ROMANIAN MATHEMATICAL MAGAZINE

SP.567 Prove that $\frac{3}{2}$ is the largest positive value of the constant k such that the inequality:

$$(a+k)^2+(b+k)^2+(c+k)^2+(d+k)^2+(e+k)^2+(f+k)^2\geq 6(1+k)^2$$

holds whenever $a\geq b\geq 1\geq c\geq d\geq e\geq f\geq 0$ satisfying

$$ab+bc+cd+de+ef+fa=6$$

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Solution by proposer

Assuming $a=b=2, c=1$ and $d=e=f=0$, the equality constrait is satisfied, while the inequality becomes

$$2(2+k)^2+3k^2\geq 5(1+k)^2$$

which is equivalent to $2k\leq 3$. To show that $\frac{3}{2}$ is the largest positive value of k, we need to prove the inequality $E\geq 150$, where

$$E=(2a+3)^2+(2b+3)^2+(2c+3)^2+(2d+3)^2+(2e+3)^2+(2f+3)^2$$

For fixed c,d,e,f, we may assume that b and E are functions of a. By differentiating the equality constraint, we get

$$(a+c)b'+b+f=0, b'=\frac{-(b+f)}{a+c}\geq -1$$

Since

$$\frac{E'(a)}{4}=2a+3+(2b+3)b'\geq 2a+3-(2b+3)=2(a-b)\geq 0,$$

$E(a)$ is increasing and has the minimum value when a is minimum, hence when $a=b$. Similarly, for fixed a,b,c,d, assume that E are functions of f. By differentiating the equality constraint, we get

$$(d+f)e'+a+e=0,\ \ e'=\frac{-(a+e)}{d+f}\leq -1$$

Since

$$\frac{E'(f)}{4}=2f+3+(2e+3)e'\leq 2f+3-(2e+3)=2(f-e)\leq 0,$$

$E(f)$ is decreasing and has the minimum value when f is maximum, hence when $f=e$. So, it suffices to consider $a=b$ and $f=e$, when we need to show that $F\geq 150$ for $b\geq 1\geq c\geq d\geq e\geq 0$ such that $b^2+bc+cd+de+e^2+be=6$, where

$$F=2(2b+3)^2+(2c+3)^2+(2d+3)^2+2(2e+3)^2$$

Now, for fixed d and e, assume that b and F are functions of c. By differentiating the equality constraint, we get

$$(2b+c+e)b'+b+d=0, b'=\frac{-(b+d)}{2b+c+e}\leq\frac{-(b+d)}{2b+c+d},$$

hence

$$\frac{F'(c)}{4}=2c+3+2(2b+3)b'\leq 2c+3-\frac{2(2b+3)(b+d)}{2b+c+d}\leq$$

$$\le 5-\frac{2(2b+3)(b+d)}{2b+1+d}=\frac{5+4b-4b^2-(4b+1)d}{2b+1+d}$$

We will show that $F'(c)\le 0$, that is

$$4b^2-4b-5+(4b+1)d\ge 0$$

From

$$6=b^2+bc+cd+de+e^2+be\le b^2+bc+cd+2d^2+bd\le b^2+b+3d+bd$$

we get

$$d\ge\frac{6-b-b^2}{b+3}$$

therefore

$$4b^2-4b-5+(4b+1)d\ge 4b^2-4b-5+\frac{(4b+1)(6-b-b^2)}{b+3}=\frac{3(b-1)(b+3)}{b+3}\ge 0$$

Since $F'(c)\le 0, F(c)$ is decreasing and has the minimum value when c is maximum, hence when $c=1$. So, it suffices to consider this case, when we need to show that $G\ge 125$ for $b\ge 1\ge d\ge e\ge 0$ such that $b^2+b+d+de+e^2+be=6$, where

$$G=2(2b+3)^2+(2d+3)^2+2(2e+3)^2$$

For fixed b, we may assume that d is a function of e. By differentiating the equality constraint, we get

$$(1+e)d'+b+d+2e=0,$$

hence

$$\frac{G(e)}{4}=2(2e+3)+(2d+3)d'=2(2d+3)-\frac{(2d+3)(b+d+2e)}{1+e}\le$$

$$\le 2(2e+3)-\frac{(2e+3)(b+d+2e)}{1+e}$$

$$=\frac{(2e+3)(2-b-d)}{1+e}.$$

From

$$6=b^2+b+d+de+e^2+be\le b^2+b+d+2d^2+bd\le(b+d)^2+(b+d)$$

we get $b+d\ge 2$, therefore $G'(e)\le 0, G(e)$ is decreasing and has the minimum value when e is maximum, hence when $e=d$. So, it suffices to consider $e=d$, when we need to show that if $b\ge 1\ge d$ such that

$$b^2+b+d+2d^2+bd=6,$$

then $2(2b+3)^2+3(2d+3)^2\ge 125$, i.e.

$$2b^2+3d^2+6b+9d\ge 20, 2b(2-d)\ge d^2-7d+8$$

Since $2b=-d-1+\sqrt{25-2d-7d^2}$, we need to show that

$$\left(-d-1+\sqrt{25-2d-7d^2}\right)(2-d)\ge d^2-7d+8,$$

i.e.

$$(2-d)\sqrt{25-2d-7d^2}\ge 10-6d$$

This is true if

$$(2-d)^2(25-2d-7d^2)\ge(10-6d^2),$$

which is equivalent to the obvious inequality

$$d(d-1)^2(12-7d)\ge 0.$$

The proof is finished. For $k=\frac{3}{2}$, the equality occurs when $a=b=c=d=e=f=1$, and also for $a=b=2, c=1$ and $d=e=f=0$.