqrt{R(R-2r)}\overset{②}{=}2R\pm 2R.\sqrt{1-4sc+4s^2}$$

Now, ① *and* ② $\Rightarrow r_a\le 2R-r+2\sqrt{R(R-2r)}$

$\Leftrightarrow 4Rsc\le 2R+2R.\sqrt{1-4sc+4s^2}\Leftrightarrow 2sc-1\overset{(*)}{\le}\sqrt{1-4sc+4s^2}$ *and* if $2sc-1<0$, *then* : $(*)$ is trivi*ally* true *and* so, we now focus on :

$$2sc-1\ge 0$$

and then : $(*)\Leftrightarrow 4s^2c^2-4sc+1\le 1-4sc+4s^2\Leftrightarrow 4s^2(c^2-1)\le 0\to$ true

$$\because c^2=\cos^2\frac{B-C}{2}\le 1\Rightarrow (*)\text{ is true }\forall\,\Delta\,ABC\therefore r_a\le 2R-r+2\sqrt{R(R-2r)}$$

Also, ① *and* ② $\Rightarrow r_a\ge 2R-r-2\sqrt{R(R-2r)}$

$\Leftrightarrow 4Rsc\ge 2R-2R.\sqrt{1-4sc+4s^2}\Leftrightarrow\sqrt{1-4sc+4s^2}\overset{(**)}{\ge}1-2sc$ *and* if $1-2sc<0$, *then* : $(**)$ is trivi*ally* true *and* so, we now focus on : $1-2sc\ge 0$

and then : $(**)\Leftrightarrow 1-4sc+4s^2\ge 4s^2c^2-4sc+1\Leftrightarrow 4s^2(1-c^2)\ge 0\to$ true

$$\because c^2=\cos^2\frac{B-C}{2}\le 1\Rightarrow (**)\text{ is true }\forall\,\Delta\,ABC\therefore r_a\ge 2R-r-2\sqrt{R(R-2r)}$$

and hence

$$2R-r-2\sqrt{R(R-2r)}\le r_a\le 2R-r+2\sqrt{R(R-2r)}\ \forall\,\Delta\,ABC\ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$4\le(r_b+r_c)\left(\frac{1}{r_b}+\frac{1}{r_c}\right)=(4R+r-r_a)\left(\frac{1}{r}-\frac{1}{r_a}\right)\Leftrightarrow 4rr_a\le(4R+r-r_a)(r_a-r)$$

$$\Leftrightarrow {r_a}^2-2(2R-r)r_a+r(4R+r)\le 0$$

$$\Leftrightarrow\left(r_a-2R+r+2\sqrt{R(R-2r)}\right)\left(r_a-2R+r-2\sqrt{R(R-2r)}\right)\le 0$$

$$\Leftrightarrow 2R-r-2\sqrt{R(R-2r)}\le r_a\le 2R-r+2\sqrt{R(R-2r)}.$$

Equality holds iff $b=c$.

2086. In any Δ ABC, the following relationship holds :

$$\frac{w_a^2}{ab}+\frac{w_b^2}{bc}+\frac{w_c^2}{ca}\ge\frac{9}{4}\cdot\sqrt{\frac{6R-3r}{5R-r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$w_a^2=\frac{4bc}{(b+c)^2}.s(s-a)=\frac{bc((b+c)^2-a^2)}{(b+c)^2}=bc-\frac{a^2bc}{(b+c)^2}\overset{A-G}{\ge}bc-\frac{a^2}{4}$$

$$\Rightarrow\frac{w_a^2}{ab}\ge\frac{c}{a}-\frac{a}{4b}\text{ and analogs}\Rightarrow\frac{w_a^2}{ab}+\frac{w_b^2}{bc}+\frac{w_c^2}{ca}\ge\sum_{cyc}\frac{c}{a}-\frac{1}{4}.\sum_{cyc}\frac{a}{b}=\frac{3}{4}.\sum_{cyc}\frac{a}{b}$$

$$\ge\frac{3}{4}.\frac{9\sum_{cyc}a^2}{\left(\sum_{cyc}a\right)^2}\overset{?}{\ge}\frac{9}{4}.\sqrt{\frac{6R-3r}{5R-r}}\Leftrightarrow\frac{9.4(s^2-4Rr-r)^2}{16s^4}\overset{?}{\ge}\frac{6R-3r}{5R-r}$$

$$\Leftrightarrow(15R-3r)(s^2-4Rr-r)^2\overset{?}{\ge}(8R-4r)s^4$$

$$\Leftrightarrow(7R+r)s^4-r(120R^2+6Rr-6r^2)s^2+r^2(240R^3+72R^2r-9Rr^2-3r^3)\overset{①}{\ge}0$$

