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WORLD’S MATH OLYMPIADS

FROM AUTHORS

In July 2016 was founded "Romanian Mathematical Magazine" (RMM) (www.ssmrmh.ro) as an Interactive Mathematical Journal.

Same date was founded "Romanian Mathematical Magazine"-Online Mathematical Journal (ISSN-2501-0099) and "Romanian Mathematical Magazine"-Paper Variant (ISSN-1584-4897).

In five years the website of RMM was visited by over 5,000,000 people from all over the world. With over 12,000 proposed problems posted, over 18,000 solutions and many math articles and math notes RMM is a big chance for young mathematicians from whole world to be known as great proposers and solvers. This book is a small part of RMM-Interactive Journal.

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CONTENT

NUMBER'S THEORY

1.1 Let $S = \{1, 2, \dots, 2017\}$. Find the minimal n with the property that there exist n distinct subsets of S such that for no two subsets their union equals S.

Gerhard Woeginger-Austrian NMO-2017

Solution: *There are 2^{2016} subsets of S which do not contains 2017. The union of any two such subsets does not contain 2017 and is thus a proper subset of S.Thus $n \geq 2^{2016}$.To show the other direction, we group the subsets of S into 2^{2016} pairs in such a way thet every subsets forms a pair with its complement. If $n \geq 2^{2016}$ then the n subsets would contain such a pair. Its union would be S, contradiction. Thus $n = 2^{2016}$.*

1.2 For a positive integer n, let $(a_0, a_1, \dots, a_n)$ be an $n+1$ –tuple of integers. For each $k = 1, 2, \dots, n$, let b_k be the number of k's in $(a_0, a_1, \dots, a_n)$, and let c_k be the number of k's in $(b_0, b_1, \dots, b_n)$. Find all $(a_0, a_1, \dots, a_n)$ such that $a_0 = c_0, a_1 = c_1, \dots, a_n = c_n$.

Korean NMO-2017

Solution: *Let $A = (a_0, a_1, \dots, a_n)$ and $B = (b_0, b_1, \dots, b_n)$ such that a_k is the number of k's in B and b_k is the number of k's in A. Clearly, they satisfy $0 \leq a_i, b_i \leq n$ for all $i = 0,1, \dots, n$ and*

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} i b_i = n+1, \quad \sum_{i=0}^{n} b_i = \sum_{i=0}^{n} i a_i = n+1.$$

Suppose that A has no zeros. Since the sum of all a_i's equals $n+1$, it implies $A = (1,1,\dots,1)$ and $B = (0, n+1, 0, \dots, 0)$, which is impossible. Therefore A should contain 0, and, similarly, B should contain 0 as well. Hence, $a_0 > 0$. Assume that there are k distinct numbers in A, denote by $x_1, x_2, \dots, x_k$. Then, there are k non-zeros and $n+1-k$ zeros in B. Therefore, $a_0 = n+1-k$ and $a_1 + a_2 + \dots + a_n = k$.

Since x_i's are distinct,

$$n+1 = a_1 + a_2 + \dots + a_k \geq x_1 + x_2 + \dots + x_k \geq$$

$$\geq 0 + 1 + \dots + (k-2) + (n+1-k) = \frac{(k-2)(k-1)}{2} + (n+1-k)$$

This implies $k^2 - 5k + 2 \leq 0$, that is, $k = 1,2,3,4$.

Case 1: $k = 1$.

Since $a_0 = n$, A contains 0 and n. So A is impossible.

Case 2: $k = 2$.

This means $a_0 = n-1 > 0, a_1 + a_2 + \cdots + a_n = 2$ *and* $a_1 + 2a_2 + \cdots + na_n = n+1$. *Suppose* $n \geq 4, A$ *contains* $0, n-1$ *and 1 (or 2). Then there are at least three distinct numbers in* A, *which is impossible. So, we have* $n = 2,3$.
If $n = 2, A$ *should be all 1, which is impossible.*
If $n = 3, A = B = (2,0,2,0)$.
Case 3: $k = 3$.
This means $a_0 = n-2 > 0$ *and* $a_1 + a_2 + \cdots + a_n = 3$ *and* $a_1 + 2a_2 + \cdots + na_n = n+1$.
There are three distinct numbers in A: $0, n-3$, *and* t.
If $n = 3$, *then* $A = B = (1,2,1,0)$.
If $n = 4$, *then* $A = B = (2,1,2,0,0)$.
If $n = 5$, *then* $A = (3,1,1,1,0,0)$ *and* $B = (2,3,0,1,0,0)$.
If $n = 6$, *then* A *is impossible.*
If $n \geq 7$ *and* $t = 1$, *then* $A = (n-2,1,0,1,0^{n-7},1,0,0,0)$ *and* $B = (n-3,3,0,0,0^{n-7},0,1,0,0)$.
If $n \geq 7$ *and* $t = 3$, *then* A *is impossible.*
Case 4: $k = 4$.
This means $a_0 = n-3 > 0$ *and* $a_1 + a_2 + \cdots + a_n = 4$ *and* $a_1 + 2a_2 + \cdots + na_n = n+1$. *There are four distinct numbers in* A: $0, n-3, t$ *and* u *under* $t < u$.
If $n = 4$, *then* A *is impossible.*
If $n = 5$, *then* $A = (2,3,0,1,0,0)$ *and* $B = (3,1,1,1,0,0)$.
If $n = 6$, *then* $A = B = (3,2,1,0,0,0)$.
If $n \geq 7$ *and* $(t,u) = (1,3)$, *then* $A = (n-3,3,0,0,0^{n-7},0,1,0,0)$ *and* $B = (n-2,1,0,1,0^{n-7},1,0,0,0)$.
If $n \geq 7$ *and* $(t,u) = (1,2)$, *then* $A = B = (n-3,2,1,0,0^{n-7},1,0,0,0)$.
Therefore, we have the answer:
(2,0,2,0),(1,2,1,0),(3,1,1,1,0,0),(2,3,0,1,0,0),(3,2,1,1,0,0,0),
$(n-2,1,0,1,0^{n-7},1,0,0,0)$ *for* $n \geq 7$, $(n-3,3,0,0,0^{n-7},0,1,0,0)$ *for* $n \geq 7$, $(n-3,2,1,0,0^{n-7},1,0,0,0)$ *for* $n \geq 7$.

1.3 Suppose that 2017 boxes are arranged in a circle in a room. A set of boxes is called "nice" if it contains at least two boxes and from each box in the set, the number of boxes to skip in order to reach another box in the set in the clockwise order is 0 or odd. Each of 30 students enters this room one by one and selects a nice set of boxes and put a slip of paper containing his or her name in each of selected boxes. Prove that if the set of all boxes containing 30 slips is not nice, then

there exists students A, B and boxes a, b satisfying the following two conditions.
(i) A selected a but not b, and B selected b but not a.
(ii) The number of boxes to skip in order to reach b from a in the clockwise order is not odd and none of skipped boxes were selected by A or B.

Korean NMO-2017

Solution: ***Remark:*** *This problem is true, even if 30 is replaced with n and 2017 is replaced with any odd positive integer.*
Let A_i be the set of boxes selected by the ith student. Suppose that there exist no students A, B and boxes a, b satisfying (i) and (ii).
Lemma 1. *If both A and B are nice, then either $A \cap B$ is nice or there exists boxes a and b such that $a \in A, a \notin A, b \in B, b \notin B$ and the number of boxes to skip in order to reach b from a in the clockwise order or counterclockwise order is not odd and none of skipped boxes were selected by A or B.*
Proof lemma 1. *Suppose that there does not exist boxes a and b such that $a \in A, a \notin B, b \in B, b \notin B$ and the number of boxes to skip in order to reach b from a in the clockwise order or counterclockwise order is not odd and none of skipped boxes were selected by A or B.*
If $A \cap B = \emptyset$, then by following the circle we encounter boxes in A or B. The circle can be partitioned into intervals so that each interval starts at and ends with a box in A or B and no other boxes in the interval are in A or B. Let us say that the length of an interval is the number of its boxes minus 1. Then the sum of length of all intervals is precisely 2017. If both ends of an interval are in A, the length is even or 1. Similarly, if both ends of an interval are in B, the length is even or 1. If an interval starts at a box in A plus the number of consecutive pairs of boxes. Then this contradicts to the assumption that A is nice.
So we may assume that $A \cap B \neq \emptyset$.Suppose that $A \cap B$ is not nice. Then there is an interval starting from a box x in $A \cap B$ and ending at a box $y \in A \cap B$ such that no internal boxes are in $A \cup B$ and the length of the interval is odd and larger than 1. (We allow $x = y$, in case $|A \cap B| = 1$. Since this interval has odd length and A is nice, it contains an odd number of pairs of consecutive boxes in A. This interval can be partitioned into subintervals by B and one of the subintervals, say J, must contain an odd number of pairs of consecutive boxes in A. By the assumption, both ends of J are in B and J has length greater than 1 and so it has even length because B is nice. However, j contains an odd number of pairs of consecutive boxes in A and so J has odd length, a contradiction. (To see this, we need to consider the case when an end of J is A

or not. In both cases, we deduce that the first subinterval of J from end to the first box in A not B has even length.)

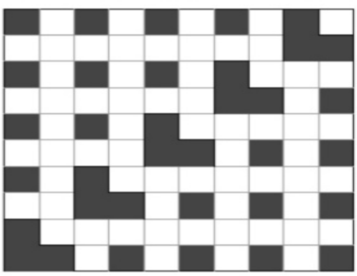
Lemma 2. *If in A, B, C fail to satify (i) and (ii), then $A \cap B$ and C fail to satisfy (i) and (ii).*

Proof Lemma 2. *Suppose there exists an interval I of odd length starting from a box a in $A \cap B$, ending at a box c in C, and no internal boxes are in $A \cap B$ or C. As A is nice, I contains an odd number of pairs of consecutive boxes in A. Similarly I contains an odd number of pairs of consecutive boxes in B closest to c is a pair of consecutive boxes $b_1 b_2$ in B. Then the interval from b_2 to c has odd length and therefore the interval J from a to b_1 has even length. However, J contains an odd number of pairs of consecutive boxes in A, contradicting the assumption that A is nine.Now, we are ready to solve this problem. We proceed by induction on the number n of students. If $n = 2$, then it is true by Lemma 1. Let us assume that $n \geq 3$. If $A_1 \cap A_2 \cap \ldots \cap A_{n-1}$ is not nice, then we are done by induction hypothesis. So we may assume that $A_1 \cap A_2 \cap \ldots \cap A_{n-1}$ is nice. By applying Lemma 2 repeatedly we deduce that $A_1 \cap A_2 \cap \ldots \cap A_{n-1}$ and A_n satisfy (i) and (ii). By lemma 1. $A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n$ is nice, contradicting the assumption.*

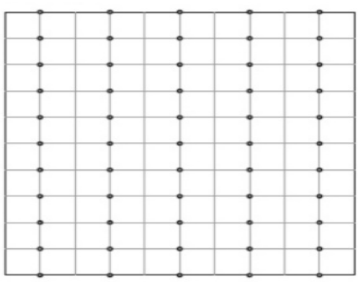
1.4 A 10×10 square is divided into 1×1 squares. A point lies at a vertex of 1×1 squares that the vertex belongs. (A point light can be positioned on a vertex on the edge of the bug square). Find the minimum number of point lights required such that all the squares are lit even if one of the point lights is not functioning.

B.Battsengel-Mongolian NMO-2017

Solution: *The minimum number of light is 55.Each 1×1 black square needs at least 2 lights and each figure that consist of there 1×1 square needs at least 3 lights. (See picture 1). So we need at least $2 \times 20 + 3 \times 5 = 55$ lights. Figure 2 shows that 55 lights could be placed as required.*



Picture 1



Picture 2

1.5 For an integer $n \geq 2$, let
$\boldsymbol{a_1 = \frac{n(2n-1)(2n+1)}{3}, a_k = \frac{(n+k-1)(n-k+1)}{2(k-1)(2k+1)} a_{k-1}}$ **for** $\boldsymbol{k = 2, 3, \dots, n}$.
(a) Show that $a_1, a_2, \dots, a_n$ are all integers.
(b) Show that both $2n-1$ and $2n+1$ are prime numbers if and only if $a_1, a_2, \dots, a_n$ are divisible by $(2n-1)$ with exactly one exception and are divisible by $2n+1$ with exactly one exception.

Korean NMO-2017

Solution: *(a) We first show that* $a_k = \binom{n+k-1}{2k-1}\frac{(2n-1)(2n+1)}{2k+1}$. *Let us use induction on* k. *The case when* $k = 1$, $a_1 = \frac{n(2n-1)(2n+1)}{3}$. *We now assume the assertion hods when* $k < i$. *Then*

$$a_i = \frac{(n+i-1)(n-i+1)}{2(i-1)(2i+1)} a_{i-1}$$
$$= \frac{(n+i-1)(n-i+1)}{2(i-1)(2i+1)}\binom{n+i-2}{2i-3}\frac{(2n-1)(2n+1)}{2i-1}$$
$$= \binom{n+i-1}{2i-1}\frac{(2n-1)(2n+!)}{2i+1}.$$

Therefore, by induction on k, $a_k = \binom{n+k-1}{2k-1}\frac{(2n-1)(2n+1)}{2k+1}$ *for all* k.
Now, we are ready to prove that a_k *is an integer. Note that* a_k *is an integer if and only if* $\binom{n+k-1}{2k-1}(2n-1)(2n+1)$ *is a multiple of* $(2k+1)$. *Thus,* $(2n-1)(2n+1) \equiv 4n^2 - 1 \equiv 4(n-k)(n-k-1)(mod\ 2k+1)$ *implies the following:*

$$\binom{n+k-1}{2k-1}(2n-1)(2n+1) \equiv \binom{n+k-1}{2k-1}4(n-k)(n-k-1)$$
$$\equiv \binom{n+k-1}{2k+1}4(2k)(2k+1) \equiv 0(mod\ 2k+1).$$

Therefore, a_k *is an integer.*
(b) We remark that $(2n-1) \nmid a_{n-1}$, $(2n-1) \mid a_n$, $(2n+1) \mid a_{n-1}$, *and* $(2n+1) \nmid a_n$ *since* $a_{n-1} = 2(n-1)(2n+1)$ *and* $a_n = 2n-1$. *Therefore, it is enough to prove thet both* $2n-1$ *and* $2n+1$ *are prime numbers if and only if* $a_1, \dots, a_{n-2}$ *are all multiples of* $(2n-1)(2n+1)$.
Necessary condition. *Suppose that* $2n-1$ *is not a prime number. We consider the prime number* p *such that* $p \mid 2n-1$.

$$\frac{a_{\frac{p-1}{2}}}{(2n-1)(2n+1)} = \frac{\left(n+\frac{p-1}{2}-1\right)\dots\left(n-\frac{p-1}{2}+1\right)}{(p-2)!\,p}.$$

$p \mid n - \frac{p-1}{2}$ *implies that* p *does not divide the numerator of the right hand side. This implies that* $\frac{1}{(2n-1)(2n+1)} a_{\frac{p-1}{2}}$ *is not an integer since dominator of the right hand side is divided by* p. *This contradicts that* $a_{\frac{p-1}{2}}$ *is a multiple of* $(2n - 1)(2n + 1)$. *Therefore, is a prime number and we skip the proof here.*
Sufficient conditions: *For an integer* k *between 1 and* $n - 2$, *we have*

$$a_k = \frac{(n + k - 1)(n + k - 2) \dots (n - k + 1)}{(2k - 1)!\,(2k + 1)}(2n - 1)(2n + 1).$$

As $2k - 1 < 2n - 1, 2k + 1 < 2n - 1$, *and both* $2n - 1$ *and* $2n + 1$ *are prime numbers, we deduce that* $(2n - 1)(2n + 1) \mid a_k$.

1.6 Anna and Berta play a game in which they take turns in removing marbles from a table. Anna takes the first turn.When at the beginning of turn there are $n \geq 1$ marbles on the table , then the player whose turn it is removes k marbles, where $k \geq 1$ either is an even number with $k \leq \frac{n}{2}$ or an odd number with $\frac{n}{2} \leq k \leq n$. A player wins the game if she removes the last marble from the table. Determine the smallest number $N \geq 100\,000$ such that Berta can enforce a victory if there are exactly N marbles on the table in the beginning.

Gerhard Woeginger-Austrian NMO-2017

Solution: *We claim that the losing situations are those with exactly* $n = 2^a - 2$ *marbles left on the table for all integers* $a \geq 2$. *All other situations are winning situations.*
Proof: *By induction for* $n \geq 1$.*For* $n = 1$ *the player wins by taking the single remaining marble.For* $n = 2$ *the only possible move is to take* $k = 1$ *marbles, and then the opponent wins in the next move.*
Induction step from $n - 1$ *to* n *for* $n \geq 3$;
1. If n *is odd, then the player takes all* n *marbles and wins.*
2. If n *is even but not of the form* $2^a - 2$, *then* n *lies between two other numbers of that form, so there eists a unique* b *with* $2^b - 2 < n < 2^{b+1} - 2$. *Because of* $n \geq 3$ *it holds that* $b \geq 2$. *Therefore all three numbers in this chain of inequalities are even, and therefore we can conclude that* $2^b < n < 2^{b+1} - 4$. *From the induction hypothesis we know that* $2^b - 2$ *is a losing situation, and by taking*

$$k = n - (2^b - 2) = n - \frac{2^{b+1} - 4}{2} \leq n - \frac{n}{2} = \frac{n}{2}$$

marbles we leave it to the opponent.

3. If n is even and of the form $n = 2^a - 2$, then the player cannot leave a losing situation with $2^b - 2$ marbles to the opponent (where $b < a$ holds because at least one marble must be removed, and $b \geq 2$ holds because after a legal move starting from an even n, at least one marble remains).
But because of $b \geq 2$ we know that k is even and strictly greater than $\frac{n}{2}$ because
$2^a - 2^b \geq 2^a - 2^{a-1} = 2^{a-1} > 2^{a-1} - 1 = \frac{2^a - 2}{2} = \frac{n}{2}$; *impossible.*
Solution: Berta can enforce a victory if and only if N is of the form $2^a - 2$. The smallest number $N \geq 100\,000$ of this form is $N = 2^{17} - 2 = 131\,070$.

1.7 A necklace contains 2016 pearls, each of which has one of the colours black, green or blue. In each step we replace simultaneously each pearl with a new pearl, where the colour of the new pearl is determined as follows: If the two original neighbours were of the same colour, the new pearl has their colour. If the new neighbours had two different colours, the new pearl has the third colour.
a) Is there such a necklace that can be transformed with such steps to a necklace of blue pearls if half of the pearls were black and half of the pearls were green at the start?
b) Is there such a necklace that can be transformed with such steps to a necklace of blue pearls if thousand of the pearls were black at the start and the rest green?
c) Is it possible to transform a necklace that contains exactly two adjacent black pearls and 2014 blue pearls a necklace than contains one green pearl and 2015 blue pearls?

Theresia Eisenkolbl-Austrian NMO-2017

Solution: *a) Since 2016 is a divisible by 4, we can alternatingly take two black and two green pearls.*

b) In the assign to each blue pearl the number 0, to each green pearl the number 1 and to each black pearl the number 2, then it holds in each step that the new colour of a pearl modulo 3 is equal to the negative sum of its two original neighbours. The new total sum of all colours modulo 3 therefore can be calculated by multiplying the odd total sum of all colours with 2 and changing the sign. But modulo 3, a multiplication with -2 is equivalent to a

multiplication with 1, therefore the total sum always remains the same modulo 3.
For a necklace with only blue pearls the total sum is 0. But for 1000 black and 1016 green peals it is 2000+1016≡1(mod3). Therefore, there does not exist an arrangement of 1000 black and 1016 green pearls that can be transformed into a necklace with only blue pearls using such steps.
c) Using the same assignment of numbers modulo 3, in each step the sum of all colours in even positions becomes the sum of the colours in odd positions, and vice versa. If these sums are A and B in the beginning, then at the end we still have these same two sums modulo 3, maybe with switched positions.
But in the beginning, we have sums 2 and 2 modulo 3, because both among the even and among the odd positions there is exactly one black pearl with value 2, and otherwise only blue pearls with value 0. However, at the end we are supposed to have sums 1 and 0 because one of the two sums is determined only by blue pearls with value 0, and the other by exactly one green pearl with value 1 and only blue pearls with value o otherwise. Therefore, is not possible.

1.8 For a positive integer $n \geq 2$, let $C(n)$ be the smallest positive real constant such thet there is a sequence of n real numbers $x_1, x_2, \dots, x_n$, not all zero, satisfying the following conditions:
(i) $x_1 + x_2 + \cdots + x_n = 0$
(ii) **for each $i = 1, 2, \dots, n$, it holds that $x_i \leq x_{i+1}$ *or***
$x_i \leq x_{i+1} + C(n)x_{i+2}$ **(the indices are taken modulo n).**
Prove that:
(a) $C(a) \geq 2$ for all n;
(b) $C(n) = 2$ if and only if n *is* even.

Dusan Djukic-Serbian TST-2017

Solution: *The sequence cannot two adjacent non-positive terms. Indeed, if $a_{i-1} > 0 \geq a_i, a_{i+1}$, then $a_{i-1} > max\ \{a_i, a_i, a_i + C(n)a_{i+1}\}$, contradicting (ii). Hence, the sequence consists of blocks os positive terms followed by a single non-positive term. Consider an arbitrary block of positive terms $a_k, a_{k+1}, \dots, a_{k+l-1}$. We call the term a_k $initial$ and $a_{k+l-1} final$. Let P be the sum of all initial terms in the sequence, K be the sum of the final ones, N be the sum of all non-positive terms, and S be the sum of positive terms which are not initial or final. Summing the inequalities $a_{k+l-1} \leq a_{k+l} + C(n)a_{k+l+1}$ over all blocks yields $K \leq N + C(n)P$. Since $N = -K - S - P$, this relation becomes: $2K \leq (C(n) - 1)P - S$; $(*)$*

Suppose now that $C(n) \leq 2$*. Summing the inequalities* $a_{k+2i} \leq a_{k+2i+1} + a_{k+2i+2}$ *for* $0 \leq i \leq \left\{\frac{l-3}{2}\right\}$ *and adding the inequality* $a_{n+l-2} \leq 2a_{k+l-1}$ *if* $2 \mid l$*, we obtain* $a_k \leq a_{k+1} + a_{k+2} + \cdots + a_{k+l-2} + 2a_{k+l-1}$*. Summing over all blocks then yields*

$$P \leq S + 2K; \quad (**)$$

Equality in $(**)$ *is possible only if the length* l *of the block is odd. Indeed, if* $2 \mid l$*, all inequalities participanting in the sum must be equalities, so in particular* $a_{k+l-2} = 2a_{k+l-1}$*, which contradicts the condition* $a_{k+l-2} \leq a_{k+l-1} > 0$*. Summing* $(*)$ *and* $(**)$ *gives us* $0 \leq (C(n) - 2)P$*, and therefore* $C(n) \geq 2$*. Morever, if* $C(n) = 2$*, all blocks must have odd lengths, which implies that* n *is even. Conversely, the example* $x_r = (-1)^r$ *shows that* $C(n) = 2$ *for even n.*

1.9 Let k be a positive integer and let n be the smallest positive integer having exactly k divisors. If n is a perfect cube, can the number k have a prime divisor of the form $3j + 2$?

Bojan Basic-Serbian TST-2017

Solution: *Suppose that such a* k *exists. Let* $p_1 < p_2 < \cdots$ *be all primes in the increasing order and let* $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ $(\alpha_m > 0)$*, where* $k = (\alpha_1 + 1) \dots (\alpha_m + 1)$ *and* $3 \mid \alpha_i$ *for all* i*. By the minimality of* n *we have* $\alpha_1 \geq \cdots \geq \alpha_m > 0$*.*
Lemma. Suppose that $\alpha_r + 1 = ab$ *for* $a, b \in \mathbb{N} - \{1\}$*. If* $p_s < p_r^a < p_{s+1}$*, then*

$$\alpha_s \geq b - 1 \geq \alpha_{s+1}$$

Proof. The number $n_1 = p_r^{(\alpha_s+1)a-1} p_s^{b-1} \prod_{i \notin \{r,s\}} p_i^{r_i}$ *also has* k *divisors, so it satisfies* $n_1 \geq n$*. However, this reduces to* $\left(\frac{p_r^a}{p_s}\right)^{\alpha_s - b+1} \geq 11$*,i.e.* $\alpha_s \geq b - 1$*.*
Similarly, $n'_1 = p_r^{(\alpha_{s+1}+1)a-1} p_{s+1}^{b-1} \prod_{i \notin \{r,s+1\}} p_i^{r_i} \geq n$ *yields* $\alpha_{s+1} \leq b - 1$*.*
Consider the largest r *such that* $\alpha_r + 1 = ab$ *for some* $a \equiv b \equiv 2 \pmod 3$*, and let* s *and* t *be such that* $p_s < p_r^a < p_{s+1}$ *and* $p_t < p_r^b < p_{t+1}$*. Observe that Bertrand,s postulate implies* $\frac{1}{2} p_r^a < p_s$ *and* $p_{s+1} < 2p_r^a$*, hence* $p_r^{a-1} < p_s < p_r^a < p_{s+1} < p_r^{a+1}$ *and analogously*

$$p_t^{b-1} < p_t < p_r^b < p_{t+1} < p_r^{b+1}$$

1.10 *300* contestants participated in a competition. Every two contestants either know each other or do not know each other and there are no three contestants who know each other. Each contestant knows at most n contestants. Find the maximum possible value of n.

B.Khoroldagva,Sh.Dorjsembe-Mongolian NMO-2017

Solution: *Let $A_1, \dots, A_{300}$ denote the 300 contestants. If A_i and A_j know each other then we connect them. Denote by $|A_i|$ the number of contestants connected to A_i. Suppose that there is a contestant who knows exactly 201 contestants. Without loss of generality we may assume that A_{202} is connected to $A_1, \dots, A_{201}$. Since there is no triangle with vertices on $A_i (1 \le i \le 202)$ we have $|A_i| \le 99$ for $i \le 201$. Hence we must have*

$$\{|A_i|: 202 \le i \le 300\} \supseteq \{100, \dots, 201\}$$

which is impossible. So $n \le 200$.

Now let us show that $n = 200$ can be attained. We connect A_{200+i} to each of $A_i, \dots, A_{200}$ for $i = \overline{1,100}$. Then $|A_{200+i}| = 201 - i$ and $|A_i| = i$ for $i = \overline{1,100}$.

1.11 Let $n \ge 3$ be an integer. Prove that there exist positive integers $x_1, \dots, x_n$ in geometric progression and positive integers $y_1, \dots, y_n$ in arithmetic progression such that $x_1 < y_1 < x_2 < \dots < x_n < y_n$.

Singapore-SMO-2017

Solution: *By the binomial theorem, we have for $k \ge 2$ and $a \le \frac{1}{k^2}$,*

$$(1+a)^k = 1 + ka + a\left(\frac{ak(k-1)}{2!} + \frac{a^2k(k-1)(k-2)}{3!} + \dots + \frac{a^{k-1}k!}{k!}\right) \le$$

$$\le 1 + ka + a\left(\frac{k(k-1)}{k^2}\cdot\frac{1}{2!} + \frac{k(k-1)(k-2)}{k^4}\cdot\frac{1}{3!} + \dots + \frac{k!}{k^{2k-2}}\cdot\frac{1}{k!}\right) \le$$

$$\le 1 + ka + a\left(\frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!}\right)$$

$$< 1 + ka + a\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(k-1)\cdot k}\right) \le$$

$$\le 1 + ka + a\left(1 - \frac{1}{k}\right) < 1 + (k+1)a.$$

Let $X_k = \left(1 + \frac{1}{n^2}\right)^k, k = 1,2, \dots, n$. Then, for $2 \le k \le n, \frac{1}{n^2} \le \frac{1}{k^2}$. Therefore

$$1 + \frac{k}{n^2} < X_k < 1 + \frac{k+1}{n^2}.$$

Let $x_k = n^{2n}\left(1 + \frac{1}{n^2}\right)^k$ and $y_k = n^{2n} + (k+1)n^{2n-2}, k = 1,2, \dots, n$.

Then $x_1 < y_1 < x_2 < \dots < x_n < y_n$.

1.12 There are *2017* distinct points in the plane. For each pair of these points, construct the midpoint of the segment joining tha pair of points. What is the minimum number of distinct midpoints among all possible ways of placing the points?

Singapore SMO-2017

Solution: *Suppose the points are placed on the* x *–axis with coordinates* $(i, 0), i = 0,1, \dots, 2016$. *Then midpoints are* $\left(\frac{i}{2}, 0\right), i = 1,2, \dots, 4031$. *Thus there are 4031 distinct midpoints. Next we shall prove that there are at least 4031 distinct midpoints. Let* $A_1, \dots, A_{2017}$ *be the points and assume that* A_1, A_2 *are the pair that are furthest apart. Consider the 4030 segments from* A_1 *and* A_2 *to* $A_3, \dots, A_{2015}$. *The midpoints are distinct. For if* X, Y *are two points so that the midpoints of* A_1X *and* A_2Y *coincide, then we have two cases. If the four points are not collinear, then they are vertices of a parallelogram with* A_1X, A_2Y *as diagonals and* A_1A_2 *as a side. This is not possible as the longer diagonal is longer than a side. Otherwise* A_1, A_2, X, Y *are collinear. Then it is easy to verify that if* X *is in the segment* A_1A_2, *then* Y *must be outside making* $A_1A_2 < A_2Y$, *a contradiction. Also none of these midpoint of* A_1A_2. *Thus we have at least 4031 distinct midpoints.In conclusion, the minimum number of midpoints is 4031.*

1.13 Five teams play in a soccer competiton where each team plays one match against each of the other four teams. A winning team gains 5 points and a losing team 0 points. For a 0-0 draw both teams gain 1 point, and for other draws (1-1,2-2,etc) both teams gain 2 points. At the end of the competition, we write down the totalpoints for each team, and we find that they form five consecutive integers.
What is the minimum number of goals scored?

Gordon Lessels-Ireland SHL-2017

Solution: ***Case (a)*** *Number of wins is odd. 3 wins yield 15 points and the other seven matches yield more than 10 points. The only possibility is one win, 8 0-0 draws and one score draw. But the team that wins must gain at least 3 points in the other matches. Hence this case is impossible.*
Case (b) *The number of wins is even and cannot be 4 as only three teams have a score of five or more and none have a score of 10. No wins means there are 5 score draws and 5 no score draws. The teams scorin 8 and 7 must be insolved in 7 score draws to achieve these totals. Hence, the only possibility is 2 wins 2 score draws and 6 no score draws. The table*

	A	*B*	*C*	*D*	*E*	*Total*
A		*5*	*1*	*1*	*1*	*8*
B	*0*		*1*	*1*	*5*	*7*
C	*1*	*1*		*2*	*2*	*6*
D	*1*	*1*	*2*		*1*	*5*
E	*1*	*0*	*2*	*1*		*4*

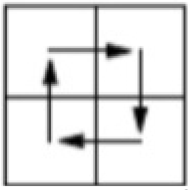
Realises this possibility. The minimum number of goals in this case is 6.
Case (c) *The number of wins is odd. No team has more than one win. If the number of wins is five, each team must win one match and all other matches are no score draws. A total of 9 is now impossible. If the number of wins is 3, there must be 3 score draws and 4 no score draws. At least 9 goals are scored in this scenario. If the number of wins is 1, there are 6 score draws and three no score draws. This gives at least 13 goals.*
Case (d) *Again we can calculate the number of wins (W), score draws (S) and no score draws (N) yields 40 points. The four possibilities are (W,S,N)=(6,1,3) or (4,4,2) or (2,7,1) or (0,10,0). The minimum number of goals is W+2S which in each case is bigger than 6.*
Case (e) *Calculating possible values of (W,S,N) we obtain (7,2,1),(5,5,0) giving more than 6 goals. Thus the minimum number of goals scored in the tournament is 6. Ten matches are played each one contributing either 2,4 or 5 points. Hence the total number of points is between 20 and 50.*
In the team scores are five consecutive integers, then the total number of points must be a multiple of 5. If the total number of points is 20, all teams will score 4 and if the total number of points is 50 all team totals will be multiples of 5. Neither of these possibilities satisfy the conditions. Therefore, we need to consider the following five cases:

1. scores are 3,4,5,6,7
2. scores are 4,5,6,7,8
3. scores are 5,6,7,8,9
1. scores are 6,7,8,9,10 and
2. scores are 7,8,9,10,11

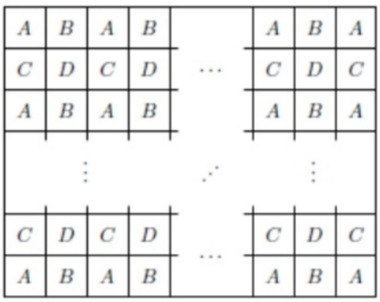
1.14 On a 2017×2017 board, some of the squares are occupied by a single ladybird; the rest of the squares are empty. The ladybirds move, never leaving the board, according to the following principles. Every second, each ladybird moves to a neighbouring square. The moves are horizontal (to the square immediately to the right or to the left of the current square), or vertical (to the square above or below the one currently occupied). A ladybird which makes a must move vertically in its next move. Similarly, a ladybird which makes a vertical move must move horizontally in its next move.
Determine the smallest number of ladybirds such that, regardless of their initial position and their chosen paths, we may be sure that two of them will eventually find themselves in the same square, at the same moment.

Croatian NMO-2017

Solution: *We claim that the required number is* $2016^2 + 1$. *Let us first show that we can find an arrangement of* 2016^2 *ladybirds and chose their paths so that no two accupy the same square at the same moment in time. We place the ladybirds in the lower left* 2016×2016 *squares of the board and let them all move in the same manner: up, right, down, left, up, right... We see that no two ladybirds will meet after the first 4 seconds. Since we arrive at the initial position after that, we see that no two ladybirds will ever occupy the same square at the same time.*



We now show that, if the board contains $2016^2 + 1$ *ladybirds, a collision must occur, regardless of the initial arrangement and the ladybirds' paths.*
We label the squares with one of the four labels, A, B, C *and* D, *so that the squares in odd rows alternate between labels* A *and* B, *while the squares in even rows alternate between* C *and* D. *We will call a square which has been labelled by* A *an* $A-$*square.*



Two observations are crucial for our solution. A ladybird occupying a $B-$*square or a* $C-$*square will after* one *second move to an* $A-$*square or a* $D-$*square. Similarly, a ladybird which occupies an* $A-$*square will after* two *seconds be sitting in a* $D-$*square.*
The board contains $2016^2 + 1$ *ladybirds, so we can assume that at least* $1008 \cdot 2016 + 1$ *of them occupy an* $A-$*or a* $D-$*square. If the opposite were true, we would have at least* $1008 \cdot 2016 + 1$ *on a* $B-$*square or a* $C-$*square, so the after one second we would arrive at the desired situation. Since the number of* $D-$*squares equals* 1008^2 *,the* $A-$*squares contain at least* $1008^2 + 1$ *ladybirds.*

All the ladybird which are now occupying the A –squares will after two seconds move to D –squares. This means that there will be at least $1008^2 + 1$ *ladybirds on* 1008^2 D *–squares, so that at least two of them will be in the same squre.*

1.15 Find all quadruples (a, b, c, d) of integers satisfying the system of equations $-a^2 + b^2 + c^2 + d^2 = 1, 3a + b + c + d = 1$. Answer: $(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, -1, -1), (1, -1, 0, -1)$ and $(1, -1, -1, 0)$

Estonian NMO-2017

Solution: *Note that* $-a^2 + b^2 + c^2 + d^2 = -2a(a + b + c + d) + (a + b)^2 + (a + c)^2 + (a + d)^2$*. By second equation,* $2a + b + c + d = 1 - a$*, so the first equation reduces to* $2a(a - 1) + (a + b)^2 + (a + c)^2 + (a + d)^2 = 1$*.If* $a > 1$ *or* $a < 0$ *then* $2a(a - 1)$ *is a positive even number, whence the Lhs of the last equation is greater then 1. If* $a = 0$ *then according to the first equation exactly one number among* b^2, c^2 *and* d^2 *is equal to 1, others are zero. Value -1 together with zeros would not satisfy the second equality, thus one* a, b, c *and* d *must be 1, while others are zero. If* $a = 1$ *then according to the first equation exactly two numbers among* b^2, c^2 *and* d^2 *are equal to 1 and the remaining number is zero. The second equation is satisfied only is both non-zero variables take value -1.*

1.16 Find all positive integer k for which the integers $1, 2, \ldots, 2017$ can be divided into k groups in such a way that the sums of numbers in these groups are k consecutive terms of an arithmetic sequence.

