ROMANIAN MATHEMATICAL MAGAZINE

UP.569 Let $\lambda \in \mathbb{N}^*$ fixed. Evaluate:

$$\lim_{n\to\infty}\left\{\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1}\right\}$$

Proposed by Marin Chirciu – Romania

Solution by proposer

We have $\left\{\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1}\right\} = \sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1} -$
$\left[\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1}\right]$

From $\lambda n < \sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1} < \lambda n + 1$ it follows $\left[\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1}\right] = \lambda n$.

We have $\{\lambda^2 n^2 + (2\lambda - 1)n + 1\} = \sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1} - \lambda n$.

Calculation

$$\lim_{n\to\infty}\left\{\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1}\right\} = \lim_{n\to\infty}\left(\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1} - \lambda n\right) =$$

$$= \lim_{n\to\infty}\frac{\lambda^2 n^2 + (2\lambda - 1)n + 1 - \lambda^2 n^2}{\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1} + \lambda n} =$$

$$= \lim_{n\to\infty}\frac{(2\lambda - 1)n + 1}{\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1} + \lambda n} = \lim_{n\to\infty}\frac{n\left(2\lambda - 1 + \frac{1}{n}\right)}{n\left(\sqrt{\lambda^2 + \frac{(2\lambda - 1)}{n} + \frac{1}{n^2}} + \lambda\right)} =$$

$$= \frac{2\lambda - 1}{\lambda + \lambda} = \frac{2\lambda - 1}{2\lambda}.$$

Conclusion:

$$\lim_{n\to\infty}\left\{\sqrt{\lambda^2 n^2 + (2\lambda - 1)n + 1}\right\} = \frac{2\lambda - 1}{2\lambda}.$$

Note.

For $\lambda = 2$ we obtain Problem UP.577 proposed by Nguyen Viet Hung from Mathematical Reflections Nr. 1/2022.

UP.577. Evaluate:

$$\lim_{n\to\infty}\left\{\sqrt{4n^2 + 3n + 2}\right\}.$$

Nguyen Viet Hung – Vietnam