$$\text{and }\because(7R+r)s^4\overset{\text{Gerretsen}}{\ge}(7R+r)(s^2-16Rr+5r^2)^2\therefore\text{ in order to prove ①,}$$
$$\text{it suffices to prove : LHS of ①}\ge(7R+r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(26R^2-11Rr-r^2)s^2\overset{②}{\ge}r(388R^3-234R^2r+6Rr^2+7r^3)$$

$$\text{Now, }(26R^2-11Rr-r^2)s^2\overset{\text{Gerretsen}}{\ge}(26R^2-11Rr-r^2)(16Rr-5r^2)\overset{?}{\ge}$$
$$r(388R^3-234R^2r+6Rr^2+7r^3)\Leftrightarrow 28t^3-72t^2+33t-2\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(20t^2+8t(t-2)+1)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow②\Rightarrow①\text{ is true}$$

$$\therefore\frac{w_a^2}{ab}+\frac{w_b^2}{bc}+\frac{w_c^2}{ca}\ge\frac{9}{4}.\sqrt{\frac{6R-3r}{5R-r}}\ \forall\Delta\text{ ABC},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{w_a{}^2}{ab}=\sum_{cyc}\frac{4cs(s-a)}{a(b+c)^2}\overset{AM-GM}{\ge}\sum_{cyc}\frac{27cs(s-a)}{\left(a+\frac{b+c}{2}+\frac{b+c}{2}\right)^3}=\frac{27}{8s^2}\sum_{cyc}c(s-a)=$$

$$=\frac{27(s^2-r^2-4Rr)}{8s^2}=\frac{27}{8}\left(1-\frac{r(4R+r)}{s^2}\right)\overset{Gerretsen}{\geq}$$

$$\geq\frac{27}{8}\left(1-\frac{r(4R+r)}{16Rr-5r^2}\right)=\frac{81(2R-r)}{4(16R-5r)}\geq$$

$$\overset{?}{\geq}\frac{9}{4}\sqrt{\frac{6R-3r}{5R-r}}\Leftrightarrow 27(2R-r)(5R-r)\overset{?}{\geq}(16R-5r)^2\ \Leftrightarrow\ (R-2r)(14R-r)\overset{?}{\geq}0,$$

which is true by Euler's inequality $R\geq 2r$. Equality holds iff ΔABC is equilateral.

2087. In ΔABC the following relationship holds:

$$R^3+s^3+r^3\geq\frac{27+\sqrt{3}}{2}Rsr$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$We\ will\ show: R^3+r^3\geq\frac{\sqrt{3}}{2}Rsr\ \ (1)$$

$$R^3+r^3\overset{Mitrinovic}{\geq}R\sqrt{3}\cdot\frac{3\sqrt{3}Rr}{4}$$

$$R^3+r^3\geq\frac{9R^2r}{4}$$

$$4R^3-9R^2r+4r^3\geq 0$$

$$4x^3-9x^2+4\overset{\frac{R}{r}=x\geq 2\ Euler}{\geq}0$$

$$(x-2)(4x^2-x-2)\geq 0$$

$$(x-2)\left(x(2x-1)+2(x^2-1)\right)\geq 0\ true\ as\ \ x\geq 2$$

$$s^3=s^2.s\geq\frac{27Rr}{2}.s\ \left(as\ s^2\geq\frac{27Rr}{2}\right)=\frac{27}{2}Rrs\ (2)$$

$$R^3+s^3+r^3 \overset{(1)\&(2)}{\geq} \frac{\sqrt{3}}{2}Rsr+\frac{27}{2}Rrs=\frac{27+\sqrt{3}}{2}Rsr$$

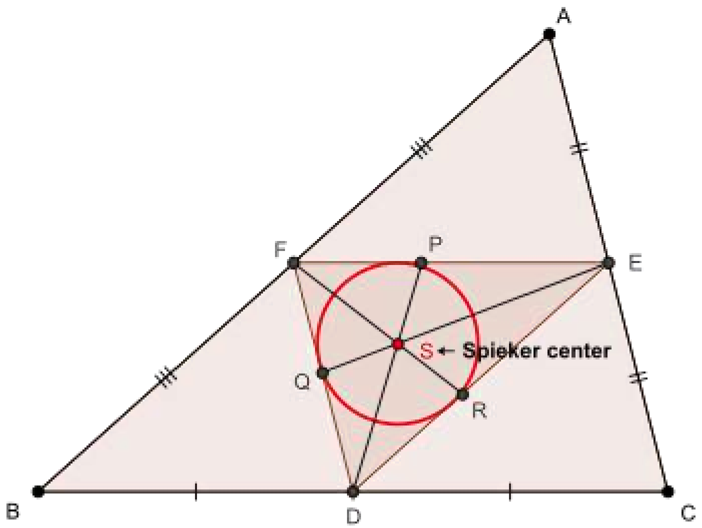
$$\textit{Equality holds for an equilateral triangle.}$$