Estonian NMO-2017

Solution: *Let the arithmetic sequence have the first term a and the common difference d. The sum of all terms equals the sum of numbers 1,2,...,2017, i.e.,* $\frac{2a+(k-1)d}{2} \cdot k = \frac{2017 \cdot 2018}{2}$*, hence* $(2a + (k - 1)d) \cdot k = 2017 \cdot 2018 = 2 \cdot 1009 \cdot 2017$*. Thus the product* $2 \cdot 1009 \cdot 2017$ *is divisible by* k*. Since 2,1009 and 2017 are primes and* $k \le 2017$ *by assumption, the only possibilities are* $k = 1, k = 2, k = 1009$ *and* $k = 2017$*. All these can occur indeed. A partition into 1 group trivially satisfies the conditions. An arbitrary partition into 2 groups also provides two consecutive terms of some arithmetic sequence. Coupling each even number with the next odd number, we get 1008 groups of size 2, whose sums are consecutive terms of the arithmetic sequence 5,9,13...*

Forming one additional group containing 1 as the only element, we obtain a partition satisfying the conditions. Finally, the conditions will also be met by the partition where every integer 1,2,...,2007 belongs to a separate group.

1.17 Find all positive integers n for which all positive divisors of n, taken without repetitions, can be placed into a rectangular table in such a way that each cell contains exactly one divisor, all row sums are equal and all column sums are equal.

Estonian NMO-2017

Solution: *Suppose that all positive divisors of n can be arranged as a rectangular table of size $k \times l$. Assume w.l.o.g. that $k \le l$(k is the number of rows). Let the sum of the numbers in each column be s, as n occurs somewhere in the table, we must have $s \ge n$, whereby equality can hold only if $k = 1$. For every $j = 1,2,\dots,l$, let d_j. As the divisors of n, are among $n, \frac{n}{2}, \frac{n}{3}, \dots$, this chain of inequalities implies $d_l \le \frac{n}{l}$. Since the average we also have $d_l \ge \frac{s}{k} \ge \frac{n}{k}$. These inequalities together imply $\frac{n}{k} \le d_l \le \frac{n}{l}$. Hence $k \ge l$. As we assumed $k \le l$, we conclude that $k = l$. Therefore all these inequalities must actually be equalities. In particular $s = n$, implying $k = l = 1$. Consequently, n has only one divisor, i.e., $n = 1$.*

1.18 Masha has an electric carousel in her garden that she rides every day. As she likes order , she always leaves the carousel in the same position after each ride. But every night three bears sneak into garden and start turning the carousel. Bear dad turns the carousel each time by $\frac{1}{7}$ of the full circle. Bear mum turns the carousel each time by $\frac{1}{9}$ of the full circle. Bear cub turns the carousel each time by $\frac{1}{32}$ of the full circle. Every bear can turn the carousel as many times as he she wants. In how many different position may Masha find the carousel in the morning?

Estonian NMO-2017

Solution: *As $7 \cdot 9 \cdot 32 = 2016$, all turns are integral multiples of $\frac{1}{2016}$ of the full turn. Thus the carousel can be in at most 2016 distinct positions. It remaining to show that all these positions are impossible. For that, we show that the bears can turn the carousel by exactly $\frac{1}{2016}$ of the full turn. Then the* *same sequence of operations can be repeat to obtain also* $\frac{2}{2016}, \frac{3}{2016}, \dots, \frac{2016}{2016}$ *of the full turn. Exactly* $\frac{1}{2016}$ *of the full turns is obtained, for instance, if bear dad turns the carousel once in one direction and both bear mum and bear cub turn the carousel once in the opposite direction since* $\frac{1}{7} - \frac{1}{9} - \frac{1}{32} = \frac{288-224-63}{2016} = \frac{1}{2016}$

1.19 Let n and m be positive integers. What is the biggest number of points that can be marked in the vertices of the squares of the $n \times m$ grid in such a way that no three of the marked points lie in the vertices of any right-angled triangle?

Estonian NMO-2017

Solution: *All vertices of squares lie on* $n+1$ *horizontal and* $m+1$ *vertical lines. Suppose that at least* $n+m+1$ *points are marked in the grid. Because* $m>0$, *the number of marked points is greater than* $n+1$. *Hence by the pigeonhole point. Hence at most* n *marked points are alone on their horizontal lines. Similarly, at most* m *marked points are alone on their vertical lines. Thus there exists a marked point that lies neither alone on its horizontal line nor alone on its vertical line. So at most* $n+m$ *points can be marked according to the conditions of the problem.*

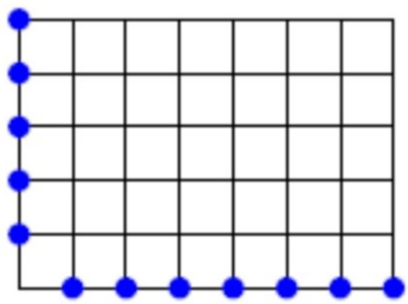


Fig. 13

By marking all vertices of squares of the grid that lie at the left and lower edge of the grid except for the lower left corner we have marked exactly $n+m$ *points (Fig.13 depicts the choice for a* 5×7 *grid). Any three of the marked points either lie on a common line or are the vertices of an obtuse triangle, so the construction satisfies the condition of the problem.*

1.20 Do there exist distinct positive integer x and y such that the number $x + y$ is divisible by 2016, the number $x - y$ is divisible by 2017 and the number xy is divisible by 2018?

Estonian NMO-2017

Solution: *For example, numbers* $x = 2016 \cdot 2015 - 2018$ *and* $y = 2018$ *meet the conditions. As* $2016 \cdot 2015 > 4 \cdot 1009 = 2 \cdot 2018$ *implies* $x > y$, *they are distinct. The sum* $2016 \cdot 2015$ *is divisible by 2016 and the product is obviously divisible by 2018. Furthermore,* $x - y = 2016 \cdot 2015 - 2 \cdot 2018 = (2017 - 1)(2017 - 2) - 2 \cdot (2017 + 1) = 2017^2 - 3 \cdot 2017 + 2 - 2 \cdot 2017 - 2 = 2017 \cdot 2012$, *where the distance of these numbers is divisible by 2017.* ***Remark.*** *This choice of the numbers is not only possible. One can prove that all suitable numbers are of the form* $x = 2016k - 2018m$ *and* $y = 2018m$, *where* k *and* m *are integers such that* $k + 2m$ *is divisible by 2017. Indeed, as* $2018 = 2 \cdot 1009$ *and 1009 is prime, one of* x*and* y *must be divisible by 1009. Also one of* x *and* y *must be even, but as* $x + y$ *is divisible by the even number 2016, both must be even. Consequently, one of these numbers must be divisible by 2018. Let w.l.o.g.* $y = 2018m$. *As* $x + y = 2016k$, *we must have* $x = 2016k - 2018m$. *Then* $x - y = 2016k - 2 \cdot 2018m = 2017(k - 2m) - (k + 2m)$, *whence* $x - y$ *is divisible by 2017 if and only if* $k + 2m$ *is divisible by 2017. The pair in the solution is obtained by taking* $k = 2015$ *and* $m = 1$.

1.21 Find all solutions of the equation $a + b + c = 61$ in natural numbers that satisfy $gcd(a, b) = 2$, $\gcd(b, c) = 3$, and

$$\gcd(c, a) = 5.$$

Estonian NMO-2017

Solution: *As* $gcd(a, b) = 2$, $gcd(b, c) = 3$, *and* $gcd(c, a) = 5$, *the number* a *is divisible by both 2 and 5, the number* b *is divisible by both 2 and 3, and the number* c *is divisible by both 3 and 5. Hence* a *is divisible by 10,* b *is divisible by 6 and* c *is divisible by 15. As 61 gives remainder 1 when divided by each 3,5 and 2, the numbers* a, b *and* c *must give remainder 1 when divided by 3,5 and 2, respectively. Since* $a, b, c \leq 61$, *the possibilities are* $a = 10$ *or* $a = 40$, $b = 6$ *or* $b = 36$, *and* $c = 15$, *or* $c = 45$. *The sum 61 appears in three cases:* $a = 10, b = 6, c = 45$; $a = 10, b = 36, c = 15$; $a = 40, b = 6, c = 15$. *A straightforward check shows that the conditions* $gcd(a, b) = 2$, $gcd(b, c) = 3$, *and* $gcd(c, a) = 5$ *are also met in all these cases.*

1.22 In the mathematics circle, Juku raised a hypothesis that, for every integer $n > 4$, at least one out of the two largest integers that are less than $\frac{n}{2}$ is relatively prime to n. Is Juku's hypothesis valid?

Estonian NMO-2017

Solution*: If n is odd then the largest integer that is less than $\frac{n}{2}$ is $\frac{n-1}{2}$. Let d be a common divisor of numbers $\frac{n-1}{2}$ and n. Then d is a common divisor of numbers $n-1$ and n, implying that $d = 1$. Hence $\frac{n-1}{2}$ and n are relatively prime, meaning that the hypothesis holds in the case of odd numbers.*
If n is even then the two largest integers that are less than $\frac{n}{2}$ are $\frac{n}{2}-1$and $\frac{n}{2}-2$. Let d_1 be a common divisor of numbers $\frac{n}{2}-1$ and n, and let d_2 be a common divisor of numbers $\frac{n}{2}-2$ and n. Then d_1 is a common divisor of numbers $n-2$ and n, and d_2 is a common divisor of numbers $n-4$ and n. Hence d_1 divides 2, i.e. is either 1 or 2, and d_2 divides 4, i.e. is either 1 or 2 or 4. If d_1 and d_2 were both larger than 1, they both should be even, whence their multiples $\frac{n}{2}-1$ and $\frac{n}{2}-2$ are consecutive integers. The contradiction shows that at least one of the divisors d_1 and d_2 equals 1. Thus one of $\frac{n}{2}-1$ and $\frac{n}{2}-2$ is relatively prime to n, meaning that the hypothesis holds in the case of even numbers, too.

1.23 Given positive integers a, b, c and d and

$$(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)=u,$$
$$ab+ac+ad+bc+bd+cd=v,$$

prove that the product uv is divisible by *3.*

Estonian NMO-2017

Solution: *If among numbers a, b, c, d there are two that give either remainders 0 and 0 remainders 1 and 2 modulo 3, the sum of these two numbers is divisible by 3. Hence u, as well as uv, is divisible by 3. Now study the case where at most one among the numbers a, b, c, d is divisible by 3 and all numbers not divisible by 3 are congruent modulo 3. If exactly one among numbers a, b, c, d is divisible by 3 then the product of this numbers with all other numbers are divisible by 3. Other numbers form 3 pairs whose products of components are congruent modulo 3. Hence the sum v of all six pairwise products are all congruent modulo 3. If none of a, b, c, d is divisible by 3 then the pairwise products are all congruent modulo 3. Again, as the number of* *pairs is divisible by 3, this implies that the sum v of the products is divisible by 3. Consequently, uv is divisible by 3 in this case, too.*

1.24 How many pairs (a, b) of positive integers are there, which satisfy the following conditions $a; b$, $ab = 29!$, a and b are relatively prime.

Japan NMO-2017

Solution: *There are 10 primes less than or equal to 29: namely 2,3,5,7,11,13,17,19,23,29. Hence there are only 10 primes factors of the number 29!. Therefore, we can write the prime factor decomposition of the number 29! in the form $p_1^{r_1}, p_2^{r_2}, \dots, p_{10}^{r_{10}}$, with $r_i \geq 1$ for each i. Since a and b are relatively prime, for each prime factor p_i of 29! , there are two separate cases: namely, either a is a multiple of $p_i^{r_i}$ or b is. Therefore, we can conclude that there are exactly 2^{10} pairs (a, b) of relatively prime integers a, b satisfying $ab = 29!$.Among such pairs, there are same number of pairs with $a < b$ and $b < a$. Since a and b are relatively prime, the case $a = b$ cannot occur in our consideration, and therefore, we conclude that the answer is $\frac{2^{10}}{2} = 512$.*

1.25 In the senior class of a certain high school, there are 30 students enrolled, and each student is assigned a distinct number chosen from 1 through 30. One day a teacher gave a test consisting of certain number of problems. When the teacher graded the test, he realized that the following two propositions concerning a subset S of the set $\{1, 2, \dots, 30\}$ are mutually equivalent:

(1) For every problem, there is some number k belonging in the set S such that the student with assigned number k gave a correct answer to that problem.

(2) Either S contains all of the multiples of 2 greater than 1 and less than or equal to 30, or S contains all of the multiples of 3 greater than 1 and less than or equal to 30, or S contains all of the multiples of 5 greater than 1 and less than or equal 30. What is the smallest possible number of the problems given in the test?

Japan NMO-2017

Solution: *Let* $U = \{1,2, \dots ,30\}$. *In the sequel, we do not distinguish the student with assigned number* k *and* k *considered as a number. Define subsets* A, B, C *of* U *as follows:*

$$A = \{k \in U/k \text{ is a multiple of } 2\}; B = \{k \in U/k \text{ is multiple of } 3\}$$
$$C = \{k \in U/k \text{ is multiple of } 5\}$$

We also denote by S^c *the complement of* $S \subset U$ *in* U.

Call a set $S \subset U$ *a good set, if* S *contains at least one of the sets* A, B, C *a bad set, if otherwise. Note that when* $S \subset T \subset U$, *if* S *is a good set, so is* T *while if* T *is a bad set, so is* S. *Call a bad set* S *a maximal bad set, if any set* T *satisfying* $S \subset T \subset U$ $(and\ S \neq T)$ *is a good set. It is clear that any bad set is contained in some maximal bad set, and that* $S \cup T$ *is a good set if* S *and* T *are distinct maximal bad sets. By assumption made for the problem, we see that if* S *is a good set, then for every test problem, there is some member of* S *who solved it correctly. On the other hand, if* S *is a bad set, there must exist least one test problem, which is not solved correctly by any member of* S. *In this case, choose one such problem, and call it* Q_S. *Now, suppose there are different maximal bad set* S *and* T *for which* Q_S *and* Q_T *coincide. If this happens, then we have the situation in which* $S \cup T$ *is a good set and there is a test problem* $Q_S(= Q_T)$ *which was not solved correctly by any member of* $S \cup T$, *which is a contradiction. Therefore, we conclude that problem* Q_S *for a maximal bad set* S *must be different for every such set* S, *and this fact implies that we the number of test problems necessary to satisfy the requirement of the problem must be at least equal to the number of maximal bad sets involved. On the other hand, if the problem* Q_S *for maximal bad set* S *is a distinct for each such* S, *and if all of the students not belonging to* S *solve the problem* Q_S, *then both of the conditions (1) and (2) of the problem are satisfied. Therefore, we conclude that the smallest possible number for the test problems necessary to satisfy the conditions of the problem must equal the number of maximal bad sets. The condition "*$S \subset U$ *is a maximal bad set" is equivalent to the following condition:* S^C *intersects with each of* A, B, C *and any non-empty set* $T \subset S^C (and\ T \neq S^C)$ *is a disjoint from at least on of* A, B, C.

Such sets S^C *can be classified as follows:*

(1) A set consisting of 3 elements chosen one each from the sets $A \cap B^C \cap C^C, A^C \cap B \cap C^C, A^C \cap B^C \cap C$; *there are* $8 \cdot 4 \cdot 2 = 64$ *ways of forming such sets.*

(2) (a) A set consisting of 2 elements chosen one each from the sets $A \cap B^C \cap C^C, A^C \cap B \cap C$; *there are* $8 \cdot 1 = 8$ *ways of forming such sets.*

1.26 Let N be a positive integer. Consider a sequence $a_1, a_2, \dots, a_N$ of positive integers, none of which is a multiple of 2^{N+1}. For integers n greater than or equal to $N+1$, determine a_n in turn as follows:

Choose k to be the number among $1, 2, \dots, n-1$ for which the remainder obtained when a_k is divided by 2^n is the smallest, and define $a_n = 2a_k$. (if there are more than one such k, choose the largest such k). Prove that there exists a positive integer M for which $a_n = a_M$ holds for every $n \geq M$.

Japan NMO-2017

Solution: *Let m be the smallest number among $a_1, a_2, \dots, a_n$, and for n greater than or equal to $N+1$, let L_n be the largest number among $a_1, a_2, \dots, a_{n-1}$. Since $a_n \geq 2m$ for any $n \geq N+1$, we note that the smallest number among $a_1, a_2, \dots, a_{n-1}$ is always m. First, let us assume that there exists M satisfying $L_M < 2^M$. Since $a_n \leq 2L_n$ holds if $n \geq N+1$, we see that $L_{n+1} \leq 2L_n < 2^{n+1}$ must holds if $L_n < 2^n$. Therefore, by induction we can conclude that $L_n < 2^n$ holds for every $n \geq M$. If $L_n < 2^n$, then it is clear than a_n is twice the smallest number among $a_1, a_2, \dots, a_{n-1}$, so that, we have $a_n = 2m$. Consequently, we have $a_n = 2m = a_M$ for every $n \geq M$.*
If we now assume that there is no M for which $L_M < 2^M$ is satisfied. Then, we conclude that for an arbitrary $n \geq N+1$, $L_n \geq 2^n$ must hold. Let us denote by A the largest non-negative number a for which $L_n \geq 2^{n+a}$ is satisfied. Then, there must exist an $l \geq N+1$ for which $2^{l+A} \leq L_l < 2^{l+A+1}$ holds. But then, we have $L_{l+1} \geq 2^{l+A+1}$, from which it follows that $L_l < 2^{l+A+1} \leq L_{l+1} = a_l$ holds. If we assume that $a_{l+1} \neq 2a_l$, then we see that it leads to $L_{l+2} < 2^{l+A+2}$, a contradiction. Therefore, we must have $a_{l+1} = 2a_l$. Since, for $n = l+1$, we have $2^{n+A} \leq L_n < 2^{n+A+l}$, we can show by induction that for every $n \geq l+1$, $a_n = 2a_{n-1}$ must hold, and therefore, $a_n = 2^{n-l}a_l$ holds for every $n \geq l$.

In view of the argument made above, we see that for $n \geq l$ the remainder obtained when a_n is divided by 2^{n+1} must less than or equal to m because of the way that a_{n+1} is determined for such n. However, if we let r be the remainder obtained when a_l is divided by 2^{l+1}, we can show by induction that the remainder obtained when for $n \geq la_n$ is divided by 2^{n+1} equals $2^{n-l}r$. Since $2^{n-l}r \leq m$ holds for every $n \geq l+1$, we conclude that $r = 0$ must hold. This implies that a_n must be a multiple of 2^{n+1} when $n \geq l$. On the other hand, by the assumption made for the problem, none of $a_1, a_2, \dots, a_N$ is a multiple of 2^{n+1}, which contradicts the fact established above.

In view of this contradiction, we conclude that exist an M for which $L_M < 2^M$ is satisfied, proving the assertion of the problem.

1.27 Bobby's booby-trapped safe requires a 3-digit code to unlock it. Alex has a prove which can test combinations without typing them on the safe. The probe responds Fail if no individual digit is correct. Otherwise it responds Close, including when all digits are correct. For example, if the correct code is 014, then the responses to 099 and 014 are both Close, but the response to 140 is Fail. If Alex is following an optimal strategy, what is smallest number of attempts needed to guarantee that he knows the correct code, whatever it is?

Paul Jeffreys-British NMO-2017

Solution (Neel Nanda): *This is a solution to the first part, in which we prove that at least 13 guesses are required. Suppose that the first six attempts are Fail. Then, there must be at least 4 possibilities remaining for each of the digits, for a total of 64. We can check that, regardless of Alex's next question, there can be more than 32 possible codes.*

- *None of the digits are possible; in other words, they have all been ruled out by previous guesses. Then an answer of Fail would leave 64 possible codes.*
- *One of the digits in the guess is possible, and the other two have been ruled out by previous guesses. Then an answer of Fail would 48 possible codes.*
- *Two of the digits in the guess are possible, and the other one has been ruled out by previous guesses. Then an answer of Fail would leave 36 possible codes.*
- *There of the digits in the guess are possible. Then an answer of Close would leave 37 possible codes.*

With the five remaining guesses, there can be at most 32 possible outcomes in total, and thus not all of the possibilities can be distinguished. Thus, at least 13 guesses are required.

1.28 Given any $2n-1$ two-element subset of set $\{1, 2, \ldots, n\}$, prove that one can always choose n of these subsets such that their union contains at most $\frac{2}{3}n+1$ elements.

Dusan Djukic NMO-2016

Solution: *We shall prove by induction on* k $\left(k \leq \frac{2n-1}{3}\right)$ *that one can always remove* $3k$ *subsets such that the cardinality of the union of the remaining subsets does not exceed* $n-k$*. The case* $k = 0$ *is trivial. Assume that* $k \geq 1$ *and that we have removed* $3(k-1)$ *subsets so that union of the remaining ones has at most* $n-k+1$ *elements. Since* $2n-1-3(k-1) < 2(n-k+1)$, *there is an element* x_k *from the union that is contained in at most three of the remaining* $2n-1-3k$ *subsets does not contain* x_k*, which finishes the induction. The problem statement for* $k = \left[\frac{n-1}{3}\right]$ *as* $n - \left[\frac{n-1}{3}\right] \leq n - \frac{n-3}{3} = \frac{2}{3}n + 1$.

1.29 A chessboard of size $(2n+1) \times (2n+1)$, with $n > 0$, is coloured so that each of its squares is either black or white. A square of the chessboard is called special if there are at least n other squares of the same colour in its row, and at least n other squares of the same colour in its column.

1. Prove that there are at least $2n+1$ special squares.

2. Give an example with at most $4n$ special squares.

3. Find the minimum possible number of special squares as a function of n.

Italian NMO-2014

Solution: *(a) This is a consequence of point (c). But for completeness we will give a simple proof. Let a square be in the majority in its row or column if it is of the same colour as at least* n *other squares in its row and its column. Obviously, a square is special if it is in the majority in both its row and its column. The number of squares in the majority in at least one direction can be found by adding the number of squares in the majority in their row to the number of special squares, which were counted twice. We know that the total cannot be greater than* $(2n+1)^2$*, and first two terms cannot be less than* $(2n+1)(n+1)$*, so the number of special squares must be at least* $2(2n+1)(n+1) - (2n+1)^2$, *which is* $2n+1$.

(b) Colour the first row from the top completely black, the first column from the left, except the top left square, completely black, and the rest of the squares in alternating colours, like a chessboard. This way all the rows except for the first have majority of black squares. The special squares must then lie on the first row or the first columns. On the other hand, it's easy to verify that

all the squares in the first row or the first columns are special except for the one in the top left corner. The total number of special squares is therefore $4n$.

(c) We will prove that there are always at least $4n$ *special squares. The previous example lets us conclude that is the minimum amount. The* $2n+1$ *rows of the chessboard are divided into* r_w *rows with a majority of white squares, and* r_b *rows with a majority of black squares. Similarly, the columns are divided into* c_w *with a white majority and* c_b *with a black majority. As we can rotate the chessboard by ninety degrees, which swaps the rows with the columns, and switch the colours around, we can assume that each of these four numbers is less than of equal to a* r_w. *In particular, we know that* r_w *is at least* $n+1$, *otherwise* r_w+r_b *could not be equal to* $2n+1$. *The* r_w *rows with a white majority contain at least* $(n+1)r_w$ *white squares, while the* c_b *columns with a black majority contain at least* $(n+1)c_b$ *black squares. Those among these* $(n+1)(r_w+c_b)$ *squares that do not lie in the intersection of a row with a white majority and a column with a black majority must therefore be special. The number of special squares cannot therefore be less than* $(n+1)(r_w+c_b)$-r_wc_b. *Since this is a degree one expression in* c_b, *and the coefficient of* c_b *is either negative or zero, it has least value when* c_b, *is as large as possible. On the other hand,* c_b *cannot exceed* r_w, *so we have at least* $(2n+2)r_w-r_w^2$ *special squares. For values of* r_w *between 2 and* $2n$ *inclusive, this last expression has value of at least* $4n$.

We are left with two cases: either $r_w\le 1$ *or* $r_w=2n+1$. *The first is clearly impossible, as* r_w *is at least* $n+1$. *We will then examine the second case. Since* $r_w=2n+1$, *there cannot be less than* $(2n+1)(n+1)$ *white squares. If* c_b *were greater than or equal to* $2n$, *there would be at least* $2n(n+1)$ *black squares. This means however that there would be more than* $(2n+1)^2$ *squares in total. It follows than* c_b *less than or equal to* $2n-1$. *In this case, then, there are at least* $(n+1)4n-(2n+1)(2n-1)=4n+1$ *special squares.*

1.30 We have three numbered boxes and 10000 red balls, 10000 blue ones and 10000 yellow ones. Balls of the same colour are undistinguishable. Determine in how many ways they can be distributed in the boxes satisfying:

- **Each box has 10000 balls**
- **No two boxes have the same amount of balls of the same color.**

- **For every two boxes A and B, there is a color c such that the number of balls of color c in A is exactly 2015 or 2016 bigger the number of balls of color c in B.**

Jose Luis Diaz Barrero-Barcelona Tech-Math Contest-2015

Solution: *Let us denote the boxes by B_1, B_2 and B_3, respectively. Let $c(B_i, B_j)$ be the color satisfying the third condition of the statement for B_i and B_j. First, we observe that $c(B_1, B_2)$ and $c(B_2, B_1)$ cannot be the same, because $c(B_1) > c(B_2)$ and at the same time $c(B_2) > c(B_1)$, which is impossible. WLOG. assume that $c(B_1, B_2)$ is red and $c(B_2, B_1)$ is blue.*
If the difference of the number of red and blue balls is the same (either 2015 or 2016), then in order for both boxes to have 10000 balls, there should be the same number of yellow balls in boxes B_1 and B_2 in contradiction with the second condition. So the difference is 2015 in one case and 2016 in the other. WLOG. B_1 has 2016 more red balls than B_2 and B_2 has 2015 more blue balls than B_1. This means that B_2has one more yellow ball than B_1.
We have that $c(B_1, B_2)$ and $c(B_3, B_1)$ are different, and so are $c(B_2, B_3)$ and $c(B_3, B_2)$. Since there are only three colors, one must be repeated.
Let x, y, z be the number of balls of this color in B_1, B_2, B_3, respectively. If it was either red or blue, we would have that $|x-y|, |y-z|, |z-x|$ are all 2015 or 2016, which is impossible. Hence the color is yellow, and $y - x = 1$.
Now, there are two options: $z - x = 2016$ and $z - y = 2015$ or $x - z = 2015$ and $y - z = 2016$. If we substitute in $x + y + z = 10000$, the first one gives $3z - 4031 = 10000$ and the second one gives $3z + 4031 = 10000$.
The second one has no integer solutions and for first one we have $z = \frac{14031}{3} = 4677, x = z - 2016 = 2661$ and $y = z - 2015 = 2662$.
Color $c(B_1, B_3)$ is either red or blue. If it is blue, B_2 would have 4031 more blue balls than B_3, but that is not possible (if we had started by considering boxes B_2 and B_3 instead B_1 and B_2, we would have obtained the same conclusion that the differences are 2015, 2016 and 1). If color $c(B_1, B_3)$ is red, there are 2015 more red balls in B_1, 2661 in B_1 and 2661 in B_3. This arrangement satisfies all the conditions of the statement.
Finally, we need to see how many configurations we discarded in the "without loss of generality" parts. In the first one there were six possible choices for the pair of colors, and in the second one we could assign the 2015 to either red or blue (two options). Therefore the total number of arrangements is $6 \times 2 = 12$.

1.31 Let $n \geq 1$ be a positive integer. Consider a pile of 3^n coins, one of which is fake. Suppose that all coins are either white or black and that if the fake coin is white, it is lighter than the others, and if the fake is black, it is heavier than the others. Furthermore, assume that the number of white coins and the number of black coins differ by at most one. Under these conditions, prove that the fake coin can be identified and classified as heavy or light by at most n weighings in a scale.

Jose Luis Dias Barrero-Barcelona Tech-Math Contest-2016

Solution

For each $n \geq 1$, let $P(n)$ be the statement to be proven. We will argue by induction. Indeed,

- ***Base step:** For $n = 1$, consider $3^1 = 3$ coins, and without loss of generality, suppose that two are black and one is white. Put a black on each pan and set the white aside. If the scale balances, then white coin is counterfeit, and since it is white, is lighter than the other coins. If the scales tips, say the left side down, then the black on the left is heavy and fake. In any case, the counterfeit coin is identified from among three coins and classified or light, so $P(1)$ holds.*
- *Next we consider the case when $n = 3$, assumng, for the moment, that $P(2)$ has been show. Consider $3^3 = 27$ coins, with, say, 14 black, and 13 white. Partition these 27 coins into three groups, say $G_1 = (4B, 5W)$, $G_2 = (5B, 4W)$, and $G_3 = (5B, 4W)$ on the left, weighed against G_3 on the right. If the scale balances, the counterfeit coin is in group G_1 and so $P(2)$ applies to a set of 9 coins, 4 black and 5 white. If the left pan goes down, either one of the 5 black from G_2 is heavy, or one of the 4 white from G_3 is light. The induction hypothesis $P(2)$ now applies to these $3^2 = 9$ coins. Similarly, if the scale tips to the right, the counterfeit is among the 4 white in G_2 or the 5 black in G_3, and again $P(2)$ applies.*
- ***Inductive step:** Fix $k \geq 1$ and assume that $P(k)$ is true. Consider a collection of 3^{k+1} coins, one of which is counterfeit. Assume that there is one more black than white (the same argument works if there is no more white than black), so suppose that $\frac{3^{k+1}+1}{2}$ are black and $\frac{3^{k+1}-1}{2}$ are white. Partition the coins into three groups, G_1, G_2 and G_3, each with 3^k coins, and each with a near balance of black and white: in G_1, put $\frac{3^k-1}{2}$ black and $\frac{3^k+1}{2}$ white coins. In both G_2 and G_3, put $\frac{3^k+1}{2}$ black*

and $\frac{3^k-1}{2}$ white coins. Putting G_1 aside, weigh G_2 on the left against G_3 on the right. If the scales balance, the counterfeit coin is in G_1 and $P(k)$ applies to G_1, finding the fake coin in an additional k weighings, $k+1$ in all, and so $P(k+1)$ is true in this case. If the scales go down on the left, either one of the $\frac{3^k+1}{2}$ black coins from G_2 is heavy, or one of the $\frac{3^k-1}{2}$ white coins from G_2 is light and these $\frac{3^k+1}{2}+\frac{3^k-1}{2}=3^k$ coins satisfy the hypothesis for $P(k)$, and so the coin is found in $k+1$ weighigs. The analogous argument works when the right pan lowers.

Finally, by mathematical induction, for each $n\geq 1$, we conclude that $P(n)$ holds.

1.32 There are $2n-1$ bulbs in a line. Initially, the central $(n-th)$ bulb is on, whereas all others oare off. A step consists of choosing a string of at least three (consecutive) bulbs, the leftmost and rightmost ones being of all between them on, and changing the states of all bulbs in the string (for instance, the configuration •◦◦◦• will turn into ◦•••◦).At most how many steps can be performed?

Dusan Djukic-Serbian NMO-2017

Solution: *The answer is $\left[\frac{2^{n+1}-5}{3}\right]$. Assign to the $i-$th bulb number $2^{|i-n|}$ and define the value of a configuration as the sum of numbers assigned to the bulbs that are on. The initial configuration has value 1. With each step, the value increases by a multiple of 3. If a step switches the $n-$th bulb, the value increases by exactly 3; we call such steps good.Since the value cannot exceed $2^{n+1}-4$ (for not all bulbs can be on), one cannot make more than $\left[\frac{2^{n+1}-5}{3}\right]$ steps. In order to show that this number can be attained, it suffices to show that at least $\frac{2^{n+1}-7}{3}$ good steps can be made.We prove by induction on n that, starting with a configuration of value at most 3, we can reach a configuration of value at least $2^{n+1}-6$ by a sequence of good steps. For $n\leq 2$ this is directly verified. Let $n\geq 3$. By the inductive hypothesis for $n-1$, it is possible to reach a configuration, other than the outer two, the bulbs that are off can be (1°) only the $n-$th, (2°) only the $n-$th and an adjacent one, or (3°) only one bulb adjacent to the $n-$th.In each of these cases, in at most three good steps we reach a configuration in which the two outer bulbs are on and the value of the rest of the configuration (not counting these two) is at most 3.*

1 n $2n-1$
•∘ … ∘∘•∘∘ … ∘•
↓
∘• … ••∘∘∘ … ∘•
↓
∘• … •∘••• … •∘
(1°)

1 n $2n-1$
•∘ … ∘∘••∘ … ∘•
↓
∘• … ••∘•∘ … ∘•
↓
∘• … •∘•∘∘ … ∘•
↓
∘• … •∘∘•• … •∘
(2°)

1 n $2n-1$
•∘ … ∘∘∘•∘ … ∘•
↓
∘• … •••∘∘ … ∘•
↓
∘• … ••∘•• … •∘
(3°)

Now we can apply the inductive hypothesis for $n-1$ again, finishing the induction.

1.33 Suppose that a positive integer a is such that, for any positive integer n, the number n^2a-1 has a divisor greater than 1 and congruent to 1 modulo n. Prove that a is perfect square.

Dusan Djukic NMO-2017

Solution: *As in the first solution, let $n^2a-1=(nx_n+1)(ny_n-1)$ i.e. $y_n-x_n=n(a-x_ny_n)=nd_n$. We distinguish three cases.*
(1°) *If $d_n>0$, then $a=d_n+x_n(x_n+nd_n)>nd_nx_n$, which is impossible for $n\geq a$.*
(2°) *If $d_n<0$, then $a=d_n+y_n(y_n-nd_n)=y_n^2-d_n(ny_n-1)>ny_n-1$, which is impossible for $n\geq a+1$.*
(3°) *If $d_n=0$, then $a=x_n^2$ is a perfect square.*

1.34 Define an n −magic square to mean an $n \times n$ square matrix of non-negative integers such that the sum of all the entries in each row and each column is $\mathrm{m}$for some $m \in \mathbb{N}^*$. Also define as n −permutation matrix to mean an $n \times n$ square matrix of $n(n-1)$ zeroes and n ones such that every row and every column of the array contains exactly one 1. Show that every n −magic square can be written as a sum of finitely many n −permutation matrices.

