ROMANIAN MATHEMATICAL MAGAZINE

SP.560 For $k \in \mathbb{N}$ fixed and $\alpha > 0$ evaluate:

$$L = \lim_{n\to\infty} \frac{1}{\sqrt{n^\alpha}} \cdot \left(\prod_{i=1}^{k} \frac{n+k+i}{n+i}\right)^{n^\alpha}$$

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution by proposers

Let be:

$$x_n = \left(\prod_{i=1}^{k} \frac{n+k+i}{n+i}\right)^{n^\alpha} = \left[\prod_{i=1}^{k} \frac{\left(1+\frac{k+i}{n}\right)^n}{\left(1+\frac{i}{n}\right)^n}\right]^{n^{\alpha-1}}$$

From:

$$\lim_{n\to\infty}\left(1+\frac{t}{n}\right)^n = e^t$$

we get:

$$\lim_{n\to\infty} x_n = \prod_{i=1}^{k} \frac{e^{k+i}}{e^i} \cdot \lim_{n\to\infty} n^{\alpha-1}$$

Therefore:

$$L = \lim_{n\to\infty} \frac{1}{\sqrt{n^\alpha}} \cdot \left(\prod_{i=1}^{k} \frac{n+k+i}{n+i}\right)^{n^\alpha} = \prod_{i=1}^{k} \frac{e^{k+i}}{e^i} \cdot \lim_{n\to\infty} n^{\alpha-1-\frac{\alpha}{2}} =$$

$$= e^{k^2} \cdot \lim_{n\to\infty} n^{\frac{\alpha}{2}-1} = \begin{cases} 0, if\ \alpha \in (0,2) \\ e^{k^2}, if\ \alpha = 2 \\ +\infty, if\ \alpha > 2 \end{cases}$$