2088. **In any Δ ABC with $p_a, p_b, p_c \to$ Spieker cevians, the following relationship holds:**

$$p_a+p_b+p_c \geq w_a+w_b+w_c+\frac{16(a^2+b^2+c^2-ab-bc-ca)}{15s}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{ Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at:

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES}, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$
$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$
$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$
$$\text{Now, } p_a-w_a \geq \frac{2s(b-c)^2}{4s^2-a^2} \Leftrightarrow p_a^2-w_a^2 \geq$$
$$\frac{4s^2(b-c)^4}{(4s^2-a^2)^2}+\frac{4s.w_a.(b-c)^2}{4s^2-a^2} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$
$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} \boxed{\overset{(\blacksquare)}{\geq}} \frac{4s.w_a}{4s^2-a^2}\ (\because (b-c)^2 \geq 0)$$
$$\text{We have : } \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} >$$
$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2a^2}{(4s^2-a^2)^2}$$
$$= \frac{s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2a^2}{(4s^2-a^2)^2} = \frac{8s^2(2s+a)(s-a)}{(4s^2-a^2)^2} > 0$$
$$\therefore (\blacksquare) \Leftrightarrow \frac{(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2(b-c)^2)^2}{(4s^2-a^2)^4}$$
$$\geq \frac{16s^2}{(4s^2-a^2)^2}.\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$
$$\Leftrightarrow \frac{16s^4(b-c)^4-8s^2(b-c)^2(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)+(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)^2}{(4s^2-a^2)^2}$$
$$\geq \frac{16s^2(s(s-a)(2s-a)^2-s(s-a)(b-c)^2)}{(2s-a)^2}$$
$$\Leftrightarrow 16s^4(b-c)^4-8s^2(b-c)^2\left(\begin{matrix}s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\\ -2s(s-a)(2s+a)^2\end{matrix}\right)$$

$$+(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)^2-16s^3(s-a)(4s^2-a^2)^2\geq 0$$
$$\Leftrightarrow 16s^4(b-c)^4-16s^3(b-c)^2(4s^3-4s^2a+sa^2+a^3)$$
$$+16a^2s^3(4s^3-4s^2a+a^3)\geq 0$$
$$\Leftrightarrow s(b-c)^4-(4s^3-4s^2a+sa^2+a^3)(b-c)^2+a^2(4s^3-4s^2a+a^3)\boxed{\overset{(\blacksquare\blacksquare)}{\geq}}0$$
and in order to prove $(\blacksquare\blacksquare)$, it suffices to prove :
$$(b-c)^2\leq\frac{(4s^3-4s^2a+sa^2+a^3)-\sqrt{\delta}}{2s}, \text{where } \delta=$$
$(4s^3-4s^2a+sa^2+a^3)^2-4sa^2(4s^3-4s^2a+a^3)$ and $\because (b-c)^2<a^2$
$\therefore$ it suffices to prove : $2sa^2\leq(4s^3-4s^2a+sa^2+a^3)$
$$-\sqrt{(4s^3-4s^2a+sa^2+a^3)^2-4sa^2(4s^3-4s^2a+a^3)}$$
$$\Leftrightarrow\sqrt{(s-a)^2(4s^2-a^2)^2}\leq 4s^3-4s^2a-sa^2+a^3=(s-a)(4s^2-a^2)\to\text{true}$$
$$\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true}\therefore p_a-w_a\geq\frac{2s(b-c)^2}{4s^2-a^2}\text{ and analogs}$$
$$\therefore p_a+p_b+p_c\overset{(\blacksquare\blacksquare\blacksquare)}{\geq}w_a+w_b+w_c+\sum_{cyc}\frac{2s(b-c)^2}{4s^2-a^2}$$
$$\text{Now,}\sum_{cyc}\left((b-c)^2(4s^2-b^2)(4s^2-c^2)\right)=$$
$$16s^4\sum_{cyc}(b-c)^2-4s^2\sum_{cyc}\left((b-c)^2\left(\sum_{cyc}a^2-a^2\right)\right)$$
$$+\sum_{cyc}\left(b^2c^2\left(\sum_{cyc}a^2-a^2-2bc\right)\right)$$
$$=32s^4(s^2-12Rr-3r^2)-16s^2(s^2-4Rr-r^2)(s^2-12Rr-3r^2)+$$
$$4s^2\sum_{cyc}\left(a^2(b^2+c^2-2bc)\right)+2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)$$
$$-48R^2r^2s^2-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)$$
$$=32s^4(s^2-12Rr-3r^2)-16s^2(s^2-4Rr-r^2)(s^2-12Rr-3r^2)+$$
$$(10s^2-8Rr-2r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-64Rrs^4-48R^2r^2s^2$$
$$-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)$$
$$=4(6s^6-(64Rr+5r^2)s^4-r^2s^2(148R^2+76Rr+12r^2)-(4Rr+r^2)^3)$$
$$\therefore\sum_{cyc}\frac{2s(b-c)^2}{4s^2-a^2}=\frac{8s\begin{pmatrix}6s^6-(64Rr+5r^2)s^4-r^2s^2(148R^2+76Rr+12r^2)\\-(4Rr+r^2)^3\end{pmatrix}}{4s^2(9s^2+6Rr+r^2)(s^2+2Rr+r^2)}$$
$$\overset{?}{\geq}\frac{16(a^2+b^2+c^2-ab-bc-ca)}{15s}$$
$$\Leftrightarrow 15(6s^6-(64Rr+5r^2)s^4-r^2s^2(148R^2+76Rr+12r^2)-(4Rr+r^2)^3)$$