India TST-2017

Solution: *Fix an $n \in \mathbb{N}$. We will prove the following statement by Induction: $P(m)$: Every n −magic square with common sum $m$can be written as a sum of m permutation matrices. For the base case, we verify that $P(1)$ is true. Since all the entries are non-negative integers, the only way possible n −magic squares with common sum 1 are exactly n −permutation matrices. Hence, every such matrices can be written as a sum of 1 n −permutation matrix, i.e. itself. For the Induction Hypothesis, assume that the statement $P(k)$ is true for some $k \in \mathbb{N}$. Hence, every n −magic square with common sum k can be written as a sum of k n −permutation matrices. Consider any n −magic square A_0 with common sum $k+1$. We will prove that A_0 can be written as a sum of n −permutation matrix and n −magic square A_1 with common sum k. Consider the graph $G = (D, E)$ defined as follows. Let $R_1, R_2, \dots, R_n$ be the rows of A_0 and let $C_1, C_2, \dots, C_n$ be the columns of A_0. Consider $D = \{R_1, R_2, \dots, R_n\} \cup \{C_1, C_2, \dots, C_n\}$. Each of the sets $R = \{R_1, R_2, \dots, R_n\}$ and $C = \{C_1, C_2, \dots, C_n\}$ are independent sets. For every $i, j \in \mathbb{N}_n, R_i, C_j$ are connected by an edge if and only if $a_{ij} > 0$, where a_{ij} is the entry of A_0 in the i^{th} row and j^{th} column. Observe that from our definition, G is bipartite graph. Consider some $S \subseteq R$. By $N(S)$, we will denote the set of vertices which are adjacent to some vertex in S. We will show that $|N(S)| \geq |S|$. For every edge having an endpoint in S, label that edge with a_{ij}. Consider the 'sum' of all such edges. Clearly, for each $R_i \in S$, the edges incident with R_i contribute m to this total. Hence, the total sum is given by $m|S|$. Now these are edges are a subset of all edges with a endpoint in $N(S)$. Hence, if we consider a similar such sum for the set $N(S)$, we get the sum to be $m|N(S)|$. As observed before, we see that $m|S| \leq m|N(S)|$. Hence, we get that $|N(S)| \geq |S|$. Since S was any arbitrary subset of R, we that this is true for all subsets of R.Since the hypothesis for Hall's Matching Theorem is satisfied, an application of it implies that there exists a matching in the above bipartite graph G. Since $|R| = |C| = n$, we see that the matching is nothing but a bijective function $\sigma: \mathbb{N}_n \to \mathbb{N}_n$ where $\sigma(i)$ is defined to be the index of the* *element in C which is matched with R_i. Note that σ can also be through of as a permutation of $\mathbb{N}_n$. We also know that $R_i \cap C_j = \{a_{ij}\}, \forall i,j \in \mathbb{N}_n$. Since σ is a matching, we know that $a_{i\sigma(i)} > 0$ for all $i \in \mathbb{N}_n$. Consider the permutation matrix P_σ where each entry p_{ij} is given by*

$$p_{ij} = \begin{cases} 1, if\ j = \sigma(i) \\ 0, if\ otherwise \end{cases} \forall i,j \in \mathbb{N}_n$$

Also consider the matrix $A_1 = A_0 - P_\sigma$. Note that since $a_{i\sigma(i)} > 0$ for all $i \in \mathbb{N}_n$, every entry of A_1 is non-negative. Also observe that since P_σ is a permutation matrix, the sum of entries in each row and each column reduces by 1. In effect, A_1 is still an n –magic square with common sum $k - 1$. This completes the inductive step. Hence by Induction, we see that the statement $P(m)$ is true for all $m \in \mathbb{N}$. The proof is complete.

1.35 Find all positive integers $a_0, a_1, a_2, b_0, b_1, b_2$ such that $a_2b_2n^2 + a_1b_1n + a_0b_0$ divide $(a_2^{2017n} + b_2)n^2 + (a_1^{2017n} + b_1)n + (a_0^{2017n} + b_0)$s for any positive integer n.

Hong Kong-PreIMO 2017-MOCK EXAM

Solution: *Suppose that*

$a_2b_2n^2 + a_1b_1n + a_0b_0 \mid (a_2^{2017n} + b_2)n^2 + (a_1^{2017n} + b_1)n + (a_0^{2017n} + b_0)$; (1)

We first claim that $a_0 = b_0 = 1$. Let n be a large multiple of a_0b_0. Since a_0b_0 divides the left-hand side (1), it also divides the right-side and hence $a_0b_0 \mid a_0^{2017n} + b_0$. This implies $a_0 \mid b_0$ and $b_0 \mid a_0^{2017n}$. This gives $a_0b_0 \mid b_0$. The only possibility is $a_0 = 1$, which implies $b_0 = 1$.

Let M be a sufficiently large integer and let n_0 be the product of all primes less than M. Choose any prime divisor p of $a_2b_2n^2 + a_1b_1n + 1$; (2)

Clearly, $(p, n_0) = 1$. Thus, $p \ge M$. In particular, we have $(p, a_1a_2) = 1$ as M is large.Consider any n such that $n \equiv n_0 (mod\ p)$. Then p divides the left-hand side of (1). This gives

$$p \mid (a_2^{2017n} + b_2)n_0^2 + (a_1^{2017n} + b_1)n_0 + 2; (3)$$

We choose n such that $p - 1 \mid n$. Such an n exists by the Chinese remainder theorem. By Fermat's little theorem, we obtain $p \mid (1 + b_2)n_0^2 + (1 + b_1)n_0 + 2$. Taking the difference with (3), as $(p, n_0) = 1$, we get $p \mid (a_2^{2017n} - 1)n_0 + (a_1^{2017n} - 1)$; (4) for any $n \equiv n_0 (mod\ p)$. By taking $n \equiv 1 (mod\ p - 1)$ and $n \equiv 2 (modp - 1)$ respectively, we have

$$p \mid (a_2^{2017\cdot 2} - 1)(a_1^{2017} - 1) - (a_1^{2017\cdot 2} - 1)(a_2^{2017} - 1) =$$

$$= (a_1^{2017} - 1)(a_2^{2017} - 1)(a_1^{2017} - a_2^{2017})$$

Since $p \ge M$ is sufficiently large, the right-hand side must be 0. If one of a_1, a_2 is 1, then (4) implies both are equal to 1. If $a_1 = a_2$, then (4) implies
$a_1 = a_2 = 1$ *or* $n_0 \equiv -1(mod\ p)$.
We first consider the case $a_1 = a_2 = 1$. In that case, (1) becomes
$b_2n^2 + b_1n + 1 \mid (1 + b_2)n^2 + (1 + b_1)n + 2$ *so that* $b_2n^2 + b_1n + 1 \mid n^2 + n + 1$*. Note that* $b_2n^2 + b_1n + 1 \ge n^2 + n + 1$*. This, equality must hold and hence $b_1 = b_2 = 1$. One easily checks that $a_0 = a_1 = a_2 = b_0 = b_1 = b_2 = 1$ is a solution.Next, it remains to consider the case $n_0 \equiv -1(mod\ p)$. As p divides (2), this yields $p \mid a_2b_2 - a_1b_1 + 1$. Since p is large, we must have*
$a_2b_2 - a_1b_1 + 1 = 0$*. Then (2) becomes*
$(a_1b_1 - 1)n_0^2 + a_1b_1n_0 + 1 = (n_0 + 1)\big((a_1b_1 - 1)n_0 + 1\big)$*. Therefore, instead of choosing any prime p dividing (2) at the beginning , we chose such a prime p dividing $(a_1b_1 - 1)n_0 + 1$. Therefore, same argument, we obtain either the same solution or $n_0 \equiv -1(mod\ p)$. In the latter case, we find that* $p \mid -(a_1b_1 - 1) + 1$*.Again, this forces $a_1b_1 = 2$ as p is large. Thus,*
$(a_1, b_1) = (1,2)$ *or* $(2,1)$*. Also, $a_2b_2 = a_1b_1 - 1 = 1$ so that $a_2 = b_2 = 1$. Now, by considering $n = 3$ in (1), we have* $16 \mid 2 \cdot 3^2 + (a_1^{2017n} + b_1) \cdot 3 + 2$*. As a_1, b_1 have different parities, the right-hand side is odd. This is impossible. Therefore, the only solution is $a_0 = a_1 = a_2 = b_0 = b_1 = b_2 = 1$.*

1.36 Prove that there are no integer pairs (x, y) satisfying

$$2560x^2 + 5x + 6 = y^6$$

Thailand NMO-2017

Solution:*Suppose for the sake of contradiction that an integer pair (x, y) satisfies* $2560x^2 + 5x + 6 = y^6$*. Thus, $6 \equiv y^5 \equiv y(mod\ 5)$, and so* $y \equiv 1(mod\ 5)$*. Let k be an integer such that $y = 5k + 1$. We now have*

$$2560x^2 + 5x + 6 = (2k + 5)^5 \equiv 1 + \sum_{j=1}^{5} \binom{5}{j} (5k)^j.$$

Dividing the above equation by 5 yields

$$512x^2 + x + 1 = 5^4k^5 + 5^4k^4 + 10 \cdot 5^2k^3 + 10 \cdot 5k^2 + 5k.$$

Taking the above equation modulo 5, we obtain $2x^2 + x + 1 \equiv 0(mod\ 5)$*. This makes* $1 \equiv 2x^2 - 4x + 2 = 2(x - 1)^2(mod\ 5)$*, and so* $3 \equiv (x - 1)^2(mod\ 5)$*. This is a contradiction since 3 is not a square modulo 5.*

1.37 Prove or disprove that there exist 2017 consecutive positive integers that cannot be written as $a^2 + b^2$ where a and b are integers.

Thailand NMO-2017

Solution: *We will prove that such a sequence exists. First we prove the following lemma.*

Lemma. *Let q be a prime such that $q \equiv 3(mod\ 4)$. If $n \equiv q(mod\ q^2)$ then n cannot be written as the sum of two squares.*

Proof. *Let q be a prime such that $q \equiv 3(mod\ 4)$, and n be an integer satisfying $n \equiv q(mod\ q^2)$. Let k be an integer where $q = 4k + 3$.*

Assume to the contrary that n can be written as $a^2 + b^2$ for some integers a and b. Suppose $q \mid a$. Then since $q \mid a^2 + b^2$, we get $q \mid b$ and so $q^2 \mid a^2 + b^2$, contradicting our assumption. Hence $q \nmid a$, and $q \nmid b$.

By Fermat's little theorem we have $a^{q-1} \equiv b^{q-1} \equiv 1(mod\ q)$. (1)

On the other hand we have $a^2 \equiv -b^2(mod\ q)$, raising this to the $(2k + 1)$ $-th$ power yields:

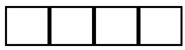
$$a^{4k+2} \equiv (a^2)^{2k+1} \equiv (-b^2)^{2k+1} \equiv -b^{4k+2}(mod\ q).$$

Since $q - 1 = 4k + 2$, this contradicts (1), thus n cannot be written as the sum of two squares.

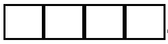
To construct the required sequence, let $q_1, q_2, \dots, q_{2017}$ be a primes congruent to 3 modulo 4, and consider the following system of congruences:

$$\begin{cases} n \equiv q_1(mod\ q_1^2) \\ n + 1 \equiv q_2(mod\ q_2^2) \\ \vdots \\ n + 2016 \equiv q_{2017}(mod\ q_{2017}^2) \end{cases}$$

Since all moduli are pairwise relatively prime, by Chinese Remainder Theorem there exists a solutions to this system modulo $\prod_{i=1}^{2017} q_i^2$. Let $N > 0$ be a solution, then the lemma impies that $N, N + 1, \dots, N + 2016$ cannot be written as the sum of two squares.

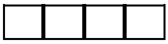
1.38 An $n \times n$ $(n \geq 4)$ square is divided into n^2 unit cells. Find all possible values of n such that this square can be covered with some layers of 4-cell figures of the following shape  **(i.e. each cell of the square must be covered with the same number of these figures) (The sides of each figure must coincide with the sides of the cells; the figures may be rotated but none of them can go beyond the bounds of the square)**

Belarusian NMO-2014

Solution: *If* $n = 2k, k \in \mathbb{N}$, *then this square can be covered with one layer (and then with any number of Layers) of the figures* .

Let $n = 2m + 1, m \geq 2$. *In this case we use chess coloring of the square.*

Suppose that the square is covered with k *layers of given figures. We write 1 and -1 in all black and in all white cells of the square respectively. Let* $B(n)$ *and* $W(n)$ *be the sums of the numbers in black and white cells of the square, respectively. Without loss of generality we can assume that* $B(n) > W(n)$. *It is evident that* $B(n) + W(n) = 1$. *Now we sum up the numbers in all cells of the square so that we count each number in the cell as many times as the number of the figures covering this cell. Let* S *be this sum. Since* k *is the number of the figures covering each cell, we have* $S = k\big(B(n) + W(n)\big) = k$.

On the other hand, any figure  *covers the same number of black and white cells, so the sum of the numbers in the cells covered with this figure is equal to 0. Therefore,* $S = 0$, *a contradiction.*

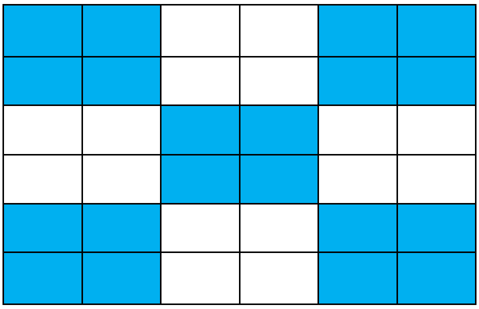
Let $n = 4m + 2$ $(m \in \mathbb{N})$. *Using four colors we paint all cells of the square as it is shown in the figure (the number in the cell corresponds to the number of the color).*

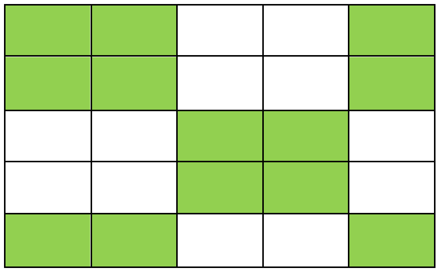
1	2	3	4	1	2
2	3	4	1	2	3
3	4	1	2	3	4
4	1	2	3	4	1
1	2	3	4	**1**	**2**
2	3	4	1	**2**	**3**

Note that there is no cell with 4 th color in the 2×2 *square in the right lower corner of the square. We place 1 in all cells painted second and third colors and place -1 in all remaining cells. As above we sum up the numbers in all cells of the square, then* $S = 2k$, *where* k *is a number of layers. On the other hand, each figure covers exactly one cell of each color, so the sum of the numbers in the cells covered with this figure is equal to 0.*

Therefore, $S = 0$, *a contradiction.*

Remark. Similar arguments can be applied for the following coloring. See Fig.1 for $n = 4m + 2$ *and Fig.2 for* $n = 2m + 1$.


(Fig.1)


(Fig.2)

1.39 In Mathcontestland there are 2017 towns. Every pair of towns is either connected by a single road, or is not connected. If we consider any subset of 2015 towns, the total number of roads connecting these towns to each other is a constant. If there are R roads in Mathcontestland, then find all possible values of R.

Jose Luiz Diaz Barrero-Barcelona Tech-MathContest-2017

Solution: *We consider the general case with n towns. Let K denote the constant number of roads connecting any subset of $n-2$ towns and let $c_{ij} \in \{0,1\}$ denote the number of roads connecting town i and town j. Finally, for $i = 1,2,\dots,n$ let d_j denote the total number of roads connected to town i. Note that*

$$R \leq \binom{n}{2} = \frac{n(n-1)}{2}$$

Clearly,

$$\sum_{i=1}^{n} d_i = 2R \ and \sum c_{ij} = R,$$

where the latter sum is over all 2-element subsets $\{i,j\}$ os $\{1,2,\dots,n\}$. This number of roads connected to at least one of the towns with number i or j is equal to $d_i + d_j - c_{ij}$. Thus for any 2-element subsets $\{i,j\} \subset \{1,2,\dots,n\}$, we have $K = R - d_i - d_j + c_{ij}$
Adding all these equations for every 2-element subset $\{i,j\}$ yields

$$\binom{n}{2} K = \binom{n}{2} R - 2(n-1)R + R$$

which may be written as $n(n-1)K = (n-2)(n-3)R$
Note that both $n(n-1)$ and $(n-2)(n-3)$ are divisible by 2, and that the only integer $k > 2$ which divides both $n(n-1)$ and $(n-2)(n-3)$ is 3, this latter case occurring if and only if n is divisible by 3. Since 3 does not divide

2017, in the situation of the given problem $\frac{n(n-1)}{2}$ *and* $\frac{(n-2)(n-3)}{2}$ *are coprime. Hence,* R *is a multiple of* $\frac{n(n-1)}{2}$*. As* $R \le \frac{n(n-1)}{2}$ *with equality when all the pairs of towns a are connected, the only possibilities are* $R = \frac{n(n-1)}{2}$ *or* $R = 0$*. Therefore, the total number of roads is* $R = 2017 \cdot 1008 = 2033136$ *or*

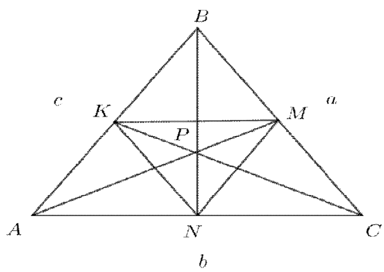
$$R = 0.$$

1.40 Determine the largest possible number of three-element sets that can be formed so that any two of these sets have exactly one common element, but there is not an element that belongs to all these sets.

Belarusian NMO-2014

Solution: *Let* N *be the required number of 3-element sets satisfying the problem condition. Suppose that* $N \ge 8$*. Let* $M = \{x, y, z\}$ *be one of these sets. Since the number of the remaining sets is greater than or equal to 7 and each such set has exactly one common element with* M*, we see that there exists an element of* M *(say,* x*) that belongs to at least [7/3]=3 sets. Let the sets* M_1, M_2, M_3 *contain* x*. By condition, there is not an element that belongs to all sets, so there exists a set (say,* M_0*) such that there is an element from* M *(say,* y*), which belongs to* M_0*. It follows that* $x \notin M_0$*. Since any two of the sets* M_1*,*M_2, M_3 *and* M *have not common elements except for* x *(and* $x \notin M_0$*), so all four elements of the intersection of* M_0 *with*M_1*,*M_2, M_3, M *are distinct. Therefore* M_0 *consist of at least four elements, a contradiction. Thus,* $N \le 7$*.*

It remains to show the example for $N = 7$*. The required sets (see the Fig.) are* $\{A, K, B\}, (A, P, M], \{A, N, C\}, \{B, M, C\}, \{B, P, N\}, \{C, P, K\}, \{K, M, N\}$



1.41 There is a lamp on each cell of a 2017×2017 square board. Each lamp is either on or off. A lamp is called bad if it has an even number of neighbours that are on. What is the smallest possible number off bad Lampson such a board?

(Two lamps are neighbours if their respective cells share a side.)

MEMO-2017

Solution: *Please consult the figures at the end of this solution.*
We divide the square in 1×1 *−squares and color the square in checkerboard fashion such that the corners are black and we call lamps on black and white squares black and white lamps, respectively. We assign the number 1 to a lamp that is on, and the number 0 to a lamp that is off.*

If we assign to coordinates (0,0) to the lamp in the center, we see that the black lamp are exactly the lamps with the coordinates (i, j) *where* $i + j$ *is even.*

Now, we assume that the minimum number is 0 that is, there is a configuration where every lamp has an odd number of neighbours that are on, and we try to get a contradiction. For every black lamp with coordinates $(i, j), i$ *and* j *even, we add the numbers associated to its neighbours, and add all these numbers. The parity of this sum* S *can be determined in the following two ways:*

On the hand, we know that every lamp has an odd number of neighbours with value 1, so we simply have to determine the number modulo 2 of lamps with i *and* j *even. Since we can group lamps at* (i, j) *with lamps* $(-i, -j)$ *and the lamp in the centeris the only one left, we get that* S *is odd.*

On the other hand, every white lamp enters the sum as often as it has neighbours with i *and* j *even. But there are exactly two such lamps because exactly one of the coordinates of the white lamp is odd and can be modified with plus or minus 1 to get a neighbour with two even coordinates. There are no problems at the boundary because this process will not change the coordinate* ± 1008 *so we will stay inside the square. Therefore,* S *is even, which is clearly a contradiction.*

So, it is impossible that all lamps have an odd number of neighbours that are on. Now, we will provide a concrete arrangement where all lamps except for the lamp at the center have an odd number of neighbours that are on.

For the black lamps, i.e. $i + j$ *even, we choose the values:*

$$f(i, j) = \begin{cases} 0, if\ max(|i|, |j|) \equiv 0,1 \ (mod\ 4) \\ 1, if\ \ max(|i|, |j|) \equiv 2,3 \ (mod\ 4) \end{cases}$$

For the white lamps, i.e. $i + j$ *odd, we choose the values:*

$$f(i, j) = \begin{cases} 0, if\ max(|i|, |j|) \equiv 0,1 \ (mod\ 4) \\ 1, if\ \ max(|i|, |j|) \equiv 2,3 \ (mod\ 4) \end{cases}$$

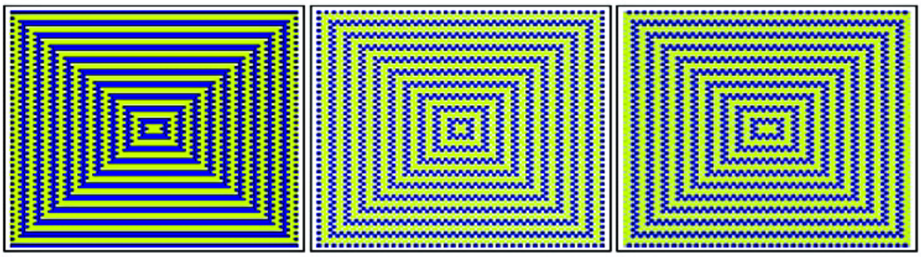
(This assignment can be found by replacing 2017 with a small number, say 17, starting with a row of zeros, using the assumptions to determine the rest and then notice that the zeros and ones for black or white lamps only from frames of depth 2 around the center.)

It is now easily checked that the condition is satisfied for all non-central lamps:

For a white lamp we assume without loss of generality $|i| < |j|$ *(equality is impossible because they have different parity). Then, for the neighbours* $(i \pm 1, j)$ *and* $(i, j \pm 1)$*, the bigger coordinates are* $|j - 1|, |j|, |j|$ *and* $|j + 1|$ *and we can check easily that an odd number of them are* $\equiv 2,3 \ (mod\ 4)$*.*

For a black lamp with $j > 0$ *or* $j < 0$*, we argue analogously. If* $j = 0$*, then* $i \neq 0$ *for a non-central lamp, therefore the maximum is* $|i|$ *and we have again the values* $|i - 1|, |i|, |i|,$
$|i + 1|$ *to check which contain and odd number of values* $\equiv 0,1 \ (mod\ 4)$*.*

Therefore, we have found an arrangement with exactly one lamp with an even number of neighbours that are on as desired.



The images show the discussed optimal arrangement for $n = 77$*. Lamps that are on are yellow, lamps that are off are blue. The first image shows all lamps. The second image shows the lamps with* $i + j$ *even and third image shows the lamps with* $i + k$ *odd.*

1.42 Let n be a positive integer and $n > 1$. The square table $ABCD$ of size n consists of n^2 unit cells, each of them is colored by one of three colors: black, white, gray. A way to color this table is called "nice" if each cell on the diagonal AC is colored by gray and each pair of cells that symmetric respect to AC are colored by the same color, both white or both black. One can fill in each gray cell by the number 0, each black cell by negative integer and each white cell by positive integer. For each positive integer k, a way to fill in the table is called "k –balanced" if it satisfies all following conditions:

- **All cells of table are filled by the integers belong to the interval $[-k, k]$.**
- **If a row meets a column at a black cell then the sets of positive integers on that row and that column are disjoint. Similarly, if a row meets a column at a white cell then the sets of negative integers on that row and that column are disjoint.**

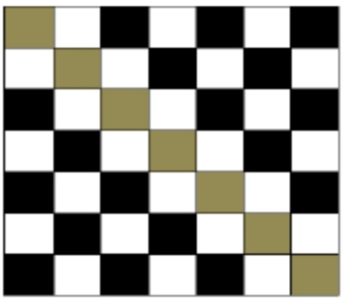
1. For $n = 5$, find the minimum value of k such that there exists a k −balanced way to fill in the table below.
2. For $n = 2017$, find the minimum value of k such that for all nice way to color the table, we can fill in the table by k −balanced way.

Vietnam NMO-2017

Solution: *(1) Let a, b, c be the number fill on the cells at position (1;2),(2;1) and (3;4),(4;3) and (4;5),(5;4). It easy to check that all a, b, c must pairwise distinct, then $k \geq 3$. We construct the way to fill in the table with $k = 3$ as follows:*

0	1	−1	−2	1
1	0	2	2	3
−1	2	0	2	−3
−2	2	2	0	3
1	3	−3	3	0

Thus the minimum value of k in this case is 3.
2) First, consider the nice coloring way as chessboard in which the cell (i, j) colored by black if and only if $i + j$ is even.



Take two white positions on the table at location (a, b) and (c, d), $1 \leq a, b, c, d \leq 2017$.

- *If $a + c$ is even then $b + d$ is also even, which implies that $a + d$ and $b + c$ are both odd. Then, one of two cells (a, b) and (c, d) will be colored by black since they cannot lie on the diagonal AC. Thus the number filled in white cells are different.*
- *If $a + c$ is odd then $b + d$ is also odd, consider cell (d, c) while filled in by the same number as (c, d) then we can apply the same argument as above the conclude that the numbers filled in the white cells are different.*

Hence, all positive numbers on the right upper of the table are pairwise distinct. This implies that

$$k \geq 2 + 4 + 6 + \cdots + 2016 = 1008 \cdot 1009 = \frac{2017^2 - 1}{4}$$

We shall prove that for all symmetric coloring ways, one can use at most $\frac{2017^2}{4}$ *positive integers to fill in the board which satisfy the given conditions (the argument is the same for negative integers).*

Consider the graph of 2017 vertices, namely $A_1, A_2, \ldots, A_{2017}$. *If row* i *and column* j *intersect at a white cell which filled in by a number* $a > 0$, *then we connect* A_iA_j *and call a as the weight of this edge. This graph is well defined since the symmetric properties and from the given condition, we can conclude that: two arbitrary vertices that not connected belongs to edges of different weights. We have to show that for all graph of* $n > 1$ *vertices, there is a way to assign at most* $\frac{n^2}{4}$ *distinct weights to the edges* $(*)$.

We shall prove $(*)$ *by induction. It is easy to check with* $n = 1,2,3$.

Consider $n \geq 4$ *and suppose that the conclusion is true for a graph of* $n - 3$ *vertices, we consider the graph of* n *vertices. There are two cases:*

1.If the number of edges in the graph is not more than $\frac{n^2}{4}$ *then we can easily assign a unique weight to an edge which satisfy the condition. If the number of edges in the graph is more than* $\frac{n^2}{4}$ *then there are three vertices* A_i, A_j, A_k *which are pairwise connected. This is true because from Mantel's theorem, we already know that: If a graph has* n *vertices and more than* $\frac{n^2}{4}$ *edges then it contains a triangle (which means three vertices are pairwise connected).*

We assign to A_iA_j, A_jA_k *and* A_kA_i *the same weight and assign no more than* $n - 3$ *distinct weights to the edges connect between one of three vertices* A_i, A_j, A_k *and* $n - 3$ *other vertices. By applying induction hypothesis, we can assign at most* $\frac{(n-3)^2}{4}$ *distinct weights to the edges connect among* $n - 3$ *other vertices. Furthermore, we have*

$$1 + (n - 3) + \frac{(n - 3)^2}{4} \leq \frac{n^2}{4}.$$

This implies that $(*)$ *also holds for* n.

For $n = 2017$, *we have* $\left[\frac{2017^2}{4}\right] = \frac{2017^2-1}{4}$ *and this exactly the minimum value of* k.

1.43 At a summer school there are 7 courses. Each participant was a student in at least one course, and each course was taken by exactly 40 students. It is known that for each 2 courses there were at most 9 students who took them both. Prove that at least 120 students participated at this summer school.

Moldova NMO-2017

Solution: Let n be the number of students $C_1, C_2, \dots, C_7$ be that 7 courses. Suppose that $n \le 119$. And let a_i be the number of courses that i^{th} student participate. We get that $\sum_{i=1}^{n} a_i = 7 \times 40 = 280$. Then consider the number of (S, C_i, C_j) for which student S participate in C_i, C_j, called X.
So $X = \sum_{i=1}^{n}\binom{a_i}{2}$. But from $n \le 119$ and $\sum_{i=1}^{n} a_i = 280$, we get that

$$X \ge 77\binom{2}{2} + 42\binom{3}{2} = 203.$$

But $X \le 9\binom{7}{2} = 189$. Contradiction. So $n \ge 120$.

1.44 Let $\mathbf{p}$ be an odd prime. Prove that the number $\left|\left(\sqrt{5}+2\right)^{\mathbf{p}} - 2^{\mathbf{p}+1}\right|$ is divisible by $\mathbf{20p}$.

Moldova NMO-2017

Solution: *First, note that the number in question is* $a_p = \left(2-\sqrt{5}\right)^p + \left(2+\sqrt{5}\right)^p - 2^{p+1}$. *Now, this is a sequence with* $a_0 = a_1 = 0, a_2 = 10$ *and* $a_{n+3} = 6a_{n+2} - 7a_{n+1} - 2a_n$ *and hence we easily see that* $20 \mid a_n$ *for all odd* n, *in particular* $20 \mid a_p$.

Now, checking the case $p = 5$ *separately, it suffices to prove that* $p \mid a_p$.

But note that $a_p = 2\sum_{k=1}^{(p-1)/2}\binom{p}{2k}5^k 2^{p-2k}$ *is clearly divisible by* p *as each of the binomial coefficients is.*

Well, first of all this clearly is an integer as can be seen e.g. by the binomial expansion (the terms with $\sqrt{5}$ *cancel). On the other hand,* $(2-\sqrt{5})$ *is a negative number of absolute value less than 1.*

Therefore $\left(2-\sqrt{5}\right)^p$ *will also be of this form and hence the result.*

1.45 Let $n \ge 2$ be a positive integer. Each square of an $n \times n$ −board is coloured red or blue. We put dominoes on the board, each covering two squares of the board. A domino is called $\mathbf{even}$ if it lies on two red or two blue squares and $\mathbf{colourful}$ if it lies on a red and a blue square. Find the largest positive integer $\mathbf{k}$ having the following

property: regardless of how the red/blue-colouring of the board is done, it is always possible to put k non-overlapping dominoes on the board that are either all even or all colourful.

Germany EGMO TST-2015

Solution: *We will prove that* $k = \left[\frac{n^2}{4}\right]$ *is the largest possible integer. Suppose that* n *is even. Then it is possible to cover the board with* $\frac{n^2}{2}$ *dominoes (without considering the colours). Because there are* $\frac{n^2}{2}$ *dominoes, each of which is either colourful or even, there are at least* $\left[\frac{n^2}{4}\right] = \frac{n^2}{4}$ *colourful or at least* $\left[\frac{n^2}{4}\right] = \frac{n^2}{4}$ *even dominoes. When* n *is odd, we can cover the board with* $\frac{n^2-1}{2}$ *dominoes. (Notice that this number is an even integer). Of these dominoes either at least* $\frac{n^2-1}{4} = \left[\frac{n^2}{4}\right]$ *are colourful, or at least* $\frac{n^2-1}{4} = \left[\frac{n^2}{4}\right]$ *are even. This proved that it is always possible to put at least* $\left[\frac{n^2}{4}\right]$ *colourful or at least* $\left[\frac{n^2}{4}\right]$ *even dominoes on the board.*

Colour the squares of the board in the colours white and black like the squares on a chess board, such that the lowerleft square is white. If n *is even, there are equally many white as black square, namely* $\frac{n^2}{2}$*. If* n *is odd, there is one black square less and the number of black squares equals* $\frac{n^2-1}{2} = \left[\frac{n^2}{2}\right]$*. In both cases this is an even number of squares, as for odd* n *we have that* $n^2 \equiv 1 \ (mod\ 4)$*. Now colour half of the black squares red and all the other squares blue. Then there are* $\left[\frac{n^2}{4}\right]$ *red squares, hence we can put at most* $\left[\frac{n^2}{4}\right]$ *non-overlapping colourful dominoes on the board as each of these dominoes covers one red square. An even domino cannot cover two red squares, because there are no pairs of adjacent squares coloured red. Hence, it must cover two blue square. One of these blue squares must have been black, hence the number of even dominoes is at most the number of black-blue squares and that is* $\left[\frac{n^2}{4}\right]$*. Hence of both the colourful as the even dominoes we can put at most* $\left[\frac{n^2}{4}\right]$ *simultaneously on the board.*

We conclude that the maximum k *is indeed* $k = \left[\frac{n^2}{4}\right]$*.*

1.46 For a positive integer n**, we define** D_n **as the largest integer that is a divisor of** $a^n + (a+1)^n + (a+2)^n$ **for all positive integers** a.

1. Show that for all positive integers n, the number D_n is of the form 3^k with $k \geq 0$ an integer.

2. Show that for all integers $k \geq 0$ there exists a positive integer n such that $D_n = 3^k$.

Germany IMO TST-2017

Solution: *1. Let p be a prime and suppose that p divides D_n. Then p divides*

$$((a+1)^n + (a+2)^n + (a+3)^n) - (a^n + (a+1)^n + (a+2)^n)$$
$$= (a+3)^n - a^n$$

for all positive integers a.

Substituting $a = p$, then it follows that $p \mid (p+3)^n - p^n$, i.e. we have $(p+3)^n - p^n \equiv 0 \pmod p$. This simply reads $3^n \equiv 0 \pmod p$, so $p = 3$. We deduce that D_n only contains prime factors equal to 3, and therefore is of the form 3^k with $k \geq 0$ an integer.

2. For $k = 0$ we take $n = 2$. We have $1^2 + 2^2 + 3^2 = 14$ and $2^2 + 3^2 + 4^2 = 29$ and these two numbers are coprime, so $D_2 = 1$. Now assume that $k \geq 1$. We show that $D_n = 3^k$ for $n = 3^{k-1}$.

We first show that $1^n + 2^n + 3^n$ for $n = 3^{k-1}$ is divisible by 3^k, but not by 3^{k+1}. For $k = 1$ we have $n = 1$, indeed, we see that 1+2+3=6 is a divisible by 3 but not by 3^2.

For $k \geq 2$ we have $n > k$, so that 3^n is divisible by 3^{k+1}. So we are reduced to showing that $1 + 2^n$ for $n = 3^{k-1}$ is divisible by 3^k but not by 3^{k+1}. We show this by induction on k.

For $k = 2$ we have $n = 3$, so indeed 1+8=9 is divisible by 9 but not by 27. Let $m \geq 2$, and suppose we have proved our claim for $k = m$. Let $n = 3^{m-1}$. Then $1 + 2^n$ is divisible by 3^m but not by 3^{m+1}. It suffices to show that $1 + 2^{3m}$ is divisible by 3^{m+1}, but not by 3^{m+2}.

Write $1 + 2^n = 3^m c$ with $3 \nmid c$. Then $2^n = 3^m c - 1$, so

$$1 + 2^{3n} = 1 + (3^m c - 1)^3 = 3^{3m}c^3 - 3 \cdot 3^{2m}c^2 + 3 \cdot 3^m c$$

Modulo 3^{m+2}, this is congruent to $3^{m+1}c$, and since $3 \nmid c$, it follows that this is divisible by 3^{m+1}, but not by 3^{m+2}, as desired. This completes our inductive argument.

Next, we show that for $n = 3^{k-1}$, we have that $(a+3)^n - a^n$ is divisible by 3^k for all positive integers a. Again, we prove this by induction on k. For $k = 1$, we have $n = 1$, so indeed we see that $(a+3) - a = 3$ is divisible by 3 for all positive integers a.

Now suppose that $m \geq 1$, and suppose that we proved our claim for $k = m$. Let $n = 3^{m-1}$. Then $(a+3)^n - a^n$ is divisible by 3^m for all positive *integers a, so we can write $(a+3)^n = a^n + 3^m c$ for some integer c. Taking third powers of both sides then yields*

$$(a+3)^{3n} = a^{3n} + 3a^{2n}\cdot 3^m c + 3a^n \cdot 3^{2m}c^2 + 3^{3m}c^3, so$$

$$(a+3)^{3n} - a^{3n} = 3a^{2n}\cdot 3^m c + 3a^n \cdot 3^{2m}c^2 + 3^{3m}c^3,$$

which is divisible by 3^{m+1}. This completes our inductive argument.

We have now shown for $n = 3^{k-1}$ that $3^k \mid 1^n + 2^n + 3^n$ and $3^k \mid (a+3)^n - a^n$ for all positive integers a, from which we, by induction on a, immediately deduce that

$3^k \mid a^n + (a+1)^n + (a+2)^n$ for all positive integers a. Therefore $3^k \mid D_n$.

As $3^{k+1} \nmid 1^n + 2^n + 3^n$, we also have $3^{k+1} \nmid D_n$. Therefore $D_n = 3^k$, as desired.

1.47 Given are positive integers r and k and an infinite sequence of positive integers $a_1 \le a_2 \le \cdots$ such that $\frac{r}{a_r} = k+1$. Prove that there is a t satisfying $\frac{t}{a_t} = k$.

