ROMANIAN MATHEMATICAL MAGAZINE

UP.564 Calculate the integral:

$$\int_0^{\infty} \frac{\arctan(x)}{\sqrt{x^4 - x^2 + 1}} dx$$

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Solution by proposer

We make the notation: $A = \int_0^{\infty} \frac{\arctan(x)}{\sqrt{x^4-x^2+1}} dx$

We also consider the integral: $B = \int_0^{\infty} \frac{\text{arccot}(x)}{\sqrt{x^4-x^2+1}} dx$

We have:

$$A + B = \int_0^{\infty} \frac{\arctan(x) + \arctan(x)}{\sqrt{x^4 - x^2 + 1}} dx = \frac{\pi}{2} \int_0^{\infty} \frac{1}{\sqrt{x^4 - x^2 + 1}} dx$$

We are going to calculate the integral:

$$C = \int_0^{\infty} \frac{1}{\sqrt{x^4 - x^2 + 1}} dx$$

We can write:

$$C = \int_0^{\infty} \frac{1}{\sqrt{x^4 - x^2 + 1}} dx = \int_0^{1} \frac{1}{\sqrt{x^4 - x^2 + 1}} dx + \int_1^{\infty} \frac{1}{\sqrt{x^4 - x^2 + 1}} dx$$

In the second integral we make the variable change $x = \frac{1}{t}$.

$$\int_1^{\infty} \frac{1}{\sqrt{x^4 - x^2 + 1}} dx = \int_1^{0} \frac{1}{\sqrt{\frac{1}{t^4} - \frac{1}{t^2} + 1}} \left(-\frac{1}{t^2}\right) dt = \int_0^{1} \frac{1}{\sqrt{t^4 - t^2 + 1}} dt$$

So we have:

$$C = 2 \int_0^{1} \frac{1}{\sqrt{t^4 - t^2 + 1}} dt$$

We will show that the C integral can be expressed using the complete elliptic integral of the first kind.

The complete elliptic integral of the first kind is defined by the relationship:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2 \sin^2 \theta}} d\theta, \text{ with } -1 < k < 1$$

Substitute:

$$t = \tan\frac{\theta}{2}, \text{ so } \sin\theta = \frac{2t}{1+t^2} \text{ and } d\theta = \frac{2}{1+t^2} dt$$

We have:

$$K(k) = \int_0^{1} \frac{1}{\sqrt{1 - k^2 \frac{4t^2}{(1+t^2)^2}}} 2 \frac{1}{1+t^2} dt = 2 \int_0^{1} \frac{1}{\sqrt{t^4 + (2 - 4k^2)t^2 + 1}} dt$$

We have:

$$C = 2\int_0^1 \frac{1}{\sqrt{t^4 - t^2 + 1}}dt$$

We put the condition:

$2 - 4k^2 = -1$, **so** $k = \frac{\sqrt{3}}{2}$ **(we choose** $k > 0$**).**

We obtain:

$$C = K\left(\frac{\sqrt{3}}{2}\right)$$

In the B **integral we make the variable change** $x = \frac{1}{t}$ **and we immediately obtain** $B = A$**.**

So we have: $2A = \frac{\pi}{2}K\left(\frac{\sqrt{3}}{2}\right)$

We obtained the value of the integral required in the problem statement:

$$A = \frac{\pi}{4}K\left(\frac{\sqrt{3}}{2}\right)$$

Thus, the problem is solved.