$$\overset{?}{\geq} 8(s^2-12Rr-3r^2)(9s^2+6Rr+r^2)(s^2+2Rr+r^2)$$

$$\Leftrightarrow 18s^6-(288Rr-61r^2)s^4-r^2s^2(12R^2-332Rr-52r^2)+$$

$$r^3(192R^3+336R^2r+108Rr^2+9r^3)\ \boxed{\overset{?}{\underset{①}{\geq}}}\ 0\ and\ \because 18(s^2-16Rr+5r^2)^3\overset{Gerretsen}{\geq}0$$

$\therefore$ in order to prove ①, it suffices to prove : LHS of ① $\geq 18(s^2-16Rr+5r^2)^3$

$$\Leftrightarrow (576R-209r)s^4-r(13836R^2-8972Rr+1298r^2)s^2+$$

$$r^2(73920R^3-68784R^2r+21708Rr^2-2241r^3)\ \boxed{\overset{②}{\geq}}\ 0\ and\ \because$$

$(576R-209r)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0\ \therefore$ in order to prove ②,

it suffices to prove : LHS of ② $\geq (576R-209r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (1149R^2-869Rr+198r^2)s^2\ \boxed{\overset{③}{\geq}}$$

$$r(18384R^3-19220R^2r+6533Rr^2-746r^3)$$

Finally, LHS of ③ $\overset{Rouche}{\geq}(1149R^2-869Rr+198r^2)\begin{pmatrix}2R^2+10Rr-r^2\\-2(R-2r).\sqrt{R^2-2Rr}\end{pmatrix}$

$$\overset{?}{\geq} r(18384R^3-19220R^2r+6533Rr^2-746r^3)$$

$$\Leftrightarrow (R-2r)(2298R^3-4036R^2r+1705Rr^2-274r^3)\overset{?}{\geq}$$

$$2(R-2r).\sqrt{R^2-2Rr}.(1149R^2-869Rr+198r^2)\ and\ \because (R-2r)\overset{Euler}{\geq}0$$

$\therefore$ in order to prove this, it suffices to prove :

$$(2298R^3-4036R^2r+1705Rr^2-274r^3)^2$$
$$> 4(R^2-2Rr)(1149R^2-869Rr+198r^2)^2$$

$$\Leftrightarrow 3309120t^4-3964248t^3+2208945t^2-620708t+75076>0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow 1326996t^4+1982124t^3(t-2)+1898591t^2+310354t(t-2)+75076>0$$

$\rightarrow$ true $\because t\overset{Euler}{\geq}2\Rightarrow ③\Rightarrow ②\Rightarrow ①$ is true $\therefore \sum_{cyc}\frac{2s(b-c)^2}{4s^2-a^2}\geq$

$$\frac{16(a^2+b^2+c^2-ab-bc-ca)}{15s}\overset{via\ (■■■)}{\Rightarrow} p_a+p_b+p_c\geq$$

$$w_a+w_b+w_c+\frac{16(a^2+b^2+c^2-ab-bc-ca)}{15s}\ \forall\ \Delta\ ABC,$$

with equality iff Δ ABC is equilateral (QED)

2089. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{1}{\frac{3r}{h_a}+2}\geq\frac{6}{\frac{3(R+r)^2}{s^2}+5}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\frac{3r}{h_a}+2}=\sum_{cyc}\frac{1}{\frac{3ra}{2rs}+2}=2s.\sum_{cyc}\frac{1}{3a+4s}$$

$$=2s.\frac{\sum_{cyc}\left((4s+3a)(4s+3b)\right)}{(4s+3a)(4s+3b)(4s+3c)}=\frac{6s\left(16s^2+3\sum_{cyc}ab+8s\sum_{cyc}a\right)}{s\left(64s^2+36\sum_{cyc}ab+48s\sum_{cyc}a+108Rr\right)}$$

$$\geq\frac{6s^2}{3(R+r)^2+5s^2}$$

$$\Leftrightarrow -7s^4+(35R^2+6Rr+28r^2)s^2+r(12R^3+27R^2r+18Rr^2+3r^3)\overset{(*)}{\geq}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\geq}\left(-7(4R^2+4Rr+3r^2)+(35R^2+6Rr+28r^2)\right)s^2$$

$$+r(12R^3+27R^2r+18Rr^2+3r^3)\overset{?}{\geq}0$$

$$\Leftrightarrow(7R^2-22Rr+7r^2)s^2+r(12R^3+27R^2r+18Rr^2+3r^3)\underset{(**)}{\overset{?}{\geq}}0$$