Germany EGMO TST-2015

Solution: *We will prove this by contradiction. Suppose that such a t does not exist. If $a_k = 1$, then $\frac{k}{a_k} = k$ would hold, contradicting our assumption. Hence, $a_k \ge 2$. We will now prove by induction to i that $a_{ik} \ge i+1$. We just proved the base case. Now suppose that for certain $i \ge 1$ we have that $a_{ik} \ge i+1$. Then we also have that $a_{(i+1)k} \ge i+1$. If $a_{(i+1)k} \ge i+1$, then $\frac{(i+1)k}{a_{(i+1)k}} = k$, which is a contradiction. Hence, $a_{(i+1)k} \ge i+2$. This finishes the induction. Now take $i = a_r$ we have $a_r k \ge a_r + 1$. Moreover, because $r = a_r(k+1)$ we have $a_r = a_{a_r(k+1)} \ge a_{a_r k} \ge a_r + 1$, which is a contradiction.*

1.48 In a country between every two cities there is a direct bus or a direct train line (all lines are two-way and they don't pass through any other city). Prove that all cities in that country can be arranged in two disjoint sets so that all cities in one set can be visited using only train so that no city is visited twice, and all cities in the other set can be visited using only bus so that no city is visited twice.

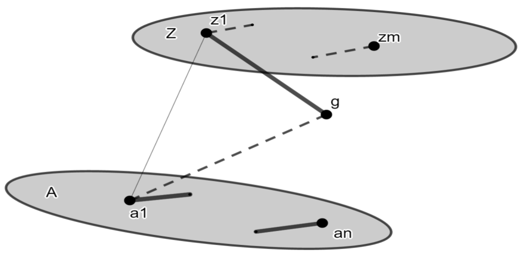
Croatian NMO-2015

Solution: *Let G be the set of all cities in the country. We call a pair (A, Z), where A and Z are disjoint subsets of G $good$ if all cities in the set A can be* *visited using only bus such that no city is visited twice and all cities in the set Z can be visited using only train such that no city is visited twice.*

Let (A, Z) be a good pair such that the set $A \cup Z$ has the maximal number of elements. If we prove $A \cup Z = G$, the statement of the problem holds.

Let us assume the opposite, i.e. there is a city g which isn't from A or Z. Without loss of generality we can assume that A and Z are non-empty, because otherwise we can transfer any city from a non-empty set to an empty one. Let n be the number of cities in the set A, and m the number of cities in the set Z. Let us arrange the cities from A in the series $a_1, \dots, a_n$ such that every two consecutive cities in that series are connected by a direct bus line. Also, let us arrange the cities from Z in the series $z_1, \dots, z_m$ such that every two consecutive cities in that series are connected by a direct train line.

Since we assume that the pair (A, Z) is maximal, the cities g and a_1 have to be connected by train (otherwise the pair $(A \cup \{g\}, Z)$ would be a good pair whose union would have more elements than $A \cup Z$), and g and z_1 have to be connected by bus (otherwise the pair $(A, Z \cup \{g\})$ would be a good pair whose union would have more elements than $A \cup Z$). The cities a_1 and z_1 have to be connected by bus or by train.



If a_1 and z_1 are connected by bus, let us put $A' = \{z_1, g, a_1, \dots, a_n\}$ and $Z' = \{z_2, \dots, z_m\}$.
Then (A', Z') is a good pair and the number of elements of $A' \cup Z'$ is greater than the number of elements of $A \cup Z$, which contradicts the assumption.

If a_1 and z_1 are connected by train, let us put $A'' = \{a_2, \dots, a_n\}$ and $Z'' = \{a_1, g, z_1, \dots, z_m\}$. Then (A'', Z'') is a good pair and the number of elements of $A'' \cup Z''$ is greater than the number of elements of $A \cup Z$, which contradicts the assumption.

Since all cases lead to contradiction, we conclude that the assumption was wrong and that every city is either in the set A or in the set Z.

1.49 The first n positive integers are written on a board $(n \geq 3)$. Ante repeats the following procedure: first he chooses two numbers on the board, and then he increases them both by the same arbitrary positive integer.Determine all positive integers n such that Ante can, by repeating this procedure, achieve that all numbers on the board are equal.

Croatian NMO-2015

Solution: *Assume $n = 4k$. Then Ante can achieve that all numbers on the board are equal in the following way: he will increase by 1 the numbers and 3,5 and 7, ... ,$4k - 3$ and $4k - 1$. By doing that, he gets that the numbers on the board are all even numbers smaller than or equal to n, and each is written twice. Finalli, he increases 2 and 2 by $n - 2$, 4 and 4 by $n - 4$, ... , $n - 2$ and $n - 2$ by 2, and he gets that all numbers on the board are equal to n.*

Assume $n = 2k + 1$. Then Annnttte can schieve that all numbers on the board are equal in the following way: he will increase by 1 the numbers 1 and n, ... , $n - 2$ and n.

By doing that, he gets that the numbers on the board are all even numbers smaller than or equal to n, each written twice, and the number $\frac{3n-1}{2}$. Finally, he increases 2 and 2 by $\frac{3n-5}{2}$, 4 and 4 by $\frac{3n-9}{2}$, ... , $n - 1$ and $n - 1$ by $\frac{n+1}{2}$. Now all numbers on the board are equal to $\frac{3n-1}{2}$.

Assume $n = 4k + 2$. Then Ante cannot achieve that all numbers are equal. The sum of all numbers on the board is initialy odd, because

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} = (2k+1)(4k+3)$$

Since there is an even number of numbers on the board, if they were equal their sum would be an even number. On the other hand, in each step the sum of numbers on the board is increased by an even, so the sum will never be even. Therefore, Ante can achieve that all numbers are equal if and only if n is not of the form $4k + 2, k \in \mathbb{N}$.

1.50 Prove that for every non-negative integer n there exist integers x, y, z with $\gcd(x, y, z) = 1$ such that $x^2 + y^2 + z^2 = 3^{2^n}$.

Czech-Polish-Slovak Match-2016

Solution: *We will use the following algebraic formula:*

$$(x^2 + y^2 + z^2)^2 = (x^2 + y^2 - z^2)^2 + (2xz)^2 + (2yz)^2.$$

This means that if a positive integer can be represented as a sum of 3 squares, then so can its square. Consequantly, if we put

$$(x_0, y_0, z_0) = (1,1,1)$$
$$(x_{n+1}, y_{n+1}, z_{n+1}) = (x_n^2 + y_n^2 - z_n^2, 2x_n z_n, 2y_n z_n),$$

then by a trivial induction we obtain that $x^2 + y^2 + z^2 = 3^{2^n}$.

It remains to show that $gcd(x_n, y_n, z_n) = 1$ *for all* n *and we proceed by induction.*

Suppose $gcd(x_n, y_n, z_n) = 1$*, but* x_{n+1}, y_{n+1} *and* z_{n+1} *have some common prime divisor* p. *Observe that since* 3^{2^n} *is odd, but* $2x_n z_n$ *and* $2y_n z_n$ *are even, we have that* $x_n^2 + y_n^2 - z_n^2$ *is odd, so* $p \neq 2$. *Hence* $p \mid x_n z_n$ *and* $p \mid y_n z_n$. *Then either* $p \mid z_n$*, or* $p \mid x_n$ *and* $p \mid y_n$. *In the latter case we could infer from* $p \mid x_n^2 + y_n^2 - z_n^2$ *that in fact also* $p \mid z_n$*, which contradicts the assumption that* $gcd(x_n, y_n, z_n) = 1$. *Hence we are left with the first case:* $p \mid z_n$.

Since $p \mid x_n^2 + y_n^2 - z_n^2$ *and* $p \mid z_n$*, we also have that* $p \mid x_n^2 + y_n^2 + z_n^2 = 3^{2^n}$. *Hence in fact* $p = 3$. *But the only quadratic residues modulo 3 are 0 and 1, so the two possibilities for a sum of three squares to be divisible by 3 is that either all or none of them is divisible by 3. The former case is excluded by the assumption that* $gcd(x, y, z) = 1$ *and the latter by* $p \mid z_n$.

1.51 Given are $2n$ people and it is known that their heights are all different. They have to stand in two rows, each with n people. How many different positions are there, if the front row person is always shorter than the back row person?

Mongolian NMO-2010

Solution*: In the first seats of the rows, we can choose 2 people* $\frac{2n(2n-1)}{2}$ *different ways. Then in the second seats of the rows, we can choose 2 people* $\frac{(2n-2)(2n-3)}{2}$ *different ways. Continuing this, we have*

$$\frac{2n(2n-1)}{2} \cdot \frac{(2n-2)(2n-3)}{2} \cdot \ldots \cdot \frac{2 \cdot 1}{2} = \frac{(2n)!}{2^n}$$

1.52 A sequence of digits is written on the board according to the following rule: each time write the last digit of the product ab, where a and b are the last two digits written. For example, if initial digits are 1;8, then the sequence is continued as follows: 1;8;8;4;...

Find the 2017^{th} digit of the sequence starting with 3;4

Ukrainian NMO-2017

Solution: Write out the sequence until it starts to repeat; it will happen when a pair of consecutive digits repeats. Since there is a finite number of such pairs, eventually it must happen.

3;4;2;8;6;8;8;4;2;...

We see that a group of six digits 4;2;8;6;8;8 repeats periodically. Moreover, the first digit 3 is outside of the period. Hence, the 4 digit is at positions 2,8,14,...,2012. So the 2017^{th} digit is 8.

1.53 Find all the positive integers $\mathbf{n}$ such that exist $\boldsymbol{n}$ sets $\boldsymbol{A_1}, \boldsymbol{A_2}, \dots, \boldsymbol{A_n}$ such that each of them has exactly 5 elements, any two of these $\mathbf{n}$ sets have exactly one common element and union of these sets consists $\mathbf{n}$ elements.

U.Batzorig Mongolian NMO-2010

Solution: *Let us assume that the union of the sets consist of integers from 1 to n. Let S_i denote the number of sets that i belongs to. Then the total number of elements of n sets is*

$$5n = S_1 + S_2 + \cdots + S_n; \quad (*)$$

Let us assume $S_1 > 5$ and $1 \in A_1, 1 \in A_2, 1 \in A_3, 1 \in A_4, 1 \in A_5, 1 \in A_6$. If $1 \in A_i, i = \overline{1,n}$ then the remaining $4n$ elements have to be different. Because $|A_i \cap A_j| = 1, i \neq j$ holds. That means, we have $4n + 1$ different elements. It contradicts to $|A_1 \cup A_2 \cup \dots \cup A_n| = n$.

Thus we can assume that $1 \notin A_7$.

Since $|A_i \cap A_j| = 1, 1 \leq i, j \leq 6$, the intersections of A_7 with $A_1, A_2, \dots, A_6$ are all different. Hence $|A_7| \geq 6$. But it contradicts to $|A_7| = 5$. This leads to $S_1 \leq 5$. Analogously, $S_k \leq 5$ for $k = \overline{1,n}$.

Considering $(*)$, $S_k = 5$ holds for $i = \overline{1,n}$. Hence the number of sets is $n = 4 \cdot 5 + 1 = 21$.

The construction is: $A_1 = \{1,2,3,4,5\}$; $A_2 = \{1,6,7,8,9\}$

$$A_3 = \{1,10,11,12,13\}; \ A_4 = \{1,14,15,16,17\}$$

$$A_5 = \{1,18,19,20,20\}; \ A_6 = \{2,6,10,14,18\}$$

A_7 = {2,7,11,15,19}; A_8 = {2,8,12,16,20}

A_9 = {2,9,13,17,12}; A_{10} = {3,6,7,8,9}

A_{11} = {3,10,11,12,13}; A_{12} = {3,14,15,16,17}

A_{13} = {3,18,19,20,21}; A_{14} = {4,6,10,14,18}

A_{15} = {4,7,11,15,19}; A_{16} = {4,8,12,16,20}

A_{17} = {4,9,13,17,21}; A_{18} = {5,6,7,8,9}

A_{19} = {5,10,11,12,13}; A_{20} = {5,14,15,16,17}

A_{21} = {5,18,19,20,21}

1.54 In a company of friends each one likes either math or computer science. Those who like math have an average age of 15, and those who like computer science have an average age of 25. One day, Andriy switched from computer science math. As a consequence, an average age of each group increased by 1. Find the number of friends in the company and give an example that demonstrates that such situation is possible.

Ukrainian NMO-2017

Solution: *Denote by* n *and* m *the number of people who like math and computer science, respectively.Then the total age of all friends can be counted in two ways:* $N = 15n + 25m = 16(m + 1) + 26(m - 1) \Rightarrow n + m = 10$.

This is indeed possible: let there be 4 mathematicians aged 15, one computer scientist aged 20, and 5 more computer scientist aged 26. Initially, average age was equal to 15 for mathematicians, and $\frac{1}{6}(20 + 26 \cdot 5) = 25$ *for computer scientist. If Andriy is 20 years old, then the average age become 26 for computer scientists, and* $\frac{1}{5}(20 + 15 \cdot 4) = 16$ *for mathematicians.*

1.55 8 small circles are arranged in a circle. Is it possible to put the numbers 1;2;...;8 in these circles so that the sum of any two neighboring numbers is not divisible by 3,5 and 7?

Ukrainian NMO-2017

Solution: *Let us write out the pairs of numbers that can be neighboring:*

$$(1;3), (1,7). (2,6), (3,5), (3,8), (4,7), (5,6), (5,8), (6,7).$$

We see that 2 can only have 6 as a neighbour. But if the necessary arrangement existed, 2 must have had two different neighbours.

1.56 In a product of 3 positive integers each multiple was decreased by 3. Could it happen that the product increases by 2016 after this change?

N.Agakhanov-Russian-Regional MO-2017

Solution: Answer, Yes. An example is $1 \cdot 1 \cdot 676$.

1.57 Initially, Brazil thinks of eight cells on a chessboard, no two of which are in the same row or in the same column. Then Pete makes a series of guesses. By a guess, he places onto the chessboard 8 rooks none of which can take another one, and then Brazil indicates those of Pere's rooks which stand on positions he thinks of. If Brazil indicates an even number of rooks, Pete wins. Otherwise, the rooks are removed from the board, and then Pete makes the next guess. Find the least number of guesses at which Pete can win for sure.

I.Bogdanov-Regional MO Russian-2017

Solution*: Answer. 2 guesses.If Pete did not win on the first move (which is possible), then on the second he swaps the rows of just two rooks; one standing on Brazil's cell, and one standing on such cell.*

1.58 In a product of 5 positive integers each multiple was decreased by 3. Could it happen that the product becomes exactly 15 times larger than the initial one?

N.Agakhanov, I.Bogdanov-Regional MO Russian-2017

Solution: Answer. Yes. An example is $1 \cdot 1 \cdot 1 \cdot 1 \cdot 48$

1.59 In a product of 7 positive integers each multiple was decreased by 3. Could it happen thet the product becomes exactly 13 times larger that the initial one?

N.Agakhanov, I.Bogdanov-Regional MO-Russian-2017

Solution: Answer. Yes. An example is $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 16 = 32$ (there are other examples)

1.60 Some pairs of cities in a country are connected with one-directional direct flights (between any two cities, there is at most one flight). We say that a city A is accessible for a city B if one may reach the city A starting at B (perhaps with zero flights or more than one flight in a chain). Assume that for any two cities P and Q there exists a city R for which each of P and Q is accessible. Prove that there exists a city for which every city is accessible.

V.Dolnikov-Russia NMO-2017

Solution: Chose the city with a maximal number of cities accessible from it.

1.61 100 dwarves whose weights are 1,2,...,100 lb came to the left bank of a river. They cannot swim, but they have a boat which can take up to 100 lb. When a boat crosses the river, one of the dwarves in it is an oarsmen; while performing one crossing, the oarsman remains the same. Due to the steam, it is difficult to oar from the right bank to the left one, so each dwarf can oar in this direction at most once. Can the whole company of dwarves reach the right bank?

A.Shapovalov, S.Usov-Russia NMO-2017

Solution*:* *Answer. No. Call the dwarves weighing at least 50 kg heavy; assume they oared backwards* d *times. Then there were at least* $51 + d$ *their forward passages, and hence at least 50 backward passages with no heavy dwarves oaring. But who could oar during these 50 passages? Answer. Yes.*

1.62 Is it possible to compose two integer numbers using each of the ten digits 0,1,...,9 exactly once, such that one of them is the square of the other? 0 cannot be the first digit in either number.

Ukrainian NMO-2015

Solution: *If a number has 3 digits, its square can contain no more than 6 digits, which gives 9 in total. If our number has 4 digits or more, its square has at least 7, therefore, we must use at least 11 digits. Therefore, no such number exist.*

1.63 A committee has 4 subcommittees, each controlled by 3 leaders from the committee. For effective coordination, each two subcommittees must have exactly one leader in common. What is the least possible number of people in the committee:

Ukrainian NMO-2015

Solution: *Answer. 6. If we consider two subcommittees, they have exactly one leader in common, therefore, together they have exactly 5 members. Hence there are at least 5 people in the committee. Denote them by* $A = \{1,2,3,4,5\}$. *However it's impossible to choose leaders for another subcommittee out of them. Therefore, the committee must have at least 6 members. Here's an example of four subcommittees and their leaders:*

$$A = \{1,2,3,4,5,6\} \to \{1,2,3\}; \{3,4,5\}; \{1,5,6\}; \{2,4,6\}$$

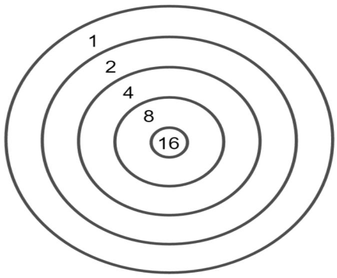
1.64 Find all integers n that have more than $\frac{n}{2}$ divisors.

Bogdan Rublyov-Ukrainian NMO-2017

Solution: *Answer:* $n \in \{1,2,3,4,6\}$.*Clearly a number cannot have divisors greater* $\frac{n}{2}$, *besides* n *itself. Therefore, in order to have more than* $\frac{n}{2}$ *divisors it must be divisible by all numbers from 1 to* $\frac{n}{2}$ *and by* n. *Denote by* m *the integer number which equals either* $\frac{n}{2}$ *or* $\frac{n-1}{2}$. *Then if* $n \geq 10$, *either* m *or* $m-1$ *is coprime with 3.*

Hence, $n \geq 3m$, because it must be divisible by both 3 and m. However, in this case $n \geq 3 \cdot \left(\frac{n-1}{2} - 1\right)$, or $2n \geq 3n - 9$, which contradicts the assumption that $n \geq 10$. All numbers $n < 10$ can be checked by hand.

1.65 The target for bow shooting looks as in shown in the figure. What is the minimum amount of shots that a sportsman has to fire to score exactly 55 points?



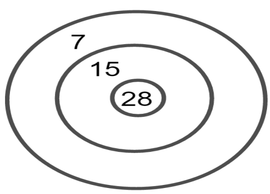
Ukrainian NMO-2015

Solution: *Clearly, for the minimum amount of shots all hits, except to be no more than 1 time. Otherwise, instead of such 2 hits, for example into 4, it is enough to hit into 8 once. In such a way, we chose the maximum possible amount of hits into 16, and then into every other point no more than 1 time:*

$$55 = 16 + 16 + 4 + 2 + 1$$

so the minimum is 6 shots.

1.66 The target for bow shooting looks as in the figure. What is the minimum amount of shots that a spotsman has to fire to score exactly 105 points?



Ukrainian NMO-2015

Solution: *Answer: 6. Suppose there is at least one hit in 15. Then*

$$105 = 7n + 15m + 28k,$$

since all the item, except $15m$, *are divisible by 7, the sum is divisible by, than* $15m$ *have to be divisible by 7, that is why* $m \geq 7$. *But if there are no hits in 15, it is enough to have exactly 6 hits:*

$$105 = 28 \cdot 3 + 7 \cdot 3,$$

and obviously it is the minimum amount of hits.

1.67 For which natural number in notation of number $\mathbf{n}! = \mathbf{1} \cdot \mathbf{2} \cdot \ldots \cdot \mathbf{n}$ are exactly 2 digits used?

Ukrainian NMO-2015

Solution*: Answer:* $n = 4$. *In case of* $n \geq 5$ *the number* $n!$ *ends with 0, so it can have one more digit. Mark it as* a, *then*

$$n! = \overline{a \ldots a0 \ldots 0a \ldots a0 \ldots 0 \ldots a0 \ldots 0} = a \cdot \overline{1 \ldots 10 \ldots 01 \ldots 10 \ldots 0 \ldots 1 \ldots 10 \ldots 0}$$

If you cross out zeros from the last factor, you will get an odd. So the amount of factors 2 in decomposition $n!$ *into prime numbers cannot be more than amount of factors 5 plus 3. Every pair* $2 \cdot 5$ *gives new 0 at the end, and superfluous 2 can from only digit* a, *and among digits 8 is divisible by the biggest power of 2. Even for* $n = 8$, $n! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$ *and the condition is not true. Since the factor 2 occurs in every second number, and 5 occurs in every fifth, then the appropriate relation can be accomplished only if* $n \leq 7$. *By a simple testing we find out that the condition is satisfied only by* $4! = 24$.

1.68 In table $\boldsymbol{n} \times \boldsymbol{n}$ the columns are renumbered from left to right by Fig.1 numbers $\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}$. Every cell is filled by $\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}$ in such way that every row and every column contains all numbers $\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}$. In every column cells are painted in grey if the number in it is bigger than the number of this column. In figure 1 the example of such painting for some arrangement of numbers for $\boldsymbol{n} = \mathbf{3}$ is shown. Is it possible that the amount of grey cells in every row is equal, if

$\boldsymbol{a}) \ \boldsymbol{n} = \mathbf{5}$ $\qquad$ $\boldsymbol{b}) \ \boldsymbol{n} = \mathbf{10}$

3	1	1
1	2	3
2	3	1

(Fig.1)

Ukrainian NMO-2015

Solution: *Answer: a) yes; b) no.a) For* $n = 5$ *it is easy to show an example (Fig.2) b) For* $n = 10$ *first row contains exactly 9 grey cells, second contains exactly* 8, … , 9^{th} *contains exactly 1 cell, 10*th *does not contain any.*

Consequently, there are $9 + 8 + \cdots + 1 = 45$ *grey cells, and this amount is not divisible by 10. Therefore every row cannot contain equal amount of grey cells.*

5	4	3	2	1
1	5	4	3	2
2	1	5	4	2
3	2	1	5	4
4	3	2	1	5

(Fig.2)

1.69 Find all integer n, such that:

$$(n - 2013)(n - 2014)(n - 2016)(n - 2017) = 4$$

Bogdan Rublyov-Ukrainian NMO-2015

Solution: *Answer:* $n = 2015$. *If integer number* n *satisfies the given condition, then 4 can be presented as product of four pairwise distinct integer. Since the* *integers divisors of this number are only* ± 1, ± 2 *and* ± 4, *we have that sought-for divisors are* ± 1 *and* ± 2. *Indeed, if the absolute value of one of divisors is equal to four then others are not less that 1 by absolute value, a contradiction. Since* $n - 2013$ *is the biggest factor, he should be equal to 2. Also we see that* $n = 2015$ *satisfies the condition of the problem.*

1.70 Find all three prime numbers (p, q, r) that satisfy the equality

$$\frac{q}{p-1} + \frac{r}{p+1} = \frac{q+r+1}{p}$$

Vyacheslav Yasinkyi=Ukrainian NMO-2015

Solution: *Answer:* $p = q = 2, r = 3$ *or* $p = q = 3, r = 2$. *Let's rewrite the given equality as follows:*

$$\frac{q}{p-1} - \frac{q}{p} = \frac{r}{p} - \frac{r}{p+1} + \frac{1}{p} \Leftrightarrow \frac{q}{p(p-1)} = \frac{r}{p(p+1)} \Leftrightarrow \frac{q}{p-1} = \frac{r}{p+1} + 1$$

$$q = \frac{(p-1)r}{p+1} + p - 1 \Leftrightarrow q = p + r - 1 - \frac{2r}{p+1}; \quad (1)$$

Since p, q, r *are prime, then the number* $\frac{2r}{p+1}$ *is a positive integer. The number* $2r$ *has only four divides:* $1, 2, r$ *and* $2r$. *Since* $p + 1 \geq 3$, *then two cases are possible:* $p + 1 = r$ *or* $p + 1 = 2r$.

1) *Suppose that* $p + 1 = r$ *which means* $p = r - 1, p + 1 \geq 3$, *so the only pair of consecutive prime numbers is 2 and 3, and thus,* $p = 2$ *and* $r = 3$. *Then from (1) we find that* $q = 2$. *After checking we make certain that* $p = q = 2, r = 3$ *is an answer.*

2) *Suppose that* $p + 1 = 2r$ *which means* $p = 2r - 1$. *Then from (1) we find that* $q = 3r - 3 \vdots 3$ *and since* q *is prime then* $q = 3$. *Further, consistently find that* $r = 2, r = 3$. *Checking shows that* $p = q = 3, r = 2$ *is an answer as well.*

1.71 Determine all prime numbers $p < q < r$ so that

$A = (r-p)(r-q)(q-p) + 1$ and $B = 3p + 5q$ equal the same prime number.

Yasinky Vyacheslav-Ukrainian NMO-2015

Solution: *Answer:* $p = 2, q = 5, r = 7$. *Let* p, q, r *be the prime numbers that satisfy the conditions of the problem. If* $p > 2$ *the all* p, q, r *are odd thus the number* $A = 3p + 5q$ *is even and the number* $B = (r-p)(r-q)(q-p) + 1$ *is odd that contradicts the conditions of the problem. Therefore,* $p = 2$, *thus:* $(r-p)(r-q)(q-p) + 1 = 6 + 5q$.

$p < q < r$, *that means* $r - 2 > q - 2$ *and* $r - q \geq 2$. *Therefore,*

$6 + 5q > 2(q-2)^2 + 1$. *After solving tha last inequality, we will have that* $q < 7$. *Prime numbers that satisfy this inequality are* $q = 3$ *and* $q = 5$.

If $q = 3$, *then* $B = 3p + 5q = 21$ *is not prime.*

If $q = 5$, *then* $B = 3p + 5q = 31$ *is prime, that implies* $A = (r-2)(r-5) \cdot 3 + 1 = 31$, *and* $r^2 - 7r = 0$, *that means* $r = 7$.

1.72 a) Determine whether there exist positive integer numbers $a_1, a_2, \dots, a_{2015}$ such that: any two of them are co-prime and $a_1 a_2 \dots a_{2015} - 1$ is a product of two consequent odd numbers?

b) Determine whether there exist positive integer numbers $a_1, a_2, \dots, a_{2015}$, such that: any two of them are co-prime and $a_1 a_2 \dots a_{2015} - 1$ is a product of two consequent even numbers?

Yasinky Vyacheslav-Ukrainian NMO-2015

Solution: *Answer: a),b) Yes, such numbers exists.*

a) Let $a_1 = p_1^2, a_2 = p_2^2, \dots, a_{2015} = p_{2015}^2$, *where* $p_1 = 2, p_2, \dots, p_{2015}$ *are first 2015 prime numbers. It is clear that every two of this numbers are co-prime and* $a_1 a_2 \dots a_{2015} - 1 = (p_1 p_2 \dots p_{2015} - 1)(p_1 p_2 \dots p_{2015} + 1)$ *is a product of two consequent odd numbers.*

b) Let $a_1 = p_1^2, a_2 = p_2^2, \dots, a_{2015} = p_{2015}^2$, *where* $p_1 = 3, p_2 = 5, \dots, p_{2015}$ *are first 2015 odd prime numbers. It is clear that every two of this numbers are co-prime and* $a_1 a_2 \dots a_{2015} - 1 = (p_1 p_2 \dots p_{2015} - 1)(p_1 p_2 \dots p_{2015} + 1)$ *is a product of two consequent even numbers.*

1.73 Some positive integers are writen on cards, at least two different numbers on each card. The same number may be written on several

cards. Two cards are called adjacent if the maximum number on one of them is equal to the minimum number on the other.

Prove that if there are no adjacent cards then all written numbers can be divided into two sets so that any card contains at leastone number from every set.

S.Chernov-Belarus NMO-2017

Solution: *We construct two required groups G_1 and G_2 as follows: we choose the smallest number on on the each card and put it in the group G_1. Therefore, every card contains at least one number from G_1. All other numbers we put in the group G_2. Since there are no adjacent cards, the largest number on any card does not coincide with the smallest numbers on other cards, so this largest number belongs to G_2. Therefore, every card contains at least one number from G_2.*

1.74 a) Given the ten digits from 0 to 9, prove that three numbers $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ can be formed by combining these digits, provided that each digit is used exactly once and $A + B = C$. Notice that 0 may not be the first digit of any of the numbers.

b) Find al possible values of the sum of the digits of C.

V.Kaskevich NMO-2017

Solution: *Answer: b) 9 or 18. a) For example* $765 + 324 = 1089$.

b) Since any integer X is congruent modulo 9 to the sum of its digits, we have: $A + B + C \equiv S(A) + S(B) + S(C) =$

$$= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 0 \equiv 0(mod\ 9).$$

By condition, we have $A + B = C$, so $2C \equiv 0(mod\ 9)$ where it follows that $C \equiv 0(mod\ 9)$ and, therefore, $S(C) \equiv 0(mod\ 9)$, i.e. the sum of the digits of C is divisible by 9. Note that $S(C) = S(A + B) \leq S(A) + S(B)$. Indeed, if the sum of the digits of A and B does not exceed 9 in all number position, then when we add A and B there is not 'carry' from any number position, so, $S(A + B) = S(A) + S(B)$. Otherwise, since the greatest possible carry is 1, if there is a carry from some number position, then the sum $S(A + B)$ decreases *by 9. Thus,* $S(C) \leq S(A) + S(B)$ *and* $S(A) + S(B) + S(C) = 45$. *Therefore,* $S(C) \leq 45:2 = 22{,}5$. *Exactly two numbers 9 and 18 are smaller than 22,5 and are divisible by 9. The following examples shows that* $S(C)$ *can admit both of these two values:* $473 + 589 = 1062$ *or* $4987 + 26 = 5013$ *(the sum of the digits is equal to 9);* $765 + 324 = 1089$ *(the sum of the digits is equal to 18).*

1.75 Let p is a prime number and let $3p + 10$ is the sum of the squares of six consecutive positive integers. Prove that $36 \mid p - 7$.

Macedonian NMO-2017

Solution: *From the conditions of the problem, we have that*

$$3p + 10 = (n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 =$$

$$= 6n^2 + 6n + 19,$$

so, we have that $3p = 6n^2 + 6n + 9$, *and*

$$p = 2n^2 + 2n + 3 = 2n(n+1) + 3.$$

If one of the numbers n *or* $n + 1$ *is divisible with 3, then, we have a contradiction with the condition that* p *is a prime number. So,* $n = 3k + 1$. *Then:* $p = 2(3k+1)(3k+1+1) + 3 = 2(3k+1)(3k+2) + 3 = 2(9k^2 + 9k + 2) + 3 = 18k(k+1) + 7$. *Since* $k(k+1)$ *is an even number, we have that* $36 \mid p - 7$.

1.76 A local supermarket is responsable for the distribution of 100 supply boxes. Each box is ought to contain 10 kilograms of rice and 30 eggs. It is known that a total of 1000 kilograms of rice and 3000 eggs are in these boxes, but in some of them the amount of either item is more or less than yhe amount required. In each step, supermarket workers can choose two arbitrary boxes and transfer any amount of rice or any number of eggs between them. At least how many steps are required so that, starting from any arbitrary initial condition, after these steps the amount of rice and the number of eggs in all these boxes is equal?

Iran NMO-2015

Solution: *The answer is 99. First consider the initial condition that all of eggs are in just one of boxes. In each step, we can transfer eggs to at most one new box and so we need at least 99 steps. We claim that 99 steps is always enough. For this end, we call a box containing exactly 30 eggs and 10 kilograms of rive a* $good\ box$, *and a box which is not good a* $bad\ box$!

In each step, consider one of bad boxes containing the most number of eggs and one of bad boxes containing the most amount rice. If these two boxes were the same, consider another arbitrary bad box (note that if all other boxes were good, this box also must be good and there is nothing to prove). Evidently, we have at least 30 number of eggs and 10 kilograms of rice in these two boxes. So by transferring eggs and rice between them, we can make one of them a good box. Therefore, after 99 steps we have at least 99 good boxes and so the last box is also good and we are done.

1.77 A positive integer is called $\mathbf{nice}$ if it is equal to the sum of the squares of its three distinct divisors. (A divisor may be equal to 1 or to the number itself).

a) Prove that any nice number is divisible by 3.

b) Are there infinitely many nice numbers?

I.Voronovich-Belarusian NMO-2017

Solution: *Answer: b) there are an infinite number of nice numbers.*

a) Let N *be a nice number, i.e.* $N = d_1^2 + d_2^2 + d_3^2$, *where* d_1, d_2, d_3 *are distinct divisors of* N. *If some divisor of* N *is divisible by 3, then* N *is divisible by 3. So, we suppose that* d_1, d_2, d_3 *are not divisible by 3. Then their squares are congruent to 1 modulo 3, i. e.* $d_i^2 = 3k_i + 1, k_i \in \mathbb{N}, i = 1,2,3$. *Therefore*

$$N = d_1^2 + d_2^2 + d_3^2 = 3(k_1 + k_2 + k_3) + 3$$

and so N *is divisible by 3.*

b) There exists a nice number N. *For example, if* $N = 30$ *and its divisors are*

$$d_1 = 1, d_2 = 2, d_3 = 5, \text{ then}$$
$$N = d_1^2 + d_2^2 + d_3^2 = 1^2 + 2^2 + 5^2 = 1 + 4 + 25 = 30 = N$$

i.e. N *is nice. Consider* $N(p) = Np^2$, *where* p *is some positive integer and* $p > 1$. *If* d_1, d_2, d_3 *are distinct divisors of* N, *then it is obviously that* d_1p, d_2p, d_3p *are distinct divisor of* $N(p)$, *and*

$$N(p) = Np^2 = [N = d_1^2 + d_2^2 + d_3^2] = (d_1^2 + d_2^2 + d_3^2)p^2$$
$$= (d_1p)^2 + (d_2p)^2 + (d_3p)^2.$$

From this equality it follows that $N(p)$ *is nice too. Since any positive integer can be used as* p, *there are infinitely many nice numbers* $N(p)$.

In this way one can get different infinite sets of nice numbers, for example, $N = 30p^2$, *or* $N = 126p^2$, *or* $N = 195p^2$, *or* $N = 170p^2$, *since*

$30 = 1^2 + 2^2 + 5^2; 126 = 3^2 + 6^2 + 9^2; 195 = 1^2 + 5^2 + 13^2; 870 = 2^2 + 5^2 + 29^2$.

1.78 Every one of six pupile attends exactly two of four hobby groups. There are no pupils attending the same two hobby groups. Each hobby group is open every day. During some consecutive days, every one of these six pupils has attend one of her/his hobby group. It has been observed that each of these days each hobby group has been attended by either one or two of the pupils. Moreover, for any chosen two days, there has been a pupil who has attended different hobby groups during these two days. Find the largest number of the days for which the situation described above is possible.

M.Karpuk- NMO-2017

Solution: *Answer: 24 days. Note that from four hobby groups one can from exactly* $4 \cdot \frac{3}{2} = 6$ *different pairs.Since we have exactly six pupils, for each pair of the hobby group there exist exactly one pupil attending just these two hobby groups. By condition, each hobby group is attended by either one or two pupils and there are exactly four hobby group, so in each of these days there are exactly two hobby groups attended by two pupils and there are exactly two hobby groups attended by one pupil.*

Consider arbitrary day. Let this day hobby group A *and* B *be attended by two pupils. Since, as was shown above, there is a pupil attending just these two hobby groups, without loss of generality we can assume that this pupil attended hobby group* A.

Each of two remained hobby groups C and D has been attended by one pupil. Again, there exists a pupil attending just these hobby groups C and D. Suppose that this pupil has attended hobby group C. Show that this information is sufficient to uniquely indicate for each hobby group who of the pupils has attended this hobby group.

Since in the considered day hobby group C has been attended by exactly one pupil (this pupil attends hobby groups C and D), the pupil attending A and C has attended hobby group A, and the pupil attending B and C has attended hobby group B. Further, since hobby group A has been attended by two pupils (one of them attend hobby groups A and B, and the other one attends hobby group A and C), the pupil attending A and D has attended hobby group D. Finally, the pupil attending hobby groups B and D has attended hobby group B, since hobby group D has been attended by exactly one pupil attending hobby group A and D.

