ROMANIAN MATHEMATICAL MAGAZINE

SP. 561 Let be the function $f:[0,1]\to\mathbb{R}$ integrable such that $f(1)=1$ and

$$\int_x^y f(t)\,dt=\frac{1}{2}\big(yf(y)-xf(x)\big),\forall x,y\in[0,1]$$

Find:

$$I=\int_0^{\frac{\pi}{4}} f(x)\cdot\tan^2 x\,dx.$$

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Solution by proposers

$$\int_x^y f(t)\,dt=\frac{1}{2}\big(yf(y)-xf(x)\big),\forall x,y\in[0,1] \qquad (*)$$

If in (*) we take $x=0$ and $y=x$, we get $\int_0^x f(t)\,dt=\frac{x}{2}f(x)$.

Let be $F(x)=\int_0^x f(t)\,dt$, how f is integrable function then f is bounded function and

$F(x)=\frac{x}{2}f(x).$ (1)

From $F'(x)=f(x)\Rightarrow xF'(x)=2F(x)$ (2)

On the other hand, $\left(\frac{F(x)}{x^2}\right)'=\frac{xF'(x)-2F(x)}{x^3}\overset{(2)}{=}0\Rightarrow F(x)=\alpha x^2,\alpha\in\mathbb{R}\Rightarrow$

$f(x)=\beta x,\beta\in\mathbb{R},x\in(0,1]$ and from $f(1)=1$, we get $\beta+1$ and hence $f(x)=x$.

Therefore,

$$I=\int_0^{\frac{\pi}{4}} f(x)\cdot\tan^2 x\,dx=\int_0^{\frac{\pi}{4}}\frac{x(1-\cos^2 x)}{\cos^2 x}dx=\int_0^{\frac{\pi}{4}}\frac{x}{\cos^2 x}dx-\int_0^{\frac{\pi}{4}}d\,dx=$$

$$=\left(x\tan x-\frac{\pi}{2}\right)\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan x\,dx=\frac{\pi}{4}-\frac{\pi^2}{32}-\frac{1}{2}\ln 2$$