Case 1: $7R^2-22Rr+7r^2\geq 0$ and then : LHS of $(**)\geq$ $r(12R^3+27R^2r+18Rr^2+3r^3)>0\Rightarrow(**)$ is true

Case 2: $7R^2-22Rr+7r^2<0$ and then : LHS of $(**)\overset{\text{Gerretsen}}{\geq}$

$$(7R^2-22Rr+7r^2)(4R^2+4Rr+3r^2)+r(12R^3+27R^2r+18Rr^2+3r^3)\overset{?}{\geq}0$$

$$\Leftrightarrow 7t^4-12t^3-3t^2-5t+6\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(7t^3+2(t^2-4)+t+5)\overset{?}{\geq}0$$

$\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)$ is true $\therefore$ combining both cases, $(**)\Rightarrow(*)$ is true

$\forall\Delta$ ABC $\therefore \sum_{cyc}\frac{1}{\frac{3r}{h_a}+2}\geq\frac{6}{\frac{3(R+r)^2}{s^2}+5}$ $\forall\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

2090.

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{1}{\frac{3r}{r_a}+2}\ge\frac{6}{\frac{3r(4R+r)}{s^2}+5}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\frac{3r}{r_a}+2}=\sum_{cyc}\frac{1}{\frac{3r(s-a)}{rs}+2}=s.\sum_{cyc}\frac{1}{5s-3a}$$

$$=s.\frac{\sum_{cyc}\left((5s-3a)(5s-3b)\right)}{(5s-3a)(5s-3b)(5s-3c)}=\frac{3s\left(25s^2+3\sum_{cyc}ab-10s\sum_{cyc}a\right)}{s\left(125s^2+45\sum_{cyc}ab-75s\sum_{cyc}a-108Rr\right)}$$

$$\ge\frac{6s^2}{3r(4R+r)+5s^2}\Leftrightarrow(4R-17r)s^2+3r(4R+r)^2\overset{(*)}{\ge}0$$

$$\text{Now, }(4R-17r)s^2+3r(4R+r)^2=(4R-8r)s^2-9rs^2+3r(4R+r)^2$$

$$\overset{\text{Gerretsen}}{\ge}(4R-8r)(16Rr-5r^2)-9r(4R^2+4Rr+3r^2)+3r(4R+r)^2\overset{?}{\ge}0$$

$$\Leftrightarrow 19R^2-40Rr+4r^2\overset{?}{\ge}0\Leftrightarrow(19R-2r)(R-2r)\overset{?}{\ge}0\to\text{true via Euler}\Rightarrow$$

$$(*)\text{ is true}\therefore\sum_{cyc}\frac{1}{\frac{3r}{r_a}+2}$$

$$\ge\frac{6}{\frac{3r(4R+r)}{s^2}+5}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

2091. In any ΔABC, the following relationship holds:

$$\frac{18R^2}{s}\le\sum_{cyc}\frac{a^3}{s(s-a)}\le\frac{8(2R^3-7r^3)}{sr}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^3}{s(s-a)}=\sum_{cyc}\frac{\left(s-(s-a)\right)^3}{s(s-a)}=$$

$$=\sum_{cyc}\frac{s^3-(s-a)^3-3s^2(s-a)+3s(s-a)^2}{s(s-a)}=$$

$$=\frac{s^2(4Rr+r^2)}{r^2s}-\frac{1}{s}.\left(\left(\sum_{cyc}(s-a)\right)^2-2\sum_{cyc}(s-b)(s-c)\right)-9s+3\sum_{cyc}(s-a)$$

$$=\frac{s(4R+r)}{r}-s+\frac{2(4Rr+r^2)}{s}-6s$$

$$\therefore\sum_{cyc}\frac{a^3}{s(s-a)}=\frac{s^2(4R-6r)+r^2(8R+2r)}{rs}\to ①$$

Via ① and Gerretsen, $\sum_{cyc}\frac{a^3}{s(s-a)}\le\frac{(4R-6r)(4R^2+4Rr+3r^2)+r^2(8R+2r)}{rs}$

$$\overset{?}{\le}\frac{8(2R^3-7r^3)}{sr}\Leftrightarrow 4r(2R^2+Rr-10r^2)\overset{?}{\ge}0\Leftrightarrow 4r(R-2r)(2R+5r)\overset{?}{\ge}0$$

$$\to \text{true via Euler}\ \therefore\sum_{cyc}\frac{a^3}{s(s-a)}\le\frac{8(2R^3-7r^3)}{sr}$$