We have exactly $4\cdot\frac{3}{2}=6$ *ways to choose the pair* (A,B) *that have been attended by two pupils, and for each such pair we have two ways to fix the hobby group has been attended by the pupil attending just these two hobby groups. For hobby groups C and D we have two ways to choose the hobby group that has been attended by the pupil attending these hobby groups C and D. Thus, we obtain* $6\cdot 2\cdot 2=24$ *different days in total.*

1.79 The central area of a town has a fprm of the $(2n+1)\times m$ rectangle, which is formed by 1×1 tiles. To iluminate the area, one-lamp lampposts are used. The lampposts are placed at the corners of some tiles, including the boundary of the area. The lamp on a lamppost illuminates all tiles with a corner at the lamppost position, and only those.Find the smallest number of the lampposts required to illuminate the whole area, even if one of the lamps should burn out.

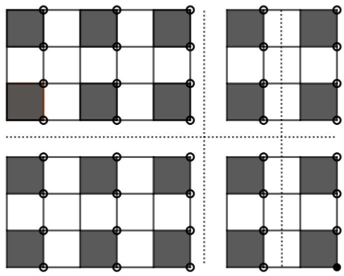
E.Barabanov, M.Karpuk, A.Voidelevich- NMO-2017

Solution: *Answer:* $2(n+2)\left[\frac{m+1}{2}\right]$, *where* $[x]$ *is the greatest integer not exceeding* x. *We paint some tiles of the town square black (if* $m=2k+1$ *is odd, then see Fig.1, if* $m=2k$ *is even, then see Fig.2).*

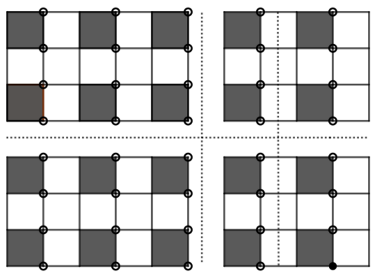
It is easy to see that any lamp can illuminate at most one painted tile. By condition, any tile must be illuminated by at least two lamps. It follows that the *minimum number of the lampposts is greater than or equal to* $2l$, *where* l *is the number of painted tiles. If the lengths of the square is odd, i.e.* $m = 2k + 1$, *where* $k \geq 0$, *then*

$$l = (n+1)(k+1) = (n+1)\left[\frac{m+1}{2}\right]$$

If $m = 2k$, *then* $l = (n+1)k = (n+1)\left[\frac{m+1}{2}\right]$. *On the other hand, if we place the lampposts as it is shown in the figures (the lamppost are indicated as uncoloured circles), then any of the square tiles will be illuminated by two lamps.*



$m = 2k + 1 \quad Fig.1$



$m = 2k \quad Fig.2$

So, $2l = 2(n+1)\left[\frac{m+1}{2}\right]$ *lampposts are sufficient to illuminate the town square so that the problem condition will hold.*

1.80 What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

Saudi Arabia-NMO-2015

Solution*:* *If the chessboard is colored black an white as usual, then any diagonal is a solid color. Sp we may consider bishops on black and white squares separately. In one direction, the lengths of the black diagonals are 2, 4, 6, 8, 6, 4, and 2. Each of these can have at most three bishops, except the first and last diagonals which can have at most two, giving a total of at most 2+3+3+3+3+3+2=19 bishops on black squares. Likewise there can be at most 19 bishops on white squares for a total of at most 38 bishops.*

●	●	●	●	●	●	●	●
●		●	●	●	●		●
●							●
●							●
●							●
●							●
●	●	●	●	●	●	●	●
●	●	●	●	●	●	●	●

Conversely, if we place 38 bishops on the four boundaries of the table and on the second and seventh rows except the second and seventh square of the second row, as shown in the picture , one ca check that this arrangement satisfies the condition of the problem.

1.81 When a factory modernized its equipament, the productivity grew by 25%. Therefore, the management decided to fire 20% of the employees. By how many % has the number of the final products in this factory changed after both actions?

(A) It has decreasd by 5%. **(B) It has decreased by 2,5%.**

(C) It has grown by 2%. **(D) It has not changed.**

(E) It has grown by 5%.

Slovenia NMO-2013

Solution: *Denote the number of products produced by the factory before the changes by x. Then they produced $\frac{120}{100}\cdot x$ products after the modernization of the equipment. After firing the employees the number has dropped to $\frac{80}{100}\cdot\frac{125}{100}\cdot x = x$ products. Hence, the number of products has not changed. The correct answer is D.*

1.82 What is the value of the product $x\cdot y$ if $3^x = a$ and $a^y = 81$?

(a) 4 **(B) 3** **(C) 3** **(D) 0** **(E) 1**

Slovenia NMO-2013

Solution: Since $81 = a^y = (3^x)^y = 3^{xy}$, we have $xy = 4$. The correct answer is A.

1.83 If at the cinema three box qffices are open, the visitors have to wait 15 min to buy a ticket. By how many minutes is the waiting time reduced if two more box offices are open?

(A) 3 (B) 5 (C) 6 (D) 7 (E) 10

Slovenia NMO-2013

Solution: With 3 box offices open the waiting time is 15 minutes. If 1 box office s open the waiting time is 3 times longer, i.e. 45 minutes. If 5 box offices are open the waiting time is reduced by 49/5=9 minutes. So, if two additional box offices are opened the waiting time is reduced by 15-9=6 minutes. The correct answer is C.

1.84 Find all natural numbers n of the form $n = \overline{23ab16c}$, such that all their digits are different and they are divisible by 9 and 11. Here, a, b and c are digits.

Slovenia NMO-2017

Solution: The number n is divisible by 99. We may write

$$n = \overline{23ab1} \cdot 100 + 60 + c = \overline{23ab1} \cdot 99 + \overline{23ab1} + 60 + c =$$

$$= \overline{23ab1} \cdot 99 + \overline{23a} \cdot 100 + 10b + 1 + 60 + c =$$

$$= \left(\overline{23ab1} \cdot 99 + \overline{23}\right) \cdot 99 + \overline{23a} + 10b + c + 61 =$$

$$= \left(\overline{23ab1} \cdot 99 + \overline{23a} + 2\right) \cdot 99 + a + 10b + c + 93,$$

so 99 divides $a + 10b + c + 93$. Since a, b and c are different digits, which are also not equal to 1, 2, 3 or 6, we get $a + 10b + c \geq 4 + 5 + 0 = 9$ and $a + 10b + c \leq 7 + 90 + 8 = 105$. So, $102 \leq a + 10b + c + 93 \leq 198$. On the other hand, $a + 10b + c + 93$ is divisible by 99, so $a + 10b + c + 93 = 198$, which implies $b = 9$ and $\{a, c\} = \{7,8\}$. There are two solutions, $n = 237918$ and $n = 2389167$.

1.85 A herd o deer consists of harts and hinds. Hinds represent 55% of the hert, and their weight is 45% of the total weight of the herd. How many times is the average weight of a hart greater than the average weight of a hind?

Slovenia NMO-2013

Solution: *Assume that the herd consists of x animals with the combined weight of y. Then there are $\frac{55}{100}x$ hinds with the combined weight of $\frac{45}{100}y$ and $\frac{45}{100}x$ harts with the combined weight of $\frac{55}{100}y$. The average weight of a hind is then $\frac{45}{100}y : \frac{55}{100}x = \frac{9y}{11x}$, and the average weight a hart is $\frac{11y}{9x} : \frac{9y}{11x} = \frac{121}{81}$ times larger than the average weight of a hind. The correct answer is C.*

1.86 Find all integer solutions of the equation $m^4 + 2n^2 = 9mn$.

Slovenia NMO-2013

Solution: *If the pair (m, n) is a solution of the equation, then so is the pair $(-m, -n)$. So, we may assume that m is non-negative. Rearrange the equation into $2n^2 - 9mn + m^4 = 0$ and treat it is a quadratic equation in n. Its discriminant is $81m^2 - 8m^4$. If the equation is to have integer solutions, the discriminant has to be a perfect square. So, $81 - 8m^2$ is a perfect square. In particular, $81 - 8m^2 \geq 0$, so $m \leq 3$. It is easy to verify that $81 - 8m^2$ is a perfect square for $m = 0, m = 2$ and $m = 3$. When$m = 0$ the solution is $n = 0$, when $m = 2$ the two solutions are $n = 1$ and $n = 8$, for $m = 3$ the only integer solution is $n = 9$. All the integer solutions of the equation are* $(-3, -9), (-2, -8), (-2, -1), (0,0), (2,1), (2,8)$ *and* $(3,9)$.

1.87 When the third grade pupil Benjamin calculated the sum $1 + 2 + 3 + \cdots + 2012$ he forgot to add some terms, and he got an incorrect sum that was divisible by 2011. When Anika calculated the sum $A = 1 + 2 + 3 + \cdots + 2013$, she forgot to add the same terms as Benjamin, and she got an incorrect sum N that was divisible by 2014. What is the ratio $\frac{N}{A}$ of the two sums?

Slovenia NMO-2013

Solution: *Let us denote the sum of the terms omitted by Benjamin by* x. *Since* $1+2+3+\cdots+2012=\frac{2012\cdot 2013}{2}=1006\cdot 2013$,

Benjamin's result was equal to $1006\cdot 2013-x$. *So, there exists a non-negative integer* m, *such that* $1006\cdot 2013-x=2011m$. *Since*

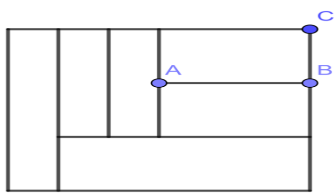
$$A=1+2+3+\cdots+2013=\frac{2013\cdot 2014}{2}=1007\cdot 2013$$

Anika's result was equal to $N=1007\cdot 2013-x$. *So, there exist a non-negative integer* n, *such that* $1007\cdot 2013-x=2014n$. *Expressing* x *from both equalities and comparing the result we get* $1006\cdot 2013-2011m=1007\cdot 2013-2014n$, *or* $2014n-2011m-2013=0$. *This last equality can be rearranged to* $2011(n-m)=2013-3n$. *Since*

$2014n=2013\cdot 1007-x\leq 2013\cdot 1007$, *we get* $n\leq\frac{2013\cdot 1007}{2014}<1007$.

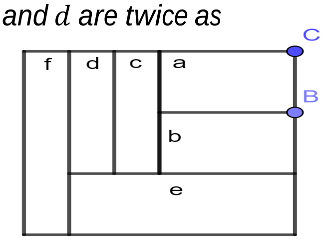
So, $-1008<2013-3n\leq 2013$. *At the same time* $2013-3n$ *is divisible by 3 and the equality implies it is also divisible by 2011. This is only possible for* $2013-3n=0$ *or* $n=671$. *Hence,* $\frac{N}{A}=\frac{2014n}{2013\cdot 1007}=\frac{2014\cdot 671}{2013\cdot 1007}=\frac{2}{3}$.

1.88 A square is divided into six rectangles, all of the same area. The length of side AB equals 5.What is the length of side BC?



Germany NMO-2015

Solution: *The six rectangles have equal areas. Rectangles* c *and* d *are twice as tall as rectangle* a *and therefore also twice as thin.*



Hence they have width $\frac{5}{2}$. *Rectangle* e *thus has a width of*

$\frac{5}{2}+\frac{5}{2}+5=10$ *and must be half as tall as rectangle* a.*This means that rectangle* f *is precisely* $\frac{5}{2}$ *times as tall as rectangle* a *and therefore has a width of* $\frac{5}{5/2}=2$. *It follows that the square has sides of length* $5+\frac{5}{2}+\frac{5}{2}+2=12$. *Because the square has a height of rectangle* a, *the height of rectangle* a *equals* $|BC|=\frac{12}{5/2}=\frac{24}{5}$.

1.89 Carl has a large number of apples and pears. He wants to pick ten pieces of fruit and place them in a row. He wants to do it in such way that there is no pear anywhere between two apples. For example, the fruit sequences AAAAAAAAAA and AAPPPPPPPP are allowed, but AAPPPPPPPA and APPPPPPPAA are not. How many sequences can Carl make?

Germany NMO-2015

Solution: *One sequence consists of pears alone. Next, we count sequences containing at least one apple. In such a sequence, all apples occur consecutively, because there can be no pear anywhere between two apples. If we want to have 8 apples, we can place them in positions 1 through 8, 2 through 9, or 3 through 10. This gives three possible sequences. In this way we find 1 sequence containing 10 apples, 2 sequences containing 9 apples, 3 sequences containing 10 apples, 2 sequences containing 9 apples, 3 sequences containing 8 apples, and so on, ending with 10 sequences containing 1 apple. In total there are* $1+2+3+\cdots+10=55$ *sequences containing at least one apple. The total number of sequences is therefore* $55+1=56$.

1.90 If you were to compute

$$\underbrace{\mathbf{999\ldots99}}_{\mathbf{2014-nines}}\times\underbrace{\mathbf{444\ldots44}}_{\mathbf{2014-fours}}$$

And then add up all digits of the resulting number, what number would the final outcome be?

Germany NMO-2015

Solution: *A good strategy is to first consider smaller examples. We find:*

$$9 \times 4 = 40 - 4 = 36, \qquad 99 \times 44 = 4400 - 44 = 4356$$

$$999 \times 444 = 444000 - 444 = 443556$$

$$9999 \times 4444 = 44440000 - 4444 = 44435556$$

The pattern should be clear. To solve the problem, observe that $999 \dots 99 = 1000 \dots 00 - 1$, *where the first number has 2014 zeroes. The product is therefore equal to*

$$\underbrace{444 \dots 44}_{2014-fours}\underbrace{000 \dots 00}_{2014-fours} - \underbrace{444 \dots 44}_{2014-fours} = \underbrace{444 \dots 44}_{2013-fours}3\underbrace{555 \dots 55}_{2013-fours}6$$

Adding these digits, we obtain $2013 \cdot 4 + 3 + 2013 \cdot 5 + 6 = 2013 \cdot 9 + 9 = 18126$.

1.91 We consider 5×5 –tables containing a number in each of the 25 cells. The same number may occur in different cells, but no row or column contains five equal numbers. Such a table is called pretty if in each row the cell in the middle contains the average of the numbers in that row, and in each column the cell in the middle contains the average of the numbers in that column. The score of a pretty table is the number of cells that contain a number that is smaller than the number in the cell in the very middle of the table. What is the smallerest possible score of a pretty table?

Germany NMO-2015

Solution:

4	4	3	4	0
4	4	3	4	0
3	3	0	3	-9
4	4	3	4	0
0	0	-9	0	-36

We first show that every pretty table has a score of at least 3. Consider such a table and let a be the number at the very middle. The five numbers in the middle row have an average of a and are not all equal to a. Hence at least one of these numbers must be smaller than a. Similarly, at least one of the numbers in the middle column must be smaller than a. Let this number be b. Since b is the average of the numbers in its row, one of *the numbers in that row must be smaller than b, and hence also smaller than b, and hence also smaller than a. Thus the table contains at least three different cells than have a number smaller than the number in the very middle. Its score is therefore at least 3.In the figure on the right you can find a pretty table with a score equal to 3. It follows than 3 is the smallest possible score.*

1.92 Note that $555555:7=79365$. Consider the number $55\dots55$ consisting of 1000 fives. What is the remainder of this number on division by 7?

A) 2 B) 3 C) 4 D) 5 E) 5 E) 6

Germany NMO-2014

Solution: C-4

1.93 A pawn is placed on a board consisting of ten squares, numbered from 1 up to 10. The pawn is allowed to move from the square it is on to square that either has a number that is two less, or a number that is twice as large. The pawn wants to make a sequence of moves that visits as many squares as possible. It may freely choose its starting point. It may visit squares multiple times. How many squares can the pawn visit in a single sequence of moves?

A) 6 B) 7 C) 8 D) 9 E) 10

Germany NMO-2014

Solution: D-9

1.94 Jan has huge square table of which the cells are numberre as in the figure. Which of the following five numbers does not occur in the leftmost column?

A) 55 B) 105 C) 172 D) 212 E) 300

Germany NMO-20172014

Solution: D-212

1.95 Brigit has a combination lock that consists of three rings next to one another, each having the digits 0 up to 9 in order. She turns the three rings until her secret combinations is visible. Aside from this combination, there are 9 more combinations visible on the three rings. Coincidentally, one of these numbers is three times the secret combination. What is Brigit's secret combination?

A) 106 B) 123 C(272 D) 318 E) 328

Germany NMO-2014

Solution: E-328

1.96 A member of a group a ten friends buys a bag of candy to share amoung the group. First he himself, who likes candy more than the rest of the group, takes a quarter of the candy. Another member grabs 30 pieces of candy. A third member grabs 10% of what is left. The remainder of the group distributes the remainder of the candy evently. The total number of pieces of candy was less than 500 and everyone got at least one piece of candy. How many pieces of candy were there in the bag?

Germany NMO-2014

Solution: 320

1.97 There are 36 balls, numbered from 1 up to 36. We want to put these into boxes in such a way that the following two conditions are satisfied:

(1) Every box contains at least 2 balls.

(2) Whenever you pick up two balls from a box, the sum of the two numbers of these balls is always a multiple of 3.

What is the smallest number of boxes for which this is possible?

Germany NMO-2017

Solution: 13

1.98 Tom and Jerry were running a race. The number of runners finishing before Tom was equal to the number of runners finishing after him. The number of runners finishing before Jerry was three times the number of runners finishing after him. In the final ranking, there are precisely 10 runners in finished at the same time.

How many runners participated in the race?

A) 22 B) 23 C) 41 D) 43 E) 45

Germany NMO-2017

Solution: *Let n be the number of runners. The number of runners that finished before Tom equals $\frac{n-1}{2}$ (half of all runners besides Tom).*

The number of runners that finished before Jerry equals $\frac{3(n-1)}{4}$. Since exactly 10 runners finished before Jerry, it follows that $\frac{3(n-1)}{4} - \frac{n-1}{2}$ equals 11 (Tom and the runners between Tom and Jerry).

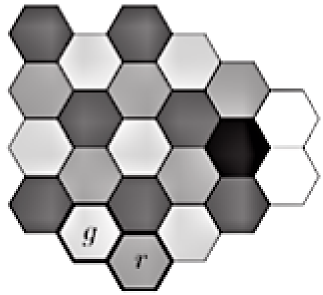
We find that $\frac{1}{4}(n-1) = 11$, hence $n = 4 \times 11 + 1 = 45$. There were 45 runners participating in the race.

1.99 A garden with a pond (the black hexagon) will be tiled using hexagonal tiles as in the figure. The tiles come in three colours: red, green and blue. No two tiles that share a side can be of the same colour. In How many ways can the garden be tiled?

A) 3 B) 6 C) 12 D) 18

Germany NMO-2017

Solution:



We start by colouring the two indicated tiles at the bottom. This can be done in six ways: there are three options for the first tile and for each option there are two possible colours for the second tile. In the figure, the colours red (r) and green (g) are chosen.

Now that these two tiles are coloured, the colours of most of the other tiles are determined as well. The tile above the red tile can only be blue. The tile above the green tile must be red and therefore the tile left of the green tile must be blue. In this way the colours of all tiles, except the two on the right (white in the figure) are fixed. For these last two tiles, there are three possible colourings. The upper and lower tile can be coloured either green and red, or blue and red, or blue and green.

Since each of the six allowed colourings of the first two tiles can be extended in three ways to a complete colouring, we find a total of $6 \times 3 = 18$ *possible colourings.*

1.100 A motorboat is moving with a speed 25 kilometres per hour, relative to the water. It is going from Arnhem to Zwolle, moving with the constant current. At a certain moment, it has travelled 42% of the total distance. From that point on, it takes the same amount of time to reach Zwolle as it would to travel back to Arnhem.What is the speed of the current (in kilometers per hour)?

A) 3 B) 4 C) $\frac{9}{2}$ D) 5 E) 6

Germany NMO-2015

Solution*: Answer B) 4. From the mentioned point, it takes the same time to go 42% of the distance upstream and to go 58% of the distance downstream. This means that the boat is $\frac{58}{42}$ times as fast going downstream as going upstream.*

If the water flows at a speed of v *kilometres per hour, then we find* $\frac{25+v}{25-v} = \frac{58}{42}$.
Hence $58 \cdot (25 - v) = 42 \cdot (25 + v)$, *or*

$1450 - 58v = 1050 + 42v$. *We find* $400 = 100v$, *hence* $v = 4$.

1.101 A piece of apple pie had been stolen, and five children are behind questioned on this. They all know who the culprit is, but not all of them are speaking the truth. Whenever one of the children lies, the next one will feel so guilty about this that he or she will tell the truth. The children make the following claims in the order shown:

- **Asim: ''Coen and I both didn't do it."**
- **Bob: "Either Coen or Dilan is the culprit"**
- **Coen:"Eva and I both didn't do it."**
- **Dilan: "Asim is the culprit."**
- **Eva:"At least two of Asim,Bob,Coen, and Dilan lied."**
  Who stole the apple pie?
  A) Asim B) Bob C) Coen D) Dilan E) Eva

Germany NMO-2014

Solution: *B) Bob*

1.102 Max has a lot white and red paint. He starts with a 2-litre bucket in which one litre of red paint and one litre of white paint. Max now repeats the following step a number of times.

Step. Max pours precisely one litre out of the bucket, into a large container. Next, he fills the

bucket back up to 2 litres of paint, using either the white paint, or the red paint. After

this, he mixes the paint in the bucket.

After a number of steps, the percentage of red paint in the bucket must be between 83 and 84 percent. What is smallest number of steps Max needs to attain this?

A) 5 B) 6 C) 7 D) 8 E) Max cannot obtain such a percentage

NMO-2017

Solution: *Answer: B) 6*

1.103 The product of three numbers $\overline{abc} \cdot \overline{ab} \cdot a = 3****7$ is a 6-digit number with first and last digits equal to 3 and 7, respectively. The digits a, b, c are not necessarily distinct. Find all possible values of the product.

Ukrainian NMO-2017

Solution: *Since the product is odd and does not end with 5, all of the digits a, b, c are odd and are not equal to 5. From the inequalities* $911 \cdot 91 \cdot 9 = 746109 > 399997$ *and* $399 \cdot 39 \cdot 3 = 56683 < 300007$ *it is clear that* $a = 7$. *The product* $abc = 7bc$ *ends with 7, so the possible cases for* (b, c) *are:*

$$(9;9), (7;3), (3;7), (1;1).$$

Let us check these four cases: $799 \cdot 79 \cdot 7 = 441847, 773 \cdot 77 \cdot 7 = 416647, 737 \cdot 73 \cdot 7 = 37660$ *and* $711 \cdot 71 \cdot 7 = 353367$.

1.104 Oleg has labelled all the columns and all the rows of a 50×50 table with 100 distinct numbers $a_1, \ldots, a_{50}$ and $b_1, \ldots, b_{50}$, respectively; exactly 50 of these numbers are rational. Then he has placed into each cell (i, j) the number $a_i + b_j$. Find the greatest possible number of rational numbers placed into the cells.

O.Podlipsky-Regional MO-Russia-2017

Solution: *Answer. 1250. Assume there are x rational numbers among the a_i. Then the total number of irrationals in the cells is at least* $x \cdot x + (50 - x) \cdot (50 - x) \geq 1250$ *(since rational + irrational=irrational).*

In an example, $x = 25$, *all irrationals among the a_i are in $\mathbb{Q} + \sqrt{2}$, and those among the b_i are in $\mathbb{Q} - \sqrt{2}$.*

1.105 We want to exchange a 200-euro bill for bills of 5, 10, and 20 euros. One possibility is to exchange it for 5 bills of 20 euros, 6 bills of 10 euros, and 8 bills of 5 euros. Another possibility is to exchange it for 20 bills of 10 euros.How many possibilities are there to exchange a 200-euro bill for bills of 5, 10, and 20 euros?

Germany NMO-2014

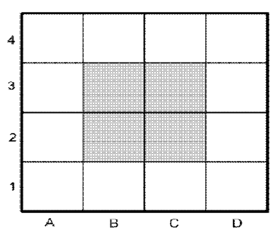
Solution: 121

1.106 By stacking small cubes (all of the same size) neatly, a larger cube is formed. Two small cubes with faces placed against one other are called neighbours. So a cube can have at most six neighbours. The number of cubes having precisely four neighbours. The number of cubes having precisely four neighbours is 96.How many small cubes are there having precisely five neighbours:

Germany NMO-2017

Solution: 384

1.107 The figure below represents a puzzle. The goal is to fill each of the 16 cells with a number from 1 up to 4. This has to be done in such a way that in each column and in each row, the four numbers are distinct. Moreover, in each of the four 2×2-squares, the four numbers have to be distinct as well. Finally, the four numbers in the grey squares also need to be distinct .How many solutions does this puzzle have?



Germany NMO-2014

Solution: 168

1.108 A grasshopper is sitting in the origin of the number line, at number 0, and then it jumps, always in the same direction. For a positive integer k, in the first jump the grasshopper jumps to number 1, and every following jump is exactly k times longer than the previous jump. There is a hole in place of all multiples of number 2015.Determine all positive integers k such that the grasshopper can jump 2015 times without falling into a hole.

Croatian NMO-2015

Solution: *Let a_n be the number the grasshopper is located at after the $n-th$ jump, i.e.* $a_1 = 1, a_n = 1 + k + \cdots + k^{n-1}, n \geq 2$.

We are looking for all numbers k such that $2015 \nmid a_n$ *for all* $n = 1,2,\ldots,2015$.

Suppose that $M(k, 2015) = d > 1$. *Then every a_n divided by d gives the reminder 1, and since* 2015 *is divisible by d we have that* $2015 \nmid a_n$ *for all n. Therefore, all positive integers which are not relatively prime to 2015 comply with the terms of the problem.*

If $M(k, 2015) = 1$, *we observe the remainders of dividing $a_1, \ldots, a_{2015}$ by 2015. If one of them is divisible by 2015, such a k is not good. Otherwise, since there are 2014 possible remainders, at least two numbers give the same remainder. Let these numbers be a_l and $a_m, m > l$. In this case, their difference is divisible by 2015. On the other hand, we have that*

$$a_m - a_l = k^l + \cdots + k^m = k^l\left(1 + \cdots + k^{m-l-1}\right) = k^l \cdot a_{m-l}.$$

From $2015 \mid k^l \cdot a_{m-l}$ *and* $M(k, 2015) = 1$, *it follows that* $2015 \mid a_{m-l}$, *which is in contradiction with the assumption that none of the numbers $a_1, \ldots, a_{2015}$ is divisible by 2015. Therefore, if* $M(k, 2015) = 1$, *the grasshopper will jump into a hole.*

To conclude, the only numbers which are suitable for the terms of the problem are those which are not relatively prime to 2015.

1.109 Pasha chose 2017 (not necessarily distinct) positive integers $a_1, a_2, \dots, a_{2017}$, and then he plays a solitare game. Initially, he has 2017 ampty large boxes and an unbounded supply of small stones. By a move, Pasha adds a_1 stones into some box by his choice, a_2 stones into any other box by his choice,..., a_{2017} stones into the remaining box. His aim is to equalize the numbers of stones in all boxes. Can he choose the initial numbers so that the aim is reachable in 43 moves, but unreachable in any smaller (nonzero) number of moves?

I.Bogdanov-Regional MO Russia-2017

Solution: *Answer. Yes. Notices that* $2017 = 43 \cdot 46 + 39$. *An example of Pasha's numbers consist of 39 twos, 46 numbers equal to 44, and ones as the remaining numbers.*

1.110 Initially $n > 1$ positive integers are written on the board. On each minute, a new number that is the sum of squares of all already written numbers appears on the board. (For example, if initial numbers were 1,2,2 then on the first minute the numbe $1^2 + 2^2 + 2^2$ appears). Prove that the 100-th new number has at least 100 different prime divisors.

I.Bogdanov, P,Kozhevnikov-Regional MO-2017

Solution: *Let S_i be the number appearing on the board on the i –th minute.*

Then $S_{i+1} = S_i(S_i + 1)$, so S_{i+1} contains all prime divisors of S_i plus at least one more.

1.111 Let p prime and m be a positive integer. Determine all pairs (p, m) satisfyng the equation: $p(p + m) + p = (m + 1)^3$.

A.Fellouris-Hellenic NMO-2014

Solution: *The given equation is written* $p(p + m + 1) = (m + 1)^3$; (1)

Therefore the prime p is a divisor of $(m + 1)^3$. Hence $p \mid (m + 1)$, which means that there exists positive integer k such that $m + 1 = kp$. Then, from (1) *we get:*

$$p(p+kp)=(kp)^3 \Leftrightarrow k^3 \mid (k+1) \Rightarrow k^3 \mid (k+1) \Rightarrow k=1.$$

Hence $p=2, m=1$ *and* $(p,m)=(2,1)$.

1.112 We color the numbers $1,2,3,\dots,20$ with two colors white and black in such a way that both colors are used. Find the number of ways we can perform this coloring if the product of white numbers and the product of black numbers have maximal common divisor equal to 1.

P.Bregiannis-Hellenic NMO-2014

Solution: Number 1 can be colored in two ways, white or blank. Number 2 also can be colored white or blank. Then all even numbers 2,4,6,8,10,12,14,16,18,20 have to be colored with the color of 2.

Also all numbers having common divisor greater than 1 with the above numbers must be colored with the color of 2. The remaining numbers greater that 10, that is , 11,13,17,19 can be colored in two ways. Therefore the coloring can b made with $2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2=2^6=64$ different ways. However we must delete the two cases we color all numbers white or blank. So we have finally 62 different ways.

1.113 Find all values of integer n for which the number $A=\frac{8n-25}{n+5}$ is equal with the cube of a rational number.

A.Fellouris-Hellenic NMO-2014

Solution: *Let* $p,q\in\mathbb{Z}, q\neq 0$, *with* $(p,q)=1$ *such that*

$$A=\frac{8n-25}{n+5}=\left(\frac{p}{q}\right)^3; \quad (1)$$

Then $(p^3,q^3)=1$, *which from relation (1) we get:*

$$q^3(8n-25)=p^3(n+5); \quad (2)$$

from which we conclude that

$p^3 \mid (8n - 25)$ *and* $q^3 \mid (n+5) \Rightarrow \exists k \in \mathbb{Z}: 8n - 25 = kp^3$ *and* $n + 5 = kq^3$; (3)

In fact, if $8n - 25 = kp^3, k \in \mathbb{Z}$, *then from (2) we find* $n + 5 = kq^3$ *and so*

$$8(n+5) - (8n-25) = k(8p^3 - q^3) \Rightarrow k(2q-p)(4q^2 + 4qp + p^2) = 65$$

Therefore, the numbers $k, 2q - p$ *and* $4q^2 + 2pq + p^2$ *are divisors of 65.*

We observe that $4q^2 + 2pq + p^2 = 3q^2 + (p+q)^2 \equiv 1 \ (mod\ 3)$. *Moreover, we have*

$4q^2 + 2pq + p^2 = 3q^2 + (p+q)^2 \geq 3$, *and so, since* $4q^2 + 2pq + p^2$ *is divisor of 65 its unique value is*

$$4q^2 + 2pq + p^2 = 13 \Leftrightarrow 3q^2 + (p+q)^2 = 13,$$

leading to the following cases:

$4q^2 + 2pq + p^2 = 13, k = \pm 1.2q - p = \pm 5$. *Then* $p = 2q \mp 5$ *abd*

$$4q^2 + 2q(2q \mp 5) + (2q \mp 5)^2 = 13 \Leftrightarrow p = 2q \mp 5, 2q^2 \mp 5q + 2 = 0 \Leftrightarrow$$

$$p = 2q \mp 5, q = \pm 2 \Leftrightarrow p = \mp 1, q = \pm 2.$$

Then for both cases we have:

$$\left(\frac{p}{q}\right)^3 = -\frac{1}{8} \Rightarrow \frac{8n-25}{n+5} = -\frac{1}{8} \Leftrightarrow 8(8n-25) - (n+5) \Leftrightarrow n = 3.$$

1.114 Let us call a year colored if the decimal representation of its number has no repeating digits. For example, all years from 2013 to 2019 are colored, unlike 2020.

a) Find the nearest chain of seven consecutive colored years in the future.

b) Can a chain of more than seven consecutive years happen in the future?

Maria Rozhkova-Ukraine NMO-2015

Solution: *Answer: a) 2103,....,2109; b) No.*

a)Let us show hat nearest sequence of 7 colored years is 2103,...,2109. First, we prove that in this century no sequence of more than six colored years can happen any longer. We see that digits 0 and 2 cannot represent unis or tens. Therefore, a chain is broken at each year ending in 0 or 2, which gives us the only way to form a chain of 7 years:

$$20*3,20*4,\ldots,20*9.$$

Here, $*$ can only be equal to 1, but his is he current chain.

In the next century, after 2100, the first chain is easy to find: 2103,...,2109.

b)Years we deal with are written with at least four digits. A colored chain cannot contain numbers ending in $**99$, hence the number of hundreds doesn't change throughout the chain. This implies that there are only 8 possible last digits.

Assume some colored chain has 8 numbers. We have two cases.

1. All years have the same number of tens. Then there are only 7 options for the last digit. Contradiction.

2. The number of tens changes with in the chain. In this case the tens digit cannot be equal to 9. Assume this digits changes from x to $x+1$. The chain has 8 numbers, thus, i should have two numbers of the type

$$\overline{abxx+1}, \overline{abx+1x},$$

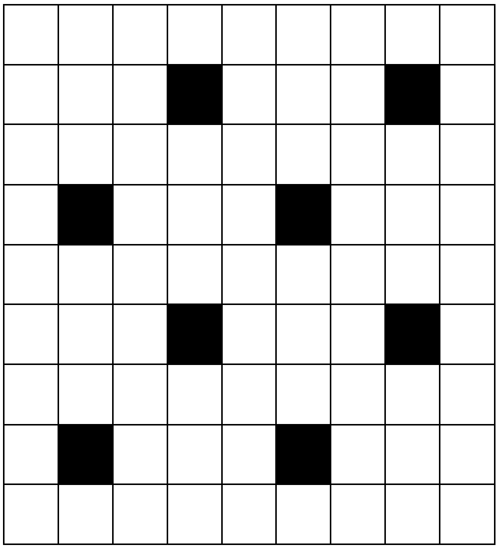
For example, they might be 2145 and 2154. But then the chain has at least 10 years (for example, 2145 to 2154). Again, we have a contradiction.

1.115 A 9×9 square is divided into 81 small $1\times\times 1$ squares, 8 of which are painted black, the rest being white. We cut a fully white rectangle (possibly, a square) out of the big 9×9 square. What is the maximal area of the rectangle that we can attain regardless of the positions of the black squares? It is only allowed to cut the rectangle along the grid lines.

Bogdan Rublyov-Ukraine NMO-2015

Solution: *Answer: 9. Cut the square into smaller* 3×3 *block's. Since there are only eight black squares, at least one of the block's doesn't contain any of them. Therefore a white square of area 9 can always be found.*

Next, we show that sometimes it is impossible to find a larger rectangle. Fig.2 is an example. Here, one can cut out either a 3×3 *square or a* 1×9 *rectangle, but not any rectangle of larger area.*



1.116 2015 candies are placed along a circle and numbered 1 to 2015 clockwise. Andriy and Olesia play the following game. In each turn, a player can take either 2 or 3 candies with consecutive numbers (1 and 2015 are also considered ''consecutive''). The player who can't make a move loses. Who has a winning strategy if Andriy plays first?

Ukrainian NMO-2015

Solution: *Answer: Olesia. Andriy takes 2 (or 3) consecutive candies. Then Olesia takes 3 (or 2) diametrically opposite candies, so that there are 1005 candies on both sides between the groups of taken candies. Then she just copies Andriy's moves on the other part. If Andriy can make a move, she can as well, so she will not lose anyway. Since the game is finite, in the end she will win.*

1.117 Three cyclists start off at the same time and ride along the sides of a triangle ABC along the route $AB \rightarrow BC \rightarrow CA$. Their speeds on each of the segments AB, BC, CA are known: he first cyclist has speeds 12, 10 and 20 mph respectively on the three sides, the second one rides 15,15 and 10 mph, the third one rides 10,20 and 12 mph, the third one rides 10, 20 and 12 mph respectively. What can be the angled measure of $\angle ABC$, if all three cyclist arrived back at the point A simultaneously?

