SPECIAL NEW TRIGONOMETRIC INEQUALITIES

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Abstract. In this paper we present new trigonometric inequalities.

Theorem 1.
If $x_k \in (0, \frac{\pi}{2}), \forall k = \overline{1,n}$, then

$$\left(\sum_{k=1}^{n}(\sin x_k + 2\tan x_k)\right)\sum_{k=1}^{n}\frac{1}{x_k} > 3n^2$$

Proof. We have:

$$\frac{\sin x + 2\tan x}{3} > 2\tan\frac{x}{2} > x, \forall x \in \left(0, \frac{\pi}{2}\right) \tag{1}$$

Indeed, we denote by $\tan\frac{x}{2} = t \in (0,1)$, and we have to prove that:

$$\frac{2t}{1+t^2} + \frac{4t}{1-t^2} > 6t, \forall t \in (0,1) \Leftrightarrow \frac{1}{1+t^2} + \frac{2}{1-t^2} > 3, \forall t \in (0,1) \Leftrightarrow 1-t^2+2+2t^2 > 3(1-t^4) \Leftrightarrow$$

$$\Leftrightarrow 3t^4 + t^2 > 0, \forall t \in (0,1), \text{ which is evidently.}$$

Therefore,

$$\sum_{k=1}^{n}(\sin x_k + 2\tan x_k) > 3\sum_{k=1}^{n} x_k, \forall x_k \in \left(0, \frac{\pi}{2}\right), k = \overline{1,n} \tag{2}$$

Multiplying the inequality (2) with $\sum_{k=1}^{n}\frac{1}{x_k} > 0$ and taking account that:

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right) \geq n^2, \text{ then the inequality (2) because:}$$

$$\left(\sum_{k=1}^{n}(\sin x_k + 2\tan x_k)\right)\sum_{k=1}^{n}\frac{1}{x_k} > 3\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right) \geq 3n^2, \text{ and we are done.}$$

□

Theorem 2.
$(F_n)_{n\geq 0}$ is the sequence of Fibonacci and $(L_n)_{n\geq 0}$ is the sequence of Lucas, then:

$$\arctan\sqrt{\frac{F_n^2 + F_{n+1}^2}{2}} + \arctan\sqrt{\frac{L_n^2 + L_{n+1}^2}{2}} \geq \arctan\frac{F_{n+2}}{2} + \arctan\frac{L_{n+2}}{2}, \forall n \in \mathbb{N}.$$

Proof.

$$f : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), f(x) = \arctan x, f'(x) = \frac{1}{1+x^2}, \forall x \in \mathbb{R} \text{ is increasing on } \mathbb{R}.$$

Since,

$$\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}, \forall x,y \in \mathbb{R}, \text{ we have } \sqrt{\frac{F_n^2+F_{n+1}^2}{2}} \geq \frac{F_n+F_{n+1}}{2} = \frac{F_{n+2}}{2} \text{ and}$$

$$\sqrt{\frac{L_n^2+L_{n+1}^2}{2}} \geq \frac{L_n+L_{n+1}}{2} = \frac{L_{n+2}}{2}, \forall n \in \mathbb{N}.$$

Therefore,

$$f\left(\sqrt{\frac{F_n^2+F_{n+1}^2}{2}}\right)+f\left(\sqrt{\frac{L_n^2+L_{n+1}^2}{2}}\right) \geq f\left(\frac{F_{n+2}}{2}\right)+f\left(\frac{L_{n+2}}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow \arctan\sqrt{\frac{F_n^2+F_{n+1}^2}{2}}+\arctan\sqrt{\frac{L_n^2+L_{n+1}^2}{2}} \geq \arctan\frac{F_{n+2}}{2}+\arctan\frac{L_{n+2}}{2}, \forall n \in \mathbb{N}$$

□

Theorem 3.
If $a, b > 0$, then $4\sqrt{ab} \cdot \frac{\sin x}{x} + b\left(\frac{\tan x}{x}\right)^2 + a > 6\sqrt{ab}$, for any $x \in (0, \frac{\pi}{2})$.

Proof.
By AM-GM inequality we have
(1)

$$4\sqrt{ab}\cdot\frac{\sin x}{x}+b\left(\frac{\tan x}{x}\right)^2+a \geq 4\sqrt{ab}\cdot\frac{\sin x}{x}+2\sqrt{ab}\cdot\frac{\tan x}{x} = 2\sqrt{ab}\left(2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}\right), \forall x \in\left(0, \frac{\pi}{2}\right)$$

Also we have $\sin x > x - \frac{x^3}{6}$ and $\tan x > x + \frac{x^3}{3}, \forall x \in (0, \frac{\pi}{2}) \Leftrightarrow$

$$\Leftrightarrow \frac{\sin x}{x} > 1 - \frac{x^2}{6} \text{ and } \frac{\tan x}{x} > 1 + \frac{x^2}{3}, \forall x \in\left(0, \frac{\pi}{2}\right).$$

(2) So, $2 \cdot \frac{\sin x}{x} + \frac{\tan x}{x} > 2 - \frac{x^2}{3} + 1 + \frac{x^2}{3} = 3, \forall x \in\left(0, \frac{\pi}{2}\right)$

From (1) and (2) yields the desired inequality. □

Theorem 4.
If $a, b > 0$, then $2\sqrt{ab} \cdot \frac{\sin x}{x} + \frac{a+b}{2} \cdot \frac{\tan x}{x} > 3\sqrt{ab}, \forall x \in (0, \frac{\pi}{2})$.

Proof.
By AM-GM inequality we have:
(1)

$$2\sqrt{ab}\cdot\frac{\sin x}{x}+\frac{a+b}{2}\cdot\frac{\tan x}{x} \geq 2\sqrt{ab}\cdot\frac{\sin x}{x}+\sqrt{ab}\cdot\frac{\tan x}{x} = \sqrt{ab}\left(2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}\right), \forall x \in\left(0, \frac{\pi}{2}\right)$$

We infer that:

$$\sin x > x - \frac{x^3}{6} \text{ and } \tan x + \frac{x^3}{3}, \forall x \in\left(0, \frac{\pi}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{\sin x}{x} > 1 - \frac{x^2}{6} \text{ and } \frac{\tan x}{x} > 1 + \frac{x^2}{3}, \forall x \in\left(0, \frac{\pi}{2}\right).$$

Therefore:

(2) $$2 \cdot \frac{\sin x}{x} + \frac{\tan x}{x} > 2 - \frac{x^2}{3} + 1 + \frac{x^2}{3} = 3, \forall x \in\left(0, \frac{\pi}{2}\right)$$

From (1) and (2) yields the desired inequality. □

Theorem 5.
If $a, b > 0$, then $(a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}>\frac{6ab}{a+b}, \forall x\in\Big(0,\frac{\pi}{2}\Big)$.

Proof.
By AM-HM inequality we have:
(1)
$$(a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}\geq\frac{4ab}{a+b}\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}=\frac{2ab}{a+b}\Big(2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}\Big), \forall x\in\Big(0,\frac{\pi}{2}\Big)$$
Since:
$$\sin x > x-\frac{x^3}{6} \text{ and } \tan x > x+\frac{x^3}{3}, \forall x\in\Big(0,\frac{\pi}{2}\Big)\Leftrightarrow$$
$$\Leftrightarrow\frac{\sin x}{x}>1-\frac{x^2}{6} \text{ and } \frac{\tan x}{x}>1+\frac{x^2}{3}, \forall x\in\Big(0,\frac{\pi}{2}\Big).$$
Therefore:
(2)
$$2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}>2-\frac{x^2}{3}+1+\frac{x^2}{3}=3, \forall x\in\Big(0,\frac{\pi}{2}\Big)$$
By (1) and (2) q.e.d. □

Theorem 6.
If $a, b, c > 0$, then:
$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}>a+b+c, \forall x\in\Big(0,\frac{\pi}{2}\Big).$$

Proof.
By AM-QM:
$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}\geq 2\cdot\frac{a+b+c}{3}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}=$$
(1)
$$=\frac{a+b+c}{3}\cdot\Big(2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}\Big), \forall x\in\Big(0,\frac{\pi}{2}\Big)$$
$$\sin x > x-\frac{x^3}{6} \text{ and } \tan x > x+\frac{x^3}{3}, \forall x\in\Big(0,\frac{\pi}{2}\Big)\Leftrightarrow$$
$$\Leftrightarrow\frac{\sin x}{x}>1-\frac{x^2}{6} \text{ and } \frac{\tan x}{x}>1+\frac{x^2}{3}, \forall x\in\Big(0,\frac{\pi}{2}\Big).$$
So,
(2)
$$2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}>2-\frac{x^2}{3}+1+\frac{x^2}{3}=3, \forall x\in\Big(0,\frac{\pi}{2}\Big)$$
By (1) and (2) q.e.d. □

Theorem 7.
If $a, b > 0$ and $x\in(0,\frac{\pi}{2})$ then $a\cdot\tan x+b\cdot\sin x>2x\sqrt{ab}$.

Proof.
By AM-GM inequality we have:
(1)
$$a\cdot\tan x+b\cdot\sin x\geq 2\sqrt{a\cdot b\cdot\tan x\cdot\sin x}, \forall x\in\Big(0,\frac{\pi}{2}\Big)$$
First we show that:
(2)
$$\tan x\cdot\sin x>x^2, \forall x\in\Big(0,\frac{\pi}{2}\Big)$$

Indeed, denoting $t = \tan \frac{x}{2}, x \in (0, \frac{\pi}{2})$ we have:

$$\tan x \cdot \sin x = \frac{2t}{1-t^2} \cdot \frac{2t}{1+t^2} = \frac{4t^2}{1-t^4} > 4t^2 = 4\tan^2 \frac{x}{2} > 4 \cdot \frac{x^2}{4} = x^2, \forall x \in \left(0, \frac{\pi}{2}\right)$$

By (1) and (2) we obtain q.e.d. □

Theorem 8.

$$a^2 \cdot \tan^k x + b^2 \cdot \sin^k x > 2 \cdot a \cdot b \cdot x^k, \forall a, b > 0, \forall x \in \left(0, \frac{\pi}{2}\right) \text{ and } k \in \mathbb{N}^*$$

Proof.
By AM-GM:

$$(1) \qquad a^2 \cdot \tan^k x + b^2 \cdot \sin^k x \geq 2 \cdot a \cdot b \cdot \sqrt{(\tan x \cdot \sin x)^k}, \forall x \in \left(0, \frac{\pi}{2}\right)$$

We prove that:

$$(2) \qquad \tan x \cdot \sin x > x^2, \forall x \in \left(0, \frac{\pi}{2}\right)$$

Indeed, denoting $t = \tan \frac{x}{2}, x \in \left(0, \frac{\pi}{2}\right)$:

$$\tan x \cdot \sin x = \frac{2t}{1-t^2} \cdot \frac{2t}{1+t^2} = \frac{4t^2}{1-t^4} > 4t^2 = 4\tan^2 \frac{x}{2} > 4 \cdot \frac{x^2}{4} = x^2, \forall x \in \left(0, \frac{\pi}{2}\right).$$

From (1) and (2) q.e.d. □

Theorem 9.
The acute triangle ABC is equilateral if and only if:

$$\frac{\tan^2 A}{\sin^2 B + \cos^2 C} + \frac{\tan^2 B}{\sin^2 C + \cos^2 A} + \frac{\tan^2 C}{\sin^2 A + \cos^2 B} = 9.$$

Proof.

" $\Rightarrow$ " If the triangle ABC is equilateral we have that:

$$U = \sum \frac{\tan^2 A}{\sin^2 B + \cos^2 C} = \sum \frac{\tan^2 A}{\sin^2 A + \cos^2 A} = \sum \frac{\tan^2 A}{1} = 3 \cdot \tan^2 \frac{\pi}{3} = 3 \cdot 3 = 9.$$

" $\Leftarrow$ " We have:

$$9 = U = \sum \frac{\tan^2 A}{\sin^2 B + \cos^2 C} = \frac{1}{\sum(\sin^2 B + \cos^2 C)} \cdot \left(\sum(\sin^2 B + \cos^2 C)\right) \cdot \sum \frac{\tan^2 A}{\sin^2 B + \cos^2 C} =$$

$$= \frac{1}{3} \cdot \left(\sum(\sin^2 B + \cos^2 C)\right) \cdot \sum \frac{\tan^2 A}{\sin^2 B + \cos^2 C},$$

where we apply Cauchy-Schwarz's inequality and we obtain that:

$$27 = 3U \geq \left(\sum \tan A\right)^2,$$

than we taking account that the function $\tan : (0, \frac{\pi}{2}) \to \mathbb{R}$ is convex, and applying Jensen's inequality we deduce that:

$$27 = 3U \geq \left(\sum \tan A\right)^2 \geq \left(3 \cdot \tan \frac{A+B+C}{3}\right)^2 = \left(3 \cdot \tan \frac{\pi}{3}\right)^2 = (3\sqrt{3})^2 = 27.$$

The last relation implies the fact that: $\tan A = \tan B = \tan C$ and because the function tan is strictly increasing on $(0, \frac{\pi}{2})$, so is injective, hence $A = B = C = \frac{\pi}{3}$, yields that the triangle ABC is equilateral. □

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