UP.563 Calculate the integral:

$$\int_1^\infty \frac{\sqrt{x}\ln^2 x}{(x+1)(x^2+1)}dx$$

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Solution by proposer

Let us denote:

$$I = \int_1^\infty \frac{\sqrt{x}\ln^2 x}{(x+1)(x^2+1)}dx$$

We make the variable change: $x = \frac{1}{y}; y = \frac{1}{x}$. We obtain:

$$I = \int_1^\infty \frac{\sqrt{x}\ln^2 x}{(x+1)(x^2+1)}dx = \int_0^1 \frac{\sqrt{y}\ln^2 y}{(y+1)(y^2+1)}dy$$

Let us denote:

$$J = \int_0^1 \frac{\sqrt{y}\ln^2 y}{(y+1)(y^2+1)}dy = \int_0^1 \frac{\sqrt{y}(1-y)\ln^2 y}{1-y^4}dy$$

We make the variable change: $y = z^2$. We obtain:

$$J = 8\int_0^1 \frac{z^2(1-z^2)\ln^2 z}{1-z^8}dz = 8\left(\int_0^1 \frac{z^2\ln^2 z}{1-z^8}dz - \int_0^1 \frac{z^4\ln^2 z}{1-z^8}dz\right)$$

We have, successively:

$$J = 8\left(\int_0^1 \sum_{n=0}^\infty x^{8n+2}\ln^2 x\,dx - \int_0^1 \sum_{n=0}^\infty x^{8n+4}\ln^2 x\,dx\right) =$$

$$= 8\sum_{n=0}^\infty \left(\int_0^1 x^{8n+2}\ln^2 x\,dx - \int_0^1 x^{8n+4}\ln^2 x\,dx\right)$$

We will to use the following relationship:

$$\int_0^1 x^a \ln^2 x\,dx = \frac{2}{(a+1)^3}$$

where $a \in \mathbb{R}, a \geq 0$. We obtain:

$$J = 8\sum_{n=0}^\infty \left[\frac{2}{(8n+3)^3} - \frac{2}{(8n+5)^3}\right] = 8\sum_{n=0}^\infty \left[\frac{\frac{2}{512}}{\left(n+\frac{3}{8}\right)^3} - \frac{\frac{2}{512}}{\left(n+\frac{5}{8}\right)^3}\right]$$

We now use the following relationship:

$$\psi_2(x) = \sum_{n=0}^{\infty} \frac{2}{(x+n)^3}$$

where $\psi_2(x)$ is the tetragamma function. We obtain the value of the integral J:

$$J = \frac{1}{64}\left[\psi_2\left(\frac{5}{8}\right) - \psi_2\left(\frac{3}{8}\right)\right]$$

Result:

$$I = J = \frac{1}{64}\left[\psi_2\left(\frac{5}{8}\right) - \psi_2\left(\frac{3}{8}\right)\right]$$

We use the reflection formula for the tetragamma function:

$$\psi_2(p) - \psi_2(1-p) = f(p)$$

where:

$$f(x) = -\pi \frac{d^2}{dx^2}\cot(\pi x)$$

We have:

$$\psi_2\left(\frac{5}{8}\right) - \psi_2\left(\frac{3}{8}\right) = 4\left(3\sqrt{2} - 4\right)\pi^3$$

We obtained the value of the integral required in the problem statement:

$$I = \frac{1}{16}\left(3\sqrt{2} - 4\right)\pi^3$$