NMO-2017

Solution: *Answer:* 60°. *Denote the sides of the triangle by* $AB = x, BC = y, CA = z$. *Then the following equality must holds:*

$$\frac{x}{12} + \frac{y}{10} + \frac{z}{20} = \frac{x}{15} + \frac{y}{15} + \frac{z}{10} = \frac{x}{10} + \frac{y}{20} + \frac{z}{12}$$

Or

$$5x + 6y + 3z = 4x + 4y + 6z = 6x + 3y + 5z$$

Hence, $x + 2y - 3z = 0$ *and* $2x - y - z = 0$, *which implies* $x = y$ *and* $z = y$.

Therefore, ΔABC *is equilateral and all its angles are equal to* 60°.

1.118 In January, Petro used to by from one to three toy cars every day. On February 1, he tried to make a rectangle of all his cars. When he arranged them into rows of 7 cars, one car remained. When he arranged the cars into rows of 10, there were 2 excessive cars. Can Petro arrange them into rows of 4 cars?

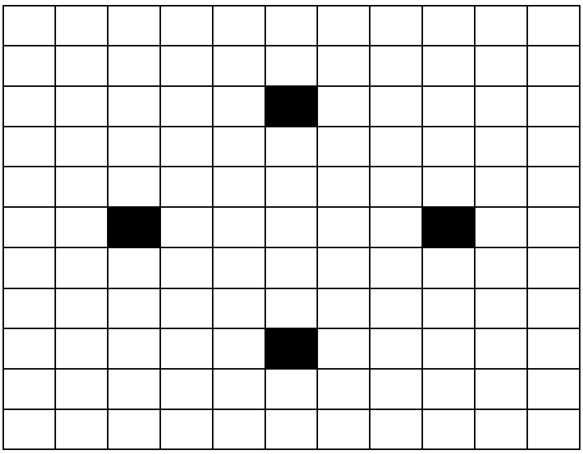
Ukrainian NMO-2015

Solution: Answer: yes. For some positive integers n, k we have: $7k + 1 = 10n = 2$ or $7k = 10n + 1$. Therefore, we need to find a number between 29 and 92 that ends with 1 and is divisible by 7. The minimal such number is 21, which is less than 30. The next one is 91. It's easy to check that no other number with these properties exists in the given interval. Therefore, Petro has $7k + 1 = 91 + 1 = 92$ cars.Since $92 = 4 \cdot 23$, he can arrange them into 4 columns.

1.119 An 11×11 square is partitioned into 121 smaller 1×1 square, 4 of which are painted black, the rest being white. We cut a fully white rectangle (possibly, a square) out of the big 11×11 square. What is the maximal area of the rectangle that we can attain regardless of the positions of the black squares? It is only allowed to cut the rectangle along the grid lines.

Bogdan Rublyov-Ukrainian NMO-2015

Solution: *Answer: 25. First paint four squares black as in Fig.5.hen it is easy to verify that a* 5×5 *square of area 25 can be cut out, and his area is maximal possible.*



Now we shall show that this area is maximal possible in the general case. Assume that there is a placement of four squares for which the answer doesn't exceed 24. It means that whichever rectangle of area 25 we can choose, it will always contain a black square. For convenience, we can number the fields of the board (Fig.5). Denote rows by numbers 1 to 11 from bottom to top, and columns by English letters $a, \dots, k$. *We shall call a unit square gray if it cannot be black in any case.*

1.120 Winnie-the Pooh and Piglet play the following game. There is a 15-inch-long stick. By his first move, Piglet breaks it into two pieces, then the players in turn break one of the existing pieces into two. The rules are that the resulting pieces must have integer length (in inces)

and can't be 1-inch-long. The player who can't make a move loses. Who has a winning strategy?

Maksym Chornyi-Ukrainian NMO-2015

Solution: *Answer: Piglet. Obviously, at the end of the game all remaining pieces will have length either 2 or 3 inches, and Piglet will win if their number is even (which means that the total number of moves was odd, thus Piglet made the last move). There are three possible outcomes:*

$$15 = 3+3+3+3+3 = 3+3+3+2+2+2$$
$$= 3+2+2+2+2+2+2.$$

This implies that in order to win Piglet needs to ensure the existence of two 3-inch and one 2-inch pieces (this will make the first and the third outcomes impossible). So, by his move he must break the stick into 5 and 10 inches. The first piece will be eventually divided into 2 and 3 inches. If Winnie breaks the 10-inch stick by his next move, Piglet must break 3 inches off the longer part, otherwise he should just break it into 3-inch pieces and one 2-inch piece, which is enough for Piglet to win.

1.121 There are 9 weights with labels 1 g, 2 g,...,9 g respectively. It's known the one of the weights is lighter than the label says, while the other eight labels are correct. Is it possible to detect the counterfeit weight using scales with no additional weights no more than twice?

Ukrainian NMO-2015

Solution*:* *Answer: yes. At the first move, put* $1+4+9$ *on the left side and* $2+5+7$ *on the right side. If the total weights are equal, the counterfeit weight is among the other three. Then put on scales the combinations* $3+4$ *and* $1+6$.

- *If* $3+4=1+6$, *they are genuine, hence, 8 is counterfeit.*
- *If* $3+4<1+6$, *then 3 if counterfeit, because it is lighter and 4 is known to be genuine.*
- *Similarly,* $3+4>1+6$ *would imply that 6 is counterfeit.*

If the total weights at the first weighing are no equal, the counterfeit weight is among the three lighter ones. The second move involves putting one potentially counterfeit weight on each side and balancing them with genuine *ones. For example,* 1 + 5 *and* 4 + 2*. Then we use the same arguments: the counterfeit one is where the total weight is less. If the scales are balanced, the unused weight is counterfeit.*

1.122 From the set of numbers $\{1; 2; \dots; 2015\}$ chose the maximum possible amount of numbers such hat the sum of any five selected numbers will be divisible by 15.

Ukrainian NMO-2015

Solution: *Answer:* $\{3; 18; 33; \dots; 2013\}$*, that contains 135 numbers.*

Let this set contain no less than 6 numbers: a, b, c, d, e, f*. Then* $a + b + c + d + e \vdots 15$ *and* $a + b + c + d + f \vdots 15$*. Hence* $e - f \vdots 15$ *or* $e \equiv f \ (mod\ 15)$*. As numbers* e, f *are arbitrary from this set, all of them have to be equal modulo 15. Let every number from this set be* k *modulo 15, then* $a + b + c + d + e \equiv 5k \equiv 0 \ (mod\ 15)$*. Consequently, all numbers have to be divisible by 15, or have the remainder 3. In the first case the set contains every fifteenth number, beginning with 15. There are* $\left[\frac{2015}{15}\right] = 134$ *numbers, the last is 2010.*

In the second case the set contains every fifteenth number, but beginning with 3, the last number is 2013, so the amount is $\left[\frac{2015}{15}\right] + 1 = 135$ *numbers, namely* $\{3; 18; \dots; 2013\}$.

1.123 Prove that number $m^4 + 1$ has no divisors in the interval $[m^2 - 2m, m^2 + 2m]$ for every natural $m > 2$.

Serdiuk Nazar-Ukrainian NMO-2015

Solution: *Suppose the opposite. Let* $m^2 + a \in [m^2 - 2m, m^2 + 2m]$ *i.e.* $m^4 + 1 \vdots m^2 + a$*. Since:*

$(m^4 + 1, m^2 + 1) = (-am^2 + 1, m^2 + a) = (a^2 + 1, m^2 + a)$*, then*

$$a^2 + 1 \vdots m^2 + a, a^2 + 1 = (m^2 + a)r.$$

Let $a \in [-2m+3,0]$, *then* $\frac{m^2+1}{m^2+a} < \frac{m^4+1}{m^2-2m+3} < m^2+2m+1$, *so* m^4+1 *has a divisor that is no less than* m^2 *and fulfils the assumption, therefore instead of* m^2+a *we can examine a divisor* $m^2+b=\frac{m^4+1}{m^2+b}$, *where* $b \geq 0$.

If $a=-2m+2$, *then* $4m^2-8m+5 \vdots m^2-2m+2$ *so* $3 \vdots m^2-2m+2$ *which is impossible in case of* $m>1$.

If $a=-2m+1$, *then* $4m^2-4m+2 \vdots (m-1)^2 \vdots m-1$, *thus* $2 \vdots m-1, m \in \{2,3\}$. *In case of* $m=2$ *and* $m=3$: $m^4+1=17$ *and* $m^4+1=2\cdot 41$ – *none of these numbers fulfils the condition.*

If $a=-2m$, *then* $1 \vdots m$ *is a contradiction.*

Hus, if number m^4+1 *has a divisor* m^2+a, *where* $a \in [-2m,2m]$, *then the number* m^4+1 *has a divisor* m^2+b, *where* $b \in [0,2m]$. *It is also clear that* $b \neq 0$. *In such case,*

$4m^2+1 \geq b^2+1=(m^2+b)r \geq m^2r$, *so* $r \leq 3$.

Case $r=3$: $b^2+1=3(m^2+b)$ –*it is impossible, because* b^2+1 *is not divisible by 3.*

Case $r=2$: $b^2+1=2(m^2+b), (b-1)^2=2m^2$ –*impossible.*

Case $r=1$: $b^2-b+1=m^2$, *but* $b^2>b^2-b+1=m^2>(b-1)^2$, *so his case is also impossible. Thus, we have the contradiction with the supposition, so the statement is proved.*

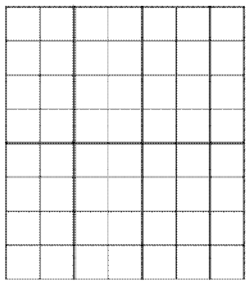
1.124 We color each unit square of a 8×8 table into green or blue such that there are a green unit squares in each 3×3 square and b green unit squares in each 2×4 rectangle. Find all possible values of $(\boldsymbol{a},\boldsymbol{b})$.

Le Anh Vinh-Saudi Arabia NMO-2015

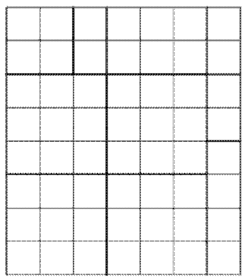
Solution: *By tiling our* 8×8 *table by eight* 2×4 *rectangles like in Tiling (1) we find that the total number of green unit squares in the table is* $8b$.

By tiling our 8×8 *able by four* 3×3 *squares, three* 2×4 *rectangles and one* 2×2 *square, like in Tiling (2), we find that the total number of green unit squares in the total table is* $4a+3b+x$, *where* x *is the number of green unit* *square in the* 2×2 *square in the left upper corner. Notice that by just rotating our tiling (2) so that the* 2×2 *square occupies at each time the four corners of the table, we deduce that the four* 2×2 *unit squares in the four corners, all have the same number of green unit squares* x.

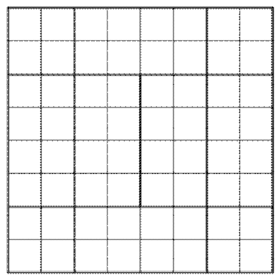
Finally, by tilling our 8×8 *table by six* 2×4 *rectangles and four* 2×2 *squares, like in Tiling (3), we find that the total number of green unit squares in the table is* $6b + 4x$



Tiling(1)



Tiling(2)



Tiling(3)

Putting together all these totals, we obtain the relations $8b = 4a + 3b + x = 6b + 4x$, *from which we deduce that* $b = 2x$ *and* $9x = 4a$. *Since* $0 \leq a \leq 9$, *either* $a = 0$ *and* $b = 0$ *or* $a = 9$ *and* $b = 8$. *In other words, either we color the whole table or we don't color at all.*

1.125 Find the number of 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ of distinct positive integers satisfying the following two conditions:

(a) $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 30$;

(b) We can write $a_1, a_2, a_3, a_4, a_5, a_6$ on sides of a hexagon such that after a finite number of time choosing a vertex of the hexagon and adding 1 to the two numbers written an two sides adjacent to the vertex, we obtain a hexagon with equal numbers on its sides.

Le Anh Vinh-Saudi Arabia NMO-2017

Solution: *We label the vertex of the hexagon by 1,2,3,4,5,6 and suppose that six numbers are written in the order* a, b, c, d, e, f *on the edges* $(1,2), (2,3), (3,4), \ldots, (6,1)$, *respectively.*

We first notice that by choosing a vertex of the hexagon and adding 1 to the two numbers written on two adjacent sides to the vertex, the difference $(a+c+e)-(b+d+f)$ *is an invariant. Thus, if we want to obtain a hexagon with equal numbers on its sides, we must have* $a+c+e=b+d+f$ *at the beginning.*

We will show that it is sufficient. Let N *be a big integer, for example, we can take* $N>30$.

We choose the vertices by $N-f, N-b, N-c$ *and* $N-e$ *times, respectively. Then, we obtain a hexagon with six numbers* $a+2N-b-f, N, N, d+2N-e-c, N, N$.

Since $a+c+e=b+d+f$, *we have* $a+2N-b-f=d+2N-e-c=M>N$ *(since* $N>30=a+b+c+d+e+f$*). We now can choose the vertices 3,6 for* $M-N$ *times each then we obtain a hexagon with numbers* M *on all of its sides.*

Now, we count how many 6-tuples (a,b,c,d,e,f) *of distinct positive integers with* $a+c+e=b+d+f=15$.

We list all triples of distinct positive integers with sum 15:
(1,2,12), (1,3,11), (1,4,10), (1,5,9), (1,6,8), (2,3,10), (2,4,9), (2,5,8), (2,6,7), (3,4,8), (3,5,7), (4,5,6)

We then check that there are exactly 19 times we can pair these triples to obtain 6 distinct numbers. Since we can permute the numbers on each tuple, the total number of 6-tuples satisfying the given conditions $19\times 6!$.

1.126 Given 2015 subsets $A_1, A_2, \ldots, A_{2015}$ of the set $\{1, 2, , \ldots, 1000\}$ such that $|A_i|\geq 2$ for every $i\geq 1$ and $|A_i\cap A_j|\geq 1$ for every $1\leq i<j\leq 2015$. Prove that $k=3$ is the smallest number of colors such that we can always color the elements of the set $\{1, 2, \ldots, 1000\}$ by k colors with the property that the subset A_i has at least two elements of different colors for every $i\geq 1$.

Le Anh Vinh-Saudi Arabia NMO-2015

Solution: *Consider the collection:*
$A_1=\{1,2\}, A_2=\{2,3\}, A_a=\{1,3\}, \ldots, A_{2015}$ *be any 2011 subsets of* $\{1,2,\ldots,1000\}$ *that contain all 1,2,3.*

One can check that this collection satisfies the given condition. For any 2-coloring of the set $\{1,2,\dots,1000\}$*, at least one of* A_1, A_2, A_3 *will contain two elements of the same color. Hence, we need more than 2 colors. Now we show that we always can color* $\{1,2,\dots,1000\}$ *by 3 colors such that the set* A_i *contains at least two elements of different color for all* i*. We choose a set* A_{i_0} *with least number of elements. We color one element of* A_{i_0} *by red and all the rest by blue. We color by green all the elements* $\{1,2,\dots,1000\}$ *which do not belong to* A_{i_0}*. It is clear that* A_{i_0} *contains two elements of different color. For any subset* A_i *with* $i \neq i_0$, $|A_i \cap A_1| \geq 1$ *so* A_i *contains at least one red or blue element. Moreover, because* $|A_i| \geq |A_{i_0}|$*, either* A_i *contains another element of* A_{i_0} *with the other color or contains a green element. Hence, the subset* A_i *contains two elements of different color.*

1.127 Prove that there exist infinitely many non prime positive integers n such that $7^{n-1} - 3^{n-1}$is divisible by n.

Le Anh Vinh-Saudi Arabian TST-2015

Solution: *We will look for integers of the form* $n = 7^a - 3^a$ *with* a *dividing* $n-1$*. Clearly, if* a *exist then* $n = 7^a - 3^a$ *divides* $7^{n-1} - 3^{n-1}$*.*

Let $a = 3^r$ *for* $n \geq 1$*. We have* $n = 7^a - 3^a \equiv (-1)^{3^r} - 3^{3^r} \equiv 4 \ (mod\ 8)$.

We deduce that n *is a composite number. On the other hand, because 3 divides 7-1, we have from the Lifting The Exponent that* $v_3(7^a - 1) = v_3(7-1) + v_a(a) = r + 1$*. We deduce that* $a = 3^r$ *divides* $7^{3^r} - 1$ *and therefore it divides* $n - 1 = 7^{3^r} - 1 + 3^{3^r}$*. But there are infinitely many such* a*. This solves the problem.*

1.128 We have 10 balls in a bowl, some of them are blue, some of them are yellow and the others are green. They can be put in a line in 360 different ways. At most how many blue balls are there in the bowl?

Slovenia NMO-2013

Solution: *Denote the numbers of blue, yellow and green balls by* b, y *and* g*, then* $b + y + g = 10$.

The balls can be put in a line in $\frac{10!}{b!y!g!}$ *different ways. So,* $\frac{10!}{b!y!g!} = 360$.

This equality can be rewritten as $10 \cdot 9 \cdot \ldots \cdot (b+1) = 360 \cdot y! \cdot g!$.

This implies $10 \cdot 9 \cdot \ldots \cdot (b+1) \geq 360 = 10 \cdot 9 \cdot 4$, *so* $b+1 \leq 8$, *or* $b \leq 7$. *It, for example, we have* $b=7, y=2$ *and* $g=1$, *then* $\frac{10!}{b!y!g!} = \frac{10!}{7!2!1!} = \frac{10 \cdot 9 \cdot 8}{2} = 360$. *There can be at most* 7 *blue balls in the bowl. The correct answer is* D.

1.129 Find all quadruples of non-zero digits, a, b, c and d such that

$$\overline{ab20} - \overline{13cd} = \overline{cdab}.$$

Slovenia NMO-2013

Solution: *Rewrite the equation as* $\overline{cdab} + \overline{13cd} = \overline{ab20}$. *Since* b *and* d *are non-zero digits the equation for the ones is* $b + d = 10$. *Substracting this on both sides of the equation we get* $\overline{cda0} + \overline{13c0} = \overline{ab10}$. *Dividing the equation by 10 we obtain* $\overline{cda} + \overline{13c} = \overline{ab1}$. *Since* a *and* c *are non-zero digits, their sun cannot be equal to 1, so the equation for the ones yields* $a + c = 11$. *Once more we substract this expression from both sides of the equation and divide by 10. We get* $\overline{cd} + \overline{13} = \overline{a(b-1)}$ *since* $b - 1 \geq 0$. *We consider two cases.*

If $d \leq 6$, *then* $d + 3 = b - 1$ *and* $c + 1 = a$. *We can conclude that* $a = 6, b = 7, c = 5$ *and* $d = 3$. *If on the other hand* $d \geq 7$, *then* $d + 3 = 10 + (b-1)$ *and* $c + 1 = a - 1$. *But in this case we get a contradiction* $c = \frac{9}{2}$. *Hence, the only solution is* $(a, b, c, d) = (6,7,5,3)$.

1.130 Zan wrote a sequence of four positive real numbers. The first term in the sequence was the number 3, and the last term was the number 9. The first three terms formed a geometric sequence, and the last three terms formed an arithmetic sequence. Determine all four terms of Zan's sequence.

Slovenia NMO-2013

Solution: *Denote the second and the third term of Zan's sequence by* x *and* y. *Then* $3, x, y$ *is a geometric sequence and* $x^2 = 3y$. *On the other hand* $x, y, 9$ *is an arithmetic sequence, so* $2y = x + 9$. *We may rewrite the second equation* *as* $y = \frac{x+9}{2}$. *Plugging this into the first equation and rearranging we get* $2x^2 - 3x - 27 = 0$, *or* $(2x - 9)(x + 3) = 0$. *Since* x *is positive we have* $x = \frac{9}{2}$. *From here we get* $y = \frac{27}{4}$. *Zan's sequence is* $\frac{3,9}{2}, \frac{27}{4}, 9$.

ALGEBRA

2.1 Determine all plynomyals $P(x) \in \mathbb{R}[x]$ satisfying the following two conditions:
a) $P(2017) = 2016$ and
b) $(P(x) + 1)^2 = P(x^2 + 1)$ for all real numbers x.

Walter Janous-Austrian NMO-2017

Solution: *Letting $Q(x) := P(x) + 1$ we get the two new conditions $Q(2017) = 2017$ and $Q(x^2 + 1) = Q^2(x) + 1, x \in \mathbb{R}$.*
We now define the sequence $(x_n)_{n\geq 0}$ recursively by $x_0 = 2017$ and $x_{n+1} = x_n^2 + 1, n \geq 0$. A straightforward induction yields $Q(x_n) = x_n, n \geq 0$, because $Q(x_{n+1}) = Q(x_n^2 + 1) = Q^2(x) + 1 = x_n^2 + 1 = x_{n+1}$.
Because of $x_0 < x_1 < x_2 < \cdots$ the two polynomials $Q(x)$ and $id(x) = x$ coincide at infinitely many arguments x. Therefore, $Q(x) = x$ and thus the unique polynomial satisfying the two conditions of our problem is
$P(x) = x - 1$.

2.2 $z_1, z_2, z_3 \in \mathbb{C}^*$, are different in pairs, $|z_1| = |z_2| = |z_3| = 1$

$$A(z_1), B(z_2), C(z_3), \sum_{cyc} \frac{1}{8 - \frac{z_1}{z_2} - \frac{z_2}{z_1} - \frac{z_1}{z_3} - \frac{z_3}{z_1}} = \frac{3}{10}$$

Prove that: $AB = BC = CA$.

Marian Ursărescu

Solution (George Florin Şerban)

$$z_1 = cos\theta_1 + isin\theta_1, z_2 = cos\theta_2 + isin\theta_2, z_3 = cos\theta_3 + isin\theta_3$$

We have: $\alpha = \frac{z_1}{z_2} + \frac{z_2}{z_1} + \frac{z_1}{z_3} + \frac{z_3}{z_1}$

$$= cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2) + cos(\theta_2 - \theta_1) + isin(\theta_2 - \theta_1) + cos(\theta_1 - \theta_3) + isin(\theta_1 - \theta_3) + cos(\theta_3 - \theta_1) + isin(\theta_3 - \theta_1)$$

$$= cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2) + cos(\theta_1 - \theta_2) - isin(\theta_1 - \theta_2) + cos(\theta_1 - \theta_3) + isin(\theta_1 - \theta_3) + cos(\theta_1 - \theta_3) - isin(\theta_1 - \theta_3) = 2cos(\theta_1 - \theta_2) + 2cos(\theta_1 - \theta_3)$$

So:

$$8 - \alpha = 8 - 2cos(\theta_1 - \theta_2) - 2cos(\theta_1 - \theta_3)$$

$$= 3 + 2cos\theta_1 cos\theta_2 + 2sin\theta_1 sin\theta_2 + 2cos\theta_2 cos\theta_3 + 2sin\theta_2 sin\theta_3 + 2cos\theta_1 cos\theta_3 + 2sin\theta_1 sin\theta_3$$

$$= (cos\theta_1 + cos\theta_2 + cos\theta_3)^2 + (sin\theta_1 + sin\theta_2 + sin\theta_3)^2 \geq 0$$

Let:

$$x = cos(\theta_1 - \theta_2); y = cos(\theta_2 - \theta_3); z = cos(\theta_1 - \theta_3)$$

$$3 + 2x + 2y + 2z \geq 0 \Rightarrow x + y + z \geq -\frac{3}{2}$$

$$\sum_{cyc} \frac{1}{8 - \frac{z_1}{z_2} - \frac{z_2}{z_1} - \frac{z_1}{z_3} - \frac{z_3}{z_1}} = \sum_{cyc} \frac{1}{8 - 2x - 2z} \overset{Bergstrom}{\geq} \frac{(1+1+1)^2}{\sum_{cyc} 8 - 2x - 2z}$$

$$= \frac{9}{24 - 4\sum x} \geq \frac{3}{10}$$

Equality for: $8 - 2x - 2y = 8 - 2x - 2z = 8 - 2y - 2z \Leftrightarrow x = y = z$

$$AB = |z_1 - z_2| = |(cos\theta_1 - cos\theta_2) + i(sin\theta_1 - sin\theta_2)|$$

$$= \sqrt{(cos\theta_1 - cos\theta_2)^2 + (sin\theta_1 - sin\theta_2)^2}$$

$$= \sqrt{2 - 2cos\theta_1 cos\theta_2 - 2sin\theta_1 sin\theta_2}$$

$$= \sqrt{2 - 2(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2)} = \sqrt{2 - 2cos(\theta_1 - \theta_2)} = \sqrt{2 - 2x}$$

Analogous: $BC = \sqrt{2 - 2y}. AC = \sqrt{2 - 2z}$

2.3 If $0 < a_1 \leq a_2 \leq \ldots \leq a_n, n > 0$, then prove:

$$\frac{a_1 \cdot a_n}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)^2 \leq \sum_{k=1}^{n}\left(\frac{a_1 + a_n}{a_k} - 1\right)$$

Florică Anastase

Solution:

$$0 \geq \sum_{k=1}^{n} \frac{1}{a_k} \cdot (a_1 - a_k)(a_n - a_k) =$$

$$= \sum_{k=1}^{n} \frac{1}{a_k} \cdot (a_1 \cdot a_n - a_k \cdot a_n - a_1 \cdot a_k + a_k{}^2) =$$

$$= \sum_{k=1}^{n}\left(\frac{a_1 \cdot a_n}{a_k} - a_1 - a_n + a_k\right) = \sum_{k=1}^{n}\left(\frac{a_1 \cdot a_n}{a_k}\right) - n(a_1 + a_n) + \sum_{k=1}^{n} a_k \Rightarrow$$

$$a_1 \cdot a_n \cdot \sum_{k=1}^{n} \frac{1}{a_k} \leq \left(n(a_1 + a_n) - \sum_{k=1}^{n} a_k\right) \overset{Am-Gm}{\leq} \left(n(a_1 + a_n) - n\sqrt[n]{\prod a_k}\right)$$

$$\frac{a_1 \cdot a_n}{n} \cdot \sum_{k=1}^{n} \frac{1}{a_k} \le a_1 + a_n - \sqrt[n]{\prod a_k} \overset{Am-Hm}{\lesssim} a_1 + a_n - \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

$$\frac{a_1 \cdot a_n}{n} \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right)^2 \le (a_1 + a_n) \sum_{k=1}^{n} \frac{1}{a_k} - n$$

$$\frac{a_1 \cdot a_n}{n} \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right)^2 \le \sum_{k=1}^{n} \left(\frac{a_1 + a_n}{a_k} - 1\right)$$

2.4 In ΔABC the following relationship holds:

$$\boldsymbol{\frac{m_a^2}{m_b} + \frac{m_b^2}{m_c} + \frac{m_c^2}{m_a} \ge s\sqrt{3}}$$

Marian Ursărescu

Solution (Rahim Shahbazov)

$$\frac{m_a^2}{m_b} + \frac{m_b^2}{m_c} + \frac{m_c^2}{m_a} \ge s\sqrt{3} \ldots (1)$$

Lemma:

$$x, y, z > 0 \; then: \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \ge 3 \cdot \sqrt{\frac{x^2 + y^2 + y^2}{3}} \overset{(1)}{\Rightarrow}$$

$$LHS \ge 3 \cdot \sqrt{\frac{m_a^2 + m_b^2 + m_c^2}{3}} \ge s\sqrt{3} \Rightarrow 3(a^2 + b^2 + c^2) \ge (a + b + c)^2$$

Prove lemma:

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \ge \frac{(x^2 + y^2 + z^2)^2}{x^2 y + y^2 z + z^2 x}$$

$$\ge \frac{(x^2 + y^2 + z^2)^2}{\sqrt{(x^2 + y^2 + z^2)(x^2 y^2 + y^2 z^2 + z^2 y^2)}}$$

$$\ge 3 \cdot \sqrt{\frac{x^2 + y^2 + y^2}{3}} \Rightarrow$$

$$(x^2+y^2+z^2)^2 \geq x^2y^2+y^2z^2+z^2y^2$$

2.5 If $a,b,c,m,n>0$ **then:**

$$\sum_{cyc}\frac{8a}{ma^2+nbc} \leq (m+n)\left(\frac{1}{m^2}+\frac{1}{n^2}\right)\left(\sum_{cyc}\frac{a}{bc}\right)$$

Florică Anastase

Solution: $From: (ma^2+nbc)^2 \geq 4mna^2bc \rightarrow \frac{a}{ma^2+nbc} \leq \frac{ma^2+nbc}{4mnabc} \rightarrow$

$$\sum_{cyc}\frac{a}{ma^2+nbc} \leq \frac{1}{4mn}\sum_{cyc}\frac{ma^2+nbc}{abc} \leq \frac{1}{4mn}\sum_{cyc}\frac{(m+n)a^2}{abc}$$

$$=\frac{m+n}{4mn}\sum_{cyc}\frac{a}{bc} \quad (i)$$

$$\frac{m+n}{4mn}=\frac{m+n}{4}\cdot\frac{1}{mn}\overset{AGM}{\leq}\frac{m+n}{4}\cdot\left(\frac{m+n}{2mn}\right)^2$$

$$=\frac{m+n}{4m^2n^2}\cdot\left(\frac{m+n}{2}\right)^2\overset{AGM}{\leq}\frac{m+n}{8}\cdot\left(\frac{m^2+n^2}{m^2n^2}\right) \quad (ii)$$

From (i),(ii) we have:

$$\sum_{cyc}\frac{8a}{ma^2+nbc} \leq (m+n)\left(\frac{1}{m^2}+\frac{1}{n^2}\right)\left(\sum_{cyc}\frac{a}{bc}\right)$$

2.6 If $x,y,z>0, xyz=1, n\in(0,2]$ **then prove:**

$$\sum_{cyc}\frac{(xy+z)(xz+y)}{(x+yz)(1+n(xy+z)(xz+y))} \leq \frac{2}{n}$$

Florică Anastase

Solution:

$$\sum_{cyc}\frac{(xy+z)(xz+y)}{(x+yz)(1+n(xy+z)(xz+y))}$$

$$=\sum_{cyc}\frac{xyz(xy+z)(xz+y)}{x(x+yz)(yz+nyz(xy+z)(xz+y))} =$$

$$= \sum_{cyc} \frac{x(y^2+1)(z^2+1)}{(x^2+1)(yz+n(y^2+1)(z^2+1))} = \sum_{cyc} \frac{\frac{x}{1+x^2}}{\frac{y}{1+y^2} \cdot \frac{z}{1+z^2} + n} \quad (1)$$

$$Let: a = \frac{x}{1+x^2}, b = \frac{y}{1+y^2}, c = \frac{z}{1+z^2} \text{ and } a, b, c \in \left(0, \frac{1}{2}\right)$$

$$We\ must\ show: \frac{a}{bc+n} + \frac{b}{ca+n} + \frac{c}{ab+n} \le \frac{2}{n} \quad (2)$$

$$We\ can\ consider\ 0 \le a \le b \le c \le \frac{1}{2} \overset{(1),(2)}{\Rightarrow}$$

$$\frac{a}{bc+n} + \frac{b}{ca+n} + \frac{c}{ab+n} \le \frac{a}{ab+n} + \frac{b}{ab+n} + \frac{c}{ab+n} \overset{c<1}{\le} \frac{a+b+1}{ab+n} \overset{?}{\le} \frac{2}{n} \leftrightarrow$$

$$n(a+b+1) \le 2ab + 2n \leftrightarrow (2-n)ab + n(1-a)(1-b)$$

$$\ge 0 \text{ true for } a, b, c \in \left(0, \frac{1}{2}\right), n \in (0,2]$$

2.7 If $a, b, c > 1$, then:

$$\sum_{cyc} \log_{a+b}(1+b^{b+1})(1+c^{c+1}) \ge 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}$$

Florică Anastase

Solution: *From Bernoulli's inequality, we have:*

$$\begin{cases} a^{a+1} = (1+a-1)^{a+1} \ge a^2 \\ b^{b+1} = (1+b-1)^{b+1} \ge b^2 \\ c^{c+1} = (1+c-1)^{c+1} \ge c^2 \end{cases}$$

$$\rightarrow \begin{cases} (1+a^{a+1})(1+b^{b+1}) \ge (1+a^2)(1+b^2) \ge (a+b)^2 \\ (1+b^{b+1})(1+c^{c+1}) \ge (1+b^2)(1+c^2) \ge (b+c)^2 \\ (1+c^{c+1})(1+a^{a+1}) \ge (1+c^2)(1+a^2) \ge (c+a)^2 \end{cases}$$

$$\sum_{cyc} log_{a+b}(1+b^{b+1})(1+c^{c+1}) \ge \sum_{cyc} log_{a+b}(b+c)^2 =$$

$$= 2\sum_{cyc} log_{a+b}(b+c) \overset{Am-Gm}{\ge} 6 \cdot \sqrt[3]{\prod_{cyc} log_{a+b}(b+c)} = 6 \quad (i)$$

$$\therefore x^x y^y z^z \ge x^z y^x z^y, \forall x, y, z > 1 \leftrightarrow (z-x)lnx + (x-y)lny + (y-z)lnz \le 0$$

$$If\ 1 \le x \le y \le z \rightarrow (lnx \le lny \le lnz, z-x \ge x-y) \overset{Chebyshev's}{\Rightarrow}$$

$$(z-x)lnx + (x-y)lny \le \frac{1}{2}(z-y)\ln(xy) = (z-y)ln\sqrt{xy}$$

$$\rightarrow (z-x)lnx + (x-y)lny + (y-z)lnz \le (z-y)ln\sqrt{xy} + (y-z)lnz$$
$$= (z-y)ln\frac{\sqrt{xy}}{z} \le 0 \therefore$$
$$From: x = b+c, y = c+a, z = a+b \rightarrow (a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a} \le 1 \quad (ii)$$
$$From\ (i),(ii) \rightarrow \sum_{cyc} log_{a+b}(1+b^{b+1})(1+c^{c+1}) \ge 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}$$

2.8 Solve the system:

$$\boldsymbol{a+b+c=0, a^2+b^2+c^2=1, a^3+b^3+c^3=4abc}$$

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Solution: *As is usual with problems of this kind, we eliminate one of the "unknowns", therebody reducing the number of equations as well.*
So, suppose a,b,c*satisfy the given equations, and eliminate* c*, say.Then, from the first, we deduce that*

$$a^3+b^3+c^3 = a^3+b^3-(a+b)^3 = a^3+b^3-(a^3+3a^2b+3ab^2+b^3) = -3ab(a+b) = 3abc.$$

This and the third equation forces $abc=0$*. Hence, one* a,b,c *is zero. Say* $c=0$*. Then, by the first and second equations,* $a=-b$*, and* $1=2a^2$*. Thus one solutions is* $a=\pm\frac{1}{\sqrt{2}}, b=\pm\frac{1}{\sqrt{2}}, c=0$ *and any permutation of this triple is a solution. Conversely, every such triple is a solution.*

2.9 Prove for all complex numbers z that:

$$\boldsymbol{|z|^2+2|z-1| \ge 1,}$$

with equality if $z=1$.

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Solution: *If* $|z|=1$*, then* $|z|^2+2|z-1| \ge 1+2|z-1| \ge 1$*, with equality if* $z=1$*. If* $|z|<1$ *then* $|z-1| \ge \big||z|-1\big| = 1-|z|$*, and so*

$$|z|^2+2|z-1|-1 \ge |z|^2+2(1-|z|)-1 = (|z|-1)^2 > 0$$

2.10 Suppose u, v are real numbers and $w = u + iv$ is a complex number. Show that the quadratic $x^2 - 2ix + w$ has precisely one real root if $v^2 + 4u = 0$.

