ROMANIAN MATHEMATICAL MAGAZINE

SP.566 If $\lambda > 0$ then find:

$$\int_0^1 \frac{(x^2e^x + (\lambda+1)x + 1)e^x}{\lambda + xe^x}dx$$

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Solution by proposer

We have $\frac{(x^2e^x+(\lambda+1)x+1)e^x}{\lambda+xe^x} = xe^x + \frac{(x+1)e^x}{\lambda+xe^x}$.

$$\int_0^1 \frac{(x^2e^x+(\lambda+1)x+1)e^x}{\lambda+xe^x}dx = \int_0^1 \left(xe^x + \frac{(x+1)e^x}{\lambda+xe^x}\right)dx = \int_0^1 xe^x dx + \int_0^1 \frac{(x+1)e^x}{\lambda+xe^x}dx \quad (1)$$

We have $\int_0^1 xe^x\,dx = (x-1)e^x|_0^1 = 1$ (2)

For the calculus $\int_0^1 \frac{(x+1)e^x}{\lambda+xe^x}dx$, we make the substitution $\lambda + xe^x = t \Rightarrow dt = (x+1)e^x dx$

$$\int_0^1 \frac{(x+1)e^x}{\lambda+xe^x}dx = \int_\lambda^{\lambda+e}\frac{dt}{t} = \ln t|_\lambda^{\lambda+e} = \ln\frac{\lambda+e}{\lambda} = \ln\left(1+\frac{e}{\lambda}\right) \quad (3)$$

From (1), (2), (3) we obtain $\int_0^1 \frac{(x^2e^x+(\lambda+1)x+1)e^x}{\lambda+xe^x}dx = 1 + \ln\left(1+\frac{e}{\lambda}\right)$.