Via ① and Gerretsen, $\sum_{cyc}\frac{a^3}{s(s-a)}\ge\frac{(4R-6r)(16Rr-5r^2)+r^2(8R+2r)}{rs}$

$$\overset{?}{\ge}\frac{18R^2}{s}\Leftrightarrow 23R^2-54Rr+16r^2\overset{?}{\ge}0\Leftrightarrow(R-2r)(23R-8r)\overset{?}{\ge}0\to \text{true via Euler}$$

$$\therefore\sum_{cyc}\frac{a^3}{s(s-a)}\ge\frac{18R^2}{s}\ \text{and so,}\ \frac{18R^2}{s}\le\sum_{cyc}\frac{a^3}{s(s-a)}\le\frac{8(2R^3-7r^3)}{sr}$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)

2092. In ΔABC the following relationship holds:

$$6r\le\sum(b+c-a)\tan\frac{A}{2}\le 3R$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ \tan\frac{A}{2}\ge\tan\frac{B}{2}\ge\tan\frac{C}{2}$$

$$\sum(b+c-a)\tan\frac{A}{2}=2\sum(s-a)\tan\frac{A}{2}\overset{Chebyshev}{\le}2\cdot\frac{1}{3}\sum(s-a)\sum\tan\frac{A}{2}=$$

$$=\frac{2}{3}s.\frac{4R+r}{s}\overset{Euler}{\le}\frac{2}{3}\frac{9R}{2}=3R$$

$$\sum(b+c-a)\tan\frac{A}{2}=2\sum(s-a)\tan\frac{A}{2}\overset{AM-GM}{\ge}$$

$$\ge 6\left(\prod (s-a)\tan\frac{A}{2}\right)^{\frac{1}{3}} = 6\left(\frac{sr^2r}{s}\right)^{\frac{1}{3}} = 6r$$

Equality holds for $a = b = c$.

2093. In ΔABC the following relationship holds:

$$18r \le \sum (b+c-a)\cot\frac{A}{2} \le \frac{2(2R-r)^2}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum (b+c-a)\cot\frac{A}{2} = \sum 2(s-a)\cot\frac{A}{2} = 2\sum (s-a)\frac{s}{r_a} =$$

$$= 2s^2\sum\frac{1}{r_a} - 2s\sum\frac{a(s-a)}{rs} = \frac{2s^2}{r} - \frac{2}{r}\left(s\sum a - \sum a^2\right) =$$

$$= \frac{2s^2}{r} - \frac{2}{r}(2s^2 - 2(s^2 - r^2 - 4Rr)) = \frac{2}{r}(s^2 - 8Rr - 2r^2) \overset{GERRETSEN}{\le}$$

$$\le \frac{2}{r}(4R^2 + 4Rr + 3r^2 - 8Rr - 2r^2) = \frac{2}{r}(4R^2 - 4Rr + r^2) = \frac{2(2R-r)^2}{r}$$

$$\sum (b+c-a)\cot\frac{A}{2} = \sum 2(s-a)\cot\frac{A}{2} = 2\sum (s-a)\frac{s}{r_a} =$$

$$= 2s\sum\frac{(s-a)}{r_a} \overset{AM-GM}{\ge} 6s(\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{r_ar_br_c}} = 6s\sqrt[3]{\frac{sr^2}{s^2r}} = 6s\sqrt[3]{\frac{r}{s}} =$$

$$= 6s\sqrt[3]{\frac{rs^2}{s^3}} \overset{Mitrinovic}{\ge} 6s\sqrt[3]{\frac{27r^3}{s^3}} = 18r$$

Equality holds for an equilateral triangle

2094. In ΔABC the following relationship holds:

$$\sum \frac{\sin^2 A}{h_b^2} \le \frac{R(R-r)}{8r^4}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$We\ have,: \sum a^4 = 2[(s^2-4Rr-3r^2)^2 - 8r^3(2R+r)] \overset{Gerretsen}{\le}$$

$$\le 2[(4R^2)^2 - 8r^3(2R+r)] = 16(2R^4 - 2Rr^3 - r^4)$$

$$We\ will\ show: 16(2R^4-2Rr^3-r^4) \le 54R^3(R-r)\ (1)$$

$$11R^4 - 27R^3r + 16Rr^3 + 8r^4 \ge 0$$

$$(R-2r)\left((R-2r)(11R^2+17Rr+24r^2)+44r^3\right) \ge 0\ true\ Euler$$

$$\sum \frac{\sin^2 A}{h_b^2} = \sum \frac{\frac{a^2}{4R^2}}{\frac{b^2c^2}{4R^2}} = \frac{1}{a^2b^2c^2}\sum a^4 \overset{(1)}{\le} \frac{54R^3(R-r)}{16R^2r^2s^2} \overset{Mitrinovic}{\le}$$