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Solution: *Suppose* $v^2 + 4u = 0$, *and let* $r = \frac{v}{2}$. *Then,* r *is real and*

$$r^2 - 2ir = \frac{v^2}{4} - iv = -u - iv = -w,$$

whence, as $w = u + iv$ *and so* $w + \overline{w} = 2u$ *and* $w - \overline{w} = 2iv, 2r^2 + 2u = 0$, $-4ir + 2iv = 0$. *Therefore,* $v^2 = 4r^2 = -4u$. *Hence, the result.*

2.11 If $a_i > 1$, $i = \overline{1, n+1}$, $n \in \mathbb{N}$, prove:

$$\sum_{cyc} \log_{a_1}^n (a_2 a_3 \cdot \ldots \cdot a_{n+1}) \geq n^{n+1}$$

Florică Anastase

Solution:

$$\log_{a_1}^n (a_2 a_3 \cdot \ldots \cdot a_{n+1}) = \left(\log_{a_1} a_2 + \log_{a_1} a_3 + \cdots + \log_{a_1} a_{n+1}\right)^n \overset{AM-GM}{\geq}$$

$$\geq n^n \cdot \log_{a_1} a_2 \log_{a_1} a_3 \cdot \ldots \cdot \log_{a_1} a_{n+1}$$

$$\sum_{cyc} \log_{a_1}^n (a_2 a_3 \cdot \ldots \cdot a_{n+1}) \geq n^n \sum_{cyc} \log_{a_1} a_2 \log_{a_1} a_3 \cdot \ldots \cdot \log_{a_1} a_{n+1} \overset{AM-GM}{\geq}$$

$$\geq n^n \cdot n \cdot \sqrt[n]{\log_{a_1} a_2 \log_{a_2} a_1 \cdot \ldots \cdot \log_{a_n} a_{n+1} \log_{a_{n+1}} a_n} = n^{n+1}$$

2.12 If $a_1, a_2, \ldots, a_n > 0$, then:

$$\prod_{i=1}^{n} \left(1 + a_i^{1+a_i^{1+a_i}}\right) \geq 2^n \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}\sum_{i=1}^{n} a_i}$$

Florică Anastase

Solution:

$$1 + a_i^{1+a_i} = 1 + (1 + a_i - 1)^{1+a_i} \overset{Benoulli}{\geq} 1 + a_i^2 \rightarrow$$

$$1+a_i^{1+a_i^{1+a_i}} \geq 1+a_i^{1+a_i^2} \overset{Am-Gm}{\geq} 2a_i^{a_i} \to \prod_{i=1}^{n}\left(1+a_i^{1+a_i^{1+a_i}}\right) \geq 2^n \prod_{i=1}^{n} a_i^{a_i} \ldots\ldots (1^{\circ})$$

$$We\ must\ show: \prod_{i=1}^{n} a_i^{a_i} \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}\sum_{i=1}^{n} a_i} \leftrightarrow \sum_{i=1}^{n} a_i log(a_i) \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} log(a_i)\right)\ true,$$

Cebyshev inequalities for sequences $(a_i)_{i\geq 1}, (log\ (a_i))_{i\geq 1}$......$(2^{\circ})$
From $(1^{\circ}), (2^{\circ})$ *we have:*

$$\prod_{i=1}^{n}\left(1+a_i^{1+a_i^{1+a_i}}\right) \geq 2^n \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}\sum_{i=1}^{n} a_i}$$

2.13 If $a_1, a_2, \ldots, a_n > 0, n \in \mathbb{N}, n > 1$. Then:

$$\sum_{cyc} \log_{1+a_1a_2}\left(1+a_2^{1+a_2}\right)\left(1+a_3^{1+a_3}\right) \geq 2n$$

Florică Anastase

Solution:

$$1+a_i^{1+a_i} = 1+(1+a_i-1)^{1+a_i} \overset{Bernoulli}{\geq} 1+a_i^2 \Rightarrow$$

$$\left(1+a_i^{1+a_i}\right)\left(1+a_j^{1+a_j}\right) \geq (1+a_i^2)(1+a_j^2) \geq (1+a_ia_j)^2 \Rightarrow$$

$$\sum_{cyc} \log_{1+a_1a_2}(1+a_2^{1+a_2})(1+a_3^{1+a_3}) \geq 2\sum_{cyc} \log_{1+a_1a_2}(1+a_2a_3) \overset{Am-Gm}{\geq}$$

$$\geq 2n\sqrt[n]{\prod_{cyc} \log_{1+a_1a_2}(1+a_2a_3)} \geq 2n$$

2.14 If $a, b, c > 0$, then:

$$\left(1+a^{1+a^{1+a}}\right)\left(1+b^{1+b^{1+b}}\right)\left(1+c^{1+c^{1+c}}\right) \geq 8a^b b^c c^a$$

Florică Anastase

Solution:

$$1+a^{1+a}=1+(1+a-1)^{1+a}\overset{Bernoulli}{\geq}1+a^2 \text{ and anlogous}$$
$$1+b^{1+b}\geq 1+b^2,\quad 1+c^{1+c}\geq 1+c^2$$
$$1+a^{1+a^{1+a}}\geq 1+a^{1+a^2}\overset{Am-Gm}{\geq}1+a^{2a}\overset{Am-Gm}{\geq}2\sqrt{a^{2a}}$$
$$=2a^a \text{ and analogous}$$
$$1+b^{1+b^{1+b}}\geq 2b^b,\quad 1+c^{1+c^{1+c}}\geq 2c^c$$
$$\left(1+a^{1+a^{1+a}}\right)\left(1+b^{1+b^{1+b}}\right)\left(1+c^{1+c^{1+c}}\right)\geq 8a^ab^bc^c\ \dots\dots(1)$$
$$a^ab^bc^c\overset{?}{\geq}a^bb^cc^a\leftrightarrow(a-b)\log(a)+(b-c)\log(b)+(c-a)\log(c)\geq 0$$
$$\therefore \text{Let } 0<a\leq b\leq c\to a-b<b-c$$
$$\text{and } \log(a)\leq\log(b)\overset{Cebyshev}{\Rightarrow}$$
$$(a-b)\log(a)-(b-c)\log(b)\geq\frac{1}{2}(a-c)\log(ab)=\log\sqrt{ab}$$
$$(a-b)\log(a)+(b-c)\log(b)+(c-a)\log(c)\geq(a-c)\log\left(\frac{\sqrt{ab}}{c}\right)$$
$$\geq 0\ \dots(2)$$

From (1) and (2) we have: $\left(1+a^{1+a^{1+a}}\right)\left(1+b^{1+b^{1+b}}\right)\left(1+c^{1+c^{1+c}}\right)\geq 8a^bb^cc^a$

2.15 Let n be positive integer and $f(x)$ be a polynomial of degree n with n distinct real positive roots. Are there positive integer $k\geq 2$ and real polynomial $g(x)$ such that

$$x(x+1)(x+2)(x+4)f(x)+1=(g(x))^k?$$

Aleksandar Ivanov-Bulgarian NMO-2017

Solution: *Let $\alpha_1<\alpha_2<\dots<\alpha_n$ be the roots of $f(x)$. Assume that*

$$x(x+1)(x+2)(x+4)f(x)+a=(g(x))^k$$

Note that $a=b^k=g^k(0)$.

If $k\geq 3$ is odd then the polynomial $g^k(x)-b^k$ has $n+4$ distinct real roots which will be also roots of $g(x)-b$. However, the degree of $g(x)-b$ is $n+4/k<n+4$, i.e. $g(x)=b$, which is impossible.

Now it is enough to prove that $k=2$ is also impossible. We have $a=b^2$, where we can assume thet $b>0$. Then

$$x(x+1)(x+2)(x+4)f(x)=g_1(x)g_2(x),$$

where $g_1(x) = g(x) + b$ and $g_2(x) = g(x) - b$. The roots of $g_1(x)$ and $g_2(x)$ are the numbers $-4, -2, -1, 0, \alpha_1, \alpha_2, \dots, \alpha_n$. Since $g_1(x) \geq g_2(x)$ for every x, the number -4 is a root of $g_1(x)$. Since the derivatives of $g_1(x)$ and $g_2(x)$ coincide, the Rolle's theorem shows that -2 and -1 are roots of $g_2(x)$ while 0 is a root of $g_1(x)$.
Let $g_1(x) = x(x+4)\prod_{j=1}^{s}(x-\alpha_j)$. Then

$$|g_1(-1)| = 3\prod_{j=1}^{s}(1+\alpha_j) < 4\prod_{j=1}^{s}(2+\alpha_j) = |g_1(-2)|,$$

which contradicts to $g_1(-1) = g_1(-2) = g(-1) + b = 2b$.

2.16 Prove that if $a, b, c, d \in \mathbb{R}$; $(a^2+b^2)(c^2+d^2) \neq 0$ then:

$$\left|\frac{a(c+d)-b(c-d)}{\sqrt{(a^2+b^2)(c^2+d^2)}}\right| \leq \left|1+\frac{(ad-bc)(ac+bd)}{(a^2+b^2)(c^2+d^2)}\right|$$

Daniel Sitaru-Joszef Wildt-International Mathematical Competition-2017

Solution (Soumava Chakraborty)

Case 1. $ad - bc = 0$

$$(ad-bc)^2 + (ac+bd)^2 = (a^2+b^2)(c^2+d^2)$$

$$\therefore |ac+bd| = \sqrt{(a^2+b^2)(c^2+d^2)}$$

$$Lhs = \frac{|ac+bd|}{\sqrt{(a^2+b^2)(c^2+d^2)}} = 1 = Rhs$$

Case 2. $ac + bd = 0$

$$(ad-bc)^2 + (ac+bd)^2 = (a^2+b^2)(c^2+d^2)$$

$$|ad-bc| = \sqrt{(a^2+b^2)(c^2+d^2)}$$

$$Lhs = \frac{|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}} = 1 = Rhs$$

Case 3. $ad - bc \neq 0, ac + bd \neq 0$

Case 3a. $(ad-bc)$ and $(ac+bd)$ are of same sign

$$|ad-bc| = psin\theta$$

$$|ac+bd| = pcos\theta$$

$$Lhs = \frac{p}{p}|cos\,\theta + sin\,\theta| \text{ and } Rhs = |1 + cos\,\theta\,sin\,\theta|$$

It suffices to prove: $(1+\cos\theta\sin\theta)^2 \geq (\cos\theta+\sin\theta)^2 \Leftrightarrow$
$\sin^2\theta\cos^2\theta \geq 0\ (true)$.
Case 3b. $(ad-bc)$ and $(ac+bd)$ *are of different.*
Then, $Lhs = \frac{p}{p}|\cos\theta - \sin\theta|$ *and* $Rhs = |1-\cos\theta\sin\theta|$
It suffices to prove: $(1-\cos\theta\sin\theta)^2 \geq (\cos\theta-\sin\theta)^2 \Leftrightarrow$
$\sin^2\theta\cos^2\theta \geq 0\ (true)$.

2.17 If $a,b,c\in(0,1)$ and $a+b+c=1$, then:

$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \leq 2$$

Florică Anastase

Solution 5:

$$Let: f:(0,1)\to\mathbf{R}, f(x) = x\cdot\log(1-x^2), f'(x) = \log(1-x^2) - \frac{2x}{1-x^2}, f''(x) = -\frac{2x\cdot(3-x^2)}{(1-x^2)^2} < 0, \forall x \in (0,1) \to f''\ concave.$$

From Jensen inequality:

$$f\left(\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\right) = f\left(\frac{a+b+c}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\right) \geq \frac{\sqrt{a}\cdot f(\sqrt{a})+\sqrt{b}\cdot f(\sqrt{b})+\sqrt{c}\cdot f(\sqrt{c})}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \leftrightarrow$$

$$\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\cdot\log\left(1-\frac{1}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\right) \geq \frac{a\cdot\log(1-a)+b\cdot\log(1-b)+c\cdot\log(1-c)}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \leftrightarrow$$

$$\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2-(a+b+c)}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2} \geq \log\left((1-a)^a\cdot(1-b)^b\cdot(1-c)^c\right) \leftrightarrow$$

$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \leq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \leftrightarrow$$

$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \leq 2(a+b+c) \leftrightarrow$$

$$(b+c)^a\cdot(c+a)^b\cdot(a+b)^c\cdot(\sqrt{a}+\sqrt{b}+\sqrt{c})^2\le 2\ldots\ldots(proved)$$

2.18 If $a,b,c\in\left(0,\frac{1}{2}\right)$, $n\in\mathbb{N}$, $a^{n+1}+b^{n+1}+c^{n+1}=1$, then:

$$\frac{(1+a^n)^a(1+b^n)^b(1+c^n)^c}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})}\le\left(\frac{1+a+b+c}{2a+2b+2c-1}\right)^{a+b+c}$$

Florică Anastase

Solution (Sanong Huayrerai)

$$Because\ a,b,c\in\left(0,\frac{1}{2}\right),n\in\mathbb{N},\rightarrow a^{n+1}<a^n<a<\frac{1}{2}\ and\ analogous$$

$$\rightarrow a+b+c\in\left(0,\frac{3}{2}\right)$$

$$Hence\ a^n(a+b+c)<a(a+b+c)<\frac{3}{4}and\ analogous$$

$$Hence\ a+b^{n+1}+c^{n+1}\ge 1+a-\frac{a}{a+b+c}\ and\ analogous$$

$$(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})\ge\left(1+a-\frac{a}{a+b+c}\right)\left(1+b-\frac{b}{a+b+c}\right)\left(1+c-\frac{c}{a+b+c}\right)$$

$$\ge\left(2-\frac{1}{a+b+c}\right)^a\left(2-\frac{1}{a+b+c}\right)^b\left(2-\frac{1}{a+b+c}\right)^c=\left(\frac{2(a+b+c)-1}{a+b+c}\right)^{a+b+c}\leftrightarrow$$

$$\frac{(2(a+b+c)-1)^{a+b+c}}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})}\le(a+b+c)^{a+b+c}\leftrightarrow$$

$$\left(\frac{a+b+c+1}{a+b+c}\right)^{a+b+c}\cdot\frac{1}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})}\le\left(\frac{a+b+c+1}{2(a+b+c)-1}\right)^{a+b+c}\leftrightarrow$$

$$\left(\frac{a+a^{n+1}+b+b^{n+1}+c+c^{n+1}}{a+b+c}\right)^{a+b+c}$$
$$\cdot\frac{1}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})}$$
$$\leq\left(\frac{a+b+c+1}{2(a+b+c)-1}\right)^{a+b+c}\leftrightarrow$$
$$\frac{(1+a^n)^a(1+b^n)^b(1+c^n)^c}{(a+b^{n+1}+c^{n+1})(b+c^{n+1}+a^{n+1})(c+a^{n+1}+b^{n+1})}$$
$$\leq\left(\frac{1+a+b+c}{2a+2b+2c-1}\right)^{a+b+c}$$

2.19 If $a,b,c\in(0,1)$ or $a,b,c\in(1,\infty)$, $ab+bc+ca=abc$ then:

$$2\cdot\sqrt[4]{\left(\prod_{cyc}\tan^{-1}a\right)\left(\sum_{cyc}\tan^{-1}a\right)}\leq\tan^{-1}\left(\frac{\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}(1-a)^2\right)}}{1-abc}\right)$$

Florică Anastase

Solution (Adrian Popa)

$$2\sqrt[4]{\tan^{-1}a\cdot\tan^{-1}b\cdot\tan^{-1}c(\tan^{-1}a+\tan^{-1}b+\tan^{-1}c)}$$
$$\overset{Am-Gm}{\leq}2\cdot\frac{2(\tan^{-1}a+\tan^{-1}b+\tan^{-1}c)}{4}$$
$$=\tan^{-1}\left(\frac{a+b+c-abc}{1-ab-bc-ca}\right)=\tan^{-1}\left(\frac{a+b+c-ab-bc-ca}{1-abc}\right)$$
$$=\tan^{-1}\left(\frac{a(1-b)+b(1-c)+c(1-a)}{1-abc}\right)$$

Let be the function: $f(x)=\tan^{-1}x, f'(x)=\frac{1}{1+x^2}>0\forall x\in\mathbb{R}\Rightarrow$

f –increasing

$$a(1-b)+b(1-c)+c(1-a)\overset{C.B.S}{\leq}\sqrt{(a^2+b^2+c^2)((1-a)^2+(1-b)^2+(1-c)^2)}$$

If $a,b,c\in(0,1)\Rightarrow\begin{cases}1-a>0\\1-b>0\\1-c>0\end{cases}$ *and* $1-abc>0\Rightarrow\frac{\sum a(1-b)}{1-abc}>0$

If $a, b, c \in (1, \infty)\ ) \Rightarrow \begin{cases} 1 - a < 0 \\ 1 - b < 0 \\ 1 - c < 0 \end{cases}$ *and* $1 - abc < 0 \Rightarrow \frac{\sum a(1-b)}{1-abc} > 0$

2.20 Let $n \geq 1$ be an integer and consider the sum

$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-4} \cdot 3^2 + \cdots$$

Show that $2x - 1, 2x, 2x + 1$ from the sides of a triangle whose area and inradius are also integers.

India NMO-2017

Solution: *Consider the binomial expansion of* $(2 + \sqrt{3})^n$. *It is easy to check that*

$$(2 + \sqrt{3})^n = x + y\sqrt{3},$$

where y *is also an integer. We also have*

$$(2 - \sqrt{3})^n = x - y\sqrt{3}.$$

Multiplying these two relations, we obtain $x^2 - 3y^2 = 1$.

Since all the terms of the expansion of $(2 + \sqrt{3})^n$ *are positive, we see that*

$$2x = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n = 2\left(2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \cdots\right) \geq 4.$$

Thus $x \geq 3$. *Hence* $2x + 1 < 2x + (2x - 1)$ *and therefore* $2x - 1, 2x, 2x + 1$ *are the sides of triangle. By Heron's formula we have*

$$\Delta^2 = 3x(x + 1)(x)(x - 1) = 3x^2(x^2 - 1) = 9x^2y^2.$$

Hence $\Delta = 3xy$ *which is an integer, Finally, its inradius is*

$$\frac{area}{perimeter} = \frac{3xy}{3x} = y,$$

which is also an integer.

2.21 A polynomial $p(x)$ with real coefficients is called a square if and only if is not a constant and there exists a polynomial $q(x)$ with real coefficients such that $p(x) = q^2(x)$. Suppose that $f(x)$ and $g(x)$ are

non-constant polynomials with real coefficients such that neither of them is a square but $f(g(x))$ is. Show that $f(g(x))$ is not a square.

India TST-2017

Solution: *We can easily extended the definition of a square polynomial to polynomials with complex coefficients. In all the arguments below we consider polynomials with complex coefficients.*

Lemma: *If $p(x)$ is $a$square and is a non-zero complex number the $p(x)-a$ is not square.*

Proof of lemma. Suppose $p(x)=q^2(x)$ and $p(x)-a=r^2(x)$, with both $q(x)$ and $r(x)$ being non-constant polynomials. The $a=\left(q(x)-r(x)\right)\left(q(x)+r(x)\right)$. Clearly, either $q(x)-r(x)$ or $q(x)+r(x)$ is not constant polynomial, and hence a contradiction. This proves lemmas.

Continuation of the solution: We can write $f(x)$ as $f_1^2(x)(x-a_1)(x-a_2)\dots(x-a_k)$, where $f_1(x)$ is polynomial and $a_1, a_k, \dots, a_k$ are distinct complex numbers. Then

$f\left(g(x)\right)=f_1^2(g(x))(g(x)-a_1)(g(x)-a_2)\dots(g(x)-a_k)$ is a square. It follows that

$(g(x)-a_1)(g(x)-a_2)\dots(g(x)-a_k)=h^2(x)$. Let β be such that $g(\beta)=a_1$. Then $h(\beta)=0$ and hence $h(x)=(x-\beta)h_1(x)$.

Note that $g(\beta)-a_i=0$ for any $i=1$. Therefore it follows that $(x-\beta)^2$ divides $g(x)-a_i$ is a square. Similarly, $g(x)-a_i$ is a square for $i=1,2\dots,k$.By the above lemma, it follows that $k=1$, so $f(x)=f_1^2(x)(x-a)$ and $g(x)=g_1^2(x)+a$ for some non-zero complex number a. Therefore $g\left(f(x)\right)=g_1^2\left(f(x)\right)+a$ and hence by the above lemma it follows that $g\left(f(x)\right)$ is not square. The completes the proof.

2.22 Let a, b, c be distinct positive real numbers such that $abc=1$. Prove that:

$$\sum_{cyc}\frac{a^6}{(a-b)(a-c)}>15$$

India TST-2017

Solution: *Let us consider a cubic polynomial whose roots are a, b, c. We get $P(x)=x^3-px^2+qx-r$, where $p=a+b+c, q=ab+bc+ca, r=abc$. We observe that:*

$$\frac{a^n}{(a-b)(a-c)}+\frac{b^n}{(b-c)(b-a)}+\frac{c^n}{(c-a)(c-b)}$$
$$=\sum_{cyc}\frac{-a^n(b-c)}{(a-b)(b-c)(c-a)}$$

Let us write

$$S_n=\sum_{cyc}\frac{-a^n(b-c)}{(a-b)(b-c)(c-a)}$$

It is easy to see that $S_1=0, S_2=0$. *Since* a,b,c *are the roots of* $P(x)=0$, *we have*

$$a^3-pa^2+qa+r=0, b^3-pb^2+qb+r=0, c^3-pc^2+qc+r=0.$$

Multiply the first by $b-c$, *the second by* $c-a$ *and the third by* $a-b$, *and adding all these and dividing the sum by* $-(a-b)(b-c)(c-a)$, *we obtain* $S_3-pS_2+qS_1=0$.

Hence $S_3=p$. *Now multiply, the first by* a, *the second by* b *and the third by* c *and divide through out by* $-(a-b0(b-c)(c-a)$ *to get* $S_4-pS_3+qS_2-rS_1=0$. *Hence* $S_4=p^2-q$. *Similarly,* $S_5=p(p^2-q)-qp+r=p^3-2pq+r$. *We also get*
$S_6-pS_5+qS_4-rS_3=0$. *This gives*

$$S_6=p(p^3-2pq+r)-q(p^2-q)+rp=p^4-3p^2q+2pr+q^2.$$

We can write it as $S_6=p^2(p^2-3q)+2pr+q^2$.
But $p^2-3q=(a+b+c)^2-3(ab+bc+ca)=a^2+b^2+c^2-ab-bc-ca>0$. *Hence* $S_6>2pr+q^2=2abc(a+b+c)+(ab+bc+ca)^2\geq 6+9=15$.

2.23 Let d be a nonnegative integer. Determine all functions $\mathrm{f}:\mathbb{R}^2\to\mathbb{R}$ such thet, for any real constants A, B, C and D, $f(At+B, Ct+D)$ is a polynomial in t of degree at most d.

Hong Kong-PreIMO 2017 MOCK EXAM

Solution: *We claim that* $f(x,y)$ *is a polynomial in* x *and* y *of degree at most* d. *It is obvious that every such polynomial satisfies the desired condition. To prove the converse, let* f *be a function satisfying the desired condition. Pick* $(d+2)$ *straight lines* $l_1, l_2, \dots, l_{d+2}$ *in* $\mathbb{R}^2$ *such that no two are parallel and no three are concurrent.*
Let the equation of l_i *be* $h_i(x,y)=0$ *where* h_i *is a linear polynomial.*
For $i<j$, *let* (a_{ij}, b_{ij}) *be the intersection of* l_i *and* l_j, *and consider the polynomial*

$$\varphi(x,y)=\sum_{1\le i<j\le d+2} f(a_{ij},b_{ij})\prod_{\substack{k=1\\ k\ne i,j}}^{d+2}\frac{h_k(x,y)}{h_k(a_{ij},b_{ij})}$$

It is easy to see that $\varphi(a_{ij},b_{ij})=f(a_{ij},b_{ij})$ *for all* $i<j$, *and that* $deg\varphi\le d$. *We shall show that* $\varphi(a,b)=f(a,b)$ *for all points* (a,b) *on* l. *Indeed, pick constants* A,B,C *and* D *such that* $t\to(At+B,Ct+D)$ *parametrizes the line. Then, note that* $\varphi(At+B,Ct+D)-f(At+B,Ct+D)$ *for all* (a,b) *on* l. *Now, for a fixed* l_i, *note that* $\varphi(a_{ij},b_{ij})=f(a_{ij},b_{ij})$ *for every* $j\ne i$, *so* $f(a,b)=\varphi(a,b)$ *for all points* (a,b) *on* l_i *from the claim. If* (c,d) *is a point not lying on any* l_i, *then we can construct a line* l *which passes through* (c,d) *does not pass through any* (a_{ij},b_{ij}), *and is not parallel to any* l_i. *Now* $f(a,b)=\varphi(a,b)$ *with* $(a,b)=l_i\cap l$ *for various* i, *so* $f(a,b)=\varphi(a,b)$ *for all* (a,b) *on* l. *In particular,* $\varphi(a,b)=f(c,d)$. *This completes the proof.*

2.24 Let k be a real parameter. Determine the number of real solutions to the system

$$\begin{cases} x^2+kxy+y^2=z\\ y^2+kyz+z^2=x\\ z^2+kzx+x^2=y\end{cases}$$

in terms of x.

Patrik Bak-Czech& Slovak NMO-2017

Solution: *We distinguish several cases.*
First, assume $x=y=z$. *Then the whole system reduces to* $(k+2)x^2=x$. *Its solutions is a triplet* $(0,0,0)$ *for any* k *and moreover triplet* $\left(\frac{1}{k+2},\frac{1}{k+2},\frac{1}{k+2}\right)$ *if* $k\ne-2$. *Let's get back to the original system. Subtracting the second equation from the first one yields*

$$(x^2-z^2)+ky(x-z)=z-x$$

which rewrites as

$$(x-z)(x+z+ky+1)=0;\quad (1)$$

Similarly, subtracting the third equation from the second one yields

$$(y-x)(y+x+kz+1)=0;\quad (2)$$

If $x\ne y\ne z\ne x$, *the equations (1),(2) reduces to*

$$x+z+ky+1=0$$
$$y+x+kz+1=0$$

Subtracting these two equations we arrive at $(y-z)(k-1)=0$ *implying that* $k=1$ *and* $x+y+z=-1$. *However that's impossible since for* $k=1$ *we get*

$$z=x^2+xy+y^2=\left(x+\frac{y}{2}\right)^2+\frac{3y^3}{4}\geq 0$$

and likewise $x\geq 1$ and $y\geq 0$ so altogether $x+y+z\geq 0$.
We found out that in every solution to the original system, some two unknowns have the same value. As the system is cyclic, let us from now on assume $x=y\neq z$ (the case $x=y=z$ has already been solved). Equation (1) then implies $x+y+ky+1=0$, that is $x=-(k+1)y-1$, and the original system to a single equation

$$(k+2)y^2+(k+1)y+1=0;\quad (3)$$

Let us remark thet any solution to equation (3) is a solution we haven't found yet, because equality $x=y$ i.e. $y=-(k+1)y-1$ is only possible for $k\neq -2$ and yields
$x=y=z=-\frac{1}{k+2}$ *which is not solution to the original system.*
For $k=-2$ the equation (3) is linear with a unique solution $y=1$. This yields solution $(0,1,1)$ and its two permutations.

For $k\neq -2$ the equation (3) is quadratic and has real solutions if and only if

$$D=(k+1)^2-4(k+2)=k^2-2k-7\geq 0,$$

which translated to $k\notin\left(1-2\sqrt{2},1+2\sqrt{2}\right)$. For $k=1\pm 2\sqrt{2}$ there is a unique solution
$y_0=-\frac{k+1}{2(k+2)}=1\pm\sqrt{2}$ *a* $x_0=\frac{k+1}{2(k+2)}-1=1$

Which yields three permutations of (x_0,y_0,y_0) as solutions to the original system.
For $k\in(-\infty,-2)\cup\left(-2,1-2\sqrt{2}\right)\cup\left(1+2\sqrt{2},\infty\right)$, the quadratic equation (3) has two distinct solutions

$$y_{1,2}=\frac{-k-1\pm\sqrt{k^2-2k-7}}{2(k+2)}$$

that give two distinct values $x_{1,2}=-(k+1)y_{1,2}-1$. The original system thus has six solutions: three permutations of (x_1,y_1,y_1) and three permutations of (x_2,y_2,y_2).

The following table summarizes the number of solutions to the given system in terms of k:

Interval for k	(0,0,0)	$\left(\frac{1}{k+2},\frac{1}{k+2},\frac{1}{k+2}\right)$	Equation (3)	Total
$(-\infty,-2)$	1	1	6	8

-2	1	0	3	4
$(-2,1-2\sqrt{2})$	1	1	6	8
$1-2\sqrt{2}$	1	1	3	5
$(1-2\sqrt{2},1+2\sqrt{2})$	1	1	0	2
$1+2\sqrt{2}$	1	1	3	5
$(1+2\sqrt{2},\infty)$	1	1	6	8

2.25 Let $k \in [0,1]$. Solve the system

$$\begin{cases} k-x^2=y \\ k-y^2=z \\ k-z^2=u \\ k-u^2=x \end{cases}$$

In real numbers.

Jaroslav Svrcek-Czech and Slovak NMO-2017

Solution: *Subtracting the third equation from the first one we get*

$$z^2-x^2=(z-x)(z+x)=y-u; \quad (1)$$

Similarly, the second and the fourth equation imply

$$y^2-u^2=(y-u)(y+u)=x-z; \quad (2)$$

Relations (1) and (2) then imply that $x=z$ holds if and only if $y=u$ holds. We distinguish two cases. Denote a) and b).

a) First, let us assume that $x=z$ and $y=u$, that is we are looking for quadruplets of the form $(x,y,z,u)=(x,y,x,y)$ with unknown x and y. The original system reduces to

$$\begin{cases} k-x^2=y \\ k-y^2=x \end{cases}$$

Subtracting the equation and rewriting we obtain

$$(y-x)(y+x-1)=0$$

We distinguish two (not quite disjoint) subcases.

If $y-x=0$, the reduced system further reduces to a single quadratic equation

$$x^2+x-k=0.$$

For any $k \in (0,1)$, this equation has two real solutions.

$$x_{1,2}=\frac{-1\pm\sqrt{4k+1}}{2}.$$

The original system therefore has at least two solutions

$$x_1 = y_1 = z_1 = u_1 = \frac{-1-\sqrt{4k+1}}{2}, x_2 = y_2 = z_2 = u_2 = \frac{-1+\sqrt{4k+1}}{2}; \quad (3)$$

If $x + y - 1 = 0$, the reduced system further reduces to quadratic equation

$$x^2 - x + (1-k) = 0.$$

Since its discriminant equals $4k - 3$, this quadratic equation has solution if and only if $k \geq \frac{3}{4}$. The solutions are $x_3 = \frac{1+\sqrt{4k-3}}{2}$ and $x_4 = \frac{1-\sqrt{4k-3}}{2}$ and the corresponding values of $y = 1 - x$ then $y_3 = \frac{1-\sqrt{4k-3}}{2}$ and $y_4 = \frac{1+\sqrt{4k-3}}{2}$.

If $k = \frac{3}{4}$, these solutions are identical and in fact identical with solutions already found in (3). On the other hand, for $\frac{3}{4} < k \leq 1$ we obtain two other distinct solutions

$(x, y, z, u) = (x_3, y_3, z_3, u_3)$ *and* $(x, y, z, t) = (x_4, y_4, z_4, u_4)$; (4)

b) Second, let us assume $x \neq z$ and $y \neq u$. In this case, plugging $x - z$ form (2) into the left-hand side of (1) and dividing by nonzero $y - u$ we arrive at

$$(x+z)(y+u) = -1; \quad (5)$$

Since the right-hand side of (5) is negative, at least one of the numbers x, y, z, u is positive and at least one of them is negative. However, this contradicts a sequence of implications

$$x \geq 0 \Rightarrow y \geq 0 \Rightarrow z \geq 0 \Rightarrow u \geq 0 \Rightarrow x \geq 0, \quad (6)$$

That prove us now:

It suffices to prove the first implication (the proofs of the others are analogous).

Assume $x > 0$. The fourth equation of the original system implies $x \leq k$ and since $k \leq 1$, we have $0 \leq x \leq k \leq 1$. This implies $x^2 \leq k$ (as $t^2 \leq t$ for any $t \in (0,1)$). The first equation of the original system now implies $y \geq 0$ as desired.

Therefore there are no solutions in case b).

Answer. If $0 \leq k \leq \frac{3}{4}$, the system has two solutions given by expressions in (3).

If $\frac{3}{4} < k \leq 1$, the system has four solutions given expressions in (3) and (4).

2.26 For all positive real numbers a, b, c which satisfy the equality:

$$ab\left(1 - \frac{c^2}{(a+b)^2}\right) = bc\left(1 - \frac{a^2}{(b+c)^2}\right) = ca\left(1 - \frac{b^2}{(c+a)^2}\right)$$

Vladislav Yurushev-Ukrainian NMO-2016

Solution: *Rewrite the given equality in the following way:*

$$\frac{a((a+b)^2-c^2)}{(a+b)^2}=\frac{c((b+c)^2-a^2)}{(b+c)^2}\ or$$

$$\frac{a(a+b-c)(a+b+c)}{(a+b)^2}=\frac{c(b+c-a)(b+c+a)}{(b+c)^2}$$

Suppose that $c\ge a+b$. Then $b+c>a$, so the left side of the equality is not positive while the right side is positive. This contradiction means that $b+c>a, a+c>b$ and $b+a>c$ so we can treat the sides a,b,c as sides of a triangle.

But the given equalities literally mean that the angle bisectors of this triangle equal. Indeed:

$w_c^2=ab-a_1b_1$, where a_1,b_1 are the lengths of two parts of the opposite side which are derived after drawing the angle bisector. Then we have: $a_1+b_1=c$ and $\frac{a_1}{b_1}=\frac{a}{b}$. It is easy to see that $a_1=\frac{ac}{a+b}$ and $b_1=\frac{bc}{b+c}$. Then $w_c^2=ab-a_1b_1=ab-\frac{abc^2}{(a+b)^2}=ab\left(1-\frac{c^2}{(a+b)^2}\right)$. Since if three angle bisectors are equal he triangle is regular, we obtain that the solutions are (t,t,t), where $t>0$.

2.27 Compare the following numbers:

$$\boldsymbol{A=11, B=\log_2 3\cdot\log_3 4\cdot\log_4 5\cdot\ldots\cdot\log_{2015}2016}$$

$$\boldsymbol{C=\log_3 2\cdot\log_4 3\cdot\log_5 4\cdot\ldots\cdot\log_{2016}2015}$$

Ukrainian NMO-2017

Solution: *Obviously, for any integer $n>0, log_{n+1}n<1$. Hence, $C<1$. Also we have that $B\cdot C=1$, so $B>1$, moreover*

$$B=\frac{lg3}{lg2}\cdot\frac{lg4}{lg5}\cdot\frac{lg5}{lg6}\cdot\ldots\cdot\frac{lg2016}{lg2015}=\frac{lg2016}{lg2}<11=A\Leftrightarrow lg2016<11lg2$$

$$\Leftrightarrow 2016<2^{11}=2048.$$

2.28 Let $f(x)=ax^2+bx+c$ be a polynomial with integer coefficients. For every integer x, $f(x)$ is divisible by N where N is a positive integer. Is it true that N necesary divisible all the coefficients of $f(x)$ if

$$\boldsymbol{a)\ N=2016\qquad b)\ N=2017?}$$

Ukrainian NMO-2017

Solution:

a) Note that for any integer x product $x(x+1)$ is even. This suggests the following example:

$$1008x(x+1)+2016=1008x^2+1008x+2016.$$

b) We have $f(x)=ax^2+bx+c$. Let us do the following substitutions:

$$x=0\Rightarrow f(0)=c \vdots 2017$$
$$x=1\Rightarrow f(1)=(a+b+c) \vdots 2017$$
$$x=-1\Rightarrow f(-1)=(a-b+c) \vdots 2017$$

Then both $a+b$ and $a-b$ are divisible by 2017, so are $2a$ and $2b$. Since 2017 is odd all the coefficients are divisible by 2017.

2.29 The sequence $a=(a_0,a_1,a_2,\dots)$ is defined by $a_0=0, a_1=2$ and $a_{n+2}=2a_{n+1}+41a_n$ for all $n\geq 0$.

Prove that a_{2016} is divisible by *2017.*

Tom Laffey-Ireland SHL-2017

Solution: *Consider the recursion modulo 2017 and go back eight steps*

$$a_{8k+8}=2a_{8k+7}+41a_{8k+6}=45a_{8k+6}+82a_{8k+5}$$
$$=172a_{8k+5}+1845a_{8k+4}$$
$$\equiv 1345a_{8k+4}+1001a_{8k+3}(mod 2017)$$
$$\equiv 1674a_{8k+2}+686a_{8k+1}(mod 2017)\equiv 56a_{8k}(mod 2017).$$

Because $a_0=0$ it now follows by induction that 2017 divides a_{8k} for every non-negative integer k. As 2016 is divisible by 8, the result follows.

2.30 For $0\leq x\leq 1$, let $f_n(x)=\sqrt[n]{x^n+(1-x)^n}, n=1,2\dots$

Prove for all $0\leq x\leq 1$ that: $f_{n+1}(x)\leq f_n(x), n=1,2\dots$

Finbar Holland-Ireland NMO-2017

Solution: *First observe that $f_n\left(\frac{1}{2}+c\right)=f_n\left(\frac{1}{2}-c\right)=\sqrt[n]{\left(\frac{1}{2}+c\right)^n+\left(\frac{1}{2}-c\right)^n}$.*

Hence, it is sufficient to consider x in the range $0\leq x\leq\frac{1}{2}$.

For such x, put $t=\frac{x}{1-x}$, so that $0\leq t\leq 1$. Define $g_n(t)=\sqrt[n]{t^n+1}$ to obtain

$f_n(x) = (1-x)g_n(t)$. *So, the problem comes down to showing that* $g_n(t) \ge g_{n+1}(t)$, *equivalently,* $(t^n+1)^{n+1} \ge (t^{n+1}+1)^n$ *whence, by binomial expansion, the claim is that*

$$\sum_{k=0}^{n+1}\binom{n+1}{k}t^{nk} \ge \sum_{k=0}^{n}\binom{n}{k}t^{(n+1)k}, i.e.$$

$$\sum_{k=0}^{n-1}\left(\binom{n+1}{k}-\binom{n}{k}\right)t^{nk}(1-t^k) + \binom{n+1}{n}t^{n^2} \ge 0.$$

But $\binom{n+1}{k}-\binom{n}{k} = \binom{n}{k-1} \ge 0$ *and* $1-t^k \ge 0$ *for* $0 \le t \le 1$, *hence the result.*

2.31 1. If $a, b > 0$ then: $b^b \cdot e^{a+\frac{1}{a}} \ge 2e^b$

2. If $a > 0, 0 < b \le 1$ then: $b^b \cdot e^{1+\frac{1}{a}} \ge 2b \cdot e^b$

Abdallah El Farissi

Solution (Soumitra Mandal)

$$1.\text{Let } a, b > 0 \text{ then } b^b \cdot e^{a+\frac{1}{a}} \ge 2e^b$$

$$\text{Now } b\ln b + a + \frac{1}{a} - \ln 2 - b = b\ln b + \left(a+\frac{1}{a}-2\right) + 2 - \ln 2 - b$$

$$\ge b\ln b + \left(a+\frac{1}{a}-2\right) + 2 + 1 - e^{\ln 2} - b \text{ since, } e^{\ln 2} \ge 1 + \ln 2$$

$$\ge b(\ln b - 1) + \left(a+\frac{1}{a}-2\right) + 1 \ge b\left(\frac{b-1}{b}-1\right) + \left(a+\frac{1}{a}-2\right) + 1$$

$$\because \ln(x+1) \ge \frac{x}{x+1}$$

$$= a + \frac{1}{a} - 2 \ge 0$$

$$\text{Hence, } b\ln b + a + \frac{1}{a} \ge \ln 2 + b \Rightarrow b^b \cdot e^{a+\frac{1}{a}} \ge 2e^b \text{ (proved)}$$

$$2.\text{Let } a > 0, 0 < b \le 1 \text{ then } b^b \cdot e^{1+\frac{1}{a}} \ge (2e)^b$$

$$\text{Now, } b\ln b + a + \frac{1}{a} - b\ln 2 - b$$

$$= b\ln b + 2 + \left(a+\frac{1}{a}-2\right) - b\ln 2 - b$$

$$\ge b\ln b + 2 + \left(a+\frac{1}{a}-2\right) + b(1-e^{\ln 2}) - b \text{ since, } e^{\ln 2} \ge 1 + \ln 2$$

$$\ge b\left(\frac{b-1}{b}\right) + 2(1-b) + \left(a+\frac{1}{a}-2\right) \text{ since, } \ln(1+x) \ge \frac{x}{x+1} \text{ for all } x \ge 0$$

$$= 1 - b + \left(a+\frac{1}{a}-2\right) \ge 0$$

$$\because 0 < b \le 1 \text{ and } a + \frac{1}{a} \ge 2$$

$$\text{Hence, } b\ln b + a + \frac{1}{a} \ge b\ln 2 + b \Rightarrow b^b \cdot e^{a+\frac{1}{a}} \ge (2e)^b \text{ (proved)}$$

2.32 If $a_1, a_2, \dots, a_n \in [0,1), n \in \mathbb{N}^*$ then:

$$\prod_{k=1}^{n} \frac{1+a_k}{1-a_k} \ge \left(\frac{1+\sqrt[n]{a_1a_2\dots a_n}}{1-\sqrt[n]{a_1a_2\dots a_n}}\right)^n$$

Regragui El Khammal

Solution:

$$1-a_1 \ge 1-a_2 \ge \dots \ge 1-a_n; \frac{1}{1-a_1} \le \frac{1}{1-a_2} \le \dots \le \frac{1}{1-a_n}$$

$$\frac{1+a_1}{1-a_1} \le \frac{1+a_2}{1-a_2} \le \dots \le \frac{1+a_n}{1-a_n}$$

$$\prod_{k=1}^{n} \frac{1+a_k}{1-a_k} \ge \left(\frac{1+a_1}{1-a_1}\right)^n = f(a_1), (1)$$

$$f:[0,1) \to \mathbb{R}, f(x) = \frac{1+x}{1-x}, f'(x) = \frac{-2x}{(1-x)^2} < 0, f \text{ decreasing}$$

$$a_1 \le \sqrt[n]{a_1a_2\dots a_n}; f(a_1) \ge f\left(\sqrt[n]{a_1a_2\dots a_n}\right)$$

By (1):

$$\prod_{k=1}^{n} \frac{1+a_k}{1-a_k} \ge \left(\frac{1+a_1}{1-a_1}\right)^n = \left(f(a_1)\right)^n \ge \left(f\left(\sqrt[n]{a_1a_2\dots a_n}\right)\right)^n$$

$$\prod_{k=1}^{n} \frac{1+a_k}{1-a_k} \ge \left(\frac{1+\sqrt[n]{a_1a_2\dots a_n}}{1-\sqrt[n]{a_1a_2\dots a_n}}\right)^n$$

2.33 Solve for real numbers:

$$3^{\sin^2 x+\sin x} + 3^{\sin^2 y+\sin y} + 3^{\sin^2 z+\sin z} = 3^{\sin x+\sin y+\sin z}$$

Daniel Sitaru

Solution(Avishek Mitra)

$$3^{\sin^2 x+\sin x} + 3^{\sin^2 y+\sin y} + 3^{\sin^2 z+\sin z} \overset{AM-GM}{\ge} 3\left(3^{\sum \sin^2 x+\sum \sin x}\right)^{\frac{1}{3}}$$

$$\Rightarrow \left(3^{\sin x+\sin y+\sin z}\right)^3 \geq 27\cdot 3^{\sum \sin^2 x+\sum \sin x}$$

$$\Rightarrow 3^{3(\sin x+\sin y+\sin z)} \geq 3^{3+\sum \sin^2 x+\sum \sin x}$$

$$\Rightarrow 3\sum \sin x \geq 3+\sum \sin^2 x+\sum \sin x \Rightarrow \sum \sin^2 x - 2\sum \sin x + 1 \leq 0$$

$$\Rightarrow (\sin x-1)^2+(\sin y-1)^2+(\sin z-1)^2 \leq 0$$

But for any real $x, y, z \Rightarrow (\sin x-1)^2+(\sin y-1)^2+(\sin z-1)^2 \geq 0$

So, this is possible if and only if

$$\Leftrightarrow (\sin x-1)^2+(\sin y-1)^2+(\sin z-1)^2 = 0$$

$$\Rightarrow (\sin x-1)^2 = (\sin y-1)^2 = (\sin z-1)^2 = 0 \Rightarrow$$

$$\sin x = \sin y = \sin z = 1$$

$$x = 2k\pi+\frac{\pi}{2},\qquad y = 2p\pi+\frac{\pi}{2},\qquad z = 2q\pi+\frac{\pi}{2}$$

$$k, p, q \in \mathbb{Z}$$

2.34 Let the sequence $(a_n)_{n\geq 1}$ be defined as:

$$a_n = \sqrt{A_{n+2}^1\sqrt[3]{A_{n+4}^2\sqrt[4]{A_{n+4}^3\sqrt[5]{A_{n+5}^4}}}}$$

where A_m^k are arrangements: $A_m^k = \binom{m}{k}\cdot k!$

Prove that $a_n < \frac{119}{120}\cdot n+\frac{7}{3}$

Moldova NMO-2017

Solution: This problem is immediately by AM-GM inequality

$$a_n = \sqrt{A_{n+2}^1\sqrt[3]{A_{n+4}^2\sqrt[4]{A_{n+4}^3\sqrt[5]{A_{n+5}^4}}}}$$

$$= \sqrt{(n+2)\sqrt[3]{(n+2)(n+3)\sqrt[4]{(n+2)(n+3)(n+4)\sqrt[5]{(n+2)(n+3)(n+4)(n+5)}}}}$$

$$= \sqrt[120]{(n+2)^{86}(n+3)^{26}(n+4)^5(n+5)}$$

$$= \frac{118n+277}{120} < \frac{119}{120}\cdot n + \frac{7}{3}.$$

2.35 For all positive real numbers x, y and z prove the following inequality:

$$\frac{x^2}{xy+z} + \frac{y^2}{yz+x} + \frac{z^2}{zx+y} \geq \frac{(x+y+z)^3}{3[x^2(y+1)+y^2(z+1)+z^2(x+1)]}$$

Tonci Kokan-Croatian NMO-2015

Solution: *By the CBS inequality we have*

$$\left(\frac{x^2}{xy+z} + \frac{y^2}{yz+x} + \frac{z^2}{zx+y}\right)[x(xy+z)+y(yz+x)+z(zx+y)]$$
$$\geq \left(x\sqrt{x}+y\sqrt{y}+z\sqrt{z}\right)^2; \quad (1)$$

By the inequality between power means we have

$$\left(\frac{x\sqrt{x}+y\sqrt{y}+z\sqrt{z}}{3}\right)^{\frac{2}{3}} \geq \frac{x+y+z}{3},$$

i. e.

$$\left(x\sqrt{x}+y\sqrt{y}+z\sqrt{z}\right)^2 \geq \frac{(x+y+z)^3}{3}; \quad (2)$$

By the AM-GM inequality we have

$$x^2+y^2+z^2 = \frac{x^2+y^2}{2} + \frac{y^2+z^2}{2} + \frac{z^2+x^2}{2} \geq xy+yz+zx; \quad (3)$$

So, from (1),(2) and (3) it follows that

$$\frac{x^2}{xy+z}+\frac{y^2}{yz+x}+\frac{z^2}{zx+y}\ge\frac{\left(x\sqrt{x}+y\sqrt{y}+z\sqrt{z}\right)^2}{x^2y+y^2z+z^2x+zx+xy+yz}$$

$$\ge\frac{(x+y+z)^3}{3(x^2y+y^2z+z^2x+zx+xy+yz)}$$

$$\ge\frac{(x+y+z)^3}{3[x^2(y+1)+y^2(z+1)+z^2(x+1)]}$$

2.36 Find all quadruplets (a,b,c,d) of real numbers satisfying the system

$$\begin{cases}(a+b)(a^2+b^2)=(c+d)(c^2+d^2)\\(a+c)(a^2+c^2)=(b+d)(b^2+d^2)\\(a+d)(a^2+d^2)=(b+c)(b^2+c^2)\end{cases}$$

Czech-Polish-Slovak Match-2016

Solution: *Let us set* $f(x,y)=(x+y)(x^2+y^2)$

We'll show that for any real x *the inequality* $y\ge z$ *implies* $f(x,y)\ge f(x,z)$. *After subtraction we see that*

$$f(x,y)-f(x,z)=\frac{1}{2}(y-z)((x+y)^2+(y+z)^2+(z+x)^2)\ge 0$$

Moreover, equality occurs when $y=z$ *or* $x=y=z=0$, *so either way it implies* $y=z$.

We can rewrite the system (implicitly using the symmetry of f*) to the form:*

$$\begin{cases}f(a,b)=f(c,d)\\f(a,c)=f(b,d)\\f(a,d)=f(b,c)\end{cases}$$

Now we can see that the system is symmetric in variables a,b,c,d *and may assume* $a=max\{a,b,c,d\}$. *We then write the chain of (in)equalities*

$$f(c,d)=f(a,b)\ge f(c,b)=f(a,d)\ge f(b,d)=f(a,c)\ge f(d,c)$$

and since we in fact have equality everywhere, we deduce that $a=b=c=d$.

All such quadruplets clearly satisfy the system so the problem is solved.

2.37 Let $a,b,c\ge 0$. Prove that

$$\sqrt[3]{(a+b)(b+c)(c+a)}\ge\sqrt[6]{\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{8}}+\sqrt[3]{abc}$$

E.Enkhzaya-Mongolian NMO-2010

Solution: *By Am-GM inequality*

$$3=\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}\right)+\left(\frac{x_1}{1+x_1}+\frac{x_2}{1+x_2}+\frac{x_3}{1+x_3}\right)\ge$$

$$\ge 3\sqrt[3]{\frac{1}{(1+x_1)(1+x_2)(1+x_3)}}+3\sqrt[3]{\frac{x_1x_2x_3}{(1+x_1)(1+x_2)(1+x_3)}}$$

for $x_1,x_2,x_3>0$.

Now, if $x_i=\frac{a_i}{b_i}, a_i, b_i>0$, *from above it implies*

$$\sqrt[3]{(a_1+b_1)(a_2+b_2)(a_3+b_3)}\ge\sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}.\quad(*)$$

Also $x+y\le\sqrt{2(x^2+y^2)}$ *holds for* $x,y>0$.

If $x=\sqrt{\frac{a^2+b^2}{2}}$ *and* $y=\sqrt{ab}$ *then it implies*

$$\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}\le a+b.\quad(1)$$

Using (1) and $(*)$, *we have*

$$\sqrt[3]{(a+b)(b+c)(c+a)}\overset{(1)}{\ge}\sqrt{\prod_{cyc}\left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}\right)}\overset{(*)}{\ge}$$

$$\ge\sqrt[3]{\sqrt{\frac{a^2+b^2}{2}}\cdot\sqrt{\frac{b^2+c^2}{2}}\cdot\sqrt{\frac{c^2+a^2}{2}}}+\sqrt[3]{\sqrt{ab}\cdot\sqrt{bc}\cdot\sqrt{ca}}=$$

$$=\sqrt[6]{\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{8}}+\sqrt[3]{abc}$$

Equality holds for $a=b=c$.

2.38 Prove that $6\nmid\left[\left(\sqrt[3]{28}-3\right)^{-n}\right]$, **for all n**

B.Amarsanaa-Mongolian NMO-2010

Solution: *Using* $a^3-b^3(a-b)(a^2+ab+b^2)$,

$$\left(\sqrt[3]{28}-3\right)^{-n}=\left(\left(\sqrt[3]{2m}\right)^2+3\cdot\sqrt[3]{28}+3^3\right)^n=\sum_{k+l+m=n}\frac{n!}{k!\,l!\,m!}\left(\sqrt[3]{28}\right)^{2k+l}3^{l+2m}$$

Let $\varepsilon = cos\frac{2\pi}{3} + i\,sin\frac{2\pi}{3}$. *Consider*

$$S = \left(\left(\sqrt[3]{2m}\right)^2 + 3\cdot\sqrt[3]{28} + 3^3\right)^n + \left(\left(\sqrt[3]{2m\varepsilon}\right)^2 + 3\cdot\sqrt[3]{28}\varepsilon + 3^3\right)^n +$$

$$+\left(\left(\sqrt[3]{2m\varepsilon^2}\right)^2 + 3\cdot\sqrt[3]{28}\varepsilon^2 + 3^3\right)^n =$$

$$= \sum_{k+l+m=n}\frac{n!}{k!\,l!\,m!}\left(\sqrt[3]{28}\right)^{2k+l}3^{l+2m}\left(1 + \varepsilon^{2k+l} + \varepsilon^{4k+2l}\right)$$

If $2k + l \not\equiv 0\ (mod\ 3)$, *then* $1 + \varepsilon^{2k+l} + \varepsilon^{4k+2l} = 0$.

If $2k + l \equiv 0\ (mod\ 3)$, *then* $1 + \varepsilon^{2k+l} + \varepsilon^{4k+2l} = 3$. *Hence we get that* $S \in \mathbb{Z}$ *and* $S \equiv 3\ (mod\ 3)$. *On the other hand, it is easy to check that*

$$A = \left|\left(\sqrt[3]{2m\varepsilon}\right)^2 + 3\cdot\sqrt[3]{28}\varepsilon + 3^3\right| < 1,\ B = \left|\left(\sqrt[3]{2m\varepsilon^2}\right)^2 + 3\cdot\sqrt[3]{28}\varepsilon^2 + 3^3\right| < 1$$

From this implies that $\left[\left(\sqrt[3]{28} - 3\right)^{-n}\right] = [S - A - B] \not\equiv 0(mod\ 6)$

2.39 Positive integers a, p satisfies: $p = 2^a - 1$. Find all a such that $\frac{1}{2}(p^2 + 1)$ is a square of integer.

Klurman Olekisy-Ukrainian NMO-2015

Solution: *For* $a = 1$ *we have* $p = 1$ *and* $\frac{1}{2}(p^2 + 1) = 1$ *satisfies the problem.*

For $a = 2$ *we have* $p = 3$ *and* $\frac{1}{2}(p^2 + 1) = 5$ *doesn't satisfy the problem.*

Assume now $a \geq 3$. *Let* $\frac{1}{2}(p^2 + 1) = p_1^2$, *then* $p^2 - 2p_1^2 = -1$.

Hence: $2^{2a} - 2^{a+1} + 1 - 2p_1^2 = -1$ *or* $2^{2a-1} - 2^a = p_1^2 - 1$. *So* $2^a(2^{a-1} - 1) = (p_1 - 1)(p_2 - 1)$. *LHS is even, so as RHS. So* $gcd(p_1 - 1, p_2 + 1) = 2$. *So, only the following cases are possible.*

1) $p_1 + 1 = 2l$ *and* $kl = 2^{a-1} - 1$.

If $k \geq 2, p_1 \geq 2^a + 1$ *and* $l \geq 2^{a-1} + 1$, *a contradiction with* $kl = 2^{a-1} - 1$.

If $k = 1, p_1 = 2^{a-1} + 1$*and* $kl = 2^{a-2} + 1 = 2^{a-1} - 1$. *Hence* $a = 3$.

2) $p_1 - 1 = 2k, p_1 + 1 = 2^{a-1}l$, *and* $kl = 2^{a-1} - 1$.

If $l \geq 2, p_1 \geq 2^a - 1$ *and* $k \geq 2^{a-1} - 1$, *so* $2^{a-1} = kl \geq 2^a - 2$, *hence* $a = 1$.

If $l = 1, p_1 = 2^{a-1} - 1$ *and* $k = 2^{a-2} - 1$ *and* $kl = 2^{a-2} - 1 = 2^{a-1} - 1$ *–contradiction.*

2.49 Find the maximum amount of 3-element sets that every two of them contain exactly one common element, but there exist no element that is in all sets simultaneously.

Ukrainian NMO-2015

Solution: *Answer: 7 sets. Suppose that there exist at least 8 such sets, let* $M = \{x, y, z\}$ *is one of such sets. Every other set (there are no less than 7 sets) contains exactly one element from* M. *Therefore there exist at least three sets* M_1, M_2 *and* M_3 *that have one common element from* M, *for example* x. *Since there is no element that is in all sets simultaneously, there exist a set* M_0 *that does not contain* x. *As this set intersect* M, *it contain another element of* M, *for example* y. *This set can not contain the same common element with two sets of* M_1, M_2 *and* M_3, *otherwise this element is the second common element of that pair of sets. Therefore this set have to contain common element with every of these sets, but not* y. *So, the set have to contain at least 4 elements, and received contradiction ends proof. The example of 7 such sets:*

$$\{1;2;3\};\{1;4;5\};\{1;6;7\};\{3;5;7\};\{3;4;6\};\{2;4;7\};\{2;5;6\}.$$

2.41 Primes p, q, r, such that $p + q < 111$, fulfill the equality $\frac{p+q}{r} = p - q + r$. Find the maximum value of the product pqr.

Ukrainian NMO-2015

Solution: *Answer: 2014. Rewrite the condition of the problem in such way:* $q(r + 1) - p(r - 1) = r^2$. *If* $r > 2$, *then* r *is odd, but in this case the left part* *is even. Thus* $r = 2$. *Hence, the initial condition becomes:* $p = 3q - 4$, *so the maximum* pqr *is in case of maximum* q. *From the condition* $p + q < 111$ *follows that* $q < 29$.*If* $q = 23$ *then* $p = 65$ *is not a prime.If* $q = 21$ *then* $p = 53$ *is a prime, so it is a required number. Hence* $pqr = 53 \cdot 19 \cdot 2 = 2014$.

2.42 Consider the numbers $a = (3^4)^5$, $b = (4^4)^4$, and $c = (5^4)^3$. If you sort a, b and c from smallest to largest, you obtain:

A) $a < b < c$ ***B)*** $a < c < b$ ***C)*** $b < a < c$ ***D)*** $c < a < b$
E) $c < b < a$

Germany NMO-20174

Solution: *D)* $c < a < b$

2.43 Can real numbers x, y, z satisfy:

$$\frac{1}{(x-y)(x+y)} + \frac{1}{(y-z)(y+z)} + \frac{1}{(z-x)(z+x)} = 0\ ?$$

Bogdan Rublyov-Ukrainian NMO-2015

Solution: *Answer: no. Denote* $a = x^2 - y^2, b = y^2 - z^2$ *then*$-a - b = z^2 - x^2$ *and the equation can be rewritten as:* $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b} \Leftrightarrow (a+b)^2 = ab \Leftrightarrow a^2 - ab + b^2 = 0 \Leftrightarrow \left(a - \frac{b}{2}\right)^2 + \frac{3b^2}{4} = 0$. *The last equality holds only with* $a = b = 0$, *which is impossible. Therefore, the equation has no solutions.*

2.44 Solve in the real numbers the system

$$\begin{cases} x^3 = \dfrac{z}{y} - \dfrac{2y}{z}, \\ y^3 = \dfrac{x}{z} - \dfrac{2z}{x} \\ z^3 = \dfrac{y}{x} - \dfrac{2x}{y} \end{cases}$$

A.Fellouris-Hellenic NMO-2014

Solution: *For* $x, y, z \in \mathbb{R}$, *such that* $xyz \neq 0$, *the system is written:*

$$\begin{cases} x^3yz = z^2 - 2y^2; & (1) \\ y^3zx = x^2 - 2z^2; & (2) \\ z^3xy = y^2 - 2x^2; & (3) \end{cases}$$

Using summation by parts we find:

$$xyz(x^2 + y^2 + z^2) = -(x^2 + y^2 + z^2) \Leftrightarrow (x^2 + y^2 + z^2)(xyz + 1) = 0.$$

Since $xyz \neq 0$ *we have* $x^2 + y^2 + z^2 > 0$, *and so* $xyz = -1$; (4)

Using equation (4) *in the system of* (1) – (3) *we get:*

$$\begin{cases} x^2 = -z^2 + 2y^2; & (5) \\ y^2 = -x^2 + 2z^2; & (6) \\ z^2 = -y^2 + 2x^2; & (7) \end{cases}$$

From (5) *and* (6) *we get* $y^2 = z^2$, *while from* (6) *and* (7) *we get* $x^2 = z^2$, *and so:* $x^2 = y^2 = z^2 \Leftrightarrow x = y = \pm z$ *or* $x = -y = \pm z$; (8)

Finally from equations (8) *and* (4) *we have the solutions:*

$$(x, y, z) \in \{(-1-1,-1); (1,1,-1); (1,-1,1); (-1,1,1)\}$$

2.45 Compare A to 0, where:

$$\boldsymbol{a)}\ \boldsymbol{A = 1 - 2 - 3 + 4 + 5 - 6 - 7 + \cdots + 2012 + 2013 - 2014 - 2015 + 2016}$$

$$\boldsymbol{b)}\ \boldsymbol{A = \frac{1}{1} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{2012} + \frac{1}{2013} - \frac{1}{2014} - \frac{1}{2015} + \frac{1}{2016}}$$

Ukrainian NMO-2015

Solution: *Answer:* $a)\ A = 0; b)\ A > 0$

a) *If we split all numbers into 504 groups of 4, from left to right, each group will have numbers of the type:* $\big((4k + 1) - (4k + 2) - (4k + 3) + (4k + 4)\big) = 0.$

As we can see, the sum of numbers in each group is 0, therefore, $A = 0$.

b) Like in the previous question, split all numbers into groups of 4:

$$\left(\frac{1}{4k+1}-\frac{1}{4k+2}-\frac{1}{4k+3}+\frac{1}{4k+4}\right)$$

Then the sum of numbers in each group is positive:

$$\frac{1}{4k+1}-\frac{1}{4k+2}-\frac{1}{4k+3}+\frac{1}{4k+4}>0\Leftrightarrow\frac{1}{4k+1}-\frac{1}{4k+2}>\frac{1}{4k+3}-\frac{1}{4k+4}$$

$$\frac{1}{(4k+1)(4k+2)}>\frac{1}{(4k+3)(4k+4)}.$$

2.46 Find all integers a, b for which there exist integers x, y such that the following equation holds:

$$\boldsymbol{8x^4+8y^4=a^4+6a^2b^2+b^4}$$

Bogdan Rublyov Ukraine NMO-2015

Solution: *Answer: all pairs a,b of the same parity. If a,b has the same parity, define x,y as: $x=\frac{a+b}{2}$, $y=\frac{a-b}{2}$. Then they are obviously integers, and by substituting them we verify that the equation does hold. If a,b don't have the same parity, the right-hand side of the equation is an odd integer. For example, if a is even and b is odd, then $a^4+6a^2b^2$ is even and b^4 is odd, therefore, the equality cannot hold.*

2.47 Find the smallest integer m for which there are positive integers $n>k>1$ satisfying the equation: $\underbrace{\mathbf{11\ldots1}}_{n}=\underbrace{\mathbf{11\ldots1}}_{k}\cdot\boldsymbol{m}$

Bogdan Rublyov-Ukrainian NMO-2015

Solution: *Answer: $m=101$. Obviously $m>9$. If $m=\overline{ab}$, where $a\geq1$, then the equality $\underbrace{11\ldots1}_{n}=\underbrace{11\ldots1}_{k}\cdot\overline{ab}$ implies that $b=1$. But in this case* *regardless of* a *the second last digit of the product* $\underbrace{11\dots1}_{k}\cdot\overline{ab}$ *is equal to* $a+1$ *if* $a<9$ *or to 0 if* $a=9$, *hence, it can't be 1. Therefore* $m\ge 100$. *Clearly* $m=100$ *doesn't satisfy the condition, because* $\underbrace{11\dots1}_{k}\cdot 100=\underbrace{11\dots1}_{k}00$.

On the other hand, $m=101$ *does, because* $101\cdot 11=1111$.

2.48 The road between A and B is 15 Km long, firstly the road goes up, then it is fla, and lasly it goes down. It is known the every part is no less than 1 km. The path made by a pedestrian takes exactly 3 hours. What are the minimum and the maximum amount of time that is taken by the path in opposite direction, if it is known that the speed of pedestrian while going up is 4 km per hour, while going straight is per hour and is 6 per hour while going down?

Bogdan Rublyov-Ukraine NMO-2015

Solution: *Answer:* $t_{max}=\frac{97}{30}, t_{min}=\frac{73}{24}$. *Mark the up, flat and down pars on the way from* A *and* B *as* x,y,z *respectively. Then* $x+y+z=15;\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=3;\ 1\le x,y,z\le 13$. *From the first equation:* $y=15-x-z$, *substitute it into the second equation:*

$$\frac{x}{4}+\frac{15-z-x}{5}+\frac{z}{6}=3\Leftrightarrow\frac{x}{4}-\frac{x}{5}=\frac{z}{5}-\frac{z}{6}\Leftrightarrow\frac{x}{20}=\frac{z}{30}\Leftrightarrow z=\frac{3}{2}x$$

Then $y=15-x-z=15-x-\frac{3}{2}x=15-\frac{5}{2}x$

So the required time is:

$$t=\frac{x}{6}=\frac{y}{5}+\frac{z}{4}=\frac{x}{6}+3-\frac{x}{2}+\frac{3}{8}x=3+\frac{x}{24}$$

The maximum (the minimum) t *can be in case of* x *is maximum of the problem.*

Put down the limitation for x, *which follow from the condition of the problem:*

$$1\le x\le 13; 1\le z=\frac{3}{2}x\le 13\Leftrightarrow\frac{2}{3}\le x\le\frac{26}{3}; 1\le y=15-\frac{5}{2}x\le 13\Leftrightarrow$$

$\frac{4}{5} \le \frac{28}{5}$. *Since all conditions have to be fulfilled simultaneously, we have such limitation for* x:

$$1 \le x \le \frac{28}{5}$$

If $x = 1$, *then* $z = \frac{3}{2}$ *and* $y = \frac{25}{2}$. *If* $x = \frac{28}{5}$, *then* $z = \frac{42}{5}$ *and* $y = 1$.

$$t_{max} = 3 + \frac{1}{24} \cdot \frac{28}{5} = 3 + \frac{7}{30} = \frac{97}{30}; t_{min} = 3 + \frac{1}{24} \cdot 1 = \frac{73}{24}$$

2.49 The const of one kilo of chocolate$-x$ UAH and a kilogram of potatoes$-y$ UAH, numbers x and y are positive integers and have not more than 2 digits. Mother said to Mary to buy 200 grams of chocolates and 1 kg of potatoes that cost exactly $N\ UAH$. Mariyko confuses all and bought 200 grams of potatoes and 1 kg of chocolates. He had to pay exactly $M > N\ UAH$. It turned out that the number M, N have no more than two digits and are formed of the same digits, bu in a different order. How much is a kilogram of potatoes and a kilogram of candies cost?

Bogdan Rublyov-Ukrainian NMO-2015

Solution: *Let the chocolate and potato are respectively* x *and* y UAH, *thus* $N = \overline{ab} = 10a + b$; $M = \overline{ba} = 10b + a$. $M > N$ *so* $b > a > 0$. *We have:*

$\frac{1}{5}x + y = 10a + b$ *and* $\frac{1}{5}y = x = 10b + a$, *or* $x + 5y = 50a + 5b$ *and* $y + 5x = 50b + 5a$. *Where* $b > a > 0$ *–digits,* $x > y$. *Then*

$$4(x - y) = 45(b - a); \quad (1)$$

$$6(x + y) = 55(b + a); \quad (2)$$

So $b - a \vdots 4$ *and* $b + a \vdots 6$. *Then:* $b = 5, a = 1$ *or* $b = 8, a = 4$.

Case 1. $b = 5, a = 1$, *then* $\begin{cases} x - y = 45 \\ x + y = 55 \end{cases}$. *So* $x = 50, y = 5$.

Case 2. $b = 8, a = 74$, *then* $\begin{cases} x - y = 45 \\ x + y = 110 \end{cases}$. *So,* x, y *–nonintegers.*

2.50 It is known that the arithmetic average of the numbers a, b is equal to the number c, so $c = \frac{1}{2}(a + b)$, and that the harmonious averge number of a, c is equal to the number b, so $b = \frac{2}{\frac{1}{a}+\frac{1}{c}}$. Is it necessary that numbers a, b, c are equal?

Bogdan Rublyov-Ukrainian NMO-2015

Solution: *Answer: no necessarily. Let's rewrite the condition of harmonious average:* $b = \frac{2ac}{a+c}$

$$2a \cdot \frac{a+b}{2} = b\left(a + \frac{a+b}{2}\right) \Leftrightarrow a^2 + ab = ba + \frac{ab + b^2}{2} \Leftrightarrow$$

$$2a^2 = ba + b^2 \Leftrightarrow (a - b)(2a + b) = 0.$$

Let's denote, for example, $a = 2$, *which means* $b = -4$ *and* $c = -1$, *hence we receive three different numbers satisfying the conditions.*

2.51 Determine the smallest integer n, for which there exist integers $x_1, x_2, \dots, x_n$ and positive integers $a_1, a_2, \dots, a_n$ so that

$$x_1 + \dots + x_n = 0, a_1x_1 + \dots + a_nx_n > 0, a_1^2x_1 + \dots + a_n^2x_n < 0.$$

Mediterranean MO-2017

Solution: *The answer is* $n = 3$. *One possible example for* $n = 3$ *is* $x_1 = 2$ *and* $x_2 = x_3 = -1$, *with* $a_1 = 4, a_2 = 1, a_3 = 6$.

For $n = 1$, *the first constraint enforces* $x_1 = 0$; *this is in contradiction with the other two constrains. For* $n = 2$, *the first constraint enforces* $x_2 = -x_1$. *Then the second constraint is equivalent to* $a_1x_2 - a_2x_1 > 0$. *If we multiply this inequality by the positive value* $a_1 + a_2$, *we get* $a_1^2x_1 - a_2^2x_1 > 0$: *this is equivalent to* $a_1^2x_1 + a_2^2x_1 > 0$ *and contradicts he third constraint.*

2.52 Let a and b be positive real numbers, such that their product is 1 and the sum of their squares is 4. Find the exact value of the expression $a^{-3} + b^{-3}$.

Slovenia NMO-2013

Solution: *From $a^2 + b^2 = 4$ and $ab = 1$ we get $(a + b)^2 = a^2 + b^2 + 2ab = 4 + 2 = 6$, or $a + b = \sqrt{6}$ since a and b are positive. This implies*

$$\frac{1}{a^3} + \frac{1}{b^3} = \frac{a^3 + b^3}{a^3b^3} = \frac{(a + b)^3 - 3ab(a + b)}{a^3 + b^3} = \frac{6\sqrt{6} - 3 \cdot 1 \cdot \sqrt{6}}{1} = 3\sqrt{6}.$$

2.53 Let $x = 2^{2013}$. Then the value of the expression

$$x - \sqrt{x^2 + 1} + \frac{1}{\sqrt{x^2 + 1} + x}$$

is equal to $(A) - 1$ $(B)0$ $(C)1$ $(D)2^{2013}$ $(E)2$

Slovenia NMO-2013

Solution: After finding the common denominator and rearranging the expression we get

$$\frac{\left(x + \sqrt{x^2 + 1}\right)\left(x - \sqrt{x^2 + 1}\right) + 1}{\sqrt{x^2 + 1} + x} = \frac{\left(x^2 - (x^2 + 1)\right) + 1}{\sqrt{x^2 + 1} + x} = 0.$$

The answer is B.

2.54 Prove that arbitrary real numbers a and b satisfy the inequality

$$(a + ab - b^2)^2 + ab^2(a + 2) \geq 0$$

When does the equality hold?

Slovenia NMO-2013

Solution: *Expanding the left-hand side of the inequality we get* $a^2 + a^2b^2 + b^4 + 2a^2b - 2ab^3 + a^2b^2$. *This can be rearranged into* $a^2(1+b)^2 + b^2(b-a)^2$, *and the desired inequality now follows. At the same time we see that the equality holds if and only if* $a = b = 0$ *or* $a = b = -1$.

2.55 Prove that the 2015-digit integer $\underbrace{11\ldots1}_{1007}2\underbrace{11\ldots1}_{1007}$ is composite.

Ukrainian NMO-2015

Solution: *Rewrite the number as follows:*

$$\underbrace{11\ldots1}_{1007}2\underbrace{11\ldots1}_{1007} = \underbrace{11\ldots1}_{1007}\underbrace{00\ldots0}_{1007} + \underbrace{11\ldots1}_{1007} = \underbrace{11\ldots1}_{1008}\underbrace{00\ldots0}_{1007} + \underbrace{11\ldots1}_{1008} \vdots \underbrace{11\ldots1}_{1008}$$

hence, it's not prime.