ROMANIAN MATHEMATICAL MAGAZINE

SP.558 In $\Delta ABC, O -$ circumcenter. If the bisector from angle A, altitude from angle B and CO circumcevian are in concurrence, then holds:

$$\sqrt[3]{-1+3\frac{r}{R}-\frac{3}{2}\left(\frac{r}{R}\right)^2} \leq \cos A \leq \sqrt[3]{\frac{1}{2}\left(\frac{r}{R}\right)^2}$$

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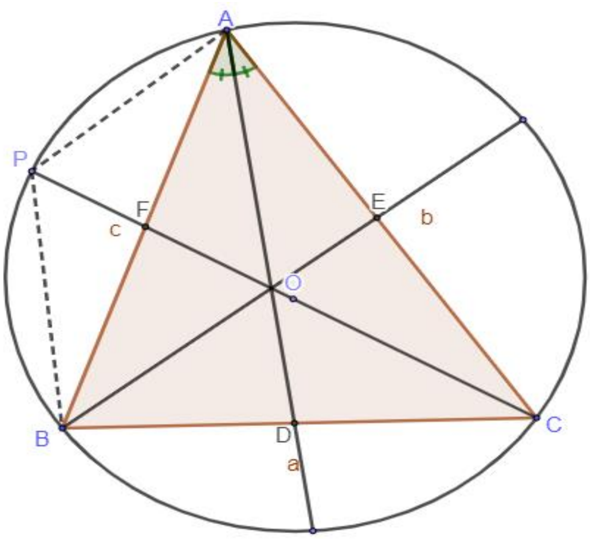
Solution by proposer



From Ceva's theorem:

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1 \qquad (1)$$

From bisector theorem:

$$\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b} \qquad (2)$$

$EC = BC \cdot \sin C$ and $AE = AB \cdot \cos A$ (3)

From Law of Sines in ΔFAC:

$$\frac{AF}{\sin(ACF)}=\frac{AC}{\sin(AFC)}$$

But: $\sin(ACF) = \sin\left(\frac{\pi}{2}-B\right) = \cos B$

$$AF = \frac{AC\cdot\cos B}{\sin(AFC)} \qquad (4)$$

$$\Delta FBD \Rightarrow \frac{FB}{\sin(BCF)} = \frac{BC}{\sin(BFC)}$$

$$\Rightarrow FB = \frac{BC\cdot\cos A}{\sin(\pi - AFC)} = \frac{BC\cdot\cos A}{\sin(AFC)} \quad (5)$$

From (1)-(5), it follows:

$$\frac{AB}{AC}\cdot\frac{BC\cdot\cos C}{AB\cdot\cos A}\cdot\frac{AC\cdot\cos B}{\sin(AFC)}\cdot\frac{\sin(AFC)}{BC\cdot\cos A} = 1$$

$$\Rightarrow \frac{\cos C\cdot\cos B}{\cos^2 A} = 1 \Rightarrow \cos^2 A = \cos B\cos C$$

$$\cos^3 A = \cos A\cos B\cos C \quad (6)$$

But: $\cos A\cos B\cos C = \frac{s^2-4R^2-4Rr-r^2}{4R^2}$ (7)

From (6) and (7), we get:

$$\cos^3 A = \frac{s^2-4R^2-4Rr-r^2}{4R^2} \quad (8)$$

But from Gerretsen: $s^2 \leq 4R^2 + 4Rr + 3r^3$ (9)

From (8) and (9):

$$\cos^3 A \leq \frac{2r^2}{4R^2} = \frac{1}{2}\left(\frac{r}{R}\right)^2 \Rightarrow \cos A \leq \sqrt[3]{\frac{1}{2}\left(\frac{r}{R}\right)}$$

From Gerretsen: $s^2 \geq 16Rr - 5r^2$ (10)

From (8) and (10):

$$s^2 \geq \frac{-4R^2+12Rr-6r^2}{4R^2} = -1 + \frac{3r}{R} - \frac{3r^2}{2R^2}$$

$$\cos A \geq \sqrt[3]{-1 + 3\frac{r}{R} - \frac{3}{2}\left(\frac{r}{R}\right)}$$