$$\le \frac{54R^3(R-r)}{16R^2r^2 27r^2} = \frac{R(R-r)}{8r^4}$$

$$Equality\ holds\ if\ \Delta ABC\ is\ an\ equilateral\ one.$$

2095. In ΔABC the following relationship holds:

$$\frac{15}{2} + \sum \frac{\sin^2 A}{\sin^2 B + \sin^2 C} \le 9\left(\frac{R}{2r}\right)^2$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$6 = \frac{6R^2}{R^2} \overset{Euler}{\le} 6\left(\frac{R}{2r}\right)^2\ (1)$$

$$\sum \frac{\sin^2 A}{\sin^2 B + \sin^2 C} = \sum \frac{a^2}{b^2+c^2} \overset{AM-HM}{\le} \frac{1}{4}\sum\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right) = \frac{1}{4}\sum\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right) =$$

$$= \frac{1}{4}\sum\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2 - 2\right) \overset{Bandila}{\le} \frac{1}{4}\sum\left(\left(\frac{R}{r}\right)^2 - 2\right) = \frac{3}{4}\left(\frac{R}{r}\right)^2 - \frac{6}{4} = 3\left(\frac{R}{2r}\right)^2 - \frac{3}{2}\ (2)$$

$$\frac{15}{2}+\sum\frac{\sin^2 A}{\sin^2 B+\sin^2 C}\overset{(2)}{\leq}\frac{15}{2}+3\left(\frac{R}{2r}\right)^2-\frac{3}{2}=3\left(\frac{R}{2r}\right)^2+6\overset{(1)}{\leq}$$

$$\leq 3\left(\frac{R}{2r}\right)^2+6\left(\frac{R}{2r}\right)^2=9\left(\frac{R}{2r}\right)^2$$

$$Equality\ holds\ for\ A=B=C$$

2096. In ΔABC the following relationship holds:

$$\sum\sqrt{\cot A}\,.\sin A\leq\frac{3}{4}\frac{R}{r}\sqrt[4]{3}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$\sum\sqrt{\cot A}\,.\sin A=\sum\sqrt{\cos A}\,.\sqrt{\sin A}\overset{CBS}{\leq}\sqrt{\left(\sum\cos A\right)\left(\sum\sin A\right)}=$$

$$=\sqrt{\left(1+\frac{r}{R}\right)\frac{s}{R}}\overset{Euler\,\&\,Mitrinovic}{\leq}\sqrt{\left(1+\frac{1}{2}\right)\frac{\frac{3\sqrt{3}R}{2}}{R}}=\frac{3}{2}\sqrt[4]{3}=$$

$$=\frac{3}{2}\sqrt[4]{3}\cdot\frac{R}{R}\overset{Euler}{\leq}\frac{3}{2}\sqrt[4]{3}\cdot\frac{R}{2r}=\frac{3}{4}\cdot\frac{R}{r}\sqrt[4]{3}$$

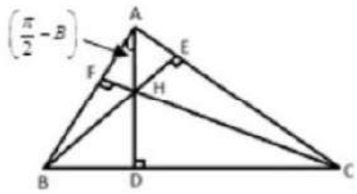
$$Equality\ holds\ for\ an\ equilateral\ triangle$$

2097. In acute $\Delta ABC, H$ –orthocenter, the following relationship holds:

$$\sec A+\sec B+\sec C\leq\frac{AH}{HD}+\frac{BH}{HE}+\frac{CH}{HF}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India



$$From\ \Delta\ AFC, AF = b\cos A\,, In\ \Delta AFH, \cos\left(\frac{\pi}{2}-B\right) = \frac{AF}{AH} = \frac{b\cos A}{AH}\ or,$$

$$AH = \frac{b\cos A}{\sin B} = 2R\cos A\,,$$

$$similarly\ CH = 2R\cos C\ and\ BH = 2R\cos B\,,$$

$$In\ \Delta AFH\ ,\tan\left(\frac{\pi}{2}-B\right) = \frac{FH}{AF} = \frac{FH}{b\cos A}\ or, FH = b\cos A\cot B$$

$$\frac{AH}{HD}+\frac{BH}{HE}+\frac{CH}{HF} = \sum\frac{CH}{HF} = \sum\frac{2R\cos C}{b\cos A\cot B} = \sum\frac{2R\cos C\sin B}{2R\sin\ B\cos A\cos B} =$$

$$= \sum\frac{\cos^2 C}{\cos A\cos B\cos C} \overset{AM-GM}{\geq} \frac{\sum\cos A\cos B}{\cos A\cos B\cos C} =$$

$$= \sum\frac{\cos A\cos B}{\cos A\cos B\cos C} = \sum\sec C = \sec A + \sec B + \sec C$$

2098. **In ΔABC the following relationship holds:**

$$\left(h_a\tan\frac{A}{2}\right)^2 + \left(h_b\tan\frac{B}{2}\right)^2 + \left(h_c\tan\frac{C}{2}\right)^2 \leq \frac{9R^2}{4}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$h_a\tan\frac{A}{2} \overset{h_a\leq\sqrt{s(s-a)}}{\leq} \sqrt{s(s-a)}\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{(s-b)(s-c)}\ (1)$$

$$\left(h_a\tan\frac{A}{2}\right)^2 + \left(h_b\tan\frac{B}{2}\right)^2 + \left(h_c\tan\frac{C}{2}\right)^2 = \sum\left(h_a\tan\frac{A}{2}\right)^2 \overset{(1)}{\leq}$$

$$\le \sum (s-b)(s-c) = \sum (s^2 - s(b+c) + bc) =$$

$$= 3s^2 - 2s(a+b+c) + \sum ab = 3s^2 - 4s^2 + s^2 + 4Rr + r^2 =$$

$$= 4Rr + r^2 \overset{Euler}{\le} 4R \cdot \frac{R}{2} + \frac{R^2}{4} = \frac{9R^2}{4}$$

Equality holds for an equilateral triangle

2099. In acute ΔABC the following relationship holds:

$$\frac{b^2+c^2-a^2}{b+c-a} + \frac{a^2+c^2-b^2}{a+c-b} + \frac{a^2+b^2-c^2}{a+b-c} \le 3R \sum_{cyc} cotA$$

Proposed by Ertan Yildirim-Turkiye

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{4FcotA}{2(s-a)} + \frac{4FcotB}{2(s-b)} + \frac{4FcotC}{2(s-c)} =$$

$$\boxed{cotA = \frac{b^2+c^2-a^2}{4F}} (True), \quad \boxed{F = (s-a).r_a} (True)$$

$$= \frac{2FcotA}{s-a} + \frac{2FcotB}{s-b} + \frac{2FcotC}{s-c} = \overset{\frown}{=}$$

$$= \frac{2(s-a)r_a}{s-a} \cdot cotA + \frac{2(s-b)r_b}{s-b} \cdot cotB + \frac{2(s-c)r_c}{s-c} \cdot cotC =$$

$$= 2(r_a cotA + r_b cotB + r_c cotC)$$

$$WLOG: \ a \le b \le c \rightarrow r_a \le r_b \le r_c \ and \ cotA \ge cotB \ge cotC$$

$$2(r_a cotA + r_b cotB + r_c cotC) \overset{CEBYSHEV}{\le} 2 \cdot \frac{1}{3}(r_a + r_b + r_c) \sum_{cyc} cotA =$$

$$= \frac{2}{3}(4R+r) \sum_{cyc} cotA = \frac{1}{3}(8R+2r) \sum_{cyc} cotA \overset{Euler}{\le} \frac{1}{3} \cdot 9R \sum_{cyc} cotA = 3R \sum_{cyc} cotA$$

Equality holds for: $a = b = c$.

2100. In any non $-$ obtuse Δ ABC, the following relationship holds :

$$\frac{2c}{w_b^2} \ge \frac{1}{2m_a} + \frac{1}{c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2c}{w_b^2}-\left(\frac{1}{2m_a}+\frac{1}{c}\right)\ge\frac{2c(c+a)^2}{4ca.s(s-b)}-\left(\frac{1}{a}+\frac{1}{c}\right)$$

$$(\because \text{ in non}-\text{obtuse }\Delta\,ABC, 4m_a^2-a^2=2(b^2+c^2-a^2)\ge 0\Rightarrow 2m_a\ge a)$$

$$=\frac{2(c+a)^2}{a((c+a)^2-b^2)}-\frac{c+a}{ca}\overset{?}{\ge}0\Leftrightarrow 2c(c+a)\overset{?}{\ge}(c+a)^2-b^2$$

$$\Leftrightarrow 2c^2+2ca\overset{?}{\ge}c^2+a^2+2ca-b^2\Leftrightarrow b^2+c^2\overset{?}{\ge}a^2\rightarrow\text{true}\because$$

$$\Delta\,ABC\text{ is non}-\text{obtuse}\therefore\frac{2c}{w_b^2}\ge\frac{1}{2m_a}+\frac{1}{c}\ \forall\text{ non}-\text{obtuse }\Delta\,ABC,$$

$$''=''\ \text{ iff }\hat{A}=90^\circ\ (QED)$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru