GEOMETRY

3.1 In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$(a+a')(b+b')(c+c') \geq 8\left(r^2s + r'^2s' + 6rr'\sqrt{ss'}\right)$$

Daniel Sitaru

Solution(Tran Hong)

$$(a+a')(b+b')(c+c') =$$

$$= \left(\left(\sqrt[3]{a}\right)^3 + \left(\sqrt[3]{a'}\right)^3\right)\left(\left(\sqrt[3]{b}\right)^3 + \left(\sqrt[3]{b'}\right)^3\right)\left(\left(\sqrt[3]{c}\right)^3 + \left(\sqrt[3]{c'}\right)^3\right) \overset{Holder}{\geq}$$

$$\geq \left(\sqrt[3]{abc} + \sqrt[3]{a'b'c'}\right)^3 \geq \left(\sqrt[3]{4Rrs} + \sqrt[3]{4R'r's'}\right)^3 \overset{R\geq 2r; R'\geq 2r'}{\geq}$$

$$\geq \left(\sqrt[3]{8r^2s} + \sqrt[3]{8r'^2s'}\right)^3 = 8\left(\sqrt[3]{r^2s} + \sqrt[3]{r'^2s'}\right)^3$$

Let: $x = \sqrt[3]{r^2s}; y = \sqrt[3]{r'^2s'}\ (x, y > 0)$

We need to prove:

$$8(x+y)^3 \geq 8\left(x^3 + y^3 + 6xy\sqrt{xy}\right) \Leftrightarrow$$

$$x^3 + y^3 + 3xy(x+y) \geq x^3 + y^3 + 6xy\sqrt{xy} \Leftrightarrow$$

$$3xy(x+y) \geq 6xy\sqrt{xy} \Leftrightarrow xy(x+y) \geq 2xy\sqrt{xy}$$

Which is clearly true, because $x, y > 0, x + y \overset{AGM}{\geq} 2\sqrt{xy} \Leftrightarrow$

$xy(x+y) \geq 2xy\sqrt{xy}$. *Proved.*

3.2 If $x, y \geq 0, x + y \leq \pi$ then:

$$2\left(\cos\frac{2x}{3} + \cos\frac{2y}{3}\right) + 1 \geq 4\cos\frac{x}{3}\cos\frac{y}{3}$$

Daniel Sitaru

Solution (Khanh Hung Vu)

Put $\cos\frac{x}{3} = a$ *and* $\cos\frac{y}{3} = b; a, b \in \left[\frac{1}{2}; 1\right]$. *We have the thing to prove is:*

$$2(2a^2 - 1 + 2b^2 - 1) + 1 \geq 4ab \text{ or } a^2 + b^2 - ab \geq \frac{3}{4}$$

We have: $x + y \leq \pi \Rightarrow 0 \leq \frac{x}{3} + \frac{y}{3} \leq \frac{\pi}{3} \Rightarrow \cos\left(\frac{x+y}{3}\right) \geq \frac{1}{2} \Rightarrow$

$$\cos\frac{x}{3}\cos\frac{y}{3}-\sin\frac{x}{3}\sin\frac{y}{3}\geq\frac{1}{2}\Rightarrow ab-\sqrt{(1-a^2)(1-b^2)}\geq\frac{1}{2}$$

So, $\sqrt{(1-a^2)(1-b^2)}\leq ab-\frac{1}{2}\Rightarrow 1-a^2-b^2+a^2b^2\leq a^2b^2-ab+\frac{1}{4}$

$$a^2+b^2-ab\geq\frac{3}{4}$$

3.3 In ΔABC the following relationship holds:

$$\prod_{cyc}(m_a^5-h_a^5+w_a^5)\leq\left(\prod_{cyc}(m_a-h_a+w_a)\right)^5$$

Daniel Sitaru

Solution (Tran Hong)

$$(x-y+z)^5=$$

$$=x^5-y^5+z^5+5xy(y^3-x^3)+5xy(x^3+z^3)+10x^2y^2(x-y)$$
$$+10x^2z^2(x+z)+10y^2z^2(z-y)$$
$$-20xyz(x^2+y^2+z^2)+30xyz(xy+yz+zx)\Rightarrow$$

$$(x-y+z)^5-(x^5-y^5+z^5)=$$

$$=5(x-y)(x+z)(z-y)[x^2+y^2+z^2-(xy+yz+zx)]\geq 0$$

Because $x\geq z\geq y>0;\ x^2+y^2+z^2-(xy+yz+zx)\geq 0$

Choose: $x=m_a; y=h_a; z=w_a; (x\geq z\geq y>0)\Rightarrow$

$$(m_a-h_a+w_a)^5\geq m_a^5-h_a^5+w_a^5\Rightarrow$$

$$\prod_{cyc}(m_a^5-h_a^5+w_a^5)\leq\left(\prod_{cyc}(m_a-h_a+w_a)\right)^5$$

3.4 Find all $x,y,z>0$ such that:

$$\boldsymbol{4\sin x\cdot\sin y\cdot\sin z\cdot\sin(x+y+z)=1}$$

Daniel Sitaru

Solution (Marian Dincă)

$$4\sin x\cdot\sin y\cdot\sin z\cdot\sin(x+y+z)\le 1$$

$$2\sin x\cdot\sin y=\cos(x-y)-\cos(x+y)$$

$$2\sin z\cdot\sin(x+y+z)=\cos(x+y)-\cos(2z+x+y)$$

$$[\cos(x-y)-\cos(x+y)][\cos(x+y)-\cos(2z+x+y)]=$$

$$=-\cos^2(x+y)+\cos(x+y)[\cos(x+y)-\cos(2z+x+y)]$$

$$-\cos(x-y)\cos(2z+x+y)-1\le 0$$

$$\cos^2(x+y)-\cos(x+y)[\cos(x+y)-\cos(2z+x+y)]$$

$$+\cos(x-y)\cos(2z+x+y)+1\ge 0$$

Let: $\cos(x+y)=t$

$$t^2-t[\cos(x+y)-\cos(2z+x+y)]+\cos(x-y)\cos(2z+x+y)+1\ge 0$$

$$\left[t-\frac{1}{2}\big(\cos(x-y)+\cos(2z+x+y)\big)\right]^2-$$

$$-\frac{1}{4}[\cos(x-y)+\cos(2z+x+y)]^2+\cos(x-y)\cos(2z+x+y)+1\ge 0$$

$$\left[t-\frac{1}{2}\big(\cos(x-y)+\cos(2z+x+y)\big)\right]^2-$$

$$+\frac{4-[\cos(x-y)-\cos(2z+x+y)]^2}{4}\ge 0$$

But: $\cos(x-y)-\cos(2z+x+y)=2\sin(z+y)\sin(z+x)$

So,

$$4-[\cos(x-y)-\cos(2z+x+y)]^2=4-4\sin^2(z+y)\sin^2(z+x)\ge 0$$

because: $0\le\sin^2(z+y)\le 1; 0\le\sin^2(z+x)\le 1$

So, the inequality

$4\sin x\cdot\sin y\cdot\sin z\cdot\sin(x+y+z)\le 1$ *is true for any* $x,y,z\in\mathbb{R}$.

Equality holds when: $\sin(z+y)=\pm 1; \sin(z+x)=\pm 1$ *result:*

$$\cos(x+z)=0; \cos(y+z)=0; t-\frac{1}{2}[\cos(x-y)+\cos(2z+x+y)]=0$$

$$2cos(x+y) = cos(x-y) + cos(2z+x+y) = 2cos(x+z)cos(y+z)$$

But: $cos(x+z) = 0; cos(y+z) = 0 \Rightarrow cos(x+y) = 0$

So, $x+z = (2a+1)\frac{\pi}{2}; y+z = (2b+1)\frac{\pi}{2}; x+y = (2c+1)\frac{\pi}{2}$

$$x = (2a-2b+2c+1)\frac{\pi}{4}; y = (-2a+2b+2c+1)\frac{\pi}{4}$$

$$z = (2a+2b-2c+1)\frac{\pi}{4}; a,b,c \in \mathbb{Z}$$

3.5 In $\Delta ABC, I$ – incenter, K – Lemoine's point the following relationship holds: $a(AI+AK)+b(BI+BK)+c(CI+CK) \geq 8S$

Daniel Sitaru

Solution:

$$aAI + bBI + cCI + aAK + bBK + cCK \geq 8s \quad (1)$$

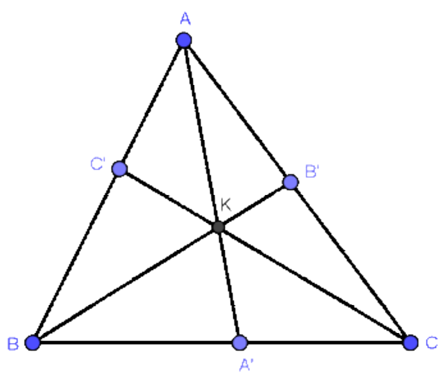
$$AI = \frac{r}{\sin\frac{A}{2}} \Rightarrow aAI + bBI + cCI = \left(\frac{a}{\sin\frac{A}{2}} + \frac{b}{\sin\frac{B}{2}} + \frac{c}{\sin\frac{C}{2}}\right) \geq r\cdot 3\sqrt[3]{\frac{abc}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}$$

(2)

But $abc = 4sRr$ *and* $\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2} = \frac{r}{4R}$ *(3)*

From (2)+(3)$\Rightarrow aAI + bBI + cCI \geq 3r\sqrt[3]{16sR^2} \overset{Mitirinovici}{\geq}$

$$\geq 3r\sqrt[3]{16\cdot s\cdot\frac{4}{27}s^2} = 3r\sqrt[3]{\frac{64s^3}{27}} = 4sr = 4S \quad (4)$$



From Van Aubel theorem we have:

$$\frac{AK}{KA'}=\frac{AC'}{C'B}+\frac{AB'}{B'C}\overset{Steiner}{=}\frac{b^2+c^2}{a^2}\Rightarrow$$

$$\frac{AK}{KA'}=\frac{b^2+c^2}{a^2}\Rightarrow\frac{AK}{sa}=\frac{b^2+c^2}{a^2+b^2+c^2}\Rightarrow$$

$AK=\frac{b^2+c^2}{a^2+b^2+c^2}sa=\frac{2bc}{a^2+b^2+c^2}\cdot m_a$ *and similarly (5)*

From (5) $\Rightarrow aAK+bBK+cCK=\frac{2abc}{a^2+b^2+c^2}(m_a+m_b+m_c)$ *(6)*

But $m_a\geq\frac{b^2+c^2}{4R}\Rightarrow m_a+m_b+m_c\geq\frac{a^2+b^2+c^2}{2R}$ *(7)*

From (6) + (7) $\Rightarrow aAK+bBK+cCK\geq\frac{abc}{R}=\frac{4sRr}{R}=4sr=4S$ *(8)*

From (1)+(4)+(8)⇒

$$a(AI+AK)+b(BI+BK)+c(CI+CK)\geq 4S+4S=8S$$

3.6 If $0<a\leq b<\frac{\pi}{5}$ then:

$$\sin\left(\frac{(4a+b)\pi}{5}\right)\sin\left(\frac{(a+4b)\pi}{5}\right)\leq\sin\left(\frac{(a+4b)\pi}{5}\right)\sin\left(\frac{(4a+b)\pi}{35}\right)$$

Daniel Sitaru

Solution (Florentin Vișescu)

$$0<a\leq b<\frac{\pi}{5}$$

$$\sin\left(\frac{(4a+b)\pi}{5}\right)\sin\left(\frac{(a+4b)\pi}{5}\right)\leq\sin\left(\frac{(a+4b)\pi}{5}\right)\sin\left(\frac{(4a+b)\pi}{5}\right)\sin\left(\frac{(4a+b)\pi}{35}\right)$$

$$\frac{1}{2}\left(\cos\frac{(27a+3b)\pi}{35}-\cos\frac{(29a+11b)\pi}{35}\right)\leq$$

$$\frac{1}{2}\left(\cos\frac{(3a+27b)\pi}{35}-\cos\frac{(11a+29b)\pi}{35}\right)\leq$$

$$\cos\frac{(27a+3b)\pi}{35}-\cos\frac{(29a+11b)\pi}{35}-\cos\frac{(3a+27b)\pi}{35}+\cos\frac{(11a+29b)\pi}{35}\leq 0$$

$$-2\sin\cdot\frac{30(a+b)\pi}{70}\sin\frac{24(a-b)\pi}{70}-2\sin\frac{30(a+b)\pi}{70}\sin\frac{18(b-a)}{70}\leq 0$$

$$-2sin\frac{30(a+b)\pi}{70}\left(sin\frac{24(a-b)\pi}{35}-sin\frac{18(a-b)\pi}{70}\right)\leq 0$$
$$-2sin\frac{3(a+b)\pi}{7}\left(sin\frac{12(a-b)\pi}{35}-sin\frac{9(a-b)\pi}{35}\right)\leq 0$$
$$-2sin\frac{3(a+b)\pi}{7}2sin\frac{3(a-b)\pi}{70}cos\frac{2(a-b)\pi}{70}\leq 0$$
$$-4sin\frac{3(a+b)\pi}{7}sin\frac{3(a-b)\pi}{70}cos\frac{3(a-b)\pi}{10}\leq 0$$
$$0<a<\frac{\pi}{5},0<b<\frac{\pi}{5},0<a+b<\frac{2\pi}{5}$$
$$0<\frac{3(a+b)\pi}{7}<\frac{2\pi}{5}\cdot\frac{3\pi}{7}=\frac{6\pi^2}{35}<\pi$$
$$a\leq b\Rightarrow a-b=0\Rightarrow\frac{3(a-b)\pi}{10}\leq 0$$
$$0<a<\frac{\pi}{5},-\frac{\pi}{5}<-b<0,\qquad -\frac{\pi}{5}<a<-b<\frac{\pi}{5}$$
$$-\frac{\pi}{2}<-\frac{3\pi^2}{50}<\frac{3(a-b)\pi}{10}\leq 0,(-25\pi<-3\pi^2\quad 25>3\pi^4)$$
$$-\frac{\pi}{2}<\frac{3(a-b)\pi}{70}\leq 0$$

3.7 If in ΔABC, $sin\,A+sinB=\left(\frac{sin^2A}{cosA}+\frac{sin^2B}{cosB}\right)tan\frac{C}{2}$ then:

$$2a^2m_am_c+c^2m_a^2\leq(2m_b+m_c)^2R^2$$

Daniel Sitaru

Solution (Tran Hong)

$$sin\,A+sinB=\left(\frac{sin^2A}{cosA}+\frac{sin^2B}{cosB}\right)tan\frac{C}{2}\ldots(*)$$
$$\therefore A,B\neq\frac{\pi}{2}$$

If in ΔABC *acute then* $cosA,cosB>0$

$$\frac{sin^2A}{cosA}+\frac{sin^2B}{cosB}\overset{CBS}{\geq}\frac{(sinA+sinB)^2}{cosA+cosB}$$
$$\Rightarrow RHS_{(*)}\geq\left(\frac{(sinA+sinB)^2}{cosA+cosB}\right)tan\frac{C}{2}$$
$$=\frac{cos\left(\frac{A+B}{2}\right)}{sin\left(\frac{A+B}{2}\right)}\frac{(sinA+sinB)^2}{cosA+cosB}\overset{(2)}{=}sinA+sinB$$

(2) is true.In fact,

$$\Leftrightarrow \frac{cosA + cosB}{sinA + sinB} = \frac{cos\left(\frac{A+B}{2}\right)}{sin\left(\frac{A+B}{2}\right)}$$

$$\Leftrightarrow \frac{2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)}{2sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)} = \frac{cos\left(\frac{A+B}{2}\right)}{sin\left(\frac{A+B}{2}\right)}$$

$$\left(\therefore true, because\ \Delta ABC\ acute \Rightarrow -\frac{\pi}{2} < \frac{A-B}{2} < \frac{\pi}{2} \Rightarrow cos\left(\frac{A-B}{2}\right) > 0\right)$$

$$\Rightarrow RHS_{(*)} \ge LHS_{(*)}, = \Leftrightarrow A = B \Rightarrow a = b$$

Now,

$$\therefore m_a = \sqrt{\frac{2b^2 + 2c^2 - a^2}{4}} = \frac{\sqrt{a^2 + 2c^2}}{2}$$

$$\therefore m_b = \sqrt{\frac{2a^2 + 2c^2 - b^2}{4}} = \frac{\sqrt{a^2 + 2c^2}}{2}$$

$$\therefore m_c = \sqrt{\frac{2a^2 + 2b^2 - c^2}{4}} = \frac{\sqrt{4a^2 - c^2}}{2}$$

$$\therefore R = \frac{abc}{4S} = \frac{ca^2}{\sqrt{(2a+c)(2a-c)c^2}} = \frac{a^2}{\sqrt{4a^2 - c^2}}$$

So, inequality$\Leftrightarrow$

$$2a^2 \cdot \frac{\sqrt{a^2 + 2c^2}}{2} \cdot \frac{\sqrt{4a^2 - c^2}}{2} + c^2 \frac{a^2 + 2c^2}{4} \le \left(2 \cdot \frac{\sqrt{a^2 + 2c^2}}{2} + \frac{\sqrt{4a^2 - c^2}}{2}\right)^2 \cdot \frac{a^4}{4a^2 - c^2} \ldots (1)$$

$$Let\ x = \sqrt{a^2 + 2c^2}; y = \sqrt{4a^2 - c^2}\ (wlog, c - max\{a; b; c\})$$

$$\Rightarrow x^2 = a^2 + 2c^2;\ y^2 = 4a^2 - c^2$$

$$\Rightarrow a^2 = \frac{x^2 + 2y^2}{9};\ c^2 = \frac{4x^2 - y^2}{9} \Rightarrow x \ge y$$

$$(1) \Leftrightarrow 2\left(\frac{x^2 + 2y^2}{9}\right)xy + \left(\frac{4x^2 - y^2}{9}\right)x^2 \le (2x + y)^2 \cdot \frac{(x^2 + 2y^2)^2}{9^2 y^2}$$

$$\Leftrightarrow 18x^3y^3 + 36xy^5 + 36x^4y^2 - 9x^2y^4 \le 16x^6 + 16x^5y + 20x^4y^2 + 16x^3y^3 + 8x^2y^4 + 4x^5y^6$$

$$\Leftrightarrow (x - y)(16x^5 + 32x^4y + 16x^3y^2 + 14x^2y^3 + 31xy^4 - y^5)_{(\Psi)} \ge 0;$$

Which is true because: $\therefore x \ge y > 0 \Rightarrow x - y \ge 0$

$$\Rightarrow x^5 \ge y^5 \Rightarrow 16x^5 > y^5 \Rightarrow 16x^5 - y^5 > 0 \Rightarrow \Psi > 0 \Rightarrow (1) true. Finish.$$

3.8 If in $\Delta ABC, m(\sphericalangle A) < 90^\circ$ then:

$$\sin^3\frac{A}{2} \le \frac{32a^3}{\left(\sqrt{b}+\sqrt{c}\right)^4(\sqrt[4]{b}+\sqrt[4]{c})^4}$$

Daniel Sitaru

Solution (Tran Hong)

$$\therefore \frac{a}{b+c} = \frac{|a|}{|b+c|} = \frac{|2R\sin A|}{|2R\sin B + 2R\sin C|} = \frac{|\sin A|}{|\sin B + \sin C|}$$

$$= \frac{\left|2\sin\frac{A}{2}\cos\frac{A}{2}\right|}{\left|2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right|} \overset{\sin\frac{A}{2}\cos\frac{A}{2}>0}{\cong} \frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}\left|\cos\left(\frac{A-B}{2}\right)\right|}$$

$$\underset{0<A<\frac{\pi}{2}}{\overset{0<\left|\cos\left(\frac{A-B}{2}\right)\right|\le 1}{\ge}} \sin\frac{A}{2} \Rightarrow \frac{a}{b+c} \ge \sin\frac{A}{2}$$

We just check:

$$\frac{a^3}{(b+c)^3} \le \frac{32a^3}{\left(\sqrt{b}+\sqrt{c}\right)^4(\sqrt[4]{b}+\sqrt[4]{c})^4}$$

$$\Leftrightarrow \left(\sqrt{b}+\sqrt{c}\right)^4(\sqrt[4]{b}+\sqrt[4]{c})^4 \le 32(b+c)^3$$

$$Let\ \sqrt[4]{b} = x;\ \sqrt[4]{c} = y\ (x; y > 0) \Rightarrow b = x^4; c = y^4$$

$$\Leftrightarrow (x^2+y^2)^4(x+y)^4 \le 32(x^4+y^4)^3$$

$$31(x^{12}+y^{12}) - 4(x^{11}y+xy^{11}) - 10(x^{10}y^2+x^2y^{10}) - 20(x^9y^3+x^3y^9) + 65(x^8y^4+x^4y^8) - 40(x^7y^5+x^5y^7) - 44x^6y^6 \ge 0$$

$$\Leftrightarrow (x-y)^2[31(x^{10}+y^{10}) + 58(x^9y+xy^9) + 72(x^7y^3+x^3y^7) + 134(x^6y^4+x^4y^6) + 155x^5y^5] \ge 0$$

Which is clearly true.Equality for $a=b$.

3.9 If $x \in \left[0, \frac{\pi}{12}\right]$ then

$$\cos^{202}x \ge \cos^{10}x \cdot \cos^{15}(2x) \cdot \cos^5(3x) \cdot \cos^6(4x) \cdot \cos 5x \cdot \cos 6x$$

Daniel Sitaru

Solution (Tran Hong)

$$\therefore\ For\ all\ x \in \left[0, \frac{\pi}{12}\right]\ we\ have:$$

$$cos\frac{\pi}{12} \le cosx \le 1 \Rightarrow cos^2\frac{\pi}{12} \le cos^2x \le 1 \Rightarrow \alpha = \frac{1}{2} + \frac{\sqrt{3}}{4} \le cos^2x \le 1$$

Now, we must show that: $\therefore\ cos6x \overset{(1)}{\overbrace{\le}} cos^{36}x, \forall\ x \in \left[0, \frac{\pi}{12}\right] \Leftrightarrow$

$$32cos^6x - 48cos^4x + 18cos^2x - 1 \le cos^{36}x$$

$$\Leftrightarrow t^{18} - 32t^3 + 48t^2 - 18t + 1 \ge 0\ (\therefore\ t = cos^2x, t \in [0,1])$$

$$\Leftrightarrow (t-1)^2(t^{16} + 2t^{15} + 3t^{14} + 4t^{13} + 5t^{12} + 6t^{11} + 7t^{10} + 8t^9 + 9t^8 + 10t^7 + 11t^6 + 12t^5 + 13t^4 + 14t^3 + 15t^2 - 16t + 1) \underbrace{\ge}_{(\Psi)} 0;$$

Which is true because: $\therefore\ (t-1)^2 \ge 0$

$$\therefore\ t \in [\alpha, 1] \Rightarrow \Psi \ge 22 - 16 = 6 > 0 \Rightarrow (1)true.$$

$$\therefore\ cos5x \overset{(2)}{\overbrace{\le}} cos^{25}x, x \in \left[0, \frac{\pi}{12}\right]$$

$$16cos^5x - 20cos^3x + 5cosx \le cos^{25}x$$

$$\Leftrightarrow t^{25} - 16t^5 + 20t^3 - 5t \ge 0\ (\therefore\ t = cosx \in [\alpha, 1])$$

$$\Leftrightarrow t(t-1)^2(t+1)^2(t^{20} + 2t^{18} + 3t^{16} + 4t^{14} + 5t^{12} + 6t^{10} + 7t^8 + 8t^6 + 9t^4 + 10t^2 - 5) \underbrace{\ge}_{(\Phi)} 0$$

Which is true because: $t(t-1)^2(t+1)^2 \ge 0, \forall t \in [\alpha, 1] \Rightarrow \Phi > 12 - 7 > 0$

$$\therefore\ cos4x \overset{(3)}{\overbrace{\le}} cos^{16}x, \forall x \in [\alpha, 1]$$

$$\Leftrightarrow 8cos^4x - 8cos^2x + 1 \le cos^{16}x$$

$$\Leftrightarrow t^8 - 8t^2 + 8t - 1 \ge 0, (\therefore\ t = cosx \in [\alpha, 1])$$

$$\Leftrightarrow (t-1)^2(t^6 + 2t^5 + 3t^4 + 4t^3 + 5t^2 + 6t - 1) \ge 0$$

Which is true because: $(t-1)^2 \ge 0$

$$t^6 + 2t^5 + 3t^4 + 4t^3 + 5t^2 + 6t - 1 \geq 17 - 1 = 16 > 0$$

$$\therefore cos3x \overset{(4)}{\leq} cos^9x, \forall x \in \left[0, \frac{\pi}{12}\right]$$

$$\Leftrightarrow 4cos^3x - 3cosx \leq cos^9x$$

$$\Leftrightarrow t^9 - 4t^3 + 3t \geq 0 \Leftrightarrow t(t-1)^2(t+1)^2(t^4 + 2t^2 + 3) \geq 0$$

Which is clearly true because: $\alpha \leq t \leq 1$.

$$\therefore cos2x \overset{(5)}{\leq} cos^4x, \forall x \in \left[0, \frac{\pi}{12}\right]$$

$$\Leftrightarrow cos^4x - 2cos^2x + 1 \geq 0 \Leftrightarrow t^2 - 2t + 1 \geq 0$$

$$(\therefore t = cos^2x) \Leftrightarrow (t-1)^2 \geq 0$$

Which is clearly true for all $x \in \left[0, \frac{\pi}{12}\right]$. *By (1),(2),(3),(4),(5) we have:*

$$cos^{10}x \cdot cos^{15}(2x) \cdot cos^5(3x) \cdot cos^6(4x) \cdot cos5x \cdot cos6x$$

$$\leq cos^{10}x \cdot cos^{60}x \cdot cos^{45}x \cdot cos^{96}x \cdot cos^{25}x \cdot cos^{36}x$$

$$= cos^{272}x = cos^{202}x \cdot cos^{70}x \overset{\alpha \leq cosx \leq 1}{\leq} cos^{202}x, x \in \left[0, \frac{\pi}{12}\right]$$

Proved. Equality for $cosx=1 \Leftrightarrow x=0$

3.10 In ΔABC the following relationship holds:

$$3(a^2 + b^2 + c^2) + 4(h_a^2 + h_b^2 + h_c^2) \geq 24\sqrt{3}S$$

Daniel Sitaru

Solution (Ravi Prakash)

$$\frac{1}{2}ah_a = S \Rightarrow h_a = \frac{2S}{a}$$

$$3a^2 + 4h_a^2 = 3a^2 + \frac{16S^2}{a^2} \geq 2\sqrt{3a^2 \cdot \frac{16S^2}{a^2}} = 8\sqrt{3}S$$

Similarly, write other two expressions and add.

3.11 If $a, b > 0, a + b \in \left(\frac{1}{\pi}, \frac{2}{\pi}\right)$, then:

$$a^a \cdot b^b \cdot \left(1 + \cos\frac{1}{a}\right)^a \cdot \left(1 + \cos\frac{1}{b}\right)^b \le \left(a\left(1 + \cos\frac{1}{a}\right) + b\left(1 + \cos\frac{1}{b}\right)\right)^{a+b}$$

Florică Anastase

Solution:

$$Let: f: \left(\frac{\pi}{2}, \pi\right) \to \mathbf{R}, f(x) = \log\left(\frac{x}{1+\cos x}\right), f'(x) = \frac{1}{x} + \frac{\sin x}{1+\cos x},$$

$$f''(x) = \frac{x^2 - \cos x - 1}{x^2(1 + \cos x)}$$

$$Let\ h: \left(\frac{\pi}{2}, \pi\right) \to \mathbf{R}, h(x) = x^2 - \cos x - 1, h'(x) = 2x + \sin x > 0 \to h(x)$$

$$> h\left(\frac{\pi}{2}\right) = \pi + 1 > 0 \to f''(x) > 0, \forall x \in \left(\frac{\pi}{2}, \pi\right)$$

$$\to f\ is\ convex. From\ Jensen\ inequality \to$$

$$f\left(\frac{1}{a+b}\right) = f\left(\frac{a}{a+b}\cdot\frac{1}{a} + \frac{b}{a+b}\cdot\frac{1}{b}\right) \le \frac{af\left(\frac{1}{a}\right) + bf\left(\frac{1}{b}\right)}{a+b} \leftrightarrow$$

$$\log\left(\frac{1}{(a+b)\cdot\left(1 + \cos\left(\frac{1}{a+b}\right)\right)}\right) \le \frac{a\cdot\log\left(\frac{1}{a(1 + \cos\frac{1}{a})}\right) + b\cdot\log\left(\frac{1}{b(1 + \cos\frac{1}{b})}\right)}{a+b}$$

$$a^a \cdot b^b \cdot \left(1 + \cos\frac{1}{a}\right)^a \cdot \left(1 + \cos\frac{1}{b}\right)^b \le (a+b)\cdot\left(\left(1 + \cos\left(\frac{1}{a+b}\right)\right)\right)^{a+b} \overset{\cos x - convexe}{\le}$$

$$\le \left(a\left(1 + \cos\frac{1}{a}\right) + b\left(1 + \cos\frac{1}{b}\right)\right)^{a+b}$$

3.12 Let $A'B'C'$ be the intouch triangle of ΔABC. Prove that:

$$A'B' + B'C' + C'A' \le \frac{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}{2} < s$$

Marian Ursărescu

Solution (Tran Hong)

In $\Delta A'B'C'$:

$$B'C'^2 = (s-a)^2 + (s-a)^2 - 2(s-a)^2 cosA$$

$$= 2(s-a)^2 - 2(s-a)^2 cosA$$

$$= 2(s-a)^2(1-cosA) = 4(s-a)^2 sin^2\frac{A}{2} \text{ then}$$

$$B'C' = 2(s-a)sin\frac{A}{2}$$

Similary: $A'B' = 2(s-c)sin\frac{C}{2}; A'C' = 2(s-b)sin\frac{B}{2}$ *then*

$$A'B' + B'C' + C'A' = 2\left((s-a)sin\frac{A}{2} + (s-c)sin\frac{C}{2} + (s-b)sin\frac{B}{2}\right)$$

$$= 2\left(\frac{r}{tan\frac{A}{2}} \cdot sin\frac{A}{2} + \frac{r}{tan\frac{B}{2}} \cdot sin\frac{B}{2} + \frac{r}{tan\frac{C}{2}} \cdot sin\frac{C}{2}\right)$$

$$= 2r\left(cos\frac{A}{2} + cos\frac{B}{2} + cos\frac{C}{2}\right) \overset{Jensen}{\leq} 2r \cdot \frac{3\sqrt{3}}{2} = 3\sqrt{3} \cdot r$$

$$\frac{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}{2} \overset{Am-G}{\geq} \frac{3\sqrt[3]{3}}{2} = \frac{3\sqrt[3]{4Rrs}}{2}$$

So, we need to prove:

$$\frac{3\sqrt[3]{4Rrs}}{2} \geq 3\sqrt{3} \cdot r \Leftrightarrow \sqrt[3]{4Rrs} \geq \sqrt{3} \cdot r \Leftrightarrow Rs \geq 6\sqrt{3} \cdot r^2$$

Which is true because $\begin{cases} R \geq 2r \\ s \geq 3\sqrt{3} \cdot r \end{cases} \Rightarrow Rs \geq 6\sqrt{3} \cdot r^2$

Hence $A'B' + B'C' + C'A' \leq \frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2}$

Lastly, using inequality: $X^2 + Y^2 + Z^2 \geq XY + YZ + ZX$

Choose $X = \sqrt{a}, Y = \sqrt{b}, Z = \sqrt{c} \Rightarrow \frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{2} \leq \frac{a+b+c}{2} = \frac{2s}{s} = s$*.Proved.*

3.13 In $\Delta ABC, \Gamma$ –is Gergonne point, BN, CM –simedians from B, C.Prove that the points B, Γ, N –are collinear if only if $\frac{r_b}{b^2} + \frac{r_c}{c^2} = \frac{r_a}{a^2}$.

Marian Ursărescu

Solution: *From tranversal theorem we have:*

B, Γ, N *–are collinear if only if* $\frac{MB}{MA}\cdot\frac{1}{s-b}+\frac{NC}{NA}\cdot\frac{1}{s-c}=\frac{1}{s-a}$ (1)

From Steiner theorem we have:

$\frac{MA}{MB}=\left(\frac{BC}{AC}\right)^2=\frac{a^2}{b^2}$ and $\frac{NC}{NA}=\left(\frac{BC}{AB}\right)^2=\frac{a^2}{c^2}$ (2)

From (1)+(2) we have: $\frac{a^2}{b^2(s-b)}+\frac{a^2}{c^2(s-c)}=\frac{1}{s-a}$

$$\frac{1}{b^2(s-b)}+\frac{1}{c^2(s-c)}=\frac{1}{a^2(s-a)}$$

But: $r_a=\frac{S}{s-a}\Rightarrow s-a=\frac{S}{r_a}$, *and analogs* $s-b=\frac{S}{r_b}$; $s-c=\frac{S}{r_c}$

So, $\frac{r_b}{b^2}+\frac{r_c}{c^2}=\frac{r_a}{a^2}$

3.14 In $\Delta ABC, AD, BE, CF$ –medians, G –centroid; $AM=MG$,

$$M\in(AG);$$

$$2cotA=cotB+cotC$$

Prove that: $DEMF$ is a cyclic quadrilateral.

Marian Ursărescu

Solution (Daniel Văcaru)

We have: $2cotA=cotB+cotC$

$$\Rightarrow\frac{2cosA}{sinA}=\frac{cosB}{sinB}+\frac{cosC}{sinC}\Rightarrow\frac{2cosA}{sinA}=\frac{sin(B+C)}{sinBsinC}\Rightarrow 2sinA\cdot sinB\cdot sinC=sin^2A$$

$$\Rightarrow 2bccosA=a^2\Rightarrow b^2+c^2=2a^2$$

For $DEMF$ is a cyclic quadrilateral it suffices to prove that:

$$MF\cdot DE+ME\cdot FD=FE\cdot MD\Rightarrow 2m_a=m_b\cdot b+m_c\cdot c$$

But $4m_b^2=2(a^2+c^2)-b^2=2a^2+2c^2-b^2=3c^2\Rightarrow m_b=\frac{c\sqrt{3}}{2}$,

$$m_c = \frac{b\sqrt{3}}{2},$$

$$4m_a^2 = 2(b^2 + c^2) - a^2 \Rightarrow m_a = \frac{a\sqrt{3}}{2}$$

It follows $m_b \cdot b + m_c \cdot c = b \cdot \left(\frac{c\sqrt{3}}{2}\right)^2 + c \cdot \left(\frac{b\sqrt{3}}{2}\right)^2 = \frac{(b^2+c^2)\sqrt{3}}{2} = \frac{2a^2\sqrt{3}}{2} =$

$$a^2\sqrt{3} = 2m_a \cdot a$$

3.15 In ΔABC the following relationship holds:

$$\frac{aA}{a+A} + \frac{bB}{b+B} + \frac{cC}{c+C} \le \frac{2s\pi}{2s+\pi}$$

Daniel Sitaru

Solution: *By Milne's inequality:*

$$\frac{aA}{a+A} + \frac{bB}{b+B} + \frac{cC}{c+C} \le \frac{(a+b+c)(A+B+C)}{a+b+c+A+B+C} = \frac{2s\pi}{2s+\pi}$$

3.16 If in $\Delta ABC; m(\sphericalangle A) < \frac{\pi}{2}$ then:

$$\sin^3\frac{A}{2} \le \frac{32a^3}{\left(\sqrt{b}+\sqrt{c}\right)^4\left(\sqrt[4]{b}+\sqrt[4]{c}\right)^4}$$

Daniel Sitaru

Solution: *By Bailey's inequality:*

$$2^{t-1}\sin^t\frac{A}{2} \le \frac{a^t}{b^t+c^t}; t \in (0,1)$$

For $t = \frac{1}{2}; t = \frac{1}{2}$ *we obtain:*

$$2^{-\frac{1}{2}}\sin^{\frac{1}{2}}\frac{A}{2} \le \frac{a^{\frac{1}{2}}}{\sqrt{b}+\sqrt{c}}; 2^{-\frac{3}{4}}\sin^{\frac{1}{4}}\frac{A}{2} \le \frac{a^{\frac{1}{4}}}{\sqrt[4]{b}+\sqrt[4]{c}}$$

$$2^{-2}\sin^2\frac{A}{2} \le \frac{a^2}{\left(\sqrt{b}+\sqrt{c}\right)^4} \quad (1)$$

$$2^{-3}\sin\frac{A}{2}\le\frac{a}{\left(\sqrt[4]{b}+\sqrt[4]{c}\right)^4}\ (2)$$

By multiplying (1); (2):

$$2^{-5}\sin^3\frac{A}{2}\le\frac{a^3}{\left(\sqrt{b}+\sqrt{c}\right)^4\left(\sqrt[4]{b}+\sqrt[4]{c}\right)^4}$$

$$\sin^3\frac{A}{2}\le\frac{32a^3}{\left(\sqrt{b}+\sqrt{c}\right)^4\left(\sqrt[4]{b}+\sqrt[4]{c}\right)^4}$$

Equality holds for $a=b=c$.

3.17 In ΔABC the following relationship holds:

$$\boldsymbol{3(a^3+b^3+8m_c^3+6abm_c)\le 2(a+b+2m_c)(3a^2+3b^2-c^2)}$$

When equality holds?

Daniel Sitaru

Solution (Tran Hong)

$$3a^2+3b^2-c^2=(a^2+b^2)+(2a^2+2b^2-c^2)$$

$$=(a^2+b^2)+4m_c^2$$

$$RHS=2(a+b+2m_c)(a^2+b^2+4m_c^2)$$

Now, we just check

$$2(a+b+2m_c)(a^2+b^2+4m_c^2)\ge 3(a^3+b^3+8m_c^3+6abm_c)$$

In $\Delta ACC': CC'=2m_a; AC=b; AC'=a.$

Let: $a=\alpha+\beta; b=\beta+\gamma; 2m_a=\gamma+\alpha\quad(\alpha,\beta,\gamma>0)$

$$\Leftrightarrow 2(\alpha^3+\beta^3+\gamma^3+3\alpha\beta\gamma)\ge 2\big(\alpha\beta(\alpha+\beta)+\beta\gamma(\beta+\gamma)+\gamma\alpha(\gamma+\alpha)\big)$$

$$\alpha^3+\beta^3+\gamma^3+3\alpha\beta\gamma\le\alpha\beta(\alpha+\beta)+\beta\gamma(\beta+\gamma)+\gamma\alpha(\gamma+\alpha)$$

Which is true Schur's inequality.

Equality$\Leftrightarrow a=b=2m_c\Leftrightarrow a^2=b^2=4m_c^2$

$$a^2=b^2=2a^2+2b^2-c^2$$

$$\Leftrightarrow \begin{cases} a = b \\ 4a^2 - c^2 = a^2 \end{cases} \Leftrightarrow \begin{cases} a = c \\ 3a^2 = c^2 \end{cases} \Leftrightarrow a = b = c\sqrt{3}$$

3.18 In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$\frac{192\sqrt{3}}{(a+a')(b+b')(c+c')} \le \frac{1}{r^3} + \frac{1}{r'^3}$$

Daniel Sitaru

Solution (Tran Hong)

$$\frac{192\sqrt{3}}{(a+a')(b+b')(c+c')} \le \frac{1}{r^3} + \frac{1}{r'^3}$$

$$\frac{1}{r^3} + \frac{1}{r'^3} \overset{Am-Gm}{\geq} 2\sqrt{\frac{1}{(rr')^3}}$$

We need to prove:

$$\frac{192\sqrt{3}}{(a+a')(b+b')(c+c')} \le 2\sqrt{\frac{1}{(rr')^3}}$$

$$\Leftrightarrow (a+a')(b+b')(c+c') \overset{Am-Gm}{\geq} 96\sqrt{3(rr')^3}$$

But: $(a+a')(b+b')(c+c') \overset{Am-Gm}{\geq} 8\sqrt{abc \cdot a'b'c'}$

So, we just check: $8\sqrt{abc \cdot a'b'c'} \geq 96\sqrt{(rr')^3}$

$$\Leftrightarrow \sqrt{abc \cdot a'b'c'} \geq 12\sqrt{3(rr')^3} \Leftrightarrow abc \cdot a'b'c' \geq 432(rr')^3$$

$$\Leftrightarrow (4Rrs)\cdot(4R'r's') \geq 432(rr')^3 \Leftrightarrow RsR's' \geq 27 \cdot r^2 r'^2 \quad (*)$$

Which is clearly true because :

$$R \geq \frac{2}{3\sqrt{3}}s \Rightarrow Rs \geq \frac{2}{3\sqrt{3}}s^2 \geq \frac{2}{3\sqrt{3}} \cdot 27r^2 = 2\cdot 3\sqrt{3}r^2 \quad (1)$$

$$R' \geq \frac{2}{3\sqrt{3}}s' \Rightarrow R's' \geq \frac{2}{3\sqrt{3}}s'^2 \geq \frac{2}{3\sqrt{3}} \cdot 27r'^2 = 2\cdot 3\sqrt{3}r'^2 \quad (2)$$

$\overset{(1),(2)}{\Longrightarrow} Rs \cdot R's' \geq 4 \cdot 27 \cdot r^2 r'^2 \geq 27 r^2 r'^2 \Rightarrow (*)$ *is true.Proved.*

3.19 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\frac{s-b}{s-c}+\frac{s-c}{s-a}+\left(\frac{s-a}{s-b}\right)^3}{\left(\frac{s-b}{s-c}\right)^9+\left(\frac{s-c}{s-a}\right)^9+\frac{s-b}{s-a}}\le\sum_{cyc}\left(\frac{s-b}{s-a}\right)^4$$

Daniel Sitaru

Solution (Tran Hong)

$$\sum_{cyc}\frac{\frac{s-b}{s-c}+\frac{s-c}{s-a}+\left(\frac{s-a}{s-b}\right)^3}{\left(\frac{s-b}{s-c}\right)^9+\left(\frac{s-c}{s-a}\right)^9+\frac{s-b}{s-a}}\le\sum_{cyc}\left(\frac{s-b}{s-a}\right)^4\quad(*)$$

Let: $x=\frac{s-a}{s-b};y=\frac{s-b}{s-c};z=\frac{s-c}{s-a}\Rightarrow xyz=1$ *then*

$$(*)\Leftrightarrow\sum_{cyc}\frac{x^3+y+z}{y^9+z^9+\frac{1}{x}}\le\sum_{cyc}x^4$$

We have: $y^9+z^9\ge(yz)^4(y+z)\overset{xyz=1}{\cong}\frac{y+z}{x^4}\Rightarrow$

$$y^9+z^9+\frac{1}{x}\ge\frac{y+z}{x^4}+\frac{1}{x}=\frac{y+z+x^3}{x^4}\Rightarrow$$

$$\sum_{cyc}\frac{x^3+y+z}{y^9+z^9+\frac{1}{x}}\le\sum_{cyc}\frac{y+z+x^3}{\frac{y+z+x^3}{x^4}}=\sum_{cyc}x^4\Rightarrow(1)\text{ is true.}$$

Lastly, we will to prove: $x^9+y^9\ge(xy)^4(x+y),\forall x,y>0$

$$\Leftrightarrow x^9+y^9\ge x^5y^4+x^4y^5,\forall x,y>0$$

$$\Leftrightarrow x^5(x^4-y^4)+y^5(x^4-y^4)\ge0,\forall x,y>0$$

$$\Leftrightarrow(x^4-y^4)(x^5-y^5)\ge0,\forall x,y>0$$

$$\Leftrightarrow(x-y)^2(x+y)(x^2+y^2)(x^4+x^3y+x^2y^2+xy^3+y^4)\ge0$$

Equality$\Leftrightarrow x=y=z=1\Leftrightarrow s-a=s-b=s-c\Leftrightarrow a=b=c$

3.20 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(r_a^2+r_b^2+r_c^2+2r_ar_b+2r_ar_c)a^2}{r_br_c}\ge28\sqrt{3}S$$

Daniel Sitaru

Solution (Tran Hong): *For all* $x,y,z>0$, *in any* ΔABC:

$$xa^2+yb^2+zc^2\geq 4\sqrt{xy+yz+zx}\cdot S$$

Choosing

$$x=\frac{\sum_{cyc} r_a^2+2r_a(r_b+r_c)}{r_br_c};\ y=\frac{\sum_{cyc} r_a^2+2r_c(r_b+r_a)}{r_br_a};$$

$$z=\frac{\sum_{cyc} r_a^2+2r_b(r_a+r_c)}{r_ar_c}$$

We must show that: $4\sqrt{xy+yz+zx}\geq 28\sqrt{3}\Leftrightarrow xy+yz+zx\geq 147;\ \ (*)$

Let: $\alpha=r_a;\ \beta=r_b;\gamma=r_c\rightarrow \alpha,\beta,\gamma>0$

$$(*)\leftrightarrow\sum_{cyc}\left(\frac{(\alpha+\beta+\gamma)^2-2\beta\gamma}{\beta\gamma}\right)\left(\frac{(\alpha+\beta+\gamma)^2-2\alpha\beta}{\alpha\beta}\right)\geq 147$$

$$\leftrightarrow\sum_{cyc}[\alpha\gamma((\alpha+\beta+\gamma)^2-2\beta\gamma)((\alpha+\beta+\gamma)^2-2\alpha\beta)]\geq 147(\alpha\beta\gamma)^2$$

$$\overset{k=\alpha+\beta+\gamma}{\Longleftrightarrow}\sum_{cyc}[\alpha\gamma(k^2-2\beta\gamma)(k^2-2\alpha\beta)]\geq 147(\alpha\beta\gamma)^2$$

$$\leftrightarrow\sum_{cyc}[\alpha\beta(k^4-2\beta(\alpha+\gamma)k^2+4\alpha\gamma\beta^2]\geq 147(\alpha\beta\gamma)^2$$

$$\leftrightarrow(\alpha\beta+\beta\gamma+\gamma\alpha)k^4-4\alpha\beta\gamma k^3+12(\alpha\beta\gamma)^2\geq 147(\alpha\beta\gamma)^2$$

$$\leftrightarrow k^3[(\alpha\beta+\beta\gamma+\gamma\alpha)k-4\alpha\beta\gamma]\geq 135(\alpha\beta\gamma)^2;\ \ (*)$$

$$k^3=(\alpha+\beta+\gamma)^3\overset{Am-Gm}{\geq}27\alpha\beta\gamma$$

$$(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)\overset{Am-Gm}{\geq}3\cdot\sqrt[3]{(\alpha\beta\gamma)^2}\cdot 3\cdot\sqrt[3]{\alpha\beta\gamma}=9\alpha\beta\gamma$$

$$\rightarrow(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)-4\alpha\beta\gamma\geq 9\alpha\beta\gamma-4\alpha\beta\gamma=5\alpha\beta\gamma$$

$\rightarrow LHS_{(*)}\geq 27\alpha\beta\gamma\cdot 5\alpha\beta\gamma=135(\alpha\beta\gamma)^2\rightarrow(*)$ *is true.Proved.*

3.21 In acute $\triangle ABC$ the following relationship holds:

$$\frac{s}{3}(cosA+cosB+cosC)\geq\frac{S}{R}$$

Florică Anastase

Solution:

$$If\ a \geq b \geq c\ \Rightarrow \begin{cases} A \geq B \geq C \\ cosA \leq cosB \leq cosC \end{cases} \overset{Chebyshev}{\Rightarrow}$$

$$\frac{s}{3}(cosA + cosB + cosC) = \frac{a+b+c}{6} \cdot (cosA + cosB + cosC)$$

$$\geq \frac{acosA + bcosB + ccosC}{2} =$$

$$= \frac{R}{2}(2sinAcosA + 2sinBcosB + 2sinCcosC) =$$

$$\frac{R}{2}(sin2A + sin2B + sin2C) = \frac{abc}{4R^2} = \frac{S}{R}$$

3.22 In ΔABC; AA_1, BB_1, CC_1 –internal bisectors and $A_2B_2C_2$ the circumcevian triangle of incenter.Prove that:

$$\left(\frac{r}{R}\right)^2 \leq \frac{[A_1B_1C_1]}{[A_2B_2C_2]} \leq \frac{1}{4}$$

Marian Ursărescu

Solution (Marian Dincă)

$$[A_1B_1C_1] = \frac{2abc}{(a+b)(b+c)(c+a)} \cdot [ABC]$$

$$= \frac{2abc}{(a+b)(b+c)(c+a)} \cdot 2R^2 sinAsinBsinC$$

$$\sphericalangle A_2 = \frac{B+C}{2}; \sphericalangle B_2 = \frac{A+C}{2}; \sphericalangle C_2 = \frac{A+B}{2}$$

$$[A_2B_2C_2] = 2R^2 sin\left(\frac{B+C}{2}\right) sin\left(\frac{A+C}{2}\right) sin\left(\frac{A+B}{2}\right)$$

$$\frac{[A_1B_1C_1]}{[A_2B_2C_2]} = \frac{\frac{2abc}{(a+b)(b+c)(c+a)} \cdot 2R^2 sinAsinBsinC}{2R^2 sin\left(\frac{B+C}{2}\right) sin\left(\frac{A+C}{2}\right) sin\left(\frac{A+B}{2}\right)}$$

$$sinAsinB \leq sin^2\left(\frac{A+B}{2}\right) \Leftrightarrow cos(B-A) - cos(A+B) \leq 1 - cos(A+B)$$

$$\Leftrightarrow cos\,(A-B) \leq 1$$

$$Similarly: sinCsinB \leq sin^2\left(\frac{C+B}{2}\right) \text{ and } sinAsinC \leq sin^2\left(\frac{A+C}{2}\right)$$

Multiplying the three inequalities and then extracting the square root we obtain the inequality:

$$sinAsinBsinC \leq sin\left(\frac{B+C}{2}\right)sin\left(\frac{A+C}{2}\right)sin\left(\frac{A+B}{2}\right)$$

And use Am-Gm result:

$$(a+b)(b+c)(c+a) \geq 8abc(Cesaro\ inequality)$$

We obtain:

$$\frac{[A_1B_1C_1]}{[A_2B_2C_2]} \leq \frac{1}{4}$$

$$\frac{[A_1B_1C_1]}{[A_2B_2C_2]} = \frac{\frac{2abc}{(a+b)(b+c)(c+a)} \cdot 2R^2 sinAsinBsinC}{2R^2 sin\left(\frac{B+C}{2}\right)sin\left(\frac{A+C}{2}\right)sin\left(\frac{A+B}{2}\right)}$$

$$= \frac{2abc}{(a+b)(b+c)(c+a)} \cdot 8sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$

$$= \frac{2abc}{(a+b)(b+c)(c+a)} \cdot \frac{2r}{R}$$

$$= \frac{4abc}{(a+b)(b+c)(c+a)} \cdot \frac{r}{R} \geq \left(\frac{r}{R}\right)^2 \Leftrightarrow \frac{4abc}{(a+b)(b+c)(c+a)} \geq \frac{r}{R} \Leftrightarrow$$

$$\frac{4abc}{(ab+bc+ca)(a+b+c)-abc} = \frac{16Rrs}{(s^2+r^2+4Rr)\cdot 2s - 4Rrs}$$

$$= \frac{8Rr}{(s^2+r^2+4Rr)-2Rr} = \frac{8Rr}{s^2+r^2+2Rr} \geq \frac{r}{R} \Leftrightarrow$$

$$8R^2 \geq s^2+r^2+2Rr \Leftrightarrow 8R^2 - r^2 - 2Rr \geq s^2$$

$$s^2 \leq 4R^2 + 4Rr + 3r^2 \ldots Gerretsen\ inequality$$

$$and: 4R^2 + 4Rr + 3r^2 \leq 8R^2 - r^2 - 2Rr \Leftrightarrow$$

$$4R^2 - 6Rr - 4r^2 \geq 0 \Leftrightarrow 2R^2 - 3Rr - 2r^2 \geq 0$$

$$\Leftrightarrow (R - 2r)(2R + r) \geq 0 \Leftrightarrow R - 2r \geq 0 \dots (Euler). Done!$$

3.23 $\boldsymbol{z_1, z_2, z_3 \in \mathbb{C} - \{0\}}$, **different in pairs**, $|\mathbf{z_1}| = |\mathbf{z_2}| = |\mathbf{z_3}| = \mathbf{1}$

$\boldsymbol{A(z_1), B(z_2), C(z_3)}$. **Prove that:**

$$\frac{\mathbf{1}}{\boldsymbol{z_1 z_2 z_3}} + \sum_{cyc} \frac{\boldsymbol{z_1}}{(\boldsymbol{z_2 - z_3})^2} = \mathbf{0} \Rightarrow \boldsymbol{AB = BC = CA}$$

Marian Ursărescu

Solution (Khaled Abd Imouti)

$$\frac{1}{z_1 z_2 z_3} + \frac{z_1}{(z_2 - z_3)^2} + \frac{z_2}{(z_1 - z_3)^2} + \frac{z_3}{(z_1 - z_2)^2} = 0;\ z_1 z_2 z_3 \neq 0$$

$$1 + \frac{z_1^2 z_2 z_3}{(z_2 - z_3)^2} + \frac{z_2^2 z_1 z_3}{(z_1 - z_3)^2} + \frac{z_3^2 z_1 z_2}{(z_1 - z_2)^2} = 0$$

$$1 + \frac{z_1^2 z_2 z_3}{z_2^2 - 2z_2 z_3 + z_3^2} + \frac{z_2^2 z_1 z_3}{z_1^2 - 2z_1 z_3 + z_3^2} + \frac{z_3^2 z_1 z_2}{z_1^2 - 2z_1 z_2 + z_2^2} = 0$$

$$1 + \frac{z_1^2}{\frac{z_2}{z_3} + \frac{z_3}{z_2} - 2} + \frac{z_2^2}{\frac{z_1}{z_3} + \frac{z_3}{z_1} - 2} + \frac{z_3^2}{\frac{z_1}{z_2} + \frac{z_2}{z_3} - 2} = 0$$

$$\begin{cases} |z_1| = 1 \\ |z_2| = 1 \\ |z_3| = 1 \end{cases} \Rightarrow \begin{cases} z_1 \overline{z_1} = 1 \\ z_2 \overline{z_2} = 1 \\ z_3 \overline{z_3} = 1 \end{cases} \Rightarrow \frac{z_2}{z_3} \cdot \frac{\overline{z_2}}{\overline{z_3}} = 1 \Rightarrow \frac{z_2}{z_3} = \frac{\overline{z_3}}{\overline{z_2}} = \overline{\left(\frac{z_3}{z_2}\right)}$$

$$\left(\frac{z_2}{z_3}\right)\overline{\left(\frac{z_2}{z_3}\right)} = 1 \Leftrightarrow \left|\frac{z_2}{z_3}\right| = 1$$

$$1 + \frac{z_1^2}{\overline{\left(\frac{z_3}{z_2}\right)} + \frac{z_3}{z_2} - 2} + \frac{z_2^2}{\overline{\left(\frac{z_3}{z_1}\right)} + \frac{z_3}{z_1} - 2} + \frac{z_3^2}{\overline{\left(\frac{z_2}{z_1}\right)} + \frac{z_2}{z_1} - 2} = 0$$

$$\begin{cases}\overline{\left(\frac{z_3}{z_2}\right)}+\frac{z_3}{z_2}-2=2cos\theta_1\\ \overline{\left(\frac{z_3}{z_1}\right)}+\frac{z_3}{z_1}-2=2cos\theta_2\\ \overline{\left(\frac{z_1}{z_2}\right)}+\frac{z_2}{z_3}-2=2cos\theta_3\end{cases}\Rightarrow\begin{cases}\theta_1=arg(z_3-z_2)\\ \theta_2=arg(z_3-z_1)\\ \theta_3=arg(z_2-z_1)\end{cases}$$

$$1+\frac{z_1^2}{2(cos\theta_1-1)}+\frac{z_2^2}{2(cos\theta_2-1)}+\frac{z_3^2}{2(cos\theta_3-1)}=0\ldots(i)$$

$$a^2=1+1-2\cdot1\cdot1\cdot cos\theta_1$$

$$-a^2=2cos\theta_1-2, analogs\ -b^2=2cos\theta_2-2; -c^2=2cos\theta_3-2$$

Subbtituted in relation (i), we obtain:

$$1+\frac{z_1^2}{-a^2}+\frac{z_2^2}{-b^2}+\frac{z_3^2}{-c^2}=0\ldots(ii)$$

$$\frac{z_1^2}{a^2}+\frac{z_2^2}{b^2}+\frac{z_3^2}{c^2}=1$$

$$1=\left|\frac{z_1^2}{a^2}+\frac{z_2^2}{b^2}+\frac{z_3^2}{c^2}\right|\le\frac{|z_1^2|}{a^2}+\frac{|z_2^2|}{b^2}+\frac{|z_3^2|}{c^2}$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge1$$

$$a^2b^2+b^2c^2+c^2a^2\ge a^2b^2c^2$$

$$a=2RsinA; R=1;\ a^2=4sin^2A$$

$$\Rightarrow16(sin^2Asin^2B+sin^2Bsin^2C+sin^2Csin^2A)\ge64sin^2Asin^2Bsin^2C$$

$$sin^2Asin^2B+sin^2Bsin^2C+sin^2Csin^2A\ge4sin^2Asin^2Bsin^2C\ldots(iii)$$

$$A=B=C=\frac{\pi}{3}$$

The triangle ABC is equaliteral.

3.24 In $\Delta ABC, AD, BE, CF$ –internal bisectors, $D\in(BC), E\in(CA)$, $F\in(AB)$.

Prove that: $[ABC]\ge\left(\frac{9}{2}-\frac{r}{R}\right)[DEF]\ge4[DEF]$

Marian Ursărescu

Solution (George Florin Șerban)

$$\Delta ABC \overset{Bisector\ theorem}{\Rightarrow} \frac{BD}{DC}=\frac{AB}{AC}\Rightarrow\frac{BD}{BC}=\frac{AB}{AB+AC}\Rightarrow BD=\frac{ac}{b+c}$$

$$\Rightarrow DC=\frac{ab}{b+c}$$

$$\frac{AE}{EC}=\frac{AB}{BC}\Rightarrow\frac{AE}{AC}=\frac{AB}{AB+BC}\Rightarrow AE=\frac{bc}{a+c}\Rightarrow CE=\frac{ab}{a+c}$$

$$\frac{AF}{BF}=\frac{AC}{BC}\Rightarrow\frac{AF}{AB}=\frac{AC}{AC+BC}\Rightarrow AF=\frac{bc}{a+b}\Rightarrow BF=\frac{ac}{a+b}$$

$$\sigma_{DEF}=\sigma_{ABC}-\sigma_{AEF}-\sigma_{DEC}-\sigma_{BFD}$$

$$\sigma_{DEF}=\sigma_{ABC}\left(1-\frac{\sigma_{AEF}}{\sigma_{ABC}}-\frac{\sigma_{DEC}}{\sigma_{ABC}}-\frac{\sigma_{BFD}}{\sigma_{ABC}}\right)$$

$$\frac{\sigma_{DEF}}{\sigma_{ABC}}=1-\frac{\frac{AE\cdot AFsinA}{2}}{\frac{AB\cdot ACsinA}{2}}-\frac{\frac{CE\cdot CDsinC}{2}}{\frac{AC\cdot BCsinC}{2}}-\frac{\frac{BF\cdot BDsinB}{2}}{\frac{AB\cdot BCsin2B}{2}}$$

$$=1-\frac{c}{a+c}\cdot\frac{b}{a+b}-\frac{a}{a+c}\cdot\frac{b}{b+c}-\frac{a}{a+b}\cdot\frac{c}{b+c}$$

$$=1-\frac{bc}{(a+c)(a+b)}-\frac{ab}{(a+c)(b+c)}-\frac{ac}{(a+b)(b+c)}\overset{?}{\leq}\frac{1}{\frac{9}{2}-\frac{r}{R}}$$

$$=\frac{2R}{9R-2r}\Rightarrow\sum_{cyc}\frac{bc}{(a+c)(a+b)}\geq 1-\frac{2R}{9R-2r}=\frac{7R-2r}{9R-2r}$$

$$\Rightarrow\frac{\sum ab(a+b)}{\prod(a+b)}\geq\frac{7R-2r}{9R-2r}\Rightarrow\frac{\sum ab(2s-c)}{\prod(a+b)}\geq\frac{7R-2r}{9R-2r}$$

$$\Rightarrow\frac{2s\sum ab-3abc}{\prod(a+b)}\geq\frac{7R-2r}{9R-2r}$$

$$\Rightarrow\frac{2s(s^2+r^2+4Rr)-3\cdot 4RS}{2s(s^2+r^2+2Rr)}\geq\frac{7R-2r}{9R-2r}$$

$$\Rightarrow\frac{2s(s^2+r^2+4Rr)-12Rrs}{2s(s^2+r^2+2Rr)}\geq\frac{7R-2r}{9R-2r}$$

$$\Rightarrow\frac{2s(s^2+r^2+4Rr-6Rr)}{2s(s^2+r^2+2Rr)}\geq\frac{7R-2r}{9R-2r}$$

$$\Rightarrow \frac{s^2+r^2-2Rr}{s^2+r^2+2Rr} \geq \frac{7R-2r}{9R-2r}$$

$$\Rightarrow (9R-2r)(s^2+r^2-2Rr) \geq (7R-2r)(s^2+r^2+2Rr)$$

$$\Rightarrow 9s^2R + 9Rr^2 - 18R^2r - 2rs^2 - 2r^3 + 4Rr^2$$

$$\geq 7s^2R + 7Rr^2 + 14R^2r - 2s^2r - 2r^3 - 4Rr^2$$

$$2s^2R + 2Rr^2 - 32R^2r + 8Rr^2 \geq 0$$

$$2R(s^2 + r^2 - 16Rr + 4r^2) \geq 0$$

$$2R(s^2 + 5r^2 - 16Rr) \geq 0, \textit{true because } R > 0, s^2$$

$$\geq 16Rr - 5r^2 (Gerretsen) \Rightarrow s^2 - 16Rr + 5r^2 \geq 0$$

$$\left(\frac{9}{2} - \frac{r}{R}\right)\sigma_{DEF} \overset{?}{\geq} 4\sigma_{DEF} \Rightarrow \frac{9}{2} - \frac{r}{R} \geq 4 \Rightarrow \frac{9}{2} - 4 \geq \frac{r}{R} \Rightarrow R \geq 2r \ldots true(Euler)$$

3.25 In ΔABC, $B' \in (AC)$ the contact point of the external circumscription circle of side AC and C' the contact point of the external circumscription circle of side AB. Prove that:

$B'C'$ is tangent of the inscribed circle in ΔABC if and only if

$$(s-b)^2 + (s-c)^2 = (s-a)^2$$

Marian Ursărescu

Solution:

$B'C'$ *–is tangent* $\Leftrightarrow$ $BCB'C$ *–tangential* $\Leftrightarrow$

$$B'C' + BC = BC' + CB' \quad (1)$$

In $\Delta A'B'C'$ *we have:* $B'C' = \sqrt{x^2 + y^2 - 2xy\cos A}$ (2)

From (1)+(2) we have:

$$\sqrt{x^2 + y^2 - 2xy\cos A} = b + c - a - (x+y) = 2(s-a) - (x+y) \Leftrightarrow$$

$$x^2 + y^2 - 2xy\cos A = 4(s-a)^2 + 4(s-a)(x-y) + x^2 + y^2 + 2xy \quad (3)$$

where $x + y < 2(s-a)$

$$4(s-a)^2 - 4(s-a)(x+y) + 2xy\left(1 + \frac{b^2+c^2-a^2}{2bc}\right) = 0 \Leftrightarrow$$

$$4(s-a)^2 - 4(s-a)(x+y) + \frac{4xy}{bc} \cdot s(s-a) = 0$$

$$s - a - x - y + \frac{xy}{bc} \cdot s = 0$$

$$b(s-c)\left(\frac{y}{b(b-y)} + \frac{1}{b}\right) + c(s-b)\left(\frac{x}{c(c-x)} + \frac{1}{c}\right) = s$$

$$(s-c)\frac{y}{b-y}+(s-b)\frac{x}{c-x}=s-(s-c)-(s-b)=b+c-s$$
$$=s-a \quad (4)$$

But B', C' –the contact points of the external circumcircle, then

$$x=s-b, y=s-c, s-y=s-a, c-x=s-a \quad (5)$$

From (4)+(5) $\Leftrightarrow \frac{(s-c)^2}{s-a}+\frac{(s-b)^2}{s-a}=s-a \Leftrightarrow (s-b)^2+(s-c)^2=(s-a)^2$

3.26 In ΔABC let the point $A' \in (BC)$ such that the circle inscribed $\Delta AA'B$ and $\Delta AA'C$ have same radius. Prove that:

$$\sqrt[3]{AA' \cdot BB' \cdot CC'} \geq 3r$$

Marian Ursărescu

Solution:

Let: r_A –the radius of circle inscribed $\Delta AA'B$ and $\Delta AA'C$.

$$S=S_{ABA'}+S_{ACA'}=s_{ABA'}\cdot r_A+s_{ACA'}\cdot r_A=r_A\cdot(s_{ABA'}+s_{ACA'})$$
$$=r_A\cdot(s+AA'); \quad (1)$$
$$\Delta II_1I_2 \sim \Delta IBC \Rightarrow \frac{I_1I_2}{BC}=\frac{r-r_A}{r}=1-\frac{r_A}{r}\Rightarrow\frac{r_A}{r}=1-\frac{I_1I_2}{a}; \quad (2)$$

Let D, E –the points of intersection with sides BC of inscribed circle

I_1I_2ED –rectangle, then: $I_1I_2=ED=DA'+A'E=s_{ABA'}-c+s_{ACA'}-b=$

$$s-b-c+AA'; \quad (3)$$

From (2)+(3) we have: $\frac{r_A}{r}=1-\frac{s-b-c-AA'}{a}=\frac{s-AA'}{a}\Rightarrow r_A=\frac{r}{a}(s-AA'); \quad (4)$

From (1)+(4) $\Rightarrow \frac{r}{a}(s-AA')(s+AA')=S\Rightarrow s^2-AA'^2=as$

$\Rightarrow AA'^2=s^2-sa\Rightarrow AA'=\sqrt{s(s-a)}$ *and analogous*

$$BB'=\sqrt{s(s-b)}; CC'=\sqrt{s(s-c)}$$
$$AA'\cdot BB'\cdot CC'=s\sqrt{s(s-a)(s-b)(s-c)}=s\cdot S=s^2r\geq 27r^3$$
$$\sqrt[3]{AA'\cdot BB'\cdot CC'}\geq 3r$$

3.27 If in $\Delta ABC, \mu(B)=2\mu(A), \mu(C)=2\mu(B)$ then:

the following relationship holds:

$$(a^2+b^2+c^2)(a^4+b^4+c^4)>15a^2b^2c^2$$

Daniel Sitaru

Solution (Adrian Popa)

$$\Delta ABC: \hat{B}=2\hat{A}; \ \hat{C}=2\hat{B}, \hat{A}+\hat{B}+\hat{C}=\pi\Rightarrow\hat{A}+2\hat{A}+4\hat{A}=\pi$$

$$\hat{A} = \frac{\pi}{7}; \hat{B} = \frac{2\pi}{7}; \hat{C} = \frac{4\pi}{7}$$

$$(a^2 + b^2 + c^2)(a^4 + b^4 + c^4) > 15a^2b^2c^2$$

$$\frac{a}{\sin A} + \frac{b}{\sin B} + \frac{c}{\sin C} = 2R \Rightarrow a = 2R\sin\frac{\pi}{7}$$

$$b = 2R\sin\frac{2\pi}{7}, c = 2R\sin\frac{4\pi}{7}$$

$$(a^2 + b^2 + c^2)(a^4 + b^4 + c^4) \overset{Bergstrom}{\geq} (a^2 + b^2 + c^2)\left(\frac{(a^2 + b^2 + c^2)^2}{3}\right)$$

$$= \frac{(a^2 + b^2 + c^2)^3}{3} = \frac{\left(4R^2\left(\sin\frac{2\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7}\right)\right)^3}{3} =$$

$$= \frac{64R^6 \cdot \left(\frac{7}{4}\right)^3}{3} = \frac{343R^6}{3}$$

$$15a^2b^2c^2 = 15 \cdot 4R^2\sin^2\frac{\pi}{7} \cdot 4R^2\sin^2\frac{2\pi}{7} \cdot 4R^2\sin^2\frac{4\pi}{7} =$$

$$= R^6 \cdot 15 \cdot 4^3\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^2 = 15 \cdot 4^3 \cdot \frac{7}{64} = 105R^6$$

$$\frac{343}{3}R^6 \overset{?}{>} 105R^6 |: R^6 \Rightarrow 343 > 3 \cdot 105$$

$$343 > 315 \quad \textit{(True)}$$

We must prove two relationships that we've used here:

1) $\sin^2\frac{\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} = \frac{9}{4}$

2) $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} = \frac{\sqrt{7}}{8}$

1) $\sin^2\frac{\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} = \frac{1-\cos2\frac{\pi}{7}}{2} + \frac{1-\cos\frac{4\pi}{7}}{2} + \frac{1-\cos\frac{8\pi}{7}}{2} =$

$$= \frac{3}{2} - \frac{\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}}{2}$$

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{3\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{5\pi}{7}}{2\sin\frac{\pi}{7}}=$$

$$=\frac{\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{2\pi}{7}+\sin\frac{6\pi}{7}-\sin\frac{4\pi}{7}}{2\sin\frac{\pi}{7}}=\frac{\sin\frac{6\pi}{7}}{2\sin\frac{\pi}{7}}=\frac{\sin\left(\pi-\frac{\pi}{7}\right)}{2\sin\frac{\pi}{7}}=\frac{1}{2}$$

$$\Rightarrow\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\left(\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}\right)=-\frac{1}{2}\Rightarrow$$

$$\Rightarrow\sin^2\frac{\pi}{7}+\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}=\frac{3}{2}+\frac{1}{4}=\frac{7}{4}$$

2) $\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\right)\left(\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right)=$

$$=\frac{\left(2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\right)\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)\left(2\sin\frac{3\pi}{7}\cos\frac{3\pi}{7}\right)}{8}=$$

$$=\frac{\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\sin\frac{6\pi}{7}}{8}=\frac{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\frac{\pi}{7}}{8}\Rightarrow$$

$$\Rightarrow\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=\frac{1}{8}$$

Now:

$$\left(2\sin^2\frac{\pi}{7}\right)\left(2\sin^2\frac{2\pi}{7}\right)\left(2\sin^2\frac{3\pi}{7}\right)=\left(1-\cos\frac{2\pi}{7}\right)\left(1-\cos\frac{4\pi}{7}\right)\left(1-\cos\frac{6\pi}{7}\right)=$$

$$=1+\frac{1}{2}\left(2\cos\frac{2\pi}{7}\cos\frac{4\pi}{2}+2\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+2\cos\frac{6\pi}{7}\cos\frac{2\pi}{7}\right)-$$

$$-\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}=$$

$$=1+\frac{1}{2}\left(\cos\frac{6\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{4\pi}{7}\right)-$$

$$-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}-\underbrace{\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\cos\frac{\pi}{7}}_{\frac{1}{8}}$$

$$\left(2\sin^2\frac{\pi}{7}\right)\left(2\sin^2\frac{2\pi}{7}\right)\left(2\sin^2\frac{3\pi}{7}\right)=$$

$$= 1 + \frac{1}{2}\left(-\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7}\right) -$$

$$-\cos\frac{2\pi}{7} - \cos\frac{4\pi}{7} - \cos\frac{6\pi}{7} - \frac{1}{8} =$$

$$= \frac{7}{8} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{\pi}{7} = \frac{7}{8}$$

So, $2\sin^2\frac{\pi}{7}\cdot 2\sin^2\frac{2\pi}{7}\cdot 2\sin^2\frac{3\pi}{7} = \frac{7}{8} \Rightarrow$

$$\Rightarrow \sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin^2\frac{3\pi}{7} = \frac{7}{64} \Rightarrow \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}$$

3.28 In acute ΔABC the altitudes AA', BB', CC' intersect at all second times the determined circle by the points A', B', C' in A'', B'', C''.

Prove that: $(2r)^{2s} \geq (A'A'')^a \cdot (B'B'')^b \cdot (C'C'')^c$

Marian Ursărescu

Solution:

$$\frac{AA''}{AA'} = \frac{AA''\cdot AA'}{AA'^2} \quad (1)$$

$$AA''\cdot AA' = \rho(A) = A\Omega^2 - \frac{R^2}{4} \quad (2)$$

From $\overset{(1)+(2)}{\Longrightarrow} AA''\cdot AA' = \frac{b^2+c^2-a^2}{4} \Rightarrow AA' = h_a = 2R\cdot sinB\cdot sinC \quad (3)$

From $\overset{(1)+(2)+(3)}{\Longrightarrow} \frac{AA''}{AA'} = \frac{b^2+c^2-a^2}{16R^2\cdot sin^2B\cdot sin^2C} = \frac{2bc\cdot cosA}{16R^2\cdot sin^2B\cdot sin^2C} = \frac{4R^2\cdot cosA\cdot sinB\cdot sinC}{8R^2\cdot sin^2B\cdot sin^2C} =$

$$\frac{2bc\cdot cosA}{2sinB\cdot sinC}$$

Therefore

$$\sum\frac{AA''}{AA'} = \frac{1}{2}\underbrace{\sum\frac{cosA}{sinBsinC}}_{1} = 1 \Rightarrow \sum\frac{AA''}{\frac{2S}{a}} = 1 \Rightarrow \sum a\cdot AA'' = 2S$$

$$\Rightarrow \sum a(h_a - A'A'') = 2S \Rightarrow \sum ah_a - \sum a\cdot A'A'' = 2S$$

$$\Rightarrow \sum a \cdot A'A" = 4S \Rightarrow \sum \frac{a \cdot A'A"}{a+b+c} = \frac{4S}{2s} = \frac{4sr}{2s} = 2r \quad (4)$$

Applying weighted mean's inequality, we have:

$$2r = \sum \frac{a}{a+b+c} \cdot A'A" \geq (A'A")^{a/2s} \cdot (B'B")^{b/2s} \cdot (C'C")^{c/2s}$$

So:

$$(2r)^{2s} \geq (A'A")^{a} \cdot (B'B")^{b} \cdot (C'C")^{c}$$

3.29 In any ΔABC the following relationship holds:

$$\frac{r}{4R} \leq \sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right) \leq \frac{1}{8}$$

Marian Ursărescu

Solution:

In any ΔABC, we have:

$$\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right) = \frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)} \quad (1)$$

$$\therefore \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R} \quad (2)$$

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2} \Rightarrow \frac{1}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}} \geq \frac{2}{3\sqrt{3}} \quad (3)$$

From (1)+(2)+(3) we have: $\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right) \geq \frac{s}{4\cdot 3\sqrt{3}R}$ (4)

From Mitrinovic: $s \geq 3\sqrt{3}r$ (5)

From (4)+(5) we get: $\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right) \geq \frac{r}{4R}$

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} \geq 3 \cdot \sqrt[3]{\cos^2\frac{A}{2} \cdot \cos^2\frac{B}{2} \cdot \cos^2\frac{C}{2}} \quad (6)$$

From (5)+(6) we have:

$$\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)\le\frac{1}{6}\cdot\sqrt[3]{\cos^2\frac{A}{2}\cdot\cos^2\frac{B}{2}\cdot\cos^2\frac{C}{2}}$$

$$=\frac{1}{3}\cdot\sqrt[3]{\frac{s^2}{16R^2}};(7)$$

From Mitrinovic$\Rightarrow s^2\le\frac{27}{4}R^2$ (8)

From (7)+(8) we get: $\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)\le\frac{1}{3}\cdot\sqrt[3]{\frac{27}{64}}=\frac{1}{8}$

3.30 In acute *ΔABC* the following relationship holds:

$$3R(s^2-r^2-4Rr)\ge(R+r)(r^2+4rR+s^2)$$

Florică Anastase

Solution:

$$Let: a\ge b\ge c\Rightarrow\begin{cases}\cos A\le\cos B\le\cos C\\ \frac{1}{\sin A}\le\frac{1}{\sin B}\le\frac{1}{\sin C}\end{cases}\overset{Chebyshev}{\Rightarrow}$$

$$\frac{a^2+b^2+c^2}{4S}=\cot A+\cot B+\cot C$$

$$\ge\frac{1}{3}(\cos A+\cos B+\cos C)\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)$$

$$\cos A+\cos B+\cos C=1+\frac{r}{R}\text{ and sinus theorem result:}$$

$$\frac{a^2+b^2+c^2}{4S}\ge\frac{1}{3}\left(1+\frac{r}{R}\right)\left(\frac{2R}{a}+\frac{2R}{b}+\frac{2R}{c}\right)\Leftrightarrow$$

$$\begin{cases}\frac{a^2+b^2+c^2}{4S}\ge\frac{2}{3}(R+r)\left(\frac{ab+bc+ca}{abc}\right)\\ a^2+b^2+c^2=2(s^2-r^2-4Rr)\\ ab+bc+ca=r^2+4Rr+s^2\end{cases}\Rightarrow$$

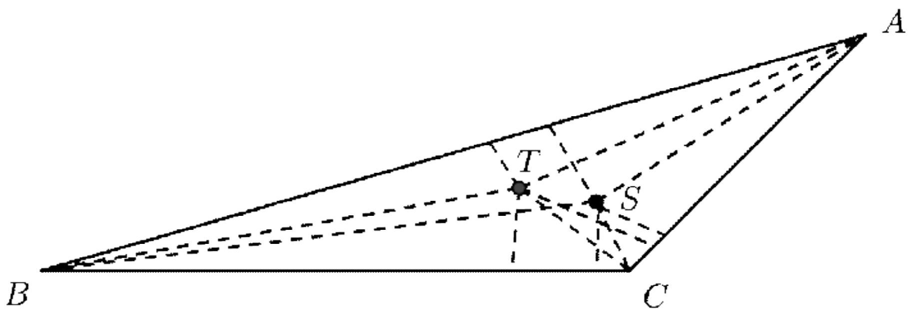
$$\frac{s^2-r^2-4Rr}{4S}\ge\frac{1}{3}(R+r)\frac{r^2+4rR+s^2}{4RS}\Leftrightarrow$$

$$3R(s^2-r^2-4rR)\ge(R+r)(r^2+4rR+s^2)$$

3.31 Let S and T be points inside the triangle ABC. The distance from S to lines AB, BC and CA is 10, 7 and 4, respectively. The distance from T to these lines is 4, 10 and 16, respectively. Determine the radius of the incircle of triangle ABC.

Stipe Vidak-Croatian NMO-2017

Solution: *Denote $a = |BC|, b = |CA|, c = |AB|$. Let P be the area, $s = \frac{1}{2}(a + b + c)$ the semiperimeter, and r the radius of the incircle of triangle ABC. We can divide triangle ABC into three smaller triangles by connecting pppoooint S to vertices A, B and C.*



By adding the areas of triangles ABS, BCS and CAS, we get the area of triangle ABC, so we have $2P = 10c + 7a + 4b$. Analogously, by observing point T we get $2P = 4c + 10a + 16b$. If we multiply the first equality by 2 and add it to the second equality, we get $6P = 24(a + b + c) = 48s, i.e. P = 8s$ Since $P = rs$, we have $r = 8$.

3.32 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{2x+y+3z}{x+2y}a^2 + \frac{2y+z+3x}{y+2z}b^2 + \frac{2z+x+3y}{z+2x}c^2 \geq 72r^2$$

Daniel Sitaru

Solution:

$$\sum_{cyc}\frac{2x+y+3z}{x+2y}a^2 = \sum_{cyc}\left(\frac{2x+y+3z}{x+2y}a^2 + a^2\right) - \sum_{cyc}a^2 =$$

$$= \sum_{cyc}\frac{2x+y+3z+x+2y}{x+2y}a^2 - \sum_{cyc}a^2 =$$

$$= 3(x+y+z)\sum_{cyc}\frac{a^2}{x+2y} - \sum_{cyc} a^2 \geq$$

$$\overset{BERGSTROM}{\geq} 3(x+y+z)\cdot\frac{(a+b+c)^2}{x+2y+y+2z+z+2x} - \sum_{cyc} a^2 =$$

$$= 3(x+y+z)\cdot\frac{(a+b+c)^2}{3(x+y+z)} - \sum_{cyc} a^2 =$$

$$= 3(a+b+c)^2 - (a^2+b^2+c^2) = 2(ab+bc+ca) =$$

$$= 2(s^2+r^2+4Rr) \overset{EULER}{\geq} 2(s^2+r^2+8r^2) \geq$$

$$\overset{MITRINOVIC}{\geq} 2(27r^2+9r^2) = 72r^2$$

3.33 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{3x+4y}{7z+4x+3y}a^2 + \frac{4y+4z}{7x+4y+3z}b^2 + \frac{3z+4x}{7y+4z+3x}c^2 \geq 2\sqrt{3}S$$

Daniel Sitaru

Solution:

$$\sum_{cyc}\frac{3x+4y}{7z+4x+3y}a^2 = \sum_{cyc}\left(\frac{3x+4y}{7z+4x+3y}a^2 + a^2\right) - \sum_{cyc} a^2 =$$

$$= \sum_{cyc}\frac{(3x+4y+7z+4x+3y)a^2}{7z+4x+3y} - \sum_{cyc} a^2 =$$

$$= \sum_{cyc}\frac{7(x+y+z)a^2}{7z+4x+3y} - \sum_{cyc} a^2 =$$

$$= 7(x+y+z)\sum_{cyc}\frac{a^2}{7z+4x+3y} - \sum_{cyc} a^2 \overset{BERGSTROM}{\geq}$$

$$\geq 7(x+y+z)\cdot\frac{(a+b+c)^2}{(7+4+3)(x+y+z)} - \sum_{cyc} a^2 =$$

$$=\frac{1}{2}(a+b+c)^2-\sum_{cyc}a^2=\frac{1}{2}\cdot 4s^2-2(s^2-r^2-4Rr)=$$

$$=2s^2-2s^2+2r^2+8Rr=2r(r+4R)\ge$$

$$\overset{DOUCET}{\ge} 2r\cdot s\sqrt{3}=2\sqrt{3}rs=2\sqrt{3}S$$

3.34 In ΔABC the following relationship holds:

$$\frac{a^{12}}{(b\cos^2 x+c\sin^2 x)^4}+\frac{b^{12}}{(c\cos^2 x+a\sin^2 x)^4}+$$

$$+\frac{c^{12}}{(a\cos^2 x+b\cos^2 x)^4}\ge\frac{64}{3}\cdot S^4; x\in(0,\pi)$$

Daniel Sitaru

Solution:

$$\sum_{cyc}\frac{a^{12}}{(b\cos^2 x+c\sin^2 x)^4}=\sum_{cyc}\left(\frac{a^3}{b\cos^2 x+c\sin^2 x}\right)^4\ge$$

$$\overset{RADON}{\ge}\frac{1}{3^3}\left(\sum_{cyc}\frac{a^3}{b\cos^2 x+c\sin^2 x}\right)^4=$$

$$=\frac{1}{27}\left(\sum_{cyc}\frac{(a^2)^2}{ab\cos^2 x+ac\sin^2 x}\right)^4\overset{BERGSTROM}{\ge}$$

$$\ge\frac{1}{27}\left(\frac{(a^2+b^2+c^2)^2}{\sum_{cyc}(ab\cos^2 x+ac\sin^2 x)}\right)^4=$$

$$=\frac{1}{27}\left(\frac{(a^2+b^2+c^2)^2}{(ab+bc+ca)(\cos^2 x+\sin^2 x)}\right)^4=$$

$$=\frac{1}{27}\cdot\left(\frac{(a^2+b^2+c^2)^2}{ab+bc+ca}\right)^4\ge\frac{1}{27}\cdot\frac{(a^2+b^2+c^2)^8}{(a^2+b^2+c^2)^4}=$$

$$=\frac{1}{27}(a^2+b^2+c)^4\overset{IONESCU-WEITZENBOCK}{\ge}\frac{1}{27}(4\sqrt{3}S)^4=$$

$$= \frac{64}{27} \cdot 9 \cdot S^4 = \frac{64}{3} S^4$$

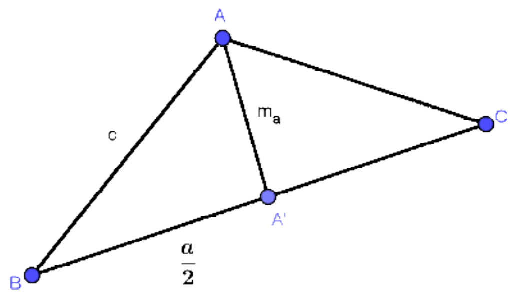
Equality holds for $a = b = c$.

3.35 In ΔABC the following relationship holds:

$$(\boldsymbol{a} + \boldsymbol{2c} + \boldsymbol{2m_a})^2 > 24\sqrt{3}\boldsymbol{S}$$

Daniel Sitaru

Solution:



$$S' = S[ABA'] = \frac{S}{2}$$

$$s' = s[ABA'] = \frac{1}{2}\left(c + \frac{a}{2} + m_a\right) = \frac{a + 2c + 2m_a}{4}$$

$$r' = \frac{S'}{s'} = \frac{S}{2s'} = \frac{2S}{a + 2c + m_a}$$

By Mitrinovic's inequality in $\Delta ABA' : s' \geq 3\sqrt{3}r'$

$$\frac{a + 2c + m_a}{4} \geq 3\sqrt{3} \cdot \frac{2S}{a + 2c + 2m_a}$$

$$(a + 2c + m_a)^2 \geq 24\sqrt{3}S$$

Equality holds for: $c = \frac{a}{2} = m_a, \ \ a = 2c$

$$m_a^2 = \frac{a^2}{4} \Rightarrow \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2 = \frac{1}{4}a^2$$

$$b^2 + c^2 = 2a^2 \Rightarrow b^2 + c^2 = 8c^2 \Rightarrow b = c\sqrt{7}$$

$$m_a = c \Rightarrow \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2 = c^2$$

$$\frac{b^2 + c^2}{2} - \frac{1}{4} \cdot 4c^2 = c^2 \Rightarrow \frac{b^2 + c^2}{2} = 2c^2 \Rightarrow b = c\sqrt{3}$$

$c\sqrt{7} = c\sqrt{3} \Rightarrow c = 0$. *False!*

Inequality is a strict one.

3.36 In ΔABC the following relationship holds:

$$\frac{1}{a^2 + bc} + \frac{1}{b^2 + ca} + \frac{1}{c^2 + ab} \leq \frac{1}{4Rr}$$

Daniel Sitaru

Solution:

$$a^2b + a^2c + b^2c + bc^2 \overset{AM-GM}{\geq} 4 \cdot \sqrt[4]{a^4b^4c^4} = 4abc$$

$$4abc \leq a^2(b + c) + bc(b + c)$$

$$4abc \leq (a^2 + bc)(b + c)$$

$$\frac{1}{a^2 + bc} \leq \frac{b + c}{4abc}$$

$$\sum_{cyc} \frac{1}{a^2 + bc} \leq \sum_{cyc} \frac{b + c}{4abc} = \frac{2(a + b + c)}{4abc} =$$

$$= \frac{4s}{4abc} = \frac{s}{abc} = \frac{s}{4Rrs} = \frac{1}{4Rr}$$

3.37 If $x \in \left[0, \frac{\pi}{12}\right]$ then:

$$\cos^{202}x \geq \cos^{10} x \cdot \cos^{15} 2x \cdot \cos^5 3x \cdot \cos^6 4x \cdot \cos 5x \cdot \cos 6x$$

Daniel Sitaru

Solution:

$$\cos^6 x = \frac{1}{32}(10 + 15\cos 2x + 6\cos 4x + \cos 6x) \geq$$

$$\overset{AM-GM}{\geq} \frac{1}{32} \cdot 32 \cdot \sqrt[32]{1^{10} \cdot (\cos 2x)^{15} \cdot (\cos 4x)^6 \cdot \cos 6x} =$$

$$= \sqrt[32]{\cos^{15} 2x \cdot \cos^6 4x \cdot \cos 6x}$$

$$\cos^{192} x \geq \cos^{15} 2x \cdot \cos^6 4x \cdot \cos 6x \quad (1)$$

$$\cos^5 x = \frac{1}{16}(10\cos x + 5\cos 3x + \cos 5x) \geq$$

$$\geq \frac{1}{16} \cdot 16 \sqrt[16]{\cos^{10} x\, (\cos 3x)^5 \cdot \cos 5x}$$

$$\cos^{80} x \geq \cos^{10} x \cdot \cos^5 3x \cdot \cos 5x \quad (2)$$

By multiplying (1); (2):

$$\cos^{202} x \geq \cos^{10} x \cdot \cos^{15} 2x \cdot \cos^5 3x \cdot \cos^6 4x \cdot \cos 5x \cdot \cos 6x$$

3.38 Let $\Delta A'B'C'$ be the circumcevian triangle of H –ortocenter in acute ΔABC. Prove that:

$$\frac{[A'B'C']}{[ABC]} \geq 4\left(\left(\frac{r}{R}\right)^2 + \frac{2r}{R} - 1\right)$$

Marian Ursărescu

Solution (Tran Hong)

$$\frac{[A'B'C']}{[ABC]} = \frac{A'B' \cdot B'C' \cdot C'A'}{4R} \cdot \frac{4R}{abc} = \frac{A'B' \cdot B'C' \cdot C'A'}{abc}$$

$$\begin{cases} \angle BAH = \angle BB'A' \\ \angle ABB' = \angle AA'B' \end{cases} \Rightarrow \Delta HAB \sim \Delta HB'A' \Rightarrow \frac{HA}{HB'} = \frac{AB}{A'B'} \Rightarrow$$

$$A'B' = \frac{HB' \cdot AB}{HA} = \frac{c \cdot HB \cdot HB'}{HA \cdot HB} = \frac{c \cdot \rho(H)}{HA \cdot HB}$$

Similary:

$$A'C' = \frac{b \cdot \rho(H)}{HA \cdot HC}; B'C' = \frac{a \cdot \rho(H)}{HB \cdot HC}$$

$$\rho(H) = R^2 - OH^2 = R^2 - \left(9R^2 - (a^2 + b^2 + c^2)\right) = a^2 + b^2 + c^2 - 8R^2$$

$$\Delta ABC \text{ –acute} \Rightarrow HA = 2R\cos A; HB = 2R\cos B; HC = 2R\cos C$$

$$\Rightarrow \frac{[A'B'C']}{[ABC]} = \frac{abc\left(\rho(H)\right)^3}{abc(HA \cdot HB \cdot HC)^2} = \frac{(a^2 + b^2 + c^2 - 8R^2)^3}{64R^6(\cos A \cdot \cos B \cdot \cos C)^2}$$

$$= \frac{8(s^2 - 4R^2 - 4Rr - r^2)^3}{4R^2(s^2 - 4R^2 - 4Rr - r^2)^2}$$

$$= \frac{2(s^2 - 4R^2 - 4Rr - r^2)}{R^2} \overset{(*)}{\geq} 4\left(\left(\frac{r}{R}\right)^2 + \frac{2r}{R} - 1\right)$$

$$(*) \Leftrightarrow s^2 - 4R^2 - 4Rr - r^2 \geq 2(r^2 + 2Rr - R^2) \Leftrightarrow s^2 \geq 2R^2 +* Rr + 3r^2$$

true by Walker's inequality, then $(*)$ *true.Proved.*

3.39 In ΔABC, BB', CC' –interior bisectors. If the circumcircle of ΔABC it is tangent to the side BC, then:

$$\frac{2b}{2a + c} + \frac{2c}{2a + b} < 1$$

Marian Ursărescu

Solution:

$$\rho(B) = BA'^2 = BC' \cdot BA;\ \rho(C) = CA'^2 = CB' \cdot CA \Rightarrow$$

$$a = BA' + A'C = \sqrt{c \cdot BC'} + \sqrt{b \cdot B'C} \quad (1)$$

$$\frac{BC'}{C'A} = \frac{a}{b} \Rightarrow \frac{BC'}{c} = \frac{a}{a+b} \Rightarrow BC' = \frac{ac}{a+b} \quad (2)$$

$$\frac{B'C}{B'A} = \frac{a}{c} \Rightarrow \frac{B'A}{b} = \frac{a}{a+c} \Rightarrow B'C = \frac{ab}{a+c} \quad (3)$$

From (1)+(2)+(3) we have: $a = \sqrt{\frac{ac^2}{a+b}} + \sqrt{\frac{ab^2}{a+c}} \Rightarrow \sqrt{a} = \frac{c}{\sqrt{a+b}} + \frac{b}{\sqrt{a+c}}$

$$\Rightarrow 1 = \frac{c}{\sqrt{a(a+b)}} + \frac{b}{\sqrt{a(a+c)}} \quad (4)$$

$$\sqrt{a(a+b)} \leq \frac{2a+b}{2} \Rightarrow \frac{1}{\sqrt{a(a+b)}} > \frac{2}{2a+b}$$

$$\sqrt{a(a+c)} \leq \frac{2a+c}{2} \Rightarrow \frac{1}{\sqrt{a(a+c)}} > \frac{2}{2a+c} \quad (5)$$

From (4)+(5) we have: $1 = \frac{c}{\sqrt{a(a+b)}} + \frac{b}{\sqrt{a(a+c)}} > \frac{2c}{2a+b} + \frac{2}{2a+c}$

3.40 Let $A'B'C'$ be the circumcevian triangle of symedians in ΔABC.

Prove that:

$$\frac{[A'B'C']}{[ABC]} \le \left(\frac{R}{2r}\right)^2$$

Marian Ursărescu

Solution (Tran Hong)

$$\frac{[A'B'C']}{[ABC]} = \frac{A'B' \cdot B'C' \cdot C'A'}{4R} \cdot \frac{4R}{abc} = \frac{A'B' \cdot B'C' \cdot C'A'}{abc}$$

$$\begin{cases} \widehat{BAK} \equiv \widehat{BB'A'} \\ \widehat{ABB'} \equiv \widehat{AA'B'} \end{cases} \Rightarrow \Delta KAB \sim \Delta K'B'A' \Rightarrow \frac{KA}{KB'} = \frac{AB}{B'A'}$$

$$\Rightarrow B'A' = \frac{AB \cdot KB'}{KA} = c \cdot \frac{KB}{KA} \cdot \frac{KB'}{KB} = \frac{c \cdot \rho(K)}{KA \cdot KB}$$

Similary $B'C' = \frac{c \cdot \rho(K)}{KB \cdot KC}$ and $A'C' = \frac{c \cdot \rho(K)}{KA \cdot KC}$

So, $\frac{[A'B'C']}{[ABC]} = \frac{abc(\rho(K))^3}{abc(KA \cdot KB \cdot KC)^3} = \frac{(\rho(K))^3}{(KA \cdot KB \cdot KC)^3}$

But: $\rho(K) = R^2 - OK^2 = \frac{3(abc)^2}{(a^2+b^2+c^2)^2}$

$$KA = \frac{b^2 + c^2}{a^2 + b^2 + c^2} \cdot s_a = \frac{b^2 + c^2}{a^2 + b^2 + c^2} \cdot \frac{2bc}{b^2 + c^2} \cdot m_a = \frac{2bcm_a}{a^2 + b^2 + c^2} \Rightarrow$$

$$KA^2 = \frac{(2bcm_a)^2}{(a^2 + b^2 + c^2)^2} \Rightarrow$$

$$\frac{[A'B'C']}{[ABC]} = \frac{\frac{27(abc)^6}{(a^2 + b^2 + c^2)^6}}{\frac{64(abc)^4}{(a^2 + b^2 + c^2)^6} \cdot (m_a m_b m_c)^2} = \frac{27}{64} \cdot \left(\frac{abc}{m_a m_b m_c}\right)^2$$

$$= \frac{27}{64} \cdot \left(\frac{4Rrs}{m_a m_b m_c}\right)^2 \overset{m_a m_b m_c \ge s^2 r}{\le} \frac{27}{64} \cdot \left(\frac{4Rrs}{s^2 r}\right)^2 = \frac{27}{4} \cdot \frac{R^2}{s^2} \overset{s \ge 3\sqrt{3}r}{\le}$$

$$\frac{27}{4} \cdot \frac{R^2}{27r^2} = \left(\frac{R}{2r}\right)^2$$

3.41 If $x \in \left(0, \frac{\pi}{2}\right)$ then:

$$(1+\sin x)^{\sin x+\sin^2 x}\cdot(1+\cos x)^{\cos x+\cos^2 x}\ge 2+\frac{1}{\sin x+\cos x}$$

Florică Anastase

Solution (Soumava Chakraborty)

Proof: weighted GM ≥ weighted HM →

$$\sqrt[\sin x+\sin^2 x+\cos x+\cos^2 x]{(1+\sin x)^{\sin x+\sin^2 x}\cdot(1+\cos x)^{\cos x+\cos^2 x}}$$

$$\ge \frac{\sin x+\sin^2 x+\cos x+\cos^2 x}{\frac{\sin x+\sin^2 x}{1+\sin x}+\frac{\cos x+\cos^2 x}{1+\cos x}}=\frac{1+\sin x+\cos x}{\sin x+\cos x}\rightarrow$$

$$(1+\sin x)^{\sin x+\sin^2 x}\cdot(1+\cos x)^{\cos x+\cos^2 x}$$

$$\ge\left(\frac{1+\sin x+\cos x}{\sin x+\cos x}\right)^{\sin x+\sin^2 x+\cos x+\cos^2 x}$$

$$=\left(\frac{1+\sin x+\cos x}{\sin x+\cos x}\right)^{1+\sin x+\cos x}\overset{Bernoulli}{\ge}1$$

$$+\left(\frac{1}{\sin x+\cos x}\right)(1+\sin x+\cos x)=2+\frac{1}{\sin x+\cos x}$$

3.42 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{a^4\sin^3 A}{b}+\frac{a^4\cos^3 A}{c}\right)\ge\frac{(s^2-r^2-4Rr)^2}{s}$$

Florică Anastase

Solution:

$$\sum_{cyc}\left(\frac{a^4\sin^3 A}{b}+\frac{a^4\cos^3 A}{c}\right)=\sum_{cyc}\left(\frac{a^4\sin^4 A}{b\sin A}+\frac{a^4\cos^4 A}{c\cos A}\right)$$

$$\overset{Bergstrom}{\ge}\sum_{cyc}\frac{a^4(\sin^2 A+\cos^2 A)^2}{b\sin A+c\cos A}=\sum_{cyc}\frac{a^4}{b\sin A+c\cos A}$$

$$\overset{BCS}{\geq}\sum_{cyc}\frac{a^4}{\sqrt{b^2+c^2}}\geq\sum_{cyc}\frac{a^4}{b+c}\overset{Bergstrom}{\geq}\frac{(a^2+b^2+c^2)^2}{2s}$$

$$=\frac{4(s^2-r^2-4Rr)^2}{2s}=\frac{(s^2-r^2-4Rr)^2}{s}$$

3.43 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{a^2\sin^3A}{b}+\frac{a^2\cos^3A}{c}\right)\geq\frac{27r^2}{2s}$$

Florică Anastase

Solution:

$$\sum_{cyc}\left(\frac{a^2\sin^3A}{b}+\frac{a^2\cos^3A}{c}\right)=\sum_{cyc}\left(\frac{a^2\sin^4A}{b\sin A}+\frac{a^2\cos^4A}{c\cos A}\right)$$

$$\overset{Bergstrom}{\geq}\sum_{cyc}\frac{(a\sin^2A+a\cos^2A)^2}{b\sin A+c\cos A}=\sum_{cyc}\frac{a^2}{b\sin A+c\cos A}$$

$$\overset{BCS}{=}\sum_{cyc}\frac{a^2}{\sqrt{b^2+c^2}}\geq\sum_{cyc}\frac{a^2}{b+c}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{2(a+b+c)}=s$$

$$=r^2\sum_{cyc}\frac{\cot^2\frac{A}{2}}{s-a}\overset{Bergstrom}{\geq}r^2\cdot\frac{\left(\sum\cot\frac{A}{2}\right)^2}{2s}=\frac{r^2}{2s}\cdot\left(\underbrace{\sum\cot\frac{A}{2}}_{\geq 3\sqrt{3}}\right)^2\geq\frac{27r^2}{2s}$$

3.44 In acute ΔABC the following relationship holds:

$$9R(bch_a+ach_b+abh_c)\geq 2r^2(s^2+r^2+4rR)$$

Florică Anastase

Solution:

$$\text{If } a\geq b\geq c\Rightarrow\begin{cases}h_a\leq h_b\leq h_c\\ \frac{1}{\sin A}\leq\frac{1}{\sin B}\leq\frac{1}{\sin C}\end{cases}\overset{Chebyshev}{\Rightarrow}$$

$$\frac{h_a}{\sin A}+\frac{h_b}{\sin B}+\frac{h_c}{\sin C}\geq\left(\frac{h_a+h_b+h_c}{3}\right)\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\Leftrightarrow$$

$$\frac{h_a}{sinA}+\frac{h_b}{sinB}+\frac{h_c}{sinC}$$

$$\geq\left(\frac{h_a+h_b+h_c}{3}\right)\left(\frac{sinAsinB+sinBsinC+sinCsinA}{sinAsinBsinC}\right)\overset{Chebyshev}{\geq}$$

$$\geq\left(\frac{h_a+h_b+h_c}{9}\right)\frac{(sinA+sinB+sinC)^2}{sinAsinBsinC}\Rightarrow$$

$$2R\left(\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}\right)\geq\frac{1}{9}\left(\frac{s^2+r^2+4rR}{2R}\right)\left(\frac{8R^3}{abc}\right)\left(\frac{r}{R}\right)^2\Leftrightarrow$$

$$9abcR\left(\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}\right)\geq 2r^2(s^2+r^2+4rR)\Leftrightarrow$$

$$9R(bch_a+ach_b+abh_c)\geq 2r^2(s^2+r^2+4rR)$$

3.45 In acute *ΔABC* the following relationship holds:

$$\mathbf{2s(s^2-r^2-4rR)(r+R)\geq 3rR[3s^2-(2R+r)(4R+r)]}$$

Florică Anastase

Solution:

$$a^2cosA+b^2cosB+c^2cosC=\frac{r}{s}[3s^2-(2R+r)(4R+r)]$$

$$\begin{cases} a^2\geq b^2\geq c^2 \\ cosA\leq cosB\leq cosC \\ a^2+b^2+c^2=2(s^2-r^2-4rR) \\ cosA+cosB+cosC=1+\dfrac{r}{R}\end{cases}\overset{Chebyshev}{\Rightarrow}$$

$$a^2cosA+b^2cosB+c^2cosC\leq\frac{a^2+b^2+c^2}{3}(cosA+cosB+cosC)\Rightarrow$$

$$2s(s^2-r^2-4rR)(r+R)\geq 3rR[3s^2-(2R+r)(4R+r)]$$

3.46 In acute *ΔABC* the following relationship holds:

$$\mathbf{\frac{9a^3b^3c^3}{b^2+c^2-a^2}+\frac{9a^3b^3c^3}{a^2+c^2-b^2}+\frac{9a^3b^3c^3}{a^2+b^2-c^2}\geq 2s[8S(R+r)]^2}$$

Florică Anastase

Solution:

$$If\ a\geq b\geq c\ \Rightarrow A\geq B\geq C\Rightarrow\begin{cases}sinA\geq sinB\geq sinC\\ \dfrac{1}{cosA}\geq\dfrac{1}{cosB}\geq\dfrac{1}{cosC}\end{cases}\overset{Chebyshev}{\Rightarrow}$$

$$tgA+tgB+tgC\geq\frac{1}{3}(sinA+sinB+sinC)\left(\frac{1}{cosA}+\frac{1}{cosB}+\frac{1}{cosC}\right)$$

$$=\frac{1}{3}(sinA+sinB+sinC)\left(\frac{cosAcosB+cosBcosC+cosCcosA}{cosAcosBcosC}\right)\quad(1)$$

$$With:\begin{cases}sinA+sinB+sinC=\dfrac{s}{R}\\ \dfrac{1}{cosAcosBcosC}\geq 8\\ tgA=\dfrac{4S}{b^2+c^2-a^2}(and\ analogs)\end{cases}\overset{(1)}{\Rightarrow}$$

$$\frac{4S}{b^2+c^2-a^2}+\frac{4S}{a^2+c^2-b^2}+\frac{4S}{a^2+b^2-c^2}\geq\frac{8s}{3R}(cosAcosB+cosBcosC+cosCcosA)$$

$$\frac{4S}{b^2+c^2-a^2}+\frac{4S}{a^2+c^2-b^2}+\frac{4S}{a^2+b^2-c^2}\geq\frac{8s}{9R}(cosA+cosB+cosC)^2$$

$$\frac{9\cdot 4RS}{b^2+c^2-a^2}+\frac{9\cdot 4RS}{a^2+c^2-b^2}+\frac{9\cdot 4RS}{a^2+b^2-c^2}\geq 8s\left(1+\frac{r}{R}\right)^2\Leftrightarrow$$

$$\frac{9\cdot abcR^2}{b^2+c^2-a^2}+\frac{9\cdot abcR^2}{a^2+c^2-b^2}+\frac{9\cdot abcR^2}{a^2+b^2-c^2}\geq 8s(R+r)^2\Leftrightarrow$$

$$\frac{9a^3b^3c^3}{b^2+c^2-a^2}+\frac{9a^3b^3c^3}{a^2+c^2-b^2}+\frac{9a^3b^3c^3}{a^2+b^2-c^2}\geq 2s[8S(R+r)]^2$$

3.47 In acute $\Delta ABC, H$ –ortocenter, AA_1, BB_1, CC_1 –altitudes. Prove that:

$$\frac{1}{HA_1^2}+\frac{1}{HB_1^2}+\frac{1}{HC_1^2}\geq\frac{12}{R^2}, A_1\in(BC), B_1\in(CA), C_1\in(AB)$$

Marian Ursărescu

Solution (Tran Hong)

In acute $\Delta ABC: AH + BH + CH = 2R(cosA + cosB + cosC) = 2(r + R)$

$$\frac{1}{HA_1^2} + \frac{1}{HB_1^2} + \frac{1}{HC_1^2} \geq \frac{1}{3}\left(\frac{1}{HA_1} + \frac{1}{HB_1} + \frac{1}{HC_1}\right)^2 \overset{CBS}{\geq} \frac{1}{3}\left(\frac{9}{HA_1 + HB_1 + HC_1}\right)^2$$

$$HA_1 + HB_1 + HC_1 = h_a + h_b + h_c - (AH + BH + CH)$$

$$= \frac{s^2 + r^2 + 4Rr}{2R} - 2(r + R) = \frac{s^2 + r^2 - 4R^2}{2R}$$

We must show that:

$$\frac{1}{3}\left(\frac{9}{HA_1 + HB_1 + HC_1}\right)^2 \geq \frac{12}{R^2} \Leftrightarrow \frac{9}{\frac{s^2 + r^2 - 4R^2}{2R}} \geq \frac{6}{R}$$

$$\Leftrightarrow 3R^2 \geq s^2 + r^2 - 4R^2 \Leftrightarrow 7R^2 - r^2 \geq s^2$$

But: $s^2 \leq 4R^2 + 4Rr + 3r^2$

So, we just check $4R^2 + 4Rr + 3r^2 \leq 7R^2 - r^2$

$$\Leftrightarrow 3R^2 - 4Rr - 4r^2 \geq 0 \Leftrightarrow (R - 2r)(3R + 2r) \geq 0 \ true\ by$$

$$R \geq 2r(Euler)$$

3.48 Prove that in every triangle there is a median whose length squared is at least $\sqrt{3}$ times the area of the triangle.

Estonian NMO-2017

Solution: *Assume w.l.o.g. that BC is the shortest side of the triangle. Then the least angle of the triangle is by vertex A. Denote $a = BC, b = CA, c = AB, \alpha = \widehat{BAC}$ and let m be the length of the median drawn from vertex A. By assumptions made at the beginning of the solution, $\alpha \leq 60^\circ$. Let D, E, F be the midpoints of sides BC, CA, AB, respectively. As $\widehat{AFD} = 180^\circ - \alpha$ because of $DF \parallel CA$, the law of cosines in triangle AFD implies $m^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2 \cdot \frac{b}{2} \cdot \frac{c}{2} cos\widehat{AFD} = \frac{b^2}{4} + \frac{c^2}{4} + \frac{bc}{2} cos\,\alpha \geq \frac{b^2}{4} + \frac{c^2}{4} + \frac{bc}{2} cos\,60^\circ = \frac{b^2+c^2+bc}{4}$. As $b^2 + c^2 \geq bc$, we obtain $m^2 \geq \frac{3bc}{4}$.*

On the other hand, let S be the area of the triangle ABC.

Then $S = \frac{1}{2}bcsin\alpha \leq \frac{1}{2}bcsin60^\circ \leq \frac{\sqrt{3}bc}{4}$.

Before we obtained the inequality $m^2 \geq \frac{3bc}{4} = \sqrt{3} \cdot \frac{\sqrt{3}bc}{4}$. *Consequently,* $m^2 \geq \sqrt{3}S$.

3.49 Let ABC be an acute triangle, and H –orthocenter. The distance from H to rays BC, CA, AB is denote by d_a, d_b, d_c. Let R be the radius of circumcenter of ΔABC and r be the radius of incenter of ΔABC. Prove that following inequality:

$$d_a + d_b + d_c < \frac{3R^2}{4r}$$

Moldova NMO-2017

Solution: *Firstly we use Euler's inequality* ($R \geq 2r$) *and turn our inequality into:* $d_a + d_b + d_c \leq \frac{3}{2}R$. *Now using Erdos-Mordell inequality we have*

$$d_a + d_b + d_c \leq \frac{HA + HB + HC}{2}$$

But $HA = 2R\cos A$ *so only have to prove that* $\cos A + \cos B + \cos C \leq \frac{3}{2}$ *and this can be easily proved using Jensen. Another way is by Carnot's Relation:*

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

3.50 In any ΔABC the following relationship holds:

$$a^2r_a + b^2r_b + c^2r_c \geq 54Rr^2$$

Marian Ursărescu

Solution (Daniel Văcaru)

We can write:

$$a^2r_a + b^2r_b + c^2r_c = \frac{a^2}{\frac{1}{r_a}} + \frac{b^2}{\frac{1}{r_b}} + \frac{c^2}{\frac{1}{r_c}} \overset{Bergstrom}{\geq} \frac{(a+b+c)^2}{\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}} =$$

$$= \frac{(a+b+c)^2}{\frac{(s-a)+(s-b)+(s-c)}{S}} = \frac{(a+b+c)^2}{\frac{1}{r}} = (a+b+c)^2 r$$

$$= \frac{(a+b+c)^3 \cdot r}{2s}$$

$$\text{But: } (a+b+c)^3 \ge 27abc \Rightarrow$$

$$a^2r_a + b^2r_b + c^2r_c \ge \frac{27abc\cdot r}{2s} = \frac{27\cdot 4RS\cdot r}{2s} = 54Rr\left(\frac{S}{s}\right) = 54Rr^2$$

3.51 In ΔABC the following relationship holds:

$$\sqrt{(8R^2-a^2)(8R^2-b^2)(8R^2-c^2)} \ge 2\sqrt{5}R^2(2R+r)$$

Marian Ursărescu

Solution (Rahim Shahbazov)

$$\sqrt{(8R^2-a^2)(8R^2-b^2)(8R^2-c^2)} \ge 2\sqrt{5}R^2(2R+r); \quad (1)$$

$$a = 2RsinA; b = 2RsinB; c = 2RsinC \overset{(1)}{\Rightarrow}$$

$$16R^2(2-sin^2A)(2-sin^2B)(2-sin^2C) \ge 5(2R+r)^2$$

$$16R^2(1+cos^2A)(1+cos^2B)(1+cos^2C) \ge 5R^2\left(2+\frac{r}{R}\right)^2$$

$$16(1+cos^2A)(1+cos^2B)(1+cos^2C) \ge 5(1+cosA+cosB+cosC)^2$$

Let: $cosA = \frac{x}{2}; cosB = \frac{y}{2}; cosC = \frac{z}{2}$ then

$$(x^2+4)(y^2+4)(z^2+4) \ge 5(x+y+z+2)^2; \quad (2)$$

$$(x+y+z+2)^2 \overset{Am-Gm}{\le} (x^2+4)\left(1+\left(\frac{y+z+2}{2}\right)^2\right) \Rightarrow$$

$$5\left(1+\left(\frac{y+z+2}{2}\right)^2\right) \le (y^2+4)(z^2+4) \Leftrightarrow 5(y+z-2)^2 \ge 0$$

3.52 If in $\Delta ABC; H$ –orthocentre; HD, HE, HF bisectors of angles BHC, CHA respectively $AHB; D \in (BC); E \in (CA); F \in (AB)$ then the following relationship holds:

$$\frac{[DEF]}{[ABC]} \ge 13\left(\frac{r}{R}\right)^2 - 3$$

Marian Ursărescu

Solution:

$$\Delta BGC \Rightarrow \frac{BD}{DC} = \frac{HB}{HC} \text{ and analogs}$$

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{HB}{HC} \cdot \frac{HC}{HA} \cdot \frac{HA}{HB} = 1; (1)$$

$$\text{Let: } \frac{HA}{HB} = m, \frac{HB}{HC} = k, \frac{HC}{HA} = p$$

$$S_{AEF} = \frac{AF \cdot AE \cdot sinA}{2} = \frac{m}{(m+1)(p+1)} \cdot \frac{bc \cdot sinA}{2} = \frac{m}{(m+1)(p+1)} \cdot S_{ABC} \text{ and analogs}$$

$$S_{DEF} = \frac{1 + kmp}{(1+k)(1+m)(1+p)} \overset{(1)}{=} \frac{2}{(1+k)(1+m)(1+p)} =$$

$$= \frac{2AH \cdot BH \cdot CH}{(AH + BH)(AH + CH)(CH + BH)} \cdot S_{ABC}$$

$$\text{But: } AH = 2RsinA \Rightarrow$$

$$S_{DEF} = \frac{2cosAcosBcosC}{(cosA + cosB)(cosB + cosC)(cosC + cosA)} \cdot S_{ABC}; (2)$$

$$\text{But: } cosAcosBcosC = \frac{s^2 - (2R+r)^2}{4R^2}; (3)$$

$$(cosA + cosB)(cosB + cosC)(cosC + cosA) = \frac{r(s^2 + r^2 + 2Rr)}{4R^3}; (4)$$

$$s^2 \le 27R^2 \cdot s^2 \ge \frac{27}{4} \cdot r^2, R \ge 2r; (5)$$

From (1)+(2)+(3)+(4)+(5) proved.

3.53 In acute ΔABC the following relationship holds:

$$\frac{\sqrt{sin2A} + \sqrt{sin2B} + \sqrt{sin2C}}{\sqrt{tanA} + \sqrt{tanB} + \sqrt{tanC}} \ge \sqrt{2\left(\frac{r}{R} + 1\right)^2 - 4}$$

Marian Ursărescu

Solution:

$$2(x + y) \ge \left(\sqrt{x} + \sqrt{y}\right)^2, \forall x, y > 0$$

$$\text{Let: } x = sin2A + sin2B - sin2C; y = sin2A - sin2B + sin2C$$

$$4sin2A \geq \left(\sqrt{sin2A + sin2B - sin2C} + \sqrt{sin2A - sin2B + sin2C}\right)^2$$

$$\sqrt{sin2A + sin2B - sin2C} + \sqrt{sin2A - sin2B + sin2C} \leq 2\sqrt{sin2A}$$

Analogous:

$$\sqrt{sin2A + sin2B - sin2C} + \sqrt{-sin2A + sin2B + sin2C} \leq 2\sqrt{sin2B}$$

$$\sqrt{sin2A - sin2B + sin2C} + \sqrt{-sin2A + sin2B + sin2C} \leq 2\sqrt{sin2C}$$

$$\sum \sqrt{sin2A + sin2B - sin2C} \leq \sqrt{sin2A} + \sqrt{sin2B} + \sqrt{sin2C};\ \ (1)$$

But: $sin2A + sin2B - sin2C = 2sin(A + B)cos(A - B) - 2sinCcosC$

$$= 2sinC[cos(A - B) - cosC] = -4sinCsin\left(\frac{A - B + C}{2}\right)sin\left(\frac{A - B - C}{2}\right)$$

$$= 4cosAcosBsinC = 4cosAcosBcosCtanC = \frac{s^2 - (2R + r)^2}{R^2}tanC;\ (2)$$

From (1),(2) we have: $\sqrt{\frac{s^2-(2R+r)^2}{R^2}}\sum\sqrt{tanA} \leq \sum\sqrt{sin2A}$

$$\sqrt{\frac{s^2 - (2R + r)^2}{R^2}} \leq \frac{\sum\sqrt{sin2A}}{\sum\sqrt{tanA}};\ (3)$$

$$\sqrt{\frac{s^2 - (2R + r)^2}{R^2}} \overset{(*)}{\geq} \sqrt{\frac{2R^2 + 8Rr + 3r^2 - 4R^2 - 4Rr - r^2}{R^2}}$$

$$= \sqrt{\frac{2r^2 + 4Rr - 2R^2}{R^2}} = \sqrt{2\left(\frac{r}{R} + 1\right)^2 - 4}$$

$$(*):\ s^2 \geq 2R^2 + 8Rr + 3r^2\ ;\ (4)$$

From (3) and (4) we have:

$$\frac{\sqrt{sin2A} + \sqrt{sin2B} + \sqrt{sin2C}}{\sqrt{tanA} + \sqrt{tanB} + \sqrt{tanC}} \geq \sqrt{2\left(\frac{r}{R} + 1\right)^2 - 4}$$

3.54 Let p be a circle with center K passing through M, q a semicircle with diameter KM and L a point inside the segment KM. A line L perpendicular to KM intersects q a point Q and p at points P_1, P_2

such that $P_1Q > P_2Q$. Line MQ intersects p for the second time at $R \neq M$. Prove that areas S_1, S_2 of triangle MP_1Q, P_2RQ satisfy

$$\mathbf{1 < \frac{S_1}{S_2} < 3 + \sqrt{8}.}$$

Sarka Gergelitsova-Czech and Slovak NMO-2017

Solution: *The circle containing the semicircle q in this image of p in homothety with center M and factor $\frac{1}{2}$, hence Q is the midpoint of RM. Since triangles MP_1Q, P_2RQ share the angle by Q, we have*

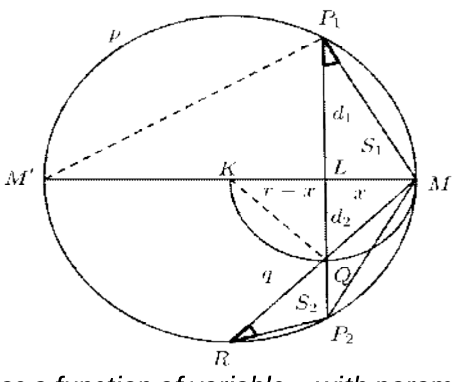
$$\frac{S_1}{S_2} = \frac{\frac{1}{2}P_1Q \cdot MQ \cdot sin\widehat{P_1QM}}{\frac{1}{2}P_2Q \cdot RQ \cdot sin\widehat{P_2QR}} = \frac{P_1Q}{P_2Q}.$$

Let $KM = r, ML = x, P_1L = d_1, QL = d_2$ $(Fig. 1)$. Points P_1, P_2 are symmetric about KM, therefore $P_1L = P_2L$ and $P_2Q = d_1 - d_2$. Denote by M' the point such that MM' is the diameter of p. Theorem (an altitude splits a right triangle into two similar triangles) we get $d_1^2 = x(2r - x)$. Similarly in right triangle KQM we get $d_2^2 = x(2r - x)$ and thus

$$\frac{S_1}{S_2} = \frac{P_1Q}{P_2Q} = \frac{d_1 + d_2}{d_1 - d_2} = \frac{(d_1 + d_2)^2}{(d_1 - d_2)(d_1 + d_2)}$$

$$= \frac{x(2r - x) + x(r - x) + 2\sqrt{x(2r - x) \cdot x(r - x)}}{rx}$$

$$= \frac{3r - 2x + 2\sqrt{(2r - x)(r - x)}}{r}$$



We view expression as a function of variable x with parameter r. The function is decreasing on $(0, r)$ (both functions $3r - 2x$ and $(2r - x)(r - x)$ are decreasing), therefore it attains its maximum $3 + 2\sqrt{2}$ at $x = 0$ and minimum 1 at $x = r$. By the problem statement, $x \in (0, r)$, thus $1 < \frac{S_1}{S_2} < 3 + 2\sqrt{2} =$

$3+\sqrt{8}$. ***Remark.*** *The inequality* $\frac{S_1}{S_2}>1$ *follows immediately from the given inequality* $P_1Q>P_2Q$ *and from* $RQ=QM$, *since* $S_1>[LMP_1]=[LMP_2]>[MQP_2]=S_2$.

3.55 In any ΔABC the following relationship holds:

$$\frac{r}{4R}\le \sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)\le\frac{1}{8}$$

Marian Ursărescu

Solution (Daniel Văcaru)

We have: $\frac{r}{4R}=\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

Observe that: $\prod_{cyc}\sin\left(\frac{\pi-A}{4}\right)=\frac{\sum_{cyc}\sin\frac{A}{2}-1}{4}\ge\frac{3\sqrt[3]{\prod_{cyc}\sin\frac{A}{2}}-1}{4}\ge\prod_{cyc}\sin\frac{A}{2};(1)$

Let's denote $\sqrt[3]{\prod_{cyc}\sin\frac{A}{2}}=x$.

Then (1) became:

$3x-1\ge4x^3\leftrightarrow(2x-1)^2(1-x)\ge0$, *true because* $x<1$.

For the right side, we have:

$$\frac{\sum_{cyc}\sin\frac{A}{2}-1}{4}\overset{t\to\sin\frac{t}{2}\ concave}{\lesseqgtr}\frac{3\sin\left(\frac{A+B+C}{6}\right)}{4}=\frac{1}{8}$$

3.56 In acute ΔABC the following relationship holds:

$$\frac{a^4}{r_a}+\frac{b^4}{r_b}+\frac{c^4}{r_c}=18R^3\left(\frac{R}{r}-1\right)$$

Marian Ursărescu

Solution(Soumava Chakraborty)

$$\sum\frac{a^4}{r_a}=\sum\frac{a^4(s-a)}{rs}\le18R^3\left(\frac{R}{r}-1\right)=\frac{18R^3(R-r)}{r}\Leftrightarrow\sum a^4(s-a)$$
$$\le18sR^3(R-r)$$

$$\Leftrightarrow 2s\sum a^4 - 2\sum a^5 \overset{(1)}{\leq} 36sR^3(R-r)$$

$$Now, \left(\sum a\right)\left(\sum a^4\right) = \sum a^5 + \sum ab(a^3+b^3) \Rightarrow$$

$$2s\sum a^4 - \sum a^5 = \sum\left\{ab\left(\sum a^3 - c^3\right)\right\} \overset{(i)}{=} \left(\sum a^3\right)\left(\sum ab\right) - abc\sum a^2$$

$$Now, \left(\sum a^2\right)\left(\sum a^3\right) = \sum a^5 + \sum\{a^2b^2(2s-c)\}$$

$$= 2s\sum a^2b^2 + \sum a^5 - abc\sum ab$$

$$\Rightarrow -\sum a^5 \overset{(ii)}{=} 2s\left(\left(\sum ab\right)^2 - 2abc\left(\sum a\right)\right) - abc\sum ab - \left(\sum a^2\right)\left(\sum a^3\right)$$

3.57 In ΔABC the following relationship holds:

$$\frac{r_a}{m_a h_a} + \frac{r_b}{m_b h_b} + \frac{r_c}{m_c h_c} \geq \frac{4(R-r)}{R^2}$$

Marian Ursărescu

Solution:

1) $\sum a^2 = 2(s^2 - r^2 - 4Rr)$

2) $\sum a^2 r_a = 4s^2(R-r)$

3) $m_a \leq \frac{a^2+b^2+c^2}{2\sqrt{3}a}$

4) $3\sqrt{3}r \leq s \leq \frac{3\sqrt{3}}{2}R$ –Mitrinovic

5) $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$ –Gerretsen

6) $R \geq 2r$ –Euler

$$(3) \rightarrow \frac{1}{m_a} \geq \frac{2\sqrt{3}a}{a^2+b^2+c^2} \rightarrow \frac{r_a}{m_a} \geq \frac{2\sqrt{3}ar_a}{a^2+b^2+c^2} \rightarrow$$

$$\frac{r_a}{m_a h_a} \ge \frac{\sqrt{3}a^2 r_a}{(a^2+b^2+c^2)S}$$

$$\sum_{cyc}\frac{r_a}{m_a h_a} \ge \sum_{cyc}\frac{\sqrt{3}a^2 r_a}{(a^2+b^2+c^2)S} = \frac{\sqrt{3}}{(a^2+b^2+c^2)S}\sum_{cyc}a^2 r_a \overset{(2)}{=}$$

$$= \frac{\sqrt{3}}{(a^2+b^2+c^2)S}\cdot 4s^2(R-r) = \frac{\sqrt{3}s^2}{S}\cdot\frac{4(R-r)}{a^2+b^2+c^2} \overset{(1)}{=}$$

$$= \frac{\sqrt{3}s^2}{sr}\cdot\frac{4(R-r)}{2(s^2-r^2-4Rr)} = \frac{\sqrt{3}s}{r}\cdot\frac{4(R-r)}{2(s^2-r^2-4Rr)} \overset{(4)}{\ge}$$

$$\ge \frac{\sqrt{3}(3\sqrt{3}r)}{r}\cdot\frac{4(R-r)}{2(s^2-r^2-4Rr)} = \frac{9\cdot 4(R-r)}{2(s^2-r^2-4Rr)} \overset{(*)}{\ge} \frac{4(R-r)}{R^2}$$

$$(*) \leftrightarrow 9R^2 \ge 2s^2 - 2r^2 - 8Rr \leftrightarrow 2s^2 \le 9R^2 + 2r^2 + 8Rr \overset{(5)}{\leftrightarrow}$$

$8R^2 + 8Rr + 6r^2 \le 9R^2 + 2r^2 + 8Rr \leftrightarrow R^2 \ge 4r^2 \leftrightarrow R \ge 2r$ true from Euler.Proved.

3.58 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\cos\frac{B}{2} + \cos\frac{C}{2} - \cos\frac{A}{2}\right)^3 \ge \frac{3s}{4R}$$

Daniel Sitaru

Solution (Tran Hong)

We have: $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}$

Let $x = \cos\frac{A}{2}; y = \cos\frac{B}{2}; z = \cos\frac{C}{2}$

$(x, y, z > 0)$

We just check:

$$\sum_{cyc}(x+y-z)^3 \ge 3xyz$$

$$\sum_{cyc}(x+y-z)^3 = (x+y-z)^3 + (x+z-y)^3 + (y+z-x)^3$$

$$= x^3 + y^3 + z^3 + 3xy^2 + 3yx^2 + 3xz^2 + 3zx^2 + 3yz^2 + 3zy^2 - 18xyz$$

So, we prove:

$$x^3 + y^3 + z^3 + 3xy^2 + 3yx^2 + 3xz^2 + 3zx^2 + 3yz^2 + 3zy^2 - 18xyz$$
$$\geq 3xyz$$

$$\Leftrightarrow \sum x^3 + 3\left(\sum xy^2 + \sum yx^2\right) \geq 21xyz$$

It is true because:

$$\sum x^3 \overset{AM-GM}{\geq} 3xyz$$

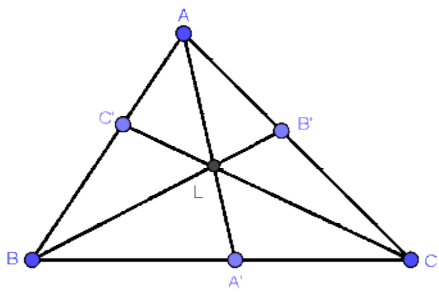
$$3(xy^2 + yx^2 + xz^2 + zx^2 + yz^2 + zy^2) \geq 3 \cdot 6\sqrt[6]{x^6y^6z^6} = 18xyz$$

3.59 If in ΔABC, K – Lemoine's point then:

$$\sum_{cyc} \frac{bc \cdot BK \cdot CK}{b \cdot BK + c \cdot CK - a \cdot AK} \geq a \cdot AK + b \cdot BK + c \cdot CK$$

Daniel Sitaru

Solution:



$$\frac{bc \cdot BL \cdot CL}{b \cdot BL + c \cdot CL - a \cdot AL} + \frac{ca \cdot CL \cdot AL}{c \cdot CL + a \cdot AL - b \cdot BL} + \frac{ab \cdot AL \cdot BL}{a \cdot AL + b \cdot BL - c \cdot CL}$$
$$\geq$$
$$\geq aAL + bBL + cCL$$

From Van Aubel theorem we have:

$$\frac{AL}{LA'}=\frac{AC'}{C'B}+\frac{AB'}{B'C}\overset{Steiner}{=}\frac{b^2+c^2}{a^2}\Rightarrow$$

$$\frac{AL}{s_a}=\frac{b^2+c^2}{a^2+b^2+c^2}\quad (1)$$

But $s_a=\frac{2bc}{b^2+c^2}m_a$ (2)

From (1)+(2)⇒

$$AL=\frac{b^2+c^2}{a^2+b^2+c^2}\cdot s_a=\frac{2bc}{a^2+b^2+c^2}\cdot m_a \text{ and similarly}\Rightarrow$$

$$\frac{bc\cdot BL\cdot CL}{b\cdot BL+c\cdot CL-aAL}=\frac{bc\cdot\frac{2ac}{a^2+b^2+c^2}\cdot m_b\cdot\frac{2ab}{a^2+b^2+c^2}\cdot m_c}{\frac{2abc}{a^2+b^2+c^2}m_b+\frac{2abc}{a^2+b^2+c^2}m_b-\frac{2abc}{a^2+b^2+c^2}m_a}$$

$$=\frac{\frac{2abc}{a^2+b^2+c^2}\cdot m_b\cdot m_c}{m_b+m_c-m_a}\Rightarrow \text{we must show:}$$

$$\frac{2abc}{a^2+b^2+c^2}\cdot\sum\frac{m_b\cdot m_c}{m_b+m_c-m_a}\ge\frac{2abc}{a^2+b^2+c^2}(m_a+m_b+m_c)\Leftrightarrow$$

$$\sum\frac{m_bm_c}{m_b+m_c-m_a}\ge m_a+m_b+m_c\quad (1)$$

Let $m_b+m_a-m_a=x$

$$m_a-m_b+m_c=y$$

$$m_a+m_b-m_c=z$$

$x,y,z>0$ and $m_a+m_b+m_c=x+y+z$ and

$m_a=\frac{y+z}{2}, m_b=\frac{x+z}{2}$ and $m_c=\frac{x+y}{2}$ (2)

From (1)+(2) we must show:

$$\sum\frac{(x+z)(x+y)}{yx}\ge x+y+z\Leftrightarrow$$

$$\sum\frac{x^2+xy+xz+zy}{x}\ge 4(x+y+z)\Leftrightarrow$$

$$\sum\left(x+y+z+\frac{zy}{x}\right)\ge 4(x+y+z)\Leftrightarrow$$

$$\frac{zy}{x}+\frac{xz}{y}+\frac{xy}{z}\ge x+y+z\Leftrightarrow$$

$$x^2y^2+x^2z^2+y^2z^2\ge xyz(x+y+z) \text{ true}$$

Because $\alpha^2 + \beta^2 + \gamma^2 \geq \alpha\beta + \alpha\gamma + \beta\gamma, \forall\, \alpha, \beta, \gamma$

3.60 In ΔABC the following relationship holds:

$$2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \leq 3\sqrt{\frac{3abc}{4Rr + r^2}}$$

Daniel Sitaru

Solution:

$$\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab} > a + b > c = \left(\sqrt{c}\right)^2 \Rightarrow$$

$\sqrt{a} + \sqrt{b} > \sqrt{c}$ *- and analogs.*

By Mitrinovic's inequality in the triangle with sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$:

$$s_1 \leq \frac{3\sqrt{3}}{2} R_1 \Leftrightarrow \frac{1}{2}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \leq \frac{3\sqrt{3}}{2} \cdot \frac{\sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c}}{4S_1} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \leq \frac{3\sqrt{3abc}}{8 \cdot \frac{1}{2}\sqrt{4Rr + r^2}} \Leftrightarrow$$

$$\Leftrightarrow 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \leq 3\sqrt{\frac{3abc}{4Rr + r^2}}$$

3.61 In ΔABC the following relationship holds:

$$\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \geq 6\sqrt{12Rr + 3r^2}$$

Daniel Sitaru

Solution: $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab} > a + b > c = \left(\sqrt{c}\right)^2 \Rightarrow$

$$\sqrt{a} + \sqrt{b} > \sqrt{c} - \text{and analogs.}$$

By Mitrinovic's inequality in the triangle with sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$:

$$s_1 \geq 3\sqrt{3} r_1 \Leftrightarrow \frac{1}{2}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \geq 3\sqrt{3} \cdot \frac{S_1}{s_1} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \ge 3\sqrt{3}\cdot\frac{\frac{1}{2}\sqrt{4Rr+r^2}}{\frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \ge 3\sqrt{12Rr+3r^2} \Leftrightarrow$$

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \ge 6\sqrt{12Rr+3r^2}$$

3.62 If a, b, c, d – sides in a bicentric quadrilateral with r – inradii then: $a^3b^3+a^3c^3+a^3d^3+b^3c^3+b^3d^3+c^3d^3 \ge 384r^6$

Daniel Sitaru

Solution (Soumava Chakraborty)

$$\because x^3+y^3 \ge xy(x+y)$$

$$\therefore a^3b^3+c^3d^3 \overset{(1)}{\ge} abcd(ab+cd)$$

$$a^3c^3+b^3d^3 \overset{(2)}{\ge} abcd(ac+bd)$$

$$a^3d^3+b^3c^3 \overset{(3)}{\ge} abcd(ad+bc)$$

$$(1)+(2)+(3) \Rightarrow LHS \overset{(4)}{\ge} abcd(ab+bc+cd+ad+ac+bd)$$

$$\overset{Ptolemy}{=} abcd\big(b(a+c)+d(a+c)+pq\big)$$

$$= abcd\big((a+c)(b+d)+pq\big) = abcd(s^2+pq)$$

$$(\because a+c=b+d=s)$$

$$\text{Now, Yiu \& Paul} \Rightarrow \frac{pq}{4r^2}-\frac{4R^2}{pq}=1 \Rightarrow \frac{p^2q^2-16R^2r^2}{4pqr^2}=1$$

$$\Rightarrow (pq)^2-4r^2(pq)-16R^2r^2=0 \Rightarrow pq=\frac{4r^2 \pm \sqrt{16r^4+64R^2r^2}}{2}$$

$$=2r^2 \pm 2r^2\sqrt{4R^2+r^2} \Rightarrow pq \overset{(i)}{=} 2r\left(r+\sqrt{4R^2+r^2}\right)$$

$$\text{Again, Radic \& Mircko} \Rightarrow s^2 \overset{(ii)}{\ge} 8r\left(\sqrt{4R^2+r^2}-r\right)$$

$$(4), (i), (ii) \Rightarrow LHS$$

$$\geq abdc\left[8r\left(\sqrt{4R^2+r^2}-r\right)+2r\left(r+\sqrt{4R^2+r^2}\right)\right]$$

$$=r^2s^2\left(10r\sqrt{4R^2+r^2}-6r^2\right)\ \left(\because \sqrt{abcd}=\Delta=rs\right)$$

$$\geq r^3\cdot 8r\left(\sqrt{4R^2+r^2}-r\right)\left(10\sqrt{4R^2+r^2}-6r\right)\ \text{(Radic \& Mirko)}$$

$$\overset{L.Fejes\ Toth}{\geq} 8r^4\left(\sqrt{4(2r^2)+r^2}-r\right)\left(10\sqrt{4(2r)^2+r^2}-6r\right)$$

$$=16\cdot 24r^6=384r^6\quad \text{(Proved)}$$

3.63 If $x\in\left(0,\frac{\pi}{2}\right)$ then:

$$\left(\frac{\sin^4 x}{\cos^4 x-\cos^2 x+1}+\frac{\cos^4 x}{\sin^4 x-\sin^2 x+1}\right)\left(\frac{\sin^6 x}{\cos^4 x-\cos^2 x+1}+\frac{\cos^6 x}{\sin^4 x-\sin^2 x+1}\right)\geq\frac{2}{9}$$

Daniel Sitaru

Solution(Șerban George Florin)

$$\cos^4 x-\cos^2 x+1=\cos^2 x(\cos^2 x-1)+1=-\sin^2 x\cos^2 x+1$$

$$\sin^4 x-\sin^2 x+1=\sin^2 x(\sin^2 x-1)+1=-\sin^2 x\cos^2 x+1$$

$$\sin^4 x+\cos^4 x=(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x=1-2\sin^2 x\cos^2 x$$

$$\sin^6 x+\cos^6 x=(\sin^2 x+\cos^2 x)(\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x)=$$
$$=1-3\sin^2 x\cos^2 x$$

$$\left(\frac{\sin^4 x}{1-\sin^2 x}+\frac{\cos^4 x}{1-\sin^2 x\cos^2 x}\right)\left(\frac{\sin^6 x}{1-\sin^2 x\cos^2 x}+\frac{\cos^6 x}{1-\sin^2 x\cos^2 x}\right)\geq\frac{2}{9}$$

$$\Rightarrow\frac{(1-2\sin^2 x\cos^2 x)(1-3\sin^2 x\cos^2 x)}{(1-\sin^2 x\cos^2 x)^2}\geq\frac{2}{9}\Rightarrow$$

$$\Rightarrow 9(1-5\sin^2 x\cos^2 x+6\sin^4 x\cos^4 x)$$
$$\geq 2(1-2\sin^2 x\cos^2 x+\sin^4 x\cos^4 x)$$

$$54\sin^4 x\cos^4 x-45\sin^2 x\cos^2 x+9$$
$$\geq 2\sin^4 x\cos^4 x-4\sin^2 x\cos^2 x+2$$

$$52\sin^4 x\cos^4 x-41\sin^2 x\cos^2 x+7\geq 0$$

Denote: $a = sin^2 x, b = cos^2 x, ab = (sin x cos x)^2 = \frac{sin^2 2x}{4} \le \frac{1}{4}$

$$\Rightarrow 52(ab)^2 - 41ab + 7 \ge 0, ab = t$$

$$\Delta = (-41)^2 - 4 \cdot 52 \cdot 7 = 1681 - 1456 = 225$$

$$t_1 = \frac{41 + 14}{104} = \frac{56}{104}, t_2 = \frac{41 - 15}{104} = \frac{26}{104} = \frac{1}{4}$$

$\Rightarrow t \in \left(-\infty, \frac{26}{104}\right] \cup \left[\frac{56}{104}, \infty\right)$, *true because*

$$t = ab \le \frac{1}{4} = \frac{26}{104}$$

3.64 In ΔABC the following relationship holds:

$$\left(\sqrt[6]{a^5} + \sqrt[6]{b^5} + \sqrt[6]{c^5}\right)^6 \left(\sqrt[8]{a^7} + \sqrt[8]{b^7} + \sqrt[8]{c^7}\right)^8 \le 3^{20} \cdot R^{12}$$

Daniel Sitaru

Solution (Tran Hong)

$$f: (0, \infty) \to (0, \infty);\ f(a) = \sqrt[6]{a^5} = a^{\frac{5}{6}}$$

$$f'(a) = \frac{5}{6} a^{-\frac{1}{6}} > 0$$

$f''(a) = -\frac{5}{36} a^{-\frac{7}{6}}$ then f is concave, we have:

$$\sum_{cyc} \sqrt[6]{x^5} \overset{Jensen}{\le} 3 \cdot \sqrt[6]{\left(\frac{a + b + c}{3}\right)^6}$$

$$\varphi: (0, \infty) \to \mathbb{R}, \varphi(a) = \sqrt[8]{a^7} = a^{\frac{7}{8}}, \varphi'(a) = \frac{7}{8} a^{-\frac{1}{8}}$$

$$\varphi''(a) = -\frac{7}{64} a^{-\frac{9}{8}} < 0, \forall a > 0$$

$$\sum_{cyc} \sqrt[8]{x^7} \overset{Jensen}{\le} 3 \cdot \sqrt[8]{\left(\frac{a + b + c}{3}\right)^7}$$

$$\left(\sum_{cyc}\sqrt[6]{x^5}\right)^6\cdot\left(\sum_{cyc}\sqrt[8]{x^7}\right)^8\le 3^6\cdot\frac{(a+b+c)^5}{3^5}\cdot 3^8\cdot\frac{(a+b+c)^7}{3^7}$$

$$=9(a+b+c)^{12}\le 9\cdot(2s)^{12}\overset{2s\le 3R\sqrt{3}}{\lesssim} 3^{20}\cdot R^{12}$$

3.65 Solve for real numbers:

$$\boldsymbol{sin2x=\left(\sqrt{2}-1\right)(sinx+cosx+1)}$$

Daniel Sitaru

Solution (Ravi Prakash)

$$sin2x=\left(\sqrt{2}-1\right)(sinx+cosx+1)\ldots(1)$$

$$Put\ sinx+cosx=t,\qquad sinx=t^2-1$$

Now, (1) becomes: $t^2-1=\left(\sqrt{2}-1\right)(t+1)$

$$t+1=0\ or\ t-1=\sqrt{2}-1,\qquad t=-1\ or\ t=\sqrt{2}$$

$$sinx+cosx=-1\ or\ sinx+cosx=\sqrt{2}$$

$$cos\left(x-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}\ or\ cos\left(x-\frac{\pi}{4}\right)=1$$

$$x-\frac{\pi}{4}=2n\pi\pm\frac{3\pi}{4}\ or\ x-\frac{\pi}{4}=2m\pi;\ m,n\in\mathbb{Z}$$

$$x\in\left\{2n\pi+\frac{\pi}{4};(2k+1)\pi;2m\pi-\frac{\pi}{2}/n,k,m\in\mathbb{Z}\right\}$$

3.66 In acute $\boldsymbol{\Delta ABC, H}$ – orthocenter, $\boldsymbol{I}$ – incenter,

$\boldsymbol{G}$ – centroid the following relationship holds:

$$\boldsymbol{am_a cosA+bm_b cosB+cm_c cosC\le\frac{3s}{2R}(HI^2+GI^2+4Rr)}$$

Daniel Sitaru

Solution (Tran Hong)

$$\bullet\ \Omega=acos^2A+bcos^2B+ccos^2C$$
$$=a(1-sin^2A)+b(1-sin^2B)+c(1-sin^2C)$$
$$=(a+b+c)-2R(sin^3A+sin^3B+sin^3C)$$

$$= 2s - 2R\left[\frac{s(s^2 - 6Rr + 3r^2 - s^2)}{2R^2}\right] = \frac{s(4R^2 + 6Rr + 3r^2 - s^2)}{2R^2}$$
$$= 2s - \frac{s(s^2 - 6Rr - 3r^2)}{2R^2} = \frac{s(4R^2 + 6Rr + 3r^2 - s^2)}{2R^2}$$

$$\bullet\ \Psi = am_a^2 + bm_b^2 + cm_c^2 = \frac{s}{2}(s^2 + 2Rr + 5r^2)\ (*)$$

$Now,$

$$LHS = \sum_{cyc}(am_a cosA) = \frac{3}{1}\cdot\frac{1}{2R}\sum_{cyc}\left(a\cdot 2RcosA\cdot\frac{2}{3}m_a\right)$$
$$\underset{\Delta ABC - acute}{\overset{AM-GM}{\leq}} \frac{3}{4R}\left(a\cdot\frac{[2RcosA]^2 + \frac{4}{9}m_a^2}{2} + b\cdot\frac{[2RcosB]^2 + \frac{4}{9}m_b^2}{2} + c\right.$$
$$\left.\cdot\frac{[2RcosC]^2 + \frac{4}{9}m_c^2}{2}\right]$$
$$= \frac{3}{8R}\left(4R^2\Omega + \frac{4}{9}\Psi\right) = \frac{3}{8R}\left[2s(4R^2 + 6Rr + 3r^2 - s^2) + \frac{4}{9}\cdot\frac{s}{2}(s^2 + 2Rr + 5r^2)\right]$$
$$= \frac{3s}{4R}\left(4R^2 + 6Rr + 3r^2 - s^2 + \frac{s^2}{9} + \frac{2R}{9} + \frac{5}{9}r^2\right)$$
$$= \frac{3s}{4R}\left(4R^2 + \frac{56}{9}Rr - \frac{8s^2}{9} + \frac{32r^2}{9}\right) = \frac{3s}{4R}\cdot\frac{36R^2 + 56Rr + 32r^2 - 8s^2}{9}$$

$Now,$

$$HI^2 = 2r^2 + 4R^2 + 4Rr - p^2; GI^2 = \frac{p^2 - 16Rr + 5r^2}{9}$$
$$\Rightarrow HI^2 + GI^2 + 4Rr = \frac{36R^2 + 32R^2 + 56Rr - 8s^2}{9}$$

$$\Rightarrow RHS = \frac{3s}{2R}\cdot\frac{36R^2 + 32R^2 + 56Rr - 8s^2}{9} \Rightarrow 2LHS \leq RHS\ (Proved)$$

3.67 In ΔABC the following relationship holds:

$$\left(m_a m_b + \frac{b^2 + c^2}{2ca}\right)\left(\frac{1}{h_a h_b} + \frac{c^2 + a^2}{2bc}\right) \geq \left(m_a(c^2 + a^2) + \frac{b^2 + c^2}{h_b}\right)\left(\frac{m_b}{2bc} + \frac{1}{2cah_a}\right)$$

Daniel Sitaru

Solution (Tran Hong):

$$\left(m_a m_b + \frac{b^2 + c^2}{2ca}\right)\left(\frac{1}{h_a h_b} + \frac{c^2 + a^2}{2bc}\right)$$

$$\geq \left(m_a(c^2 + a^2) + \frac{b^2 + c^2}{h_b}\right)\left(\frac{m_b}{2bc} + \frac{1}{2cah_a}\right)$$

$$\frac{m_a m_b}{h_a h_b} + \frac{m_a m_b(c^2 + a^2)}{2bc} + \frac{b^2 + c^2}{2cah_a h_b} + \frac{(b^2 + c^2)(c^2 + a^2)}{4abc^2} \geq$$

$$\geq \frac{m_a m_b(c^2 + a^2)}{2bc} + \frac{m_b(b^2 + c^2)}{2bch_b} + \frac{m_a(c^2 + a^2)}{2cah_a} + \frac{b^2 + c^2}{2cah_a h_b}$$

$$\frac{m_a m_b}{h_a h_b} + \frac{(b^2 + c^2)(c^2 + a^2)}{4abc^2} \geq \frac{m_b(b^2 + c^2)}{2bch_b} + \frac{m_a(c^2 + a^2)}{2cah_a}$$

$$\frac{m_a m_b}{h_a h_b} + \frac{m_a m_b}{s_a s_b} \geq \frac{m_a m_b}{s_a h_b} + \frac{m_a m_b}{h_a s_b} \text{ because } \left(s_a = \frac{2bc}{b^2+c^2} \cdot m_a\right)$$

$$\Leftrightarrow h_a h_b + s_a s_b \geq h_a s_b + h_b s_a \Leftrightarrow h_a(h_b - s_b) + s_a(s_b - h_b) \geq 0$$

$$\Leftrightarrow (s_b - h_b)(s_a - h_a) \geq 0.$$

Which is clearly true,because: $s_b \geq h_b; s_a \geq h_a$*.Proved.*

3.68 In ΔABC the following relationships holds:

$$R^2 + \frac{4}{a^2 b^2 c^2}\left(\sum_{cyc} bc(s-a)^2\right)^2 + 11r^2 \geq 21Rr$$

Daniel Sitaru

Solution (Tran Hong)

$$\sum_{cyc} bc(s-a)^2 = bc(s-a)^2 + ca(s-b)^2 + ab(s-c)^2 =$$

$$= bc(s^2 - 2sa + a^2) + ca(s^2 - 2sb + b^2) + ab(s^2 - 2sc + c^2) =$$

$$= s^2(ab + bc + ca) - 6abcs + abc(a + b + c) =$$

$$= s^2(s^2 + 4Rr + r^2) - 16Rrs^2 = s^2(s^2 - 12Rr + r^2) \Rightarrow$$

$$\frac{4}{a^2b^2c^2}\left(\sum_{cyc} bc(s-a)^2\right)^2 = \frac{4}{(4Rrs)^2}\left(s^2(s^2-12Rr+r^2)\right)^2 =$$

$$= \frac{s^2(s^2-12Rr+r^2)^2}{4R^2r^2} \overset{s^2\geq 16Rr-5r^2}{\geq} \frac{(16Rr-5r^2)(4Rr-4r^2)^2}{4R^2r^2} =$$

$$= \frac{4r(16R-5r)(R-r)^2}{R^2} = 4(16Rr-5r^2)\left(1-\frac{r}{R}\right)^2$$

Let: $\frac{R}{r} = t \geq 2$. *We need to prove:*

$$t^2 + 4(16t-5)\left(1-\frac{1}{t}\right)^2 + 11 \geq 21t \Leftrightarrow$$

$$t^4 + 4(16t-5)(t-1)^2 - 21t^3 + 11t^2 \geq 0$$

$$t^4 + (64t-20)(t-1)^2 - 21t^3 + 11t^2 \geq 0$$

$$t^4 + 43t^3 - 137t^2 + 104t - 20 \geq 0$$

$$(t-2)(t^3-45t^2-47t+10) \geq 0$$

Which is clearly true, because: $t \geq 2 \Rightarrow t-2 \geq 0$ *and*

$$t^3 - 45t^2 - 47t + 10 \geq 2(2^3 - 45\cdot 2^2 - 47\cdot 2 + 10 = 104 > 0.$$

3.69 In ΔABC the following relationship holds:

$$\sum_{cyc} a(sin3B - sin3C) \leq 8s\sum_{cyc} sin(B-C)$$

Daniel Sitaru

Solution(Tran Hong)

For all $x, y \in \mathbb{R}$ *we have identity:*

$$sinxsin3y - sin3xsiny = 4sinxsinysin(x+y)sin(x-y) \Rightarrow$$

$$\sum_{cyc} a(sin3B - sin3C) =$$

$$= 2R[sinA(sin3B - sin3C) + sinB(sin3C - sin3A)$$
$$+ sinC(sin3A - sin3B) =$$

$$= 2R[4sinAsinBsin(A+B)sin(A-B) + 4sinCsinAsin(C+A)sin(C-A) + 4sinBsinCsin(B+C)sin(B-C)]$$

$$= 8RsinAsinBsinC[sin(A-B) + sin(B-C) + sin(C-A)] =$$

$$= \frac{4sr}{R}[sin(A-B) + sin(B-C) + sin(C-A)] = \Omega$$

$$0 < A,B,C < \pi \Rightarrow -\pi < A-B, B-C, C-A < \pi \Rightarrow$$

$$sin(A-B) + sin(B-C) + sin(C-A) \geq 0 \overset{\frac{r}{R}\leq\frac{1}{2}}{\Longrightarrow}$$

$$\Omega \leq \frac{4s}{2}[sin(A-B) + sin(B-C) + sin(C-A)] \leq$$

$$\leq 8s[sin(A-B) + sin(B-C) + sin(C-A)]$$

3.70 In ΔABC the following relationship holds:

$$R\left(16Rcos^2\frac{A}{2}cos^2\frac{B}{2}cos^2\frac{C}{2} + r\right) \leq 5R^2 + 9r^2$$

Daniel Sitaru

Solution (Tran Hong)

$$cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2} = \frac{s}{4R} \Rightarrow R\left(16Rcos^2\frac{A}{2}cos^2\frac{B}{2}cos^2\frac{C}{2} + r\right) =$$

$$= R\left(16R\left(\frac{s}{4R}\right)^2 + r\right) = R\left(\frac{s^2}{R} + r\right) = s^2 + Rr \overset{(*)}{\leq} 5R^2 + 9r^2$$

$$(*) \Leftrightarrow s^2 \leq 5R^2 - Rr + 9r^2$$

But: $s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)} \overset{(*)}{\leq} 5R^2 - Rr + 9r^2$

$$(**) \Leftrightarrow 3R^2 + 11Rr - 10r^2 \geq 2(R-2r)\sqrt{R(R-2r)} \Leftrightarrow$$

$$(R-2r)(3R-5r) \geq 2(R-2r)\sqrt{R(R-2r)} \Leftrightarrow$$

$$(R-2r)\left[3R - 5r - 2\sqrt{R(R-2r)}\right] \geq 0$$

Because $R \geq 2r$ *(Euler)*$\Rightarrow R - 2r \geq 0$

We just check: $3R - 5r - 2\sqrt{R(R-2r)} \geq 0 \overset{3R\geq 6r>5r}{\Longleftrightarrow}$

$$(3R-5r)^2>4(R^2-2Rr)\Leftrightarrow 9R^2-30Rr+25r^2>4R^2-8Rr$$

$$5R^2-22Rr+25r^2>0\overset{t=\frac{R}{r}\geq 2}{\Longleftrightarrow}5t^2-22t+25>0$$

$5\left(t-\frac{11}{5}\right)^2+\frac{4}{5}>0$ *true for* $t\geq 2\Rightarrow(**)$ *is true* $\Rightarrow(*)$ *is true.Proved.*

3.71 In any $\Delta ABC,\Delta A'B'C'$ the following relationship holds:

$$\sum_{cyc}(a^2+a'^2)+2\cdot\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a'^2\right)}\geq 36(r+r')^2$$

Daniel Sitaru

Solution (Adrian Popa)

$$\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a'^2\right)=(a^2+b^2+c^2)(a'^2+b'^2+c'^2)\overset{CBS}{\geq}$$

$$\geq(aa'+bb'+cc')^2\Rightarrow$$

$$\sum_{cyc}(a^2+a'^2)+2\cdot\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a'^2\right)}\geq$$

$$\geq a^2+b^2+c^2+a'^2+b'^2+c'^2+2aa'+2bb'+2cc'=$$

$$=(a+a')^2+(b+b')^2+(c+c')^2\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a+b+c+a'+b'+c')^2}{3}=\frac{(2s+2s')^2}{3}=\frac{4(s+s')^2}{3}\overset{Mitrinovic}{\geq}$$

$$\geq\frac{4\left(3\sqrt{3}r+3\sqrt{3}r'\right)^2}{3}=\frac{4\cdot 9\cdot 3(r+r')^2}{3}=36(r+r')^2$$

3.72 Solve for real numbers:

$$4(sinx+2cosy)+3(cosx+2siny)=15$$

Daniel Sitaru

Solution(Adrian Popa)

$$4(sinx + 2cosy) + 3(cosx + 2siny) = 15$$

$$4xinx + 3cosx + 8cosy + 6siny = 15$$

$$(4sinx + 3cosy)^2 \overset{CBS}{\leq} (4^2 + 3^2)(sin^2x + cos^2x) = 25$$

$$4sinx + 3cosx \leq 5; (1)$$

$$(8cosy + 6siny)^2 \overset{CBS}{\leq} (8^2 + 6^2)(sin^2y + cos^2y) = 100$$

$$8cosy + 6siny \leq 10; (2)$$

From $(1),(2) \Rightarrow 4sinx + 3cosx + 8cosy + 6siny \leq 15$

Equality holds $\frac{4}{sinx} = \frac{3}{cosx} \Rightarrow \frac{4}{3} = \frac{sinx}{cosx} = tanx \Rightarrow x = tan^{-1}\frac{4}{3} + k\pi, k \in \mathbb{Z}$

And $\frac{8}{cosy} = \frac{6}{siny} \Rightarrow \frac{8}{6} = \frac{cosy}{siny} \Rightarrow tany = \frac{3}{4} \Rightarrow y = tan^{-1}\frac{3}{4} + q\pi, q \in \mathbb{Z}$

3.73 In ΔABC the following relationship holds:

$$2(\sqrt{a} + \sqrt{b} + \sqrt{c}) \leq 3\sqrt{\frac{3abc}{4Rr + r^2}}$$

Daniel Sitaru

Solution:

$(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab} > a + b > c = (\sqrt{c})^2 \Rightarrow$

$$\sqrt{a} + \sqrt{b} > \sqrt{c} - and\ analogs.$$

By Mitrinovic's inequality in the triangle with sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$:

$$s_1 \leq \frac{3\sqrt{3}}{2}R_1 \Leftrightarrow \frac{1}{2}(\sqrt{a} + \sqrt{b} + \sqrt{c}) \leq \frac{3\sqrt{3}}{2} \cdot \frac{\sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c}}{4S_1} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}(\sqrt{a} + \sqrt{b} + \sqrt{c}) \leq \frac{3\sqrt{3abc}}{8 \cdot \frac{1}{2}\sqrt{4Rr + r^2}} \Leftrightarrow$$

$$\Leftrightarrow 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \le 3\sqrt{\frac{3abc}{4Rr+r^2}}$$

3.74 In ΔABC the following relationship holds:

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \ge 6\sqrt{12Rr+3r^2}$$

Daniel Sitaru

Solution:

$$\left(\sqrt{a}+\sqrt{b}\right)^2 = a+b+2\sqrt{ab} > a+b > c = \left(\sqrt{c}\right)^2 \Rightarrow$$

$$\sqrt{a}+\sqrt{b} > \sqrt{c} - \text{and analogs.}$$

By Mitrinovic's inequality in the triangle with sides $\sqrt{a},\sqrt{b},\sqrt{c}$:

$$s_1 \ge 3\sqrt{3}r_1 \Leftrightarrow \frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \ge 3\sqrt{3}\cdot\frac{S_1}{s_1} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \ge 3\sqrt{3}\cdot\frac{\frac{1}{2}\sqrt{4Rr+r^2}}{\frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \ge 3\sqrt{12Rr+3r^2} \Leftrightarrow$$

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \ge 6\sqrt{12Rr+3r^2}$$

3.75 Find $\Omega = x+y+z$ such that:

$$\begin{cases} \tan x(\tan y+\tan z) = 5 \\ \tan z(\tan x+\tan y) = 9 \\ \tan y(\tan z+\tan y) = 8 \end{cases}$$

Daniel Sitaru

Solution:

Denote: $a = \tan x\tan y$; $b = \tan x\tan z$; $c = \tan y\tan z$

$$\begin{cases} a+b=5 \\ b+c=9 \\ c+a=8 \end{cases} \Rightarrow \begin{cases} a=2 \\ b=3 \\ c=6 \end{cases} \Rightarrow \begin{cases} \tan x\tan y=2 \\ \tan x\tan z=3 \\ \tan y\tan z=6 \end{cases}$$

$$\Rightarrow \begin{cases} \tan x = 1 \\ \tan y = 2 \\ \tan z = 3 \end{cases} \Rightarrow \begin{cases} x = \tan^{-1} 1 \\ y = \tan^{-1} 2 \\ z = \tan^{-1} 3 \end{cases}$$

$$x + y + z = \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$$

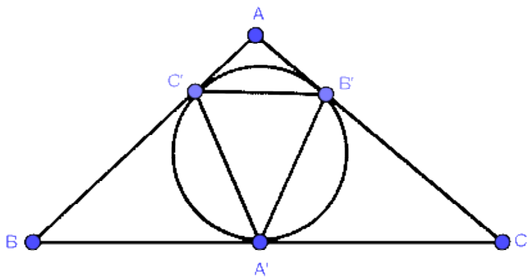
3.76 Let $\Delta ABC \wedge A' \in (BC), B' \in (AC) \wedge C' \in (AB)$ the contact points of the incircle with the sides of ΔABC. Prove that:

$$A'B' + B'C' + C'A' \leq \frac{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}{2}$$

(a refinement of the inequality: $AB' + B'C' + C'A' \leq s$), $s = \frac{a+b+c}{2}$

Marian Ursărescu

Solution:



From sine theorem in $\Delta A'B'C' \Rightarrow$

$$\frac{B'C'}{\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)} = \frac{A'C'}{\sin\left(\frac{\pi}{2} - \frac{B}{2}\right)} = \frac{A'B'}{\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)} = 2r \Rightarrow$$

$$B'C' = 2r\cos\frac{A}{2}, A'C' = 2r\cos\frac{B}{2}, B'C' = 2r\cos\frac{C}{2} \Rightarrow$$

The inequality becomes: $\forall \left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) \leq \frac{\sqrt{ab}+\sqrt{ac}+\sqrt{bc}}{4}$ *(1)*

We will prove that: $r\cos\frac{A}{2} \leq \frac{\sqrt{bc}}{4}$ *(2)*

$$(2) \Leftrightarrow 4r\sqrt{\frac{s(s-a)}{bc}} \leq \sqrt{bc} \Leftrightarrow 4 \cdot \sum_s \sqrt{s(s-a)} \leq bc \Leftrightarrow$$

$$\frac{4\sqrt{s(s-a)(s-b)(s-c)}}{s}\sqrt{s(s-a)} \le bc \Leftrightarrow 16(s-a)^2(s-c) \le b^2c^2 \quad (3)$$

But $s-a = x, x-b = y, s-c = z \Rightarrow$ *(3)* $\Leftrightarrow 16x^2yz \le (x+z)^2(x+y)^2$

Which is true because: $\begin{matrix}(x+z)^2 \ge 4xz \\ (x+y)^2 \ge 4xy\end{matrix} \Rightarrow$ *(3) (true)*

But (2) $\Rightarrow r\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) \le \frac{\sqrt{ab}+\sqrt{ac}+\sqrt{bc}}{4}$

3.77 In ΔABC the following relationship holds:

$$\frac{\frac{1}{2S}\left(\frac{a^2}{\frac{1}{b}+\frac{1}{c}} + \frac{b^2}{\frac{1}{c}+\frac{1}{a}} + \frac{c^2}{\frac{1}{a}+\frac{1}{b}}\right)}{m_a + m_b + m_c} \ge \frac{3R}{2s - 6\sqrt{3}r + 9r}$$

Daniel Sitaru

Solution (Soumava Chakraborty)

$$\frac{a^2}{\frac{1}{b}+\frac{1}{c}} + \frac{b^2}{\frac{1}{c}+\frac{1}{a}} + \frac{c^2}{\frac{1}{a}+\frac{1}{b}} = \sum\frac{a^2bc}{b+c} = 4Rrs\sum\frac{a}{b+c} \overset{Nesbitt}{\ge} 6Rrs$$

$$\Rightarrow \frac{\frac{1}{2S}\left(\frac{a^2}{\frac{1}{b}+\frac{1}{c}} + \frac{b^2}{\frac{1}{c}+\frac{1}{a}} + \frac{c^2}{\frac{1}{a}+\frac{1}{b}}\right)}{m_a + m_b + m_c} \ge \frac{3R}{\sum m_a} \overset{?}{\ge} \frac{3R}{2s - 6\sqrt{3}r + 9r}$$

$$\Leftrightarrow \sum m_a \overset{?}{\le} 2s - 6\sqrt{3}r + 9r$$

$$\Leftrightarrow \left(\sum m_a\right)^2 \overset{?}{\le} 4s^2 + \left(6\sqrt{3}-9\right)^2 r^2 - 4sr\left(6\sqrt{3}-9\right)$$

$$\Leftrightarrow \left(\sum m_a\right)^2 + 4sr\left(6\sqrt{3}-9\right) \underset{(1)}{\overset{?}{\le}} 4s^2 + \left(6\sqrt{3}-9\right)^2 r^2$$

$$Now, \left(\sum m_a\right)^2 + 4sr\left(6\sqrt{3}-9\right) \overset{Chu\ and\ Yang,\ Blundon}{\le} 4s^2 - 16Rr + 5r^2$$

$$+ 8Rr\left(6\sqrt{3}-9\right) + 4\left(3\sqrt{3}-4\right)\left(6\sqrt{3}-9\right)r^2 \overset{?}{\le} 4s^2 + \left(6\sqrt{3}-9\right)^2 r^2$$

$$\Leftrightarrow 16R-8R(6\sqrt{3}-9)\overset{?}{\gtreqless}(6\sqrt{3}-9)\{4(3\sqrt{3}-4)-(6\sqrt{3}-9)\}r+5r$$

$$\Leftrightarrow 8R(11-6\sqrt{3})\overset{?}{\gtreqless}\{(6\sqrt{3}-9)(6\sqrt{3}-7)+5\}r$$

$$\Leftrightarrow 8R(11-6\sqrt{3})\overset{?}{\gtreqless}(176-96\sqrt{3})r \Leftrightarrow R(11-6\sqrt{3})\overset{?}{\gtreqless}2(11-6\sqrt{3})r$$

$$\Leftrightarrow (11-6\sqrt{3})(R-2r)\overset{?}{\gtreqless}0$$

$$\to true \because R \overset{Euler}{\gtreqless} 2r \text{ and } (11-6\sqrt{3})>0 \Rightarrow (1) \text{ is true}$$

$$\therefore \frac{\frac{1}{2S}\left(\frac{a^2}{\frac{1}{b}+\frac{1}{c}}+\frac{b^2}{\frac{1}{c}+\frac{1}{a}}+\frac{c^2}{\frac{1}{a}+\frac{1}{b}}\right)}{m_a+m_b+m_c} \geq \frac{3R}{2s-6\sqrt{3}r+9r} \quad (Proved)$$

3.78 In acute ΔABC the following relationship holds:

$$\sum_{cyc}(sinA)^{2tan^2A}+\sum_{cyc}(sinA)^{2sec^2A}<3$$

Daniel Sitaru

Solution (Tran Hong)

$$sec^2A=\frac{1}{cos^2A}=1+tan^2A$$

$$sec^2B=\frac{1}{cos^2B}=1+tan^2B$$

$$sec^2C=\frac{1}{cos^2C}=1+tan^2C$$

$$\sum_{cyc}(sinA)^{2tan^2A}+\sum_{cyc}(sinA)^{2sec^2A}=$$

$$=\sum_{cyc}(sinA)^{2tan^2A}+\sum_{cyc}(sinA)^{2(1+tan^2A)}=\sum_{cyc}(1+sin^2A)(sinA)^{2tan^2A}$$

$$=\sum_{cyc}(1+sin^2A)(sin^2A)^{tan^2A}=\Omega$$

For $0<x<\frac{\pi}{2}$, let: $f(x)=(1+sin^2x)(sin^2x)^{tan^2x}$

$$f'(x)=2tanx\cdot sec^2x\cdot(sin^2x)^{tan^2x}[2cos^2x+(1+sin^2x)log(sin^2x)]=$$

$$=2tanx\cdot sec^2x\cdot(sin^2x)^{tan^2x}[2(1-sin^2x)+(1+sin^2x)log(sin^2x)]=$$

$$=2tanx\cdot sec^2x\cdot(sin^2x)^{tan^2x}\cdot\varphi(sin^2x)$$

$$0<x<\frac{\pi}{2}\to tanx>0, sec^2x>0, sin^2x>0, tan^2x>0\to$$

$$2tanx\cdot sec^2x\cdot(sin^2x)^{tan^2x}>0$$

$$\varphi(t)=2(1-t)+(1+t)logt;\ \left(\therefore t=sin^2x, t\in(0,1)\right)$$

$$\varphi'(t)=-2+logt+\frac{t+1}{t}$$

$$\varphi''(t)=\frac{1}{t}-\frac{1}{t^2}=\frac{t-1}{t^2}<0, t\in(0,1)$$

$$\varphi'(t)\downarrow(0,1)\Rightarrow\varphi'(t)>\varphi'(1)=-2+0+2=0\Rightarrow\varphi(t)\uparrow(0,1)$$

$$\Rightarrow\varphi(t)<\varphi(1)=2(1-1)+(1+1)\cdot 0=0$$

Hence, $f'(x)<0\ \left(0<x<\frac{\pi}{2}\right)\Rightarrow f(x)\downarrow(0,1)$

$$f(x)<f(0^+)=\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(1+sin^2x)\left[(sin^2x)^{sin^2x}\right]^{\frac{1}{cos^2x}}=1$$

$$\to f(x)<1, \left(0<x<\frac{\pi}{2}\right)$$

So, $\Omega<1+1+1=3$

3.79 In ΔABC the following relationship holds:

$$7s\sum_{cyc}s_a^3>(2\sqrt{2}+1)\left(\sum_{cyc}s_a^2\right)\left(\sum_{cyc}h_a^2\right)$$

Daniel Sitaru

Solution (Tran Hong)

$$\sum_{cyc}s_a^3=\sum_{cyc}\frac{(s_a^2)^2}{s_a}\overset{Bergstrom}{\geq}\frac{(\sum s_a^2)^2}{\sum s_a}=\frac{(\sum s_a^2)(\sum s_a^2)}{\sum s_a}\overset{s_a\geq h_a}{\geq}$$

$$\geq\frac{(\sum s_a^2)(\sum h_a^2)}{\sum s_a}\to$$

$$7s\sum_{cyc} s_a^3 \ge \frac{7s}{\sum s_a}\cdot\left(\sum s_a^2\right)\left(\sum h_a^2\right)\overset{(*)}{\ge}(2\sqrt{2}+1)\left(\sum_{cyc} s_a^2\right)\left(\sum_{cyc} h_a^2\right)$$

$$(*)\leftrightarrow \frac{7s}{\sum s_a} > 2\sqrt{2}+1 \leftrightarrow 7s > (2\sqrt{2}+1)\sum_{cyc} s_a$$

$$\sum_{cyc} s_a \le \sum_{cyc}\sqrt{s(s-a)} \overset{BCS}{\le} \sqrt{3[s(s-a)+s(s-b)+s(s-c)]} = s\sqrt{3}$$

So, we need to prove:

$$7s > (2\sqrt{2}+1)\cdot s\sqrt{3} \leftrightarrow 7 > (2\sqrt{2}+1)\sqrt{3} \leftrightarrow (7-\sqrt{3})^2 > (2\sqrt{6})^2 \leftrightarrow$$

$2 > \sqrt{3} \leftrightarrow 4 > 3$. True.$\to$ $(*)$ is true.Proved.

3.80 In ΔABC the following relationship holds (F_n-Fibonacci numbers):

$$\frac{r_a^2 F_{n+2}}{a(bF_n + cF_{n+1})} + \frac{r_b^2 F_{n+2}}{b(cF_n + aF_{n+1})} + \frac{r_c^2 F_{n+2}}{c(aF_n + bF_{n+1})} \ge \left(\frac{3r}{R}\right)^2$$

Daniel Sitaru

Solution (Adrian Popa)

$$F_{n+2}\left(\frac{r_a^2}{a(bF_n + cF_{n+1})} + \frac{r_b^2}{b(cF_n + aF_{n+1})} + \frac{r_c^2}{c(aF_n + bF_{n+1})}\right)\overset{BCS}{\ge}$$

$$\ge F_{n+2}\cdot\frac{(r_a + r_b + r_c)^2}{(F_n + F_{n+1})(ab + bc + ca)} = \frac{(4R + r)^2}{s^2 + (4R + r)r} \ge$$

$$\overset{Gerretsen}{\ge} \frac{(4R + r)^2}{4R^2 + 4Rr + 3r^2 + 4Rr + r^2} = \frac{(4R + r)^2}{4R^2 + 8Rr + 4r^2} \ge$$

$$\underset{R\ge 2r}{\overset{Euler}{\ge}} \frac{(9r)^2}{4R^2 + 8R\cdot\frac{R}{2} + 4\cdot\frac{R^2}{4}} = \frac{81r^2}{9R^2} = \left(\frac{3r}{R}\right)^2$$

3.81 If $0 \leq a, b, c \leq 1$ then:

$$27\sum_{cyc} sina \cdot cos^2c \leq \sum_{cyc} b(3-a)^3$$

Daniel Sitaru

Solution (Florentin Vişescu)

For all $x \in [0,1] \subset \left[0, \frac{\pi}{2}\right], sinx \leq x$ *and* $0 \leq cosx \leq 1 - \frac{x}{2}, x \in [0,1]$

We show that: $cosx \leq 1 - \frac{x}{2}, \forall x \in [0,1]$

Let: $f: [0,1] \to \mathbb{R}, f(x) = cosx - 1 + \frac{x}{2}$

$$f'(x) = -sinx + \frac{1}{2}; f'(x) = 0 \Leftrightarrow sinx = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{6} \in [0,1]$$

x	0		$\frac{\pi}{6}$		1
$f'(x)$		$- - - -$	0	$+ + + +$	
$f(x)$	0	↗↗↗	$f(\frac{\pi}{6})$	↘↘↘	1

So, $0 \leq cosx \leq 1 - \frac{x}{2}, x \in [0,1]$

$$sina \cdot cos^2c \leq a \cdot cos^2c = a \cdot 1 \cdot cosc \cdot cosc \leq a\left(\frac{1 + 2cosc}{3}\right)^3$$

$$\leq a\left(\frac{1 + 2\left(1 - \frac{c}{2}\right)^2}{3}\right)^3 = a\left(\frac{3-c}{3}\right)^3 = \frac{a(3-c)^3}{27}$$

$$\Rightarrow 27sina \cdot cos^2c \leq a(3-c)^3 \Rightarrow 27\sum_{cyc} sina \cdot cos^2c \leq \sum_{cyc} b(3-a)^3$$

3.82 In ΔABC the following relationship holds:

$$\left(\frac{a^4m_a^2}{m_bm_c}\right)^5+\left(\frac{b^4m_b^2}{m_cm_a}\right)^5+\left(\frac{c^4m_c^2}{m_am_b}\right)^5\ge\frac{(4S)^{10}}{81}$$

Daniel Sitaru

Solution (Tran Hong)

$$\sum_{cyc}\left(\frac{a^4m_a^2}{m_bm_c}\right)^5\overset{Am-Gm}{\ge}3\sqrt[3]{\frac{(a^4b^4c^4)^5m_a^2m_b^2m_c^2}{(m_am_bm_c)^2}}=3\sqrt[3]{(abc)^{20}}\overset{(*)}{\ge}\frac{(4S)^{10}}{81}$$

$$(*)\Leftrightarrow 27(4RS)^{20}\ge\frac{(4S)^{30}}{81^3}\Leftrightarrow 27R^{20}\ge\frac{(4S)^{20}}{81^3}=\frac{(4sr)^{10}}{81^3}$$

$$\Leftrightarrow 27R^{10}R^{10}\ge\frac{(4sr)^{10}}{81^3}$$

From $R\ge 2r$ (Euler) and $3\sqrt{3}R\ge 2s$ (Mitrinovic), we have:

$$27R^{10}R^{10}\ge 27(2r)^{10}\left(\frac{2s}{3\sqrt{3}}\right)^{10}=\frac{(4sr)^{10}}{81^3}\Rightarrow (*)\text{ is true.Proved.}$$

3.83 In $\Delta ABC, K$ –Lemoine's point, the following relationship holds:

$$\frac{aAK+bBK+cCK}{m_a+m_b+m_c}\le\frac{2R\sqrt{3}}{3}$$

Daniel Sitaru

Solution:

From Van Aubel theorem we have: $\frac{AK}{KA'}=\frac{AC'}{C'B}+\frac{AB'}{B'C}$; (1)

From Steiner theorem we have: $\frac{AC'}{C'B}=\frac{b^2}{a^2};\ \frac{AB'}{B'C}=\frac{c^2}{a^2}$; *(2)*

From (1),(2) we have:

$$\frac{AK}{KA'}=\frac{b^2+c^2}{a^2}\Rightarrow\frac{AK}{S_a}=\frac{b^2+c^2}{a^2+b^2+c^2}\Rightarrow AK=\frac{b^2+c^2}{a^2+b^2+c^2}\cdot S_a$$

$$=\frac{b^2+c^2}{a^2+b^2+c^2}\cdot\frac{2bc}{b^2+c^2}\cdot m_a=\frac{2bcm_a}{a^2+b^2+c^2}$$

$$\Rightarrow aKA = \frac{2abc}{a^2+b^2+c^2} \cdot m_a \text{ and simillary.}$$

$$\frac{aAK + bBK + cCK}{m_a + m_b + m_c} = \frac{2abc}{a^2 + b^2 + c^2}$$

We must show:

$$\frac{2abc}{a^2 + b^2 + c^2} \le \frac{2R\sqrt{3}}{3} \Leftrightarrow \frac{4Rrs}{a^2 + b^2 + c^2} \le \frac{R\sqrt{3}}{3} \Leftrightarrow$$

$$12sr \le \sqrt{3}(a^2 + b^2 + c^2); \quad \text{(3). From Mitrinovic:}$$

$$s \ge 3\sqrt{3}r \Rightarrow r \le \frac{s}{3\sqrt{3}}; \quad (4). \text{ From (3),(4) we must show:}$$

$$4s^2 \le 3(a^2 + b^2 + c^2) \Leftrightarrow (a + b + c)^2 \le 3(a^2 + b^2 + c^2) \text{ true}$$

because it's Cauchy inequality.Proved.

3.84 In ΔABC the following relationship holds:

$$\boldsymbol{a^3 + b^3 + c^3 \ge 8\sqrt[4]{3S^6}}$$

Daniel Sitaru

Solution (Adrian Popa)

$$\frac{a^3}{1} + \frac{b^3}{1} + \frac{c^3}{1} \overset{Radon}{\ge} \frac{(a + b + c)^3}{9} = \frac{(2s)^3}{9} = \frac{8s^3}{9}$$

$$\frac{8s^3}{9} \overset{?}{\ge} 8\sqrt[4]{3S^6} \Leftrightarrow s^3 \ge 9\sqrt[4]{3S^6}\Big|^4 \Leftrightarrow s^{12} \ge 3^9 S^6 \Leftrightarrow s^2 \ge 3\sqrt{3}S$$

$$a^2 + b^2 + c^2 \ge 4\sqrt{3}S \text{ (Ionescu-Weitzenbock) ;(1)}$$

$$ab + bc + ca \ge 4\sqrt{3}S \Rightarrow 2(ab + bc + ca) \ge 8\sqrt{3}S \text{ ; (2)}$$

$$\text{From (1),(2) we get: } (a + b + c)^2 \ge 12\sqrt{3}S \Leftrightarrow (2s)^2 \ge 12\sqrt{3}S$$

$$\Leftrightarrow 4s^2 \ge 12\sqrt{3}S \Leftrightarrow s^2 \ge 3\sqrt{3}S. \text{ Proved.}$$

3.85 In ΔABC the following relationship holds:

$$\boldsymbol{\frac{(m_b + m_c)sinA}{m_a sinB sinC} + \frac{(m_c + m_a)sinB}{m_b sinC sinA} + \frac{(m_a + m_b)sinC}{m_c sinA sinB} \ge 4\sqrt{3}}$$

Daniel Sitaru

Solution (Tran Hong)

$$\frac{(m_b+m_c)sinA}{m_a sinBsinC}+\frac{(m_c+m_a)sinB}{m_b sinCsinA}+\frac{(m_a+m_b)sinC}{m_c sinAsinB}$$

$$=\left(\frac{m_b}{m_a}+\frac{m_c}{m_a}\right)\frac{sinA}{sinBsinC}+\left(\frac{m_c}{m_b}+\frac{m_a}{m_b}\right)\frac{sinB}{sinCsinA}$$

$$+\left(\frac{m_a}{m_c}+\frac{m_b}{m_c}\right)\frac{sinC}{sinAsinB}$$

$$=\sum_{cyc}\left(\frac{m_b}{m_a}\cdot\frac{sinA}{sinBsinC}+\frac{m_a}{m_b}\cdot\frac{sinB}{sinCsinA}\right)$$

$$\overset{Am-Gm}{\geq} 2\sum_{cyc}\sqrt{\frac{m_b m_a}{m_a m_b}\cdot\frac{1}{sin^2C}}=2\left(\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\right)$$

$$\overset{C-B-S}{\geq} 2\cdot\frac{9}{sinA+sinB+sinC}\geq\frac{18}{\frac{3\sqrt{3}}{2}}=4\sqrt{3}$$

Because: $sinA+sinB+sinC=\frac{3\sqrt{3}}{2}$

3.86 If in ΔABC, $R<2(r+1)$ then:

$$w_a w_b w_c<(2+h_a)(2+h_b)(2+h_c)$$

Daniel Sitaru

Solution (Tran Hong)

$$w_a=\frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)}\overset{AM-GM}{\leq}1\cdot\sqrt{s(s-a)}=\sqrt{s(s-a)}$$

Similarly:

$$w_b\leq\sqrt{s(s-b)};w_c\leq\sqrt{s(s-c)}$$

$$\Rightarrow w_a w_b w_c\leq s\sqrt{s(s-a)(s-b)(s-c)}=s\cdot S=s\cdot s\cdot r=s^2r$$

$$RHS=(2+h_a)(2+h_b)(2+h_c)\quad (*)$$

$$> (1 + h_a)(1 + h_b)(1 + h_c)$$
$$= (h_a + h_b + h_c) + (h_a h_b + h_b h_c + h_c h_a) + h_a h_b h_c$$
$$> h_a h_b h_c + h_a h_b + h_c h_a + h_b h_c$$
$$= \frac{2s^2 r}{R} + \frac{2s^2 r^2}{R} = \frac{2s^2 r + 2s^2 r^2}{R}$$

We must show that: $s^2 r < \frac{2s^2 r + 2s^2 r}{R} \Leftrightarrow Rs^2 r < 2s^2 r + 2s^2 r^2$

Which is true because:

$$\because Rs^2 r \overset{R<2(r+1)}{<} 2s^2 r(1 + r) = 2s^2 r + 2s^2 r^2$$

3.87 In $\Delta\ ABC$ the following relationship holds:

$$(b + c)m_a + (c + a)m_b + (a + b)m_c \leq 6sR$$

Marian Ursărescu

Solution (Soumava Chakraborty)

$$LHS \overset{CBS}{\leq} \sqrt{\sum(a + b)^2}\sqrt{\sum m_a^2} = \sqrt{2\sum a^2 + 2\sum ab}\sqrt{\frac{3}{4}\sum a^2}$$
$$= \sqrt{3(s^2 - 4Rr - r^2)(3s^2 - 4Rr - r^2)} \overset{?}{\leq} 6sR$$
$$\Leftrightarrow 3(s^2 - 4Rr - r^2)(3s^2 - 4Rr - r^2) \overset{?}{\leq} 36s^2 R^2$$
$$\Leftrightarrow 3s^4 - 4s^2(4Rr + r^2) + r^2(4R + r)^2 \overset{?}{\leq} 12s^2 R^2$$
$$\Leftrightarrow 3s^4 + r^2(4R + r)^2 \underset{(1)}{\overset{?}{\leq}} 12s^2 R^2 + 4s^2(4Rr + r^2)$$

Now, LHS of (1) $\overset{Gerretsen}{\leq} 3s^2(4R^2 + 4Rr + 3r^2) + r^2(4R + r)^2$

$$\overset{?}{\leq} 12s^2 R^2 + 4s^2(4Rr + r^2) \Leftrightarrow s^2(4Rr - 5r^2) \underset{(2)}{\overset{?}{\geq}} r^2(4R + r)^2$$

Now, LHS of (2) $\overset{Gerretsen}{\geq} (16Rr - 5r^2)(4Rr - 5r^2) \overset{?}{\geq} r^2(4R + r)^2$

$$\Leftrightarrow 4R^2 - 9Rr + 2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R - 2r)(4R - r) \overset{?}{\geq} 0 \rightarrow true$$

⇒ (2) is true (proved)

3.88 Let ABC be an acute, non isosceles triangle with (O) is its circumcircle. Denote H as the orthocenter and BE, CF as the altitudes of triangle ABC. Suppose that AH intersects (O) at D differs from A.

1. Let I be the midpoint of AH, EI meets BD at M and FI meets CD at N. Prove that MN is perpendicular to OH.

2. The lines DE, DF intersect (O) at P, Q respectively (P and Q differ from D). The circle (AEF) intersects (O) and AO at R, S respectively (R and S differ from A). Prove that BP, CQ, RS are concurrent.

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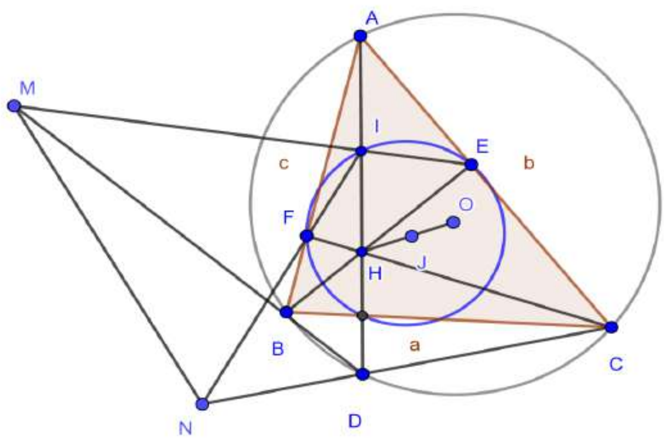
Solution: *1) Denote J as the center of nine points circllle of triangle ABC then (J) passes through E, I, F and point J is also the midpoint of segment OH. It is easy to see that D and H are symmetric with respect to the line BC then triangle BDH is isosceles with $BD = BH$. Since triangle IEH has $IE = IH$ then*

$$\angle IEH = \angle IHE = \angle BHD = \angle BDH,$$

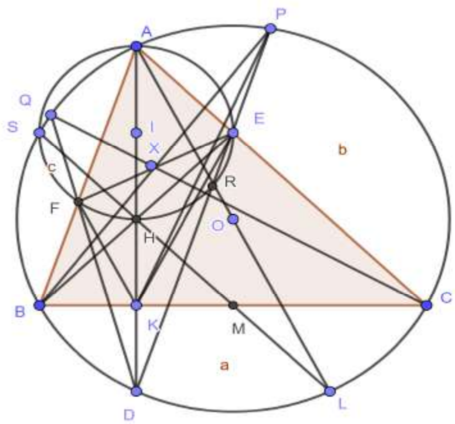
which implies that $BDEI$ is a cyclic quadrilateral. But DB cuts EI at M then

$$\overline{ME} \cdot \overline{MI} = \overline{MB} \cdot \overline{MD}.$$



Thus the power of point M to circles (J) and (O) are equal. Similarly, the power of point N to circles (J) and (O) are also equal. So we can conclude that MN is the radical axis of (O), thus $MN \perp OJ$. But O, H, J are collinear then $MN \perp OH$.

2). Let X be the midpoint of EF and K be the intersection of AH and BC. It is easy to see that two triangles BFE and KHE are similar, which implies that *two triangle BFX and DHE are also similar, thus $\angle FBX = \angle HDE = \angle FBP$. Then three points B, P, X are collinear; similar to three points C, Q, X.*



Denote AL as the diameter of circle (O) then we can see that SH passes through L and quadrilateral $HBLC$ is a parallelogram, which implies that HL passes through the midpoint M of BC. It is easy to check that two triangles SEC and SFB are similar then two triangle SEF and SCB are also similar. These triangles have the medians SX and SM respectively then $\angle FSX = \angle BSM$. We also have two triangles SFB and SRL are similar then two triangles SFR and SBL are also similar. Thus $\angle FSR = \angle BSL = \angle BSM = \angle FSX$. From this we can conclude that three points S, X, R are collinear or SR passes through X. Therefore, three lines BP, CQ and RS are concurrent at the midpoint X of the segment EF.

3.89 Let ABC be an acute triangle inscribed in the circle (O) and I is the circumcenter of triangle OBC. Point G belongs to the arc BC (not contains O) of (I). The circle ABG intersects AB at F (points E, F differ from A).

1. Denote K as the intersection of BE and CF. Prove that AK, BC and OG are concurrent.

2. Let D be a fixed point on the arc BC that contains O of (I) and GB meets CD at M, GC meets BD at N. Suppose that MN intersects

(O) at P, Q. Prove that when G moves on (I), the circumcircle of triangle GPQ always pass through two certain fixed points.

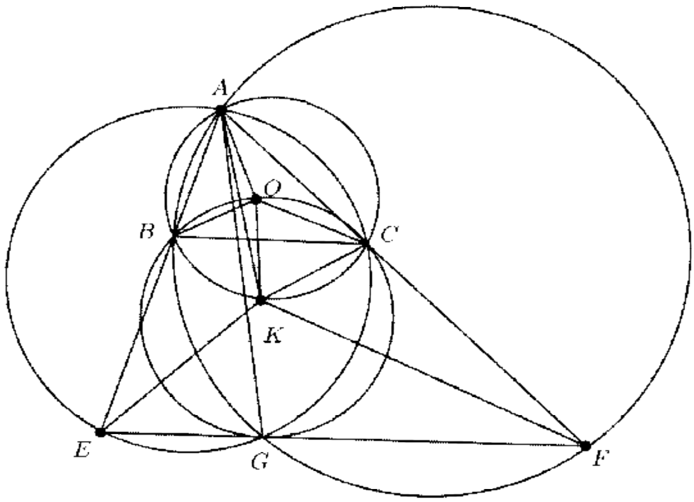
Vietnam NMO-2017

Solution:

1) We have

$$\angle EGF = \angle BGE + \angle CGF - \angle EGF = 360^\circ - 2\angle BAC - (180^\circ - 2\angle BAC) = 180^\circ$$

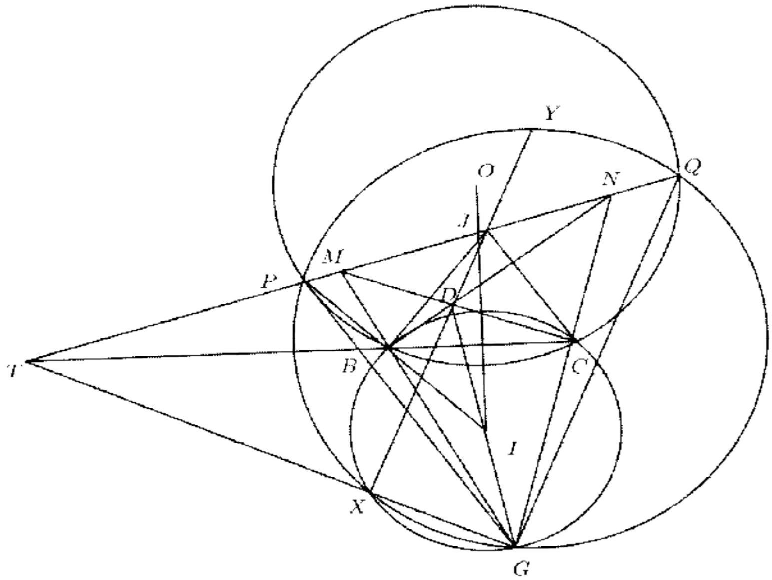
then three points E, G, F are collinear. Since $\angle ABK + \angle ACK = \angle AGE + \angle AGF = 180^\circ$ then K belongs to the circle (O).



It is easy to check that G is the Miquel point then two triangles GBA and GKC are similar, which implies that $\angle BGA = \angle KGC$. We also have GO is the angle bisector of $\angle BGC$ then GO is the angle bisector of $\angle AGK$. Combine with $OA = OK$, we can conclude that $AOKG$ is cyclic quadrilateral. Consider the radical axis circles (O), $(AOKG)$, (BOC), we can see that AK, OG and BC are concurrent.

2) In this part, we just need (O), (I) are two fixed circles that pass through B, C, point D is fixed on (I) while G moves on (I). By applying Pascal's theorem for the tuple $\begin{pmatrix} B & C & D \\ C & B & G \end{pmatrix}$, one can check that the line MN passes through the intersection of the tangent line at B, C, namely J of the fixed circle (I). Suppose that JD meet (I) at the second point X then X is the fixed point and the *quadrilateral $BCDX$ is harmonic then $G(BC, DX) = -1$. Denote T as the intersection of MN and BC then $G(BC, DT) = -1$, which means GX passes through T. Thus*

$$\overline{TX} \cdot \overline{TG} = \overline{TB} \cdot \overline{TC} = \overline{TP} \cdot \overline{TQ}$$



This implies that (GPQ) passes through the fixed point X. Suppose that Y is the intersection of (GPQ) and DX (which differs from X) then

$$\overline{JX} \cdot \overline{JY} = \overline{JP} \cdot \overline{JQ} = P_{J/(O)}$$

which is a constant then Y is fixed. Therefore, the circles (GPO) passes through two fixed points X, Y.

3.90 Let ω be the circumcircle of the acute non-isosceles triangle ΔABC. Point P lies on the altitude from A. Let E and F be the feet of the altitudes from P to CA, B respectively. Circumcircle of triangle ΔAEF intersects the circle ω in G, different from A. Prove that the lines GP, BE and CF are concurrent.

Moldova NMO-2017

Solution: *Let A' be the antipode of $Awrt\omega$. Let BE and CF concur at T. Note that $\angle AGP = 90^\circ$, so G, P, A' are collinear , so it is enough to show that $A'T$ passes through P. A is on the radical axes of circle $(BDPF)$ and $(CDPE)$, so by PoP we have that $BCEF$ is concyclic. Now we have that*

$$\angle PET = 90^\circ - \angle CEB = 90^\circ - \angle BFC = \angle PFT.$$

$$\angle PBA' = \angle PBC + \angle CBA' = \angle PCB + \angle CBA'.$$

By applying sine rule on triangles BPA' and CPA', and then applying trig Ceva in triangle ETF we get the desired result.

3.91 In a triangle ABC the point D is the intersection of the interior angle bisector of $\angle BAC$ and side BC. Let P be the second intersection point of the exterior angle bisector of $\angle BAC$ with the circumcircle of ΔABC. A circle through A and P intersects line segment BP internally in E and line segment CP internally in F. Prove that $\angle DEP = \angle DFP$.

Germany EGMO TST-2015

Solution*: We consider the configuration in which the points A, B, C and P lie in that order on the circumcircle. The other case is analogous.*
By the inscribed angle theorem for the circumcircle of ΔABC, we have

$$\angle ABE = \angle ABP = \angle ACP = \angle ACF.$$

Moreover, by the inscribed angle theorem for the circle through A, P, E and F:

$$\angle AEB = 180^\circ - \angle AEP = 180^\circ - \angle AFP = \angle AFC.$$

We therefore see (AA) that $\Delta ABE \sim \Delta ACF$. From this, it is follows that

$$\frac{|AB|}{|AC|} = \frac{|BE|}{|CF|}$$

The angle bisector theorem then implies that

$$\frac{|AB|}{|AC|} = \frac{|DB|}{|DC|}, \text{ so therefore } \frac{|BE|}{|CF|} = \frac{|DB|}{|DC|}, \quad (1)$$

Choose Z on PA such that A lies between P and Z. As AP is the external angle bisector of $\angle BAC$, we have $\angle PAB = \angle ZAC = 180^\circ - \angle PAC$. So using the inscribed angle theorem and the fact that $ACBP$
Is a cyclic quadrilateral, we see that

$$\angle DCF = \angle PCB = \angle PAB = 180^\circ - \angle PAC = \angle PBC = \angle EBD.$$

If we combine this with (1), we obtain $\Delta BED \sim \Delta CFD$ (SAS). Therefore $\angle BED = \angle CFD$, hence we have $\angle DEP = 180^\circ - \angle BED = 180^\circ - \angle CFD = \angle DFP$.

3.92 Let Γ_1 and Γ_2 be circles with respective centres O_1 and O_2 that intersect each other in A and B. The line O_1A intersects Γ_2 in A and C and the line O_2A intersects Γ_1 in A and D. The line through B parallel to AD intersects Γ_1 in B and E. Suppose that O_1A is parallel to DE. Show that CD is perpendicular to O_2C.

Germany IMO TST-2017

Solution: *We consider only the configuration in which A, B, E, and D lie in that order on a circle, in which O_1, A and C lie in that order on a line, and O_2, A and D lie in that order on a line; the proof is analogous for the other configurations.As $ABED$ is a cyclic quadrilateral, we have $\angle BED = 180^\circ - \angle DAB$. Moreover, using the parallel lines, we see that $\angle BED = \angle DAO_1$ and as $|O_1A| = |O_1D|$ we have $\angle DAO_1 = \angle ADO_1$. We deduce that $180^\circ - \angle DAB = \angle ADO_1$. Therefore DO_1 and AB are parallel.*

We already know that $\angle ADO_1 = \angle DAO_1$. Since $|O_2A| = |O_2C|$, it follows that $\angle DAO_1 = \angle O_2AC = \angle O_2CA$. So $\angle O_2DO_1 = \angle ADO_1 = \angle O_2CA = \angle O_2CO_1$, so O_1DCO_2 is a cyclic quadrilateral.

The line O_1O_2 is the perpendicular bisector AB, therefore is also perpendicular to DO_1, as this line is parallel to AB. Therefore $\angle O_2O_1D = 90^\circ$.

As O_1DCO_2 is a cyclic quadrilateral, we now also have $\angle O_2CD = 90^\circ$.

3.93 An equilateral triangle ABC is given. On the line through B parallel to AC there is a point D, such that D and C are on the same side of the line AB. The perpendicular bisector of CD intersects the line AB in E. Prove that triangle CDE is equilateral.

Germany IMO TST-2015

Solution: *We consider the configuration in which E lies between A and B. The case in which B lies between A and E is treated analogously. (Because of the condition that D and C lie on the same side of AB, it is impossible that A lies between B and E, hence we have treated all cases.)*

As E lies on the perpendicular bisector of CD, we have $|EC| = |ED|$. Hence, it is sufficient to prove that $\angle CED = 60^\circ$. First suppose that $E = B$. Then we have $\angle CED = \angle CBD = \angle ACB = 60^\circ$ because of alternating (Z) angles, hence we are done. Now suppose that $E \neq B$.

As BD is parallel to AC, we have $\angle CBD = \angle ACB = 60^\circ = \angle CBA$. Hence, the point E is the intersection point of the perpendicular bisector of CD and the exterior angle bisector of $\angle CBD$. This means that E lies on the circumcircle of triangle CBD. (This is a known fact, it is also possible to prove it as follows. Let E' be the intersection point of the exterior angle bisector of $\angle CBD$ and the circumcircle of ΔCBD. Because BE' is the exterior angle bisector, we have $\angle CBE' = 180^\circ - \angle DBE'$. Hence, chords CE' and DE' have the same length, which means that E' lies on the perpendicular bisector of CD). We conclude that $CEBD$ is a cyclic quadrilateral. Hence, $\angle CED = \angle CBD = 60^\circ$

3.94 Circles k_1 and k_2 intersect in points A and B. Line l intersect circle k_1 in points C and E, and circle k_2 in points D and F in such a way that D is between C and E, and E is between D and F. Lines CA and BF intersect in point G, and lines DA and BE in point H.
Prove that $CF \parallel HG$.

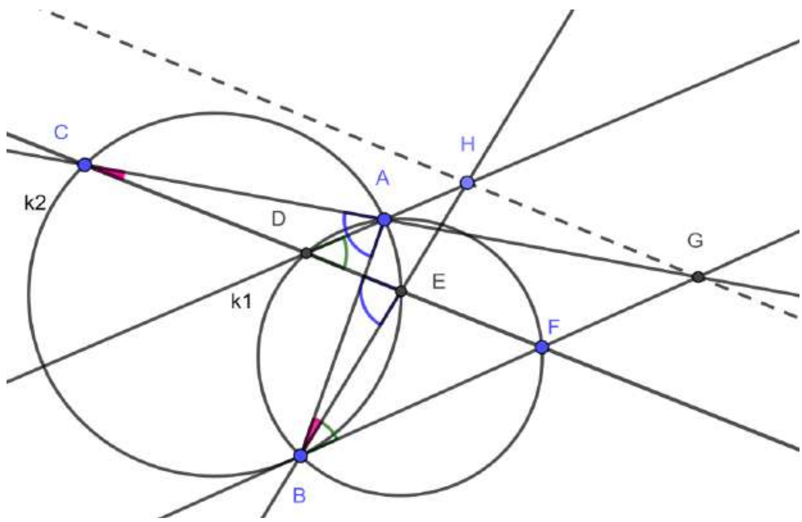
Croatian NMO-2015

Solution: *It is suffices to show that $\angle ECA = \angle HGA$.*
Since the quadrilateral $ACBE$ is cyclic, we have $\angle ECA = \angle EBA$, so it suffices to show that $ABGH$ is a cyclic quadrilateral.
From triangle DEH we have $\angle DHE = 180^\circ - \angle EDH - \angle HED$, i.e.

$$\angle AHB = \angle DHE = 180^\circ - \angle FDA - (180^\circ - \angle CEB)$$

so by using that $\angle CEB = \angle CAB$ (which holds because $ACBE$ is cyclic) we get that

$$\angle AHB = 180^\circ - \angle FDA - \left(180^\circ - \angle CAB\right) = 180^\circ - \angle FDA - \angle BAG.$$



Quadrilateral $ABDF$ is cyclic, so we have $\angle FDA = \angle FBA = \angle GBA$. It follows that $\angle AHB = 180^\circ - \angle FDA - \angle BAG = 180^\circ - \angle GBA - \angle BAG = \angle AGB$. Therefore, $ANGH$ is a cyclic quadrilateral, which finishes the proof.

3.95 Let ABC be a right triangle with the right angle at C. Let A', B' and C' be the pedals of the perpendiculars from the centroid of the triangle ABC on to the lines BC, CA and AB respectively.

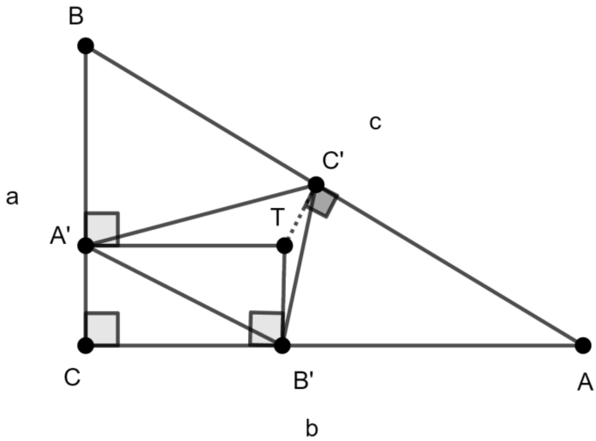
Determine the ratio of the areas of the triangles $A'B'C'$ and ABC.

Croatian NMO-2015

Solution: *Let us denote by T the centroid of the triangle ABC, by a, b and c the lengths of the sides $\overline{BC}, \overline{CA}$ and $\overline{AB}$ respectively and by v the length of the altitude to the side $\overline{AB}$. Let us denote $\alpha = \angle BAC$ and $\beta = \angle CBA$.*

Since T is the centroid of the triangle ABC, it follows that

$$|TA'| = |B'C| = \frac{1}{3}|CA| = \frac{1}{3}b, |TB'| = |A'C| = \frac{1}{3}a, |TC'| = \frac{1}{3}v.$$



We have

$$P(A'B'C') = P(A'B'T) + P(B'C'T) + P(C'A'T) =$$

$$= \frac{1}{2}(|TA'| \cdot |TB'| + |TB'| \cdot |TC'| \cdot sin(\pi - \alpha) + |TC'| \cdot |TA'| \cdot sin(\pi - \beta)) =$$

$$= \frac{1}{18}(ab + av \cdot sin\alpha + bv \cdot sin\beta)$$

Since $v = asin\beta, v = bsin\alpha, a = csin\alpha, b = csin\beta$ and $c^2 = a^2 + b^2$ hold, we get

$$P(A'B'C') = \frac{1}{18}(ab + a^2 sin\alpha sin\beta + b^2 sin\alpha sin\beta) =$$

$$= \frac{1}{18}(ab + c^2 sin\alpha sin\beta) = \frac{1}{18}(ab + ab) = \frac{1}{9}ab = \frac{2}{9}P(ABC)$$

Hence $\frac{P(A'B'C')}{P(ABC)} = \frac{2}{9}$.

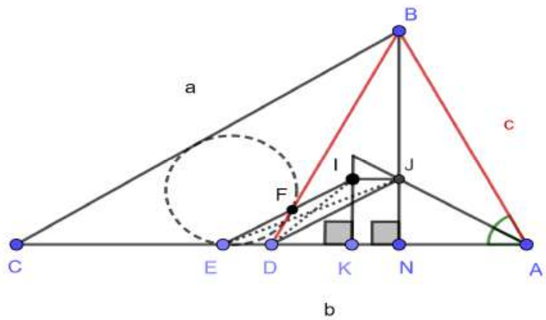
3.96 Let I be the incentre of the triangle ABC and let D be the point on side $\overline{AC}$ such that $|AD| = |DB|$. Incircle of the triangle BCD touches the lines AC and BD in points E and F, respectively. Prove that the line EF passes through the midpoint of the segment $\overline{DI}$.

Croatian MEMO TST-2015

Solution: *We denote by a, b and c the lengths of the sides $\overline{BC}, \overline{CA}$ and $\overline{AB}$ respectively. Let K be the point at which the incircle of the triangle ABC touches the line AC. Let N be the food of the altitude from B in triangle ABC and let J be the intersection of line EF and the altitude $\overline{BN}$.*

Note that $\angle JNE = \angle AKI = 90^\circ$ and

$$\angle NEJ = \frac{1}{2}(180^\circ - \angle FDE) = \frac{\angle ADB}{2} = \frac{\angle BAC}{2} = \angle IAK.$$



Let $d = |CD|$. Then $|ED| = \frac{c+d-a}{2}$ and

$$|EN| = |ED| + |DN| = \frac{c+d-a}{2} + \frac{b-d}{2} = \frac{b+c-a}{2} = |AK|.$$

Hence the triangles AIK and EJN are congruent. It follows that $|IK| = |JN|$, so $|IJ| = |KN|$. Furthermore,

$$|IJ| = |KN| = |AK| - |AN| = \frac{b+c-a}{2} - \frac{b-d}{2} = \frac{c+d-a}{2} = |ED|.$$

From this we conclude that $EDJI$ is a parallelogram and hence its diagonals have the same midpoint.

3.97 Let ABC be an acute-angled triangle with $AB < AC$. Tangent to its circumcircle Ω at A intersects the line BC at D. Let G be the centroid of ΔABC and let AG meet Ω again at $H \neq A$. Suppose the line DG intersects the lines AB and AC at E and F, respectively. Prove that $\angle EHG = \angle GHF$.

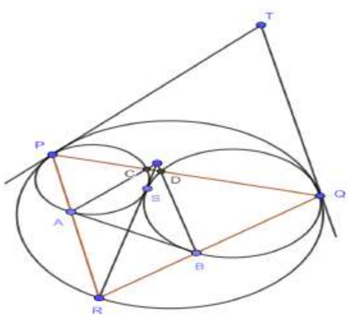
Czech-Polish-Slovak Match-2016

Solution: *Let p be the line parallel to BC that passes through A and let $P = p \cap DG$. We denote the midpoint of BC by T. We project the harmonic ratio $(BTC\infty) = -1$ from A onto the line DG and learn that $(EGFP) = -1$. Therefore by a well-known Apollonian property of harmonic ratios it suffices to prove $\angle PHA = 90^\circ$. Now let Q be the orthogonal projection of D onto AH. The homothety centered at G with factor $-\frac{1}{2}$ maps the line p onto the line BC and hence it maps P to D. Moreover, it leaves the line AH intact, so we just need to prove that it maps H to Q. As H lies on the circumcircle of ABC, which is mapped to the nine-point circle ω of ABC by the considered homothety, we just need to verify that Q lies on ω and that Q is not the „wrong" intersection point of ω and AH. But this other point is the midpoint of BC, which does not coincide with Q when $AB \neq BC$. So our current claim is just that Q lies on ω. To verify it, we denote the orthogonal projection of A to BC by R and the midpoint of AB by S. It is known that R, S and T lies on ω. Further, the points Q and R lies on the circle with diameter AD. Hence: $\angle RQT = \angle BDA = \beta - \angle BAD = \beta - \gamma = \alpha - (180^\circ - 2\beta) = \angle BST - \angle BSR = \angle RST$ and we may conclude.*

3.98 Let γ_1 and γ_2 be externally tangent circles and S be the point of tangency. Let ω be a circle that touches internally γ_1 and γ_2 at P and Q respectively. Denote by R one of the intersection points of ω and the common tangent line of γ_1 and γ_2 that passes through S. Furthermore, the lines RP and RQ intersects γ_1 and γ_2 at A and B respectively and PQ intersects γ_1 and γ_2 at C and D respectively. Prove that the lines RS, AC and BD have a common point.

U.Batzorig Mongolian NMO-2010

Solution: *Let $(AC) \cap (BD) = M$. If suffices to prove that M is on the radical axis of γ_1 and γ_2.*



It's equivalent to $MC \cdot MA = MD \cdot MB$. *This is equivalent to* $ACDB$ *is a cyclic quadrilateral. Since* PT *and* TQ *are tangents to* ω, $\measuredangle TPQ = \measuredangle TQP = \alpha$. *From this* $\measuredangle PAC = \alpha = \measuredangle QBD$ *implies. Since* R *is on the radical axis of* γ_1 *and* γ_2, $RA \cdot RP = RB \cdot RQ$. *This means* $ABQP$ *is cyclic quadrilateral and* $\measuredangle BAD + \measuredangle BQP = 180^\circ$; $\alpha + \measuredangle BAC + \measuredangle BQP = 180^\circ$

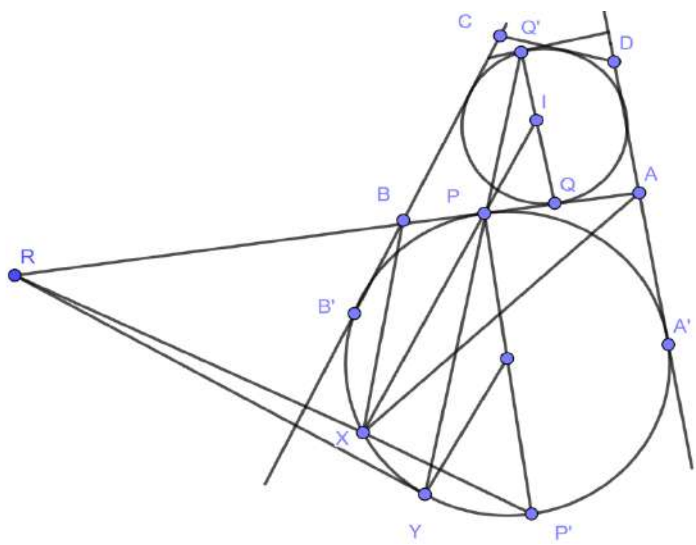
$$x + \measuredangle BQP = \measuredangle QBD + \measuredangle BQP = 180^\circ - \measuredangle QDB.$$

Hence $\measuredangle BAC = \measuredangle QDB$. *This means* $ABDC$ *is cyclic, as desired.*

3.99 Let $ABCD$ be a tangential quadrilateral. Let ω be externally inscribed circle in $ABCD$, tangent to AD, BC and AB. Denote by X_{AB} the point of tangency of ω and the circle that passes through A and B and internally tangent to ω. Let us define X_{BC}, X_{CD} and X_{DA}, analogously. Prove that the bisectors of the angles $\measuredangle AX_{AB}B$, $\measuredangle BX_{BC}C$, $\measuredangle CX_{CD}D$ and $\measuredangle DX_{DA}A$ are concurrent.

G.Batzaya-Mongolian NMO-2010

Solution: *Let* I *be center of incircle of* $ABCD$, *and* AB *touches incircle at* Q *and* ω *at* P. *It will be sufficient to show that* X_{AB}, P, I *are collinear and* $\measuredangle BX_{AB}P = \measuredangle AX_{AB}P$ *(all passes through* I*). Let* $(IP) \cap \omega = X$, PP' *and* QQ' *are diameters and* $(PQ') \cap \omega = Y$, $(XP') \cap (AB) = R$.



$Q'I = IQ, QQ' \parallel PP'$ follows that $(PP'YX)$ is harmonic division.

Therefore, tangent line to ω at Y passes through R. Tangent line at Q' is parallel to AB. From this BC, AD, PQ' are concurrent. If ω touches BC and AD at B' and A' respectively, we have $(A'YB'P)$ harmonic division and $R \in (B'A')$. According to Nagel point BA', AB', PQ' lines are concurrent. This implies that $(RBPA)$ harmonic division. Considering $\measuredangle PXP' = 90^\circ, \measuredangle BXP = \measuredangle AXP$ holds. If $(XB) \cap \omega = B'', (XA) \cap\cap\, \omega = A''$, then $A'' \parallel AB$. Hence $\omega(XBA)$ is tangent to ω, as needed.

3.100 Rectangle of perimeter 2016 was cut into four rectangles I,II,III,IV, and two regions A and B (fig.1). The perimeters of rectangles are in the ratio $P_I : P_{II} : P_{III} : P_{IV} = 1:3:5:7$. What is the sum of perimeters of regions A and B?

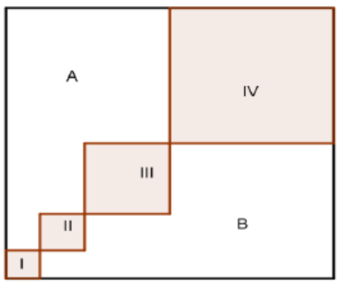


Fig. 1

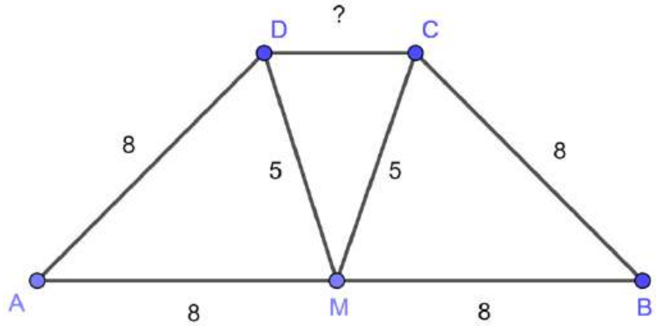
Ukrainian NMO-2017

Solution: *The sum of perimeters of rectangles is equal to the perimeter of the outer rectangles (it is easily verified by considering their sides):*

$P_I + P_{II} + P_{III} + P_{IV} = 2016$. Let $x = P_I$. Than the last equality can be rewritten as: $16x = 2016$ or $x = 126$.

3.101 In the figure, a qudrilateral $ABCD$ is drawn. The midpoint of side AB is called M. The four line segments AM, BM, BC and AD each have length 8, and the line segments DM and CM both have length 5.

What is the length of line segment CD?



Beware: the figure is not drawn to scale.

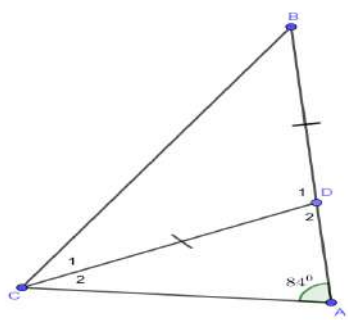
A) 3 **B)** $\frac{40}{13}$ **C)** $\frac{25}{8}$ **D)** $\frac{16}{5}$ **E)** $\frac{13}{4}$

Germany NMO-2015

Solution: *Answer: C)* $\frac{25}{8}$. *Observe that* AMB *is a straight angle. This implies that* $\angle AMD + \angle DMC + \angle CMB = 180^\circ$. *Since triangles* AMD *and* BMC *are equal (three equal sides), we see that* $\angle CMB = \angle MDA$.

Hence $\angle DMC = 180^\circ - \angle AMD - \angle MDA = \angle DAM$, *because the angles of triangle* AMD *sums to 180 degrees. It follows that* DMC *and* DAM *are isosceles triangles with equal apex angles. Hence these two triangles are equal up to scaling. This means that* $\frac{|CD|}{|DM|} = \frac{|DM|}{|AD|}$. *Therefore, the length of* CD *equals* $\frac{5}{8} \cdot 5 = \frac{25}{8}$.

3.102 In a triangle ABC, we have $\angle A = 84^\circ$. Moreover, D is a point on the line segment AB such that $\angle D_1 = 3 \cdot \angle C_2$ and such that the line segments DC and DB have equal lengths.What is $\angle C_1$?



A) 27° **B)** 28° **C)** 30° **D)** 32°

E) 36°

Germany NMO-2014

Solution: Answer: *A)* 27°

3.103 Three equal rectangles $ABCD$, $MNPQ$ and $BPXY$ are arranged as fig.2 shows. Rectangle $NCGF$ is common for all of them and has an area of 17. Determine the sides of equal rectangles if it is known that these sides can be expressed as positive integers.

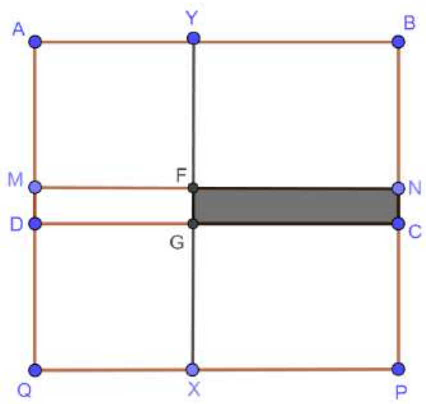
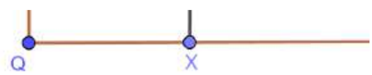


Fig.2

Ukrainian NMO-2017

Solution: *Answer: 17 and 33.*



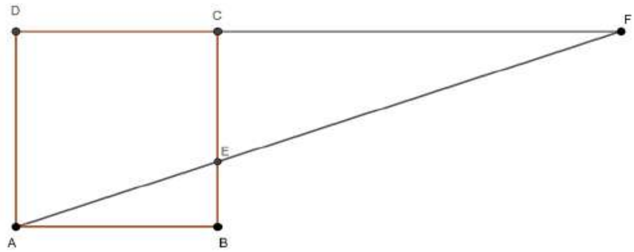
Denote the sides of equal rectangles by $a \geq b$. *Let us find the lengths of some segments formed by intersecting the rectangles (see fig.2).* $AB = BP = a, BY = BC = PN = b, BN = CP = a - b, CN = b - (a - b) = 2b - a$. *Hence* $S_{FNCG} = b(2b - a) = 17$. *Since 17 is prime number, only two cases are possible:*

Case 1: $b = 1; 2b - a = 17$, *which is impossible.*

Case 2: $b = 17; 2b - a = 1 \Rightarrow a = 33$.

3.104 We are given a square $ABCD$. A is dawn through A that intersects the segment BC in E, and the line through C and D in F. The

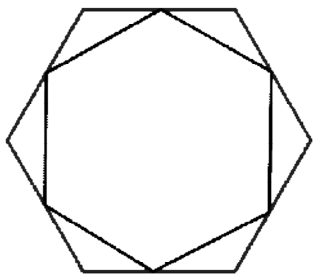
ratio of the lengths of the segments BE and EC is $\frac{1}{2}$. The area of the grey area is 60. What is the area of the square?



Germany NMO-2014

Solution: 36

3.105 Mies has drawn a regular hexagon with area 1. She notices that the midpoints of the six sides also form a regular hexagon. What is the area of this small hexagon?



Germany NMO-2015

3.106 An equilateral triangle ABC is inscribed into a circle Ω and circumscribed about a circle ω. Points P and Q are chosen on the sides

AC and AB, respectively, so that the segment PQ is tangent to ω. A circle Ω_b centered at P passes through B, and a circle Ω_c centered at Q passes through C. Prove that the circles Ω, Ω_b and Ω_c have a common point.

A.Akopyan, P.Kozhevnikov-Regional MO Russian-2017

Solution: *The second meeting point of Ω and Ω_b is symmetric to B about OP; similar statement holds for that of Ω and Ω_c. To prove that these points coincide, one needs just to verify that $\mu(\widehat{POQ}) = 60^\circ$.*

3.107 A circle centered at I is inscribed into a quadrilateral $ABCD$. Rays BA and CD meet at P, and rays AD and BC meet at Q. Given that P lies on the circle (AIC), prove that Q also lies on this circle.

A.Kuznetsov-Regional MO Rusia-2017

Solution: *Prove that $\angle DAI = \angle BAI = \angle DCI = \angle BCI$*

3.108 The circle ω is circumscribed about an acute-angled triangle ABC. Points D and E are chosen on the sides AB and BC, respectively, so that $AC \parallel DE$. Points P and Q are chosen on the smaller arc AC of ω so that $DP \parallel EQ$. Rays QA and PC meet DE at X and Y, respectively. Prove that $\angle XBY + \angle PBQ = 180^\circ$.

A.Kuznetsov-Regional NMO Russia-2017

Solution: *Since $ABCQ$ is cyclic and $AC \parallel DE$, we have $\angle BEX = \angle BCA = \angle BQA = \angle BQX$. Therefore, $XBEQ$ is cyclic; similarly, $YBDP$ is also cyclic. Thus $\angle XBQ = \angle XEQ = \angle DEQ$ and $\angle PBY = \angle PDE$.*

3.109 A quadrilateral $ABCD$ is inscribed into a circle Γ centered at O. Its diagonals AC and BD are perpendicular to each other; let P be their meeting point (the point O lies inside the triangle BPC). A point H is chosen on the segment O so that $\angle BHP = 90^\circ$. The circumcircle ω of the triangle PHD meets again the segment PC at Q. Prove that $AP = CQ$.

A. Kuznetsov-Regional MO Russia-2017

Solution: *Let BT be a diameter of Ω; then the points P, H, T, and D are concyclic. Thus $\angle PDQ = 90^\circ$, and the perpendicular bisectors to PQ, TD, and AC coincide.*

3.110 Let AM be a median of an acute triangle ABC, and let BH be the altitude. The line through M perpendicular to AM meets the ray HB at K. Given that $\mu(\widehat{MAC}) = 30^\circ$, prove that $AK = BC$.

B.Obukhov-Regional MO Russia-2017

Solution: *The points A, H, M, and K lie on a circle ω with diameter AK. Since $\mu(\widehat{MAH}) = 30^\circ$, we have $HM = \frac{AK}{2}$. On the other hand: $BC = 2HM$*

3.111 Determine if there exist a triangle whose side lengths x, y, z satisfy : $x^3 + y^3 + z^3 = (x + y)(y + z)(z + x)$.

V.Senderov-Regional MO-Russia-2017

Solution: Answer: No.

$$(x + y)(y + z)(z + x) = x^2(y + z) + y^2(x + z) + z^2(x + y) + 2xyz$$
$$> x^2 \cdot x + y^2 \cdot y + z^2 \cdot z + 0$$

3.112 Let $ABCD$ be an isisceles trapezoid where $BC \parallel AD$ and $AB \parallel CD$. A circle ω passes through B and C and meets again the segmants AB and BD at X and Y, respectively. The tangent to ω at C meets the ray AD at Z. Prove that the points X, Y and Z are collinear.

A.Kuznetsov-Russia NMO-2017

Solution: Notice that $CYDZ$ is cyclic.

3.113 Let Ω be the circumcircle of a scalene triangle ABC with $\angle ACB = 60^\circ$. The points A' and B' are chosen on the internal angle bisectors of the angles BAC and ABC, respectively, so that $AB' \parallel BC$ and $BA' \parallel AC$. The line $A'B'$ meets ω at points D and E. Prove that the triangle CDE is isosceles.

A.Kuznetsov-Russian NMO-2017

Solution: *Let N be the midpoint of the arc ABC, and T be the midpoint of the lesser arc NC. Prove that N and T lie on $A'B'$. For this purpose, notice that $AN = BN = AB = A'B = B'A$.*

3.114 In the Cartesian plane, two graphs Γ_1 and Γ_2 of monic quadratic trinomials and two non-parallel lines l_1 and l_2 are drawn. Assume that Γ_1 and Γ_2 cut out segments of equal lengths on l_1, and that they cut out segments of equal lengths on l_2. Prove that Γ_1 and Γ_2 coincide.

A.S.Golovanov-Russian NMO-2017

Solution: *The unique vector mapping Γ_1 into Γ_2 should be parallel to both l_1 and l_2.*

3.115 Let O be the circumcenter of an acute-angled isosceles triangle ABC with $AB = AC$. The rays BO and CO meet the sides AC and AB at B' and C', respectively. Let l be the line through C' paralel to AC. Prove that l is tangent to the circumcircle of tha triangle $B'OC$.

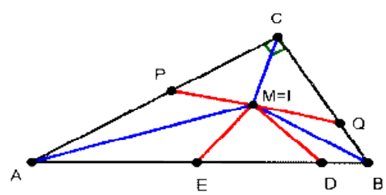
A.Kuznetsov-Russian NMO-2017

Solution: *Prove that $T = AO \cap l$ is the required tangency point.*

3.116 Let ABC be a right triangle with $\widehat{C} = 90^\circ$. Points D and E on the hypotenuse AB are such that $AD = AC$ and $BE = BC$. Points P and Q on AC and BC respectively are such that $AP = AE$ and $BQ = BD$. Let M be the midpoint of segment PQ. Find $\widehat{AMB}$.

Argentina NMO-2016

Solution: *We show that M coincides with the incenter I of the triangle. Since $\hat{A} + \hat{B} = 90^\circ$, this implies $\widehat{AMB} = \widehat{AIB} = 180^\circ - \frac{1}{2}(\hat{A} + \hat{B}) = 135^\circ$*



By hypothesis $AD = AC$, meaning that D is the reflection of C in the bisector AI of $\hat{A}$. Likewise Q is the reflection of D in the bisector BI and P is the reflection of E in the bisector AI, hence $CI = EI = PI$.

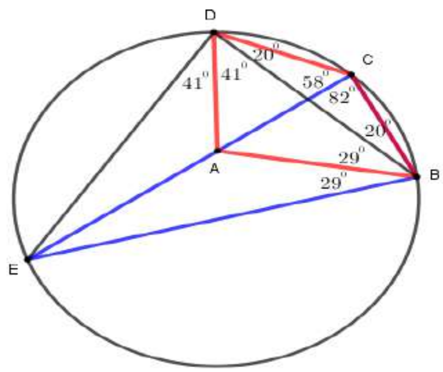
We obtain $CI = PI = QI$. Also $\widehat{PCI} = \widehat{QCI} = 45^\circ$ since CI bisects $\hat{C} = 90^\circ$. Therefore $\widehat{CIP} = \widehat{CIQ} = 90^\circ$. In conclusion P, Q and I are collinear , and I is the midpoint of PQ as $PI = QI$. Thus M and I coincide, as started.

3.117 Find the angles of a convex quadrilateral $ABCD$ such that

$\widehat{ABD} = 29^\circ, \widehat{ADB} = 41^\circ, \widehat{ACB} = 82^\circ$ **and** $\widehat{ACD} = 58^\circ$.

Argentina NMO-2016

Solution:



We have $\widehat{BAD} = 180^\circ - (29^\circ + 41^\circ) = 110^\circ, \widehat{BCD} = 82^\circ + 58^\circ = 140^\circ$. Consider the circumcircle γ of triangle BCD. Since $\widehat{BAD} + \widehat{BCD} > 180^\circ$, point A is interior to γ. Extend CA beyond A to meet γ at E. By inscribed angles $\widehat{ABD} = \widehat{ECD} = \widehat{ACD} = 58^\circ$, $\widehat{EDB} = \widehat{ECB} = \widehat{ACB} = 82^\circ$.

Given that $\widehat{ABD} = 29^\circ, \widehat{ADB} = 41^\circ$ we obtain that BA and DA are bisectors of $\widehat{EBD}$ and $\widehat{EDB}$ respectively. Hence A is the incenter of triangle BDE, implying that EA is the bisector of $\widehat{BED}$. From the cyclic quadrilateral $BCDE$ we have:

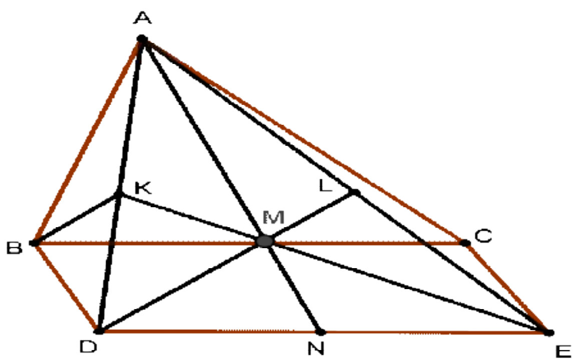
$$\widehat{BED} = 180^\circ - 140^\circ = 40^\circ.$$

Therefore $\widehat{BDC} = \widehat{BEC} = \frac{1}{2}\widehat{BED} = 20^\circ$ *and analogously* $\widehat{DBC} = 20^\circ$. *In conclusion* $\widehat{ABC} = 29^\circ + 20^\circ = 49^\circ$, $\widehat{ADC} = 41^\circ + 20^\circ = 61^\circ$.

3.118 Let ABC be a triangle and let M the middle of the side BC. Externally of the triangle we consider parallelogram $BCDE$, such that $BE \parallel AM$ and $BE = \frac{AM}{2}$. Prove that the line EM passes from the middle point of the segment AD.

S.Brazitikos-Hellenic NMO-2014

Solution: *We extend AM till it meets ED at point N. Then $BMNE$ and $MCDN$ are parallelograms and hence $EN = BM = MC = ND$. Hence N is the middle of ED. Moreover we observe that $\frac{AM}{MN} = 2$ and M lie on the median of the triangle EAD. Hence M is the centroid of the triangle AED. Therefore the line EM is the line of the median of the triangle AED passing from the vertex E, and so it intersects the side AD in the middle.*



3.119 Let ABC be an acute-angled triangle such that, denoting with H the foot of the altitude from C, one has $AH = 3 \cdot HB$. Furtermore, let

- **M be the midpoint of AB;**
- **N be the midpoint of AC;**
- **P be the point in the other half-plane than B with respect to the line AC such that $NP = NC$, $PC = CB$.**

Prove that $\angle APM = \angle PBA$.

Italian NMO-2017

Solution: *Since* $AH = 3 \cdot HB$, H *is the midpoint of* MB; CH *is then both altitudes and median of the triangle* CMB, *which is therefore isosceles. This means that* M *leis on the circle with centre* C *and radius* CB. *By the inscribed angle theorem it follows that* $\widehat{MPB} = \frac{\widehat{MCB}}{2} = \widehat{HCB}$. *The triangle* CPB *is by definition isosceles with base* PB, *and therefore* $\widehat{CPB} = \widehat{CBP}$.

Furthermore $90^\circ = \widehat{CPB} + \widehat{BPM} + \widehat{MPA}$, *and at the same time sum of the internal angles of the triangle* CHB *is* $90^\circ = \widehat{HCB} + \widehat{CBP} + \widehat{PBA}$. *Since, as shown above,* $\widehat{CPB} = \widehat{CBP}$ *and* $\widehat{BPM} = \widehat{HCB}$, *it follows that* $\widehat{MPA} = \widehat{PBA}$.

3.120 Cut a square by straight lines into 3 pieces so thet one could recompose the pieces into an obtuse triangle. You are not allowed to move the resulting pieces after the first cut.

Ukrainian NMO-2015

Solution: *Answer: one possible solution is in Fig.1. In Fig.1,* D *and* E *are the midpoints of the respective sides of the square. Then* $\Delta ACD = \Delta FCE$ *and* $\Delta ADG = \Delta BEC$, *which means thet* ΔABF, *which is clearly obtuse, can be constructed from the three resulting parts.*

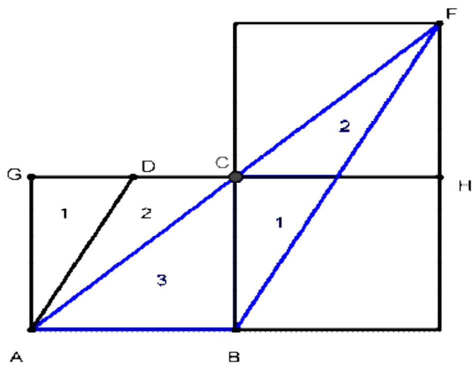


Fig. 1

3.121 Let AD be a bisector in an isosceles triangle ABC $(AB = BC)$, and let DE be another bisector in the triangle ABD. Find out the measures of all angles in ABC if the bisectors of ABD and AED intersect on the straight line AD.

Ukrainian NMO-2015

Solution: *Answer:* $\angle BAC = \angle BCA = 80^\circ, \angle ABC = 20^\circ$. *Let* K *be the intersection point for the bisectors of the angles* ABD *and* AED *(Fig.3). Then this point lies on the segment* AD *and is equidistant from rays* BA *and* BC, *as well as from* EA *and* ED. *Hence, it's equidistant from rays* DE *and* DC. *Then* DA *is the bisector of* $\angle CED$ *(in other words,* K *is an excenter of the triangle* EBD*). With the initial conditions, this implies* $\angle ADC = 60^\circ$. *Since* $\angle DCA = 2\angle DAC$, *we have that* $\angle DCA = 80^\circ$. *Finally, we can write* $\angle BAC = \angle BCA = 80^\circ, \angle ABC = 20^\circ$.

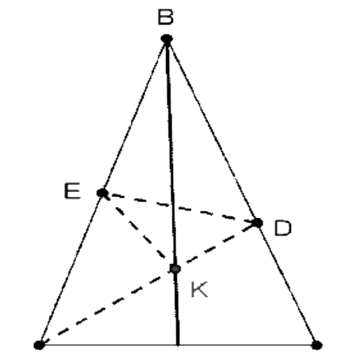


Fig.3

3.122 Is it possible to construct a triangle with sides x, y, z satisfying the condition: $3x^2y^2 + 3y^2z^2 + 3z^2x^2 = x^4 + y^4 + z^4$?

Bogdan Rublyov-Ukrainian NMO-2015

Solution: *Answer: no. Rewrite the equation as:*

$$2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4 = x^2y^2 + y^2z^2 + z^2x^2$$

The left-hand side can be decomposed as:

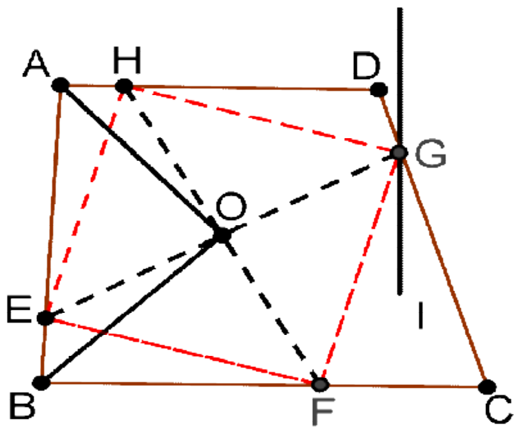
$$(x + y + z)(x + y - z)(y + z - x)(z + x - y) = -(x^2y^2 + y^2z^2 + z^2x^2).$$

Hence, the left-hand side is negative, therefore, at least one of the multipliers is negative too. The first one is always positive, then, without loss of generality, we can assume the second one is negative, i.e. $x + y - z < 0$, *which contradicts the triangle inequality.*

3.123 A right trapezoid is given with the following property: a square can be inscribed into it such that all its vertices lie on different edges of the trapezoid and none of them coincide with any vertex of the trapezoid. Construct this square with a ruler and a compass.

Mariya Rozhkova-Ukraine NMO-2015

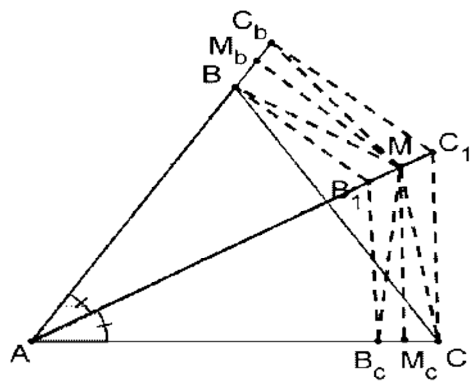
Solution: *Let $ABCD$ be our trapezoid with right angles A and B. Let $EFGH$ be the required square centred at O, and suppose $E \in AB, F \in BC$. In the quadrilateral $EBFO$ two opposite angles are right, hence, it's cyclic. This implies that $\angle EFO = \angle EBO = 45^\circ$ as they intercept the same arc EO (Fig.4). Likerwise , $\angle EAO = 45^\circ$. Thus O is the intersection point for bisectors of angles A and B in the trapezoid. Also, E and G are symmetric with respect to O.*
Construction: *Construct O as the intersection point of two bisectors drawn from A and B. Then we can construct a straight line l symmetric to AB with respect to O. The line l intersects the segment CD at a unique point G, which is a vertex of our square. Having the center and one vertex of the square, we can reconstruct it in a unique way.*



3.124 Points B_1, C_1 are chosen on the bisector BAC of a triangle ABC so that $BB_1 \perp AB, CC_1 \perp AC$. Let M be the midpoint of B_1C_1. Prove that $MB = MC$.

Ukrainian NMO-2015

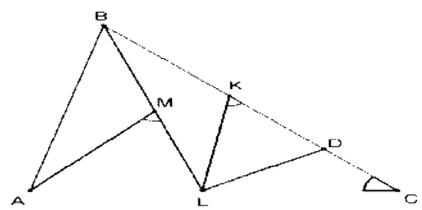
Solution: *Assume the perpendicular dropped from C_1 on AB intersects it at some point C_b. Define B_c in the same way so that $BB_b \perp AC$. Let $M_b.M_c$ be the projections of M onto lines AB and AC respectively. Without loss of generality, suppose the points are located as Fig. Then: $\Delta AB_1B_c \sim \Delta AMM_c \sim \Delta AC_bC$ and $\Delta AC_1C_b \sim \Delta AMM_b \sim \Delta AB_cB$.Then from basic properties of a trapezoid we have $B_cM_c = M_cC$, $C_bM_b = M_bB$. Hence $B_cM = MC$ and $C_bM = MB$. Since $\angle MB_1B_c = \angle BB_1M$ and $BB_1 = B_1B_c$, this implies $\Delta MBB_1 = \Delta MB_1B_c$, hence $BM = MB_c$, and $BM = MB_c = CM$.*



3.125 An acute-angled triangle ABC has the side $BC > AB$, and the bisectrix $BL = AB$. On the segment BL there exists point M, for which $\angle AML = \angle BCA$. Prove that $AM = LC$.

Ukrainian NMO-2015

Solution: *On the segment BC put a point such as $BD = BL$. Then $\Delta ABL = \Delta BLD$, mark $\angle LAB = \angle BLA = \angle BLD = \angle BDL$. On the segment BD choose a point K, such that $\angle ALM = \angle DLK$. Then $\Delta ALM = \Delta DLK$ because of $LD = AL$, but then $\angle AML = \angle LKD = \angle BCA$, that's why $KL = AM = LC$, and it is exactly what we have to prove.*



3.126 Points X, Y are chosen on the sides AB and AD of a convex quadrilateral $ABCD$ respectively so that $CX \parallel DA$, $DX \parallel CB$, $BY \parallel CD$ and $CY \parallel BA$. Find the ratio $\frac{AX}{BX}$.

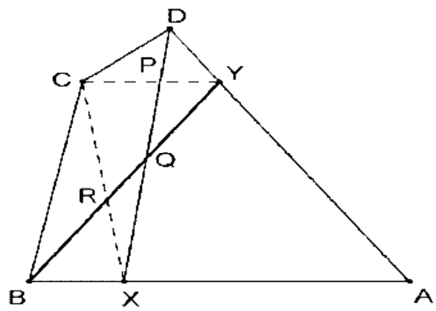
Ukrainian NMO-2015

Solution: *Answer:* $\frac{\sqrt{5}+1}{2}$. *Denote this ratio by* $\lambda = \frac{AX}{BX}$. *Since* $RYCD$ *is a parallelogram, by Thales' theorem we obtain* $\lambda = \frac{AX}{BX} = \frac{YR}{RB} = \frac{CD}{RB}$

Likewise,

$$\lambda = \frac{AX}{BX} = \frac{CY}{BX} = \frac{CY}{CP} = \frac{BY}{BQ} = \frac{BY}{CD} = \frac{YR + BR}{CD} = \frac{CD + BR}{CD} = 1 + \frac{BR}{CD} = 1 + \frac{1}{\lambda}$$

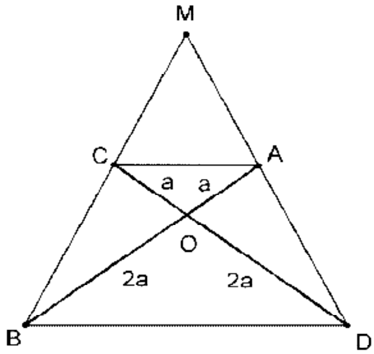
By solving this equation for λ*, we obtain the answer.*



3.127 The segments AD and CD intersect in the point O and are divided into ratio $\frac{AO}{OB} = \frac{CO}{OD} = \frac{1}{2}$. The lines AD and BC intersect in the point M. Prove that $DM = MB$.

Ukrainian NMO-2015

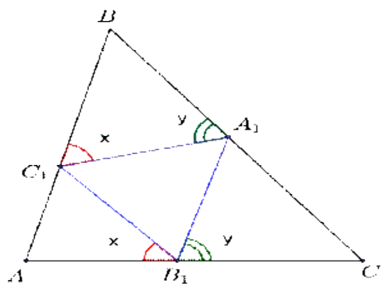
Solution: *Let the length of the segments be* $AB = CD = 3a$, *then* $AO = CO = a$ *and* $OB = OD = 2a$. *Since* $\angle AOD = \angle COB$ *as vertical, then* $\Delta AOD = \Delta COB$. *Hence* $\angle ADO = \angle CBO$. *As* ΔBOD *is isosceles, then* $\angle BDO = \angle DBO$, *consequently* $\angle MDB = \angle MBD$ *as the sum of the equal angles. In other words* ΔMDB *is isosceles, from where the equality we need.*



3.128 On the sides AB, BC, CA of triangle ABC the points C_1, A_1, B_1 are chosen respectively, and these points are noot the vertex. $\Delta A_1B_1C_1$ is equilateral and $\angle BC_1A_1 = \angle C_1B_1A$ and $\angle BA_1C_1 = \angle A_1B_1C$. If is ΔABC required to be equilateral?

Ukrainian NMO-2017

Solution: *Answer: yes. Mark angles $\angle BC_1A_1 = \angle C_1B_1A = x$ and $\angle BA_1C_1 = \angle A_1B_1C = y$. $\angle AB_1C$ is straight, so $x + y = 120^\circ$. Then $\angle ABC = 60^\circ$, hence $\angle B_1A_1C = x$, so $\angle ACB = 60^\circ$, in other words ΔABC is equilateral.*



3.129 In the convex quadrilateral $ABCD$ with angles ABC and BCD equal to 120°, O is the intersection of diagonals, M is the midpoint of BC, K is the point of the intersection of MO and AD. It happens hat $\angle BKC = 60^\circ$. Prove that $\angle BKA = \angle CKD = 60^\circ$.

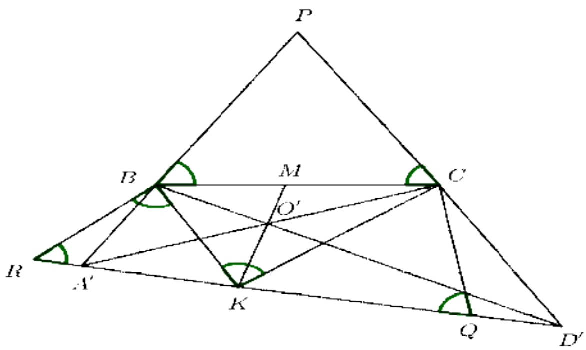
Serduk Nazar-Ukrainian NMO-2015

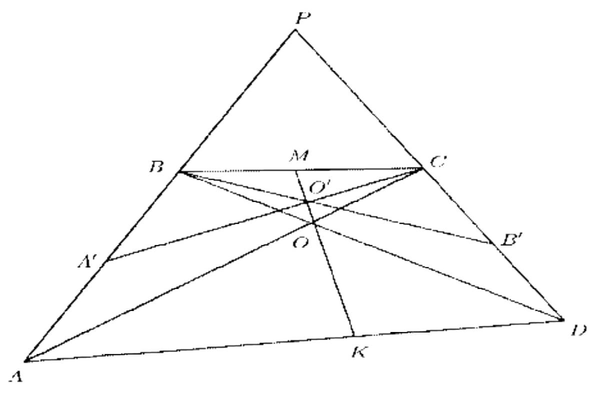
Solution: *Let us draw external bisector of* $\angle BKC$. *Let it intersects lines* BA *and* CD *at* A' *and* D'. *We draw on sides* ΔBKC *outside equilateral triangles* BPC, BRK *and* CQR. *Obviously that* R *and* Q *are on* $A'D'$, *and* P *is an intersection of* AB *and* CD. *Consider* ΔBKC. *We saw, that*

$$\angle CBR = \angle CBK + 60^\circ = 180^\circ - (120^\circ - \angle CBK) = \angle KBA'$$

It implies that BR *and* BA' *are isogonal, also* KR *is external bisector. So* A' *and* R *are isogonal, hence* CA' *and* CR *are isogonal. Similarly we have that segments* BD' *and* BQ *are isogonal. It is easy to see that* P *–the point of intersection of tangents to circumcircle of* ΔBKC, *so* $\angle KPC = \angle BKC = \angle PCB = 60^\circ$. *For this* KP *is symedian of* ΔBKC, *so* KP *and* KM *are isogonal.*

As it known KP, BQ *and* CR *intersect in the same point, namely at Fermat point, so as lines* KM, BD' *and* CA' *intersect in the same point. Let is be point* O'. *Assume that* $MO > MO'$. *Then points* A' *and* D' *are on sides* BA *and* CD, *but in this case* AD *and* $A'D'$ *don't intersect each other. Similarly if* $MO < MO'$. *Hence* $MO = MO' \Rightarrow A = A', D = D' \Rightarrow \angle AKB = \angle DKC$.

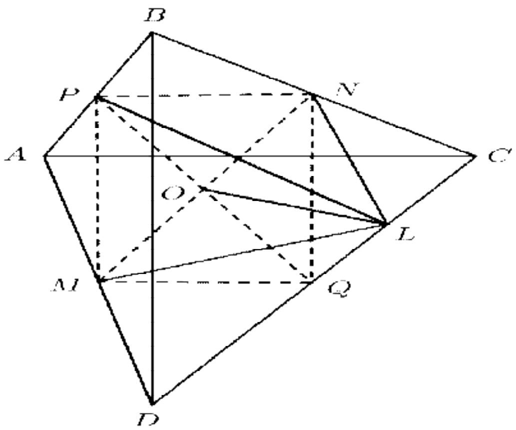




3.130 In the trapezoid $ABCD$ with perpendicular diagonals points P, M, N, Q are the middles of sides AB, BC, CD, DA respectively. On the base CD there is a point L(different from the point Q) for which the angle $\angle MNL$ is straight. Find the angle $\angle LPA$.

Bogdan Rublyov-Ukrainian NMO-2017

Solution: *Answer:* 90°. *By Varignon's Theorem quadrangle $MPNQ$ is a parallelogram and its sides are parallel to the diagonals of trapezoid $ABCD$, so it is a rectangle (Fig.). Denote O the intersection point of AC and BD. Then from the properties of rectangular ΔMNL we get that: $OL = OM = ON \Rightarrow OL = OP = OQ$, so ΔPQL is rectangular. That means $PL \perp LQ$, and since $AB \parallel CD$ then $PL \perp AB$.*

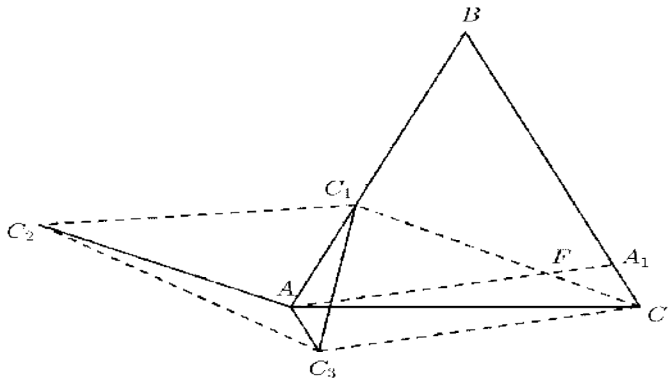


3.131 Let A_1 and C_1 be the points on the sides BC and AB of he triangle ABC respectively, so that $AA_1 = CC_1$. Segments AA_1 and CC_1 meet at the point F. Prove that if $\angle CFA_1 = 2\angle ABC$ then $AA_1 = AC$.

Andrii Gogolev-Ukrainian NMO-2015

Solution: *Consider translation by vector $\overrightarrow{A_1A}$. Thus AA_1CC_3 is a parallelogram, $CC_1C_2C_3$ is a rhombus. Then $\angle C_1CC_3 = \angle C_1FA = \angle CFA_1 = 2\alpha$, therefore $\angle C_1AC_3 = 180^\circ - \angle ABC = 180^\circ - \frac{1}{2}\angle CFA_1 = 180^\circ - \alpha$.*

If we built a circle where the point C is its centre, then the central angle $\angle C_1CC_3 = 2\alpha$, thus the inscribed angle, that subtends the arc C_1C_3, equals α, therefore the point A belongs to the circle. Therefore, $AC = CC_3 = AA_1$, this completes the proof.

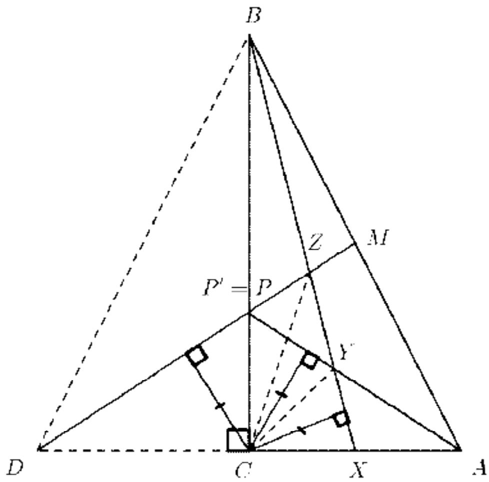


3.132 Let M be the midpoint of the hypotenuse AB of the right triangle ABC. Point P is chosen on the cathetus CB so that $\frac{CP}{PB} = \frac{1}{2}$. The straight line passing through B meets the segment AC, AP and PM at points X, Y, and Z, respectively. Prove that the bisector of the angle PZY passes through point C if and only if the bisector of the angle PYX also passes through C.

I.Voronovich-Belarus NMO-2017

Solution*:*

Let D symmetric to A with respect to the vertex C. The segment BC is the altitude and the median in the triangle ABD so the triangle ABD is isosceles. Likewise, the triangle ADP is isosceles. Let P' be the intersection point of the medians DM and BC of triangle ABD. Since the point of intersection divides the medians in the ratio $\frac{2}{1}$, points P and P' coincide. Since C belongs to the bisector of the angle $\angle DPA$, we see that C is equidistant to the lines DP and PA. If C is equidistant to the bisector of the angle $\angle PZY$, then C is equidistant to the lines PZ and ZY, therefore, C is equidistant to the lines YP and YX, i.e. C belongs to the bisector of the angle $\angle PYX$. In the same way, one can prove the converse proposition.



3.133 Point M is marked inside a convex quadrilateral $ABCD$. It appears that $AM = BM$, $CM = DM$, and $\angle AMB = \angle MD = 60^{\circ}$. Let K, L and N be the points of the segments BC, AM, and DM, respectively. Find the value of the angle LKN.

S.Mazanik-Belarus NMO-2017

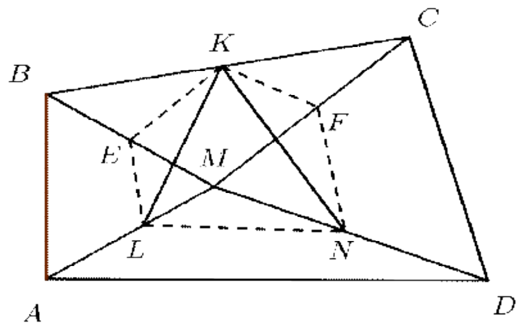
Solution: *Answer:* 60°. *Let E and F be the midpoints of the segments BM and CM, respectively. Since $\angle AMB = \angle CMD = 60^{\circ}$, we have*

$$\angle BMC = 360^{\circ} - \angle AMB - \angle CMD - \angle LMN = 360^{\circ} - 60^{\circ} - 60^{\circ} - \angle LMN =$$

$$= 240^{\circ} - \angle LMN.$$

Since KF is the midline in the triangle CMB, we have $KF \parallel BM$, therefore, $\angle KFC = \angle BMC$. Then $\angle KFM = 180^{\circ} - \angle KFC = 180^{\circ} - \angle BMC = 180^{\circ} - (240^{\circ} - \angle LMN) = \angle LMN = 60^{\circ}$. By the condition , $CM = DM$ and $\angle CMD = 60^{\circ}$, so the triangle CMD is equilateral. Since FN is the midline in the triangle CMD, we see that the triangle FMN is equilateral too and $FM = NM = FN$, $\angle MFN = 60^{\circ}$. Therefore,

$\angle KFN = \angle KFM + \angle MFN = [\angle MFN = 60^{\circ}] = \angle LMN - 60^{\circ} + 60^{\circ} = \angle LMN$. *In the same way, one can show that: $\angle KEL = \angle LMN$ and $EM = ML = EL$. Thus, $\Delta KEL = \Delta NFK = \Delta NML$ (by two sides and the angle* *between them), so $KL = KN = LN$. Therefore, the triangle KLN is equilateral and $\angle LKN = 60^\circ$.*



3.134 Given a convex hexagon H with obtuse inner angles and pairwise parallel opposite sides.

a) Prove that there exists a pair of the opposite sides of H which possesses the following property: there exists a straight line that is perpendicular to these sides and intersects each of them.

b) Is it true that there exist two pairs of the opposite sides of H, each of which possesses the same property, as described in item a)?

P.Irzhavskii-Belarus NMO-2017

Solution: *Answer: b) is not true.*

a)Let $ABCDEF$ be a hexagon with all obtuse inner angles and parallel opposite sides ($AB \parallel DE, BC \parallel EF, CD \parallel FA$). Consider the greatest side of this hexagon (one of such sides if there are more than one). Let it be the side AB. Since the hexagon $ABCDEF$ is convex, it completely lines in one of the half-planes with respect to the line AB; denote this half-plane by π.

Suppose hat none of the lines intersecting and perpendicular to the side AB intersect the side DE. Construct the perpendiculars h_A and h_B in the half-plane π. By our assumption, there are no points of the side DE in the half-strip P_0. In particular. Either point D lies to the right of the ray h_A for point E lies to the left of the ray h_B. Without loss of generality suppose that D lies to the right of

h_a. Then the vertex C lies neither to the right of the ray h_A or to the left of the ray h_B. Indeed, otherwise we obtain $BC > BA$ (Fig.3) and $CD > BA(Fig.4)$, respectively. But any of these inequalities contradicts to the choice of the side AB as the greatest side of the hexagon.

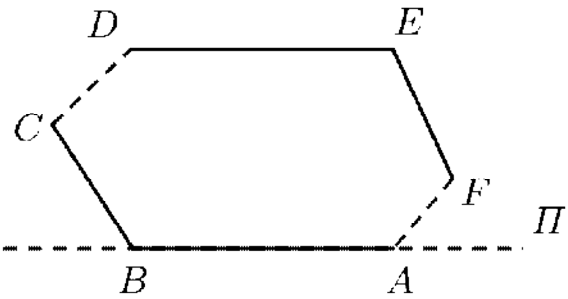


Fig.1

So the vertex C lies in the half-strip P_0, but in this case the angle ABC is not obtuse, contrary to the problem condition. Therefore, there exists a straight line such that it is perpendicular to the sides AB and DE and intersects both of them.

b)Construct the convex hexagon $ABCDEF$ satisfying the problem condition (all inner angles are obtuse and opposite sides are parallel) such that there exists exactly one pair of the opposite sides by the perpendicular straight line.

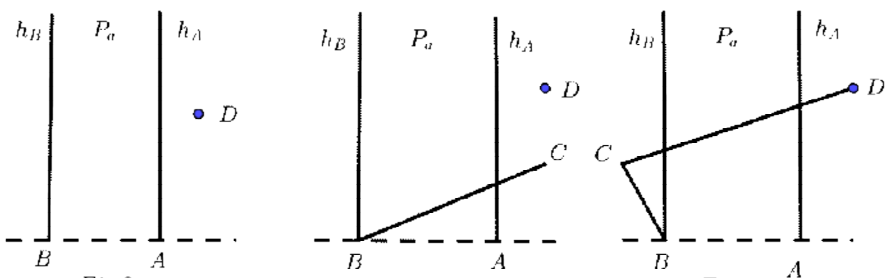


Fig.2 Fig.3 Fig.4

Consider the isosceles triangle KLM with tha base LM and the acute angle LKM (see Fig.5). Let KH be the altitude of the triangle KLM. Let points P and Q be the inner points of the segments MH and LH, respectively, such that they are symmetric with respect to H. Let E be the foot of perpendicular from P into KL, and let D be the foot of perpendicular from Q into KM (see Fig.5). Then

$DE \parallel ML$. Further, mark some point A on the segment LQ and some point B on the segment MP. Construct the line $l(A)$ passing trough A parallel to KM, and construct the line $l(B)$ passing trough B parallel to KL. Let F be the intersection point of l_A and KL, and let C be the intersection point of l_B and KM (see Fig.5). Then (see Fig.6) it is easy to see that the hexagon $ABCDEF$ is required.

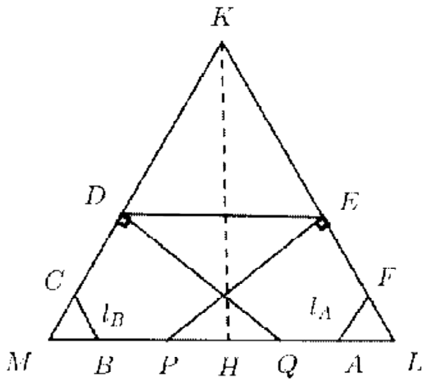


Fig.5

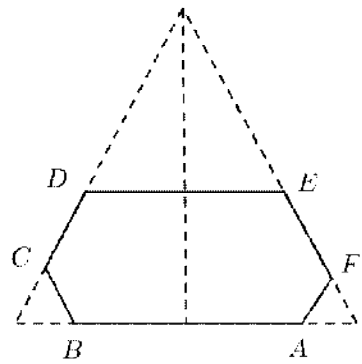
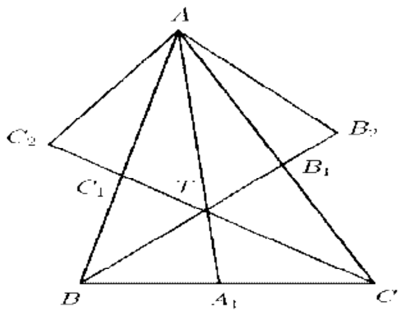


Fig.6

3.135 Let be given ΔABC and let AA_1, BB_1 and CC_1 are the medians in the triangle which intersect in the point T and $\overrightarrow{BA_1} = \overrightarrow{A_1T}$. On the continuation of CC_1 we choose a point C_2 such that $\overrightarrow{C_1C_2} = \frac{\overrightarrow{CC_1}}{3}$, and on the continuation of BB_1 we choose a point C_2 such that $\overrightarrow{B_1B_2} = \frac{\overrightarrow{BB_1}}{3}$. Prove that the quadrilateral TB_2AC_2 is a rectangle.

Macedonian NMO-2017

Solution: *Since AA_1 is a median in the ΔABC and $\overrightarrow{BA_1} = \overrightarrow{A_1T}$, we get that $\overrightarrow{A_1T} = \frac{\overrightarrow{BC}}{2}$ i.e. A_1 is the circumcenter of the circumcircle of ΔBCT. So according to the Thales theorem $\angle BTC = 90^\circ$. We have $\angle B_2TC_2 = 90^\circ$ (as vertically opposite angles). Since T is the barycentre of ΔABC we have $\overrightarrow{C_1T} = \frac{\overrightarrow{CC_1}}{3} = \overrightarrow{C_1C_2}$. From $\overrightarrow{BC_1} = \overrightarrow{C_1A}$ we get that the quadrilateral $BTAC_2$ is a parallelogram. Then $BT \parallel AC_2$, so $\angle TC_2A = \angle CTB = 180^\circ - \angle B_2TC_2 = 90^\circ$ (as angles on the transversal). With analogy, we can prove that the quadrilateral TCB_2A is a parallelogram i.e. $\angle TB_2A = \angle C_2TB_2 = 90^\circ$ (as* *angles on the transversal). So, we get hat* $\angle C_2AB_2 = 360^\circ - 270^\circ = 90^\circ$ *i.e. the quadrilateral is a rectangle.*



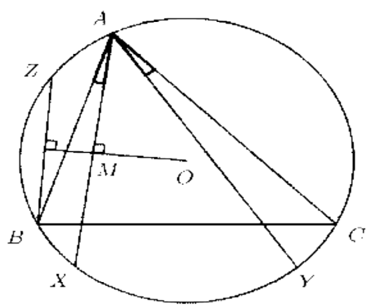
3.136 Le be given the ΔABC. On the arc $\widehat{BC}$ of the circumcircle of ΔABC, which does not conttain the point A, points X and Y are choosen, such that $\angle BAX = \angle CAY$. Let M be the middle point of the chord AX. Prove that $\overrightarrow{BM} + \overrightarrow{CM} > \overrightarrow{AY}$.

Macedonian NMO-2017

Solution: *Let O be the circumcenter of the circumcircle of ΔABC. Then $OM \perp AX$. We draw a normal line from the point Bat OM and let it intersect the circumcircle in the point Z. Since $BZ \perp OM$ we have that OM is a line of symmetry of BZ. According to this, $\overrightarrow{MZ} = \overrightarrow{MB}$. Now, from the triangle inequality we have that $\overrightarrow{BM} + \overrightarrow{MC} = \overrightarrow{ZM} + \overrightarrow{MC} > \overrightarrow{CZ}$.*

But, $BZ \parallel AZ$, so $\widehat{AZ} = \widehat{BX} = \widehat{CY}$ where from we get

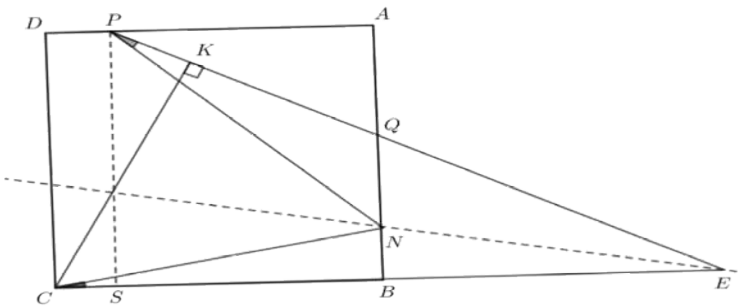
$\widehat{ZAC} = \widehat{ZA} + \widehat{AC} = \widehat{YCA}$ *i.e.* $\widehat{CZ} = \widehat{AY}$. *That is why* $\overrightarrow{BM} + \overrightarrow{CM} > \overrightarrow{AY}$.



3.137 Square $ABCD$ is given. Points N and P are selected on sides AB and AD, respectively, such that $PN = NC$, and point Q is selected on segment N such that $\angle NCB = \angle QPN$. Prove that $\angle BCQ = \frac{1}{2}\angle PQA$.

Arian Mohammadi-Iran NMO-2015

Solution: *Let E be the intersection point of PQ and BC. According to the problem assumption, $PN = NC$ and so $\angle NPC = \angle PCN$. From these we conclude that EPC is an isosceles triangle. Therefore, its altitudes PS and CK have equal length. So $CK = PS = AB = BC$ and therefore right-angled triangles QBC and QKC are congruent. So QC is the bisector of angles $\angle KCB$ and $\angle KQB$. Hence, $\angle BCQ = \frac{1}{2}\angle KCB$ $(*)$.*



On the other hand, since $\angle QBC + \angle QKC = 90^\circ + 90^\circ = 180^\circ$, we get that $QBCK$ is a cyclic quadrilateral which implies $\angle BCK = \angle AQP$. This together with $(*)$ completes the proof.

3.138 Point D is the intersection point of the angle bisector of vertex A with side BC of triangle ABC, and point E is the tangency point of the inscribed circle of triangle ABC with side BC. A_1 is a point on the circumcircle of triangle ABC such that $AA_1 \parallel BC$. If we denote by T the second intersection point of line EA_1 with the circumcircle of triangle AED and by I incenter of triangle ABC, prove that $IT = IA$.

Ali Zamani-Iran NMO-2015

Solution:

Let E_1 be the reflection of E with respect to the midpoint of BC and X the intersection point of AE_1. We claim that $IX \parallel BC$. For this reason, we have (suppose that R is the radius of circumcircle of ABC and $\angle B \geq \angle C$).

$$\frac{AX}{XE_1} = \frac{AA_1}{EE_1} = \frac{AA_1}{AC - AB}$$

$$\frac{AI}{IE} = \frac{AB}{BD} = \frac{AC}{CD} = \frac{AB + AC}{BC}$$

So referring to the Thales' theorem, we must prove $AA_1 \cdot BC = AC^2 - AB^2$.

We have $AA_1 = BC - 2BH = BC - 2ABcosB$, where H is the foot of perpendicular from A to BC and on the other hand, by the Law of Cosines we have $AC^2 - AB^2 = BC^2 - 2AB \cdot BCcosB$. Therefore, he claim is proved.

Now since the quadrilateral $DEAT$ is cyclic and $AA_1 \parallel IX$, we get that the quadrilateral $IATX$ is cyclic. Also, since pairs (E, E_1) and (A, A_1) are symmetric with respect to the perpendicular bisector of the side BC, we have $XE = XE_1$ and so $\angle ATI = \angle AXI = \angle XE_1E = \angle XEE_1 = \angle IXE = \angle TAI$. Thus, $IT = IA$.

3.139 In an isosceles triangle ABC with $AB = BC$, points K and M are the midpoints of the sides AB and AC, respectively. The circumscribed circle of the triangle CKB meets the line BM at point N different from M. The line passing through N aparallel to the side AC meets the circumscribed circle of the triangle ABC at poins $A_1$and C_1.Prove that the triangle A_1BC_1 is equilateral.

V.Karamzin-Belarusian NMO-2017

Solution: Let O be the center of the circumcircle of the triangle ABC. Since $KBCN$ is an inscribed quadrilateral and NB is the bisector of the angle KBC, we have $KN = NC$ (see Fig.1). Moreover, since N lies on the perpendicular bisector of the segment AC, we have $AN = NC$, therefore, N is the circumcenter of the triangle AKC.

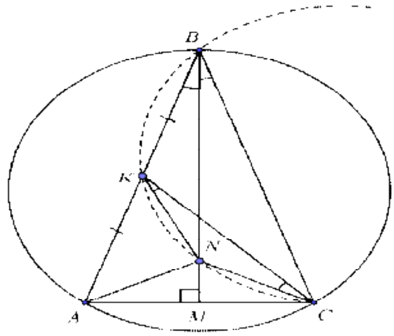


Fig.1

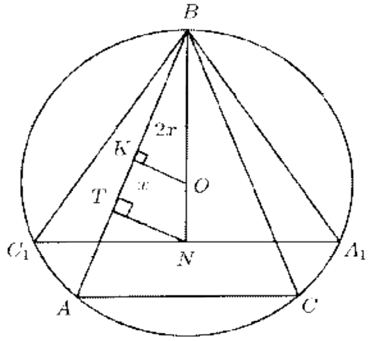


Fig.2

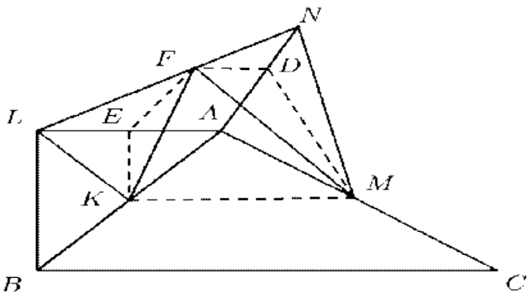
Construct the perpendicular from N and O into line AB (see Fig.2). Since the circumcenter is the intersection point of the perpendicular bisector, we see that K and T are the feet of perpendiculars from O and N, respectively. So, $KT = \frac{1}{2}KA = \frac{1}{2}KB$.It follows that $\frac{BK}{K} = \frac{2}{1}$. Now by the Thales theorem, it follows that $\frac{BO}{ON} = \frac{BK}{KT} = \frac{2}{1}$. Thus, the circumcenter O of the isosceles triangle A_1BC_1 coincides with the point of intersection of the medians, so this triangle is equilateral, as required.

3.140 Points K and M are the midpoints of the sides AB and AC of triangle ABC, respectively. The equilateral triangles AMN and BKL are constructed on the sides AM and BK to the exterior of the triangle ABC. Point F is the midpoint of the segment LN. Find the value of the angle KFM.

S.Mazanik-Belarusian NMO-2017

Solution:

Answer: 90°. *Le points A, B and C lie in the same half-plane with respect to the line LN. Let E and D be the midpoints of the segments AL and AN, respectively (see the Fig.). By condition, the triangle BKL is equilateral and K is the midpoint of the side AB, so $BK = KA = KL$. Therefore, the triangle BLA is a right-angled triangle (the median LK is half as long as the side AB) and $\angle BLA = 90^\circ$. By condition, $\angle KBL = 60^\circ$, so $\angle LAB = 30^\circ$. By condition, the triangle ANM is equilateral, then $\angle NAM = 60^\circ$.*



Therefore: $\angle LAN = 360^\circ - \angle LAK - \angle NAM - \angle KAM = 360^\circ - 30^\circ - 60^\circ - \angle KAM = 279^\circ - \angle KAM$; (1). *Since* FD *and* FE *are the midlines of the triangle* LAN, *we have* $FD \parallel LA$ *and* $FE \parallel NA$, *so* $EFDA$ *is a parallelogram.*

Then: $\angle FDA = \angle FEA = 180^\circ - \angle LAN \overset{(1)}{=} \angle KAM - 90^\circ$; (2)

Since D *is midpoint of the side* AN *of the equilateral triangle* ANM, *we have* $\angle MDA = 90^\circ$. *Now from (2) it follows that:*

$$\angle FDM = \angle FDA + \angle MDA = \angle KAM - 90^\circ + 90^\circ = \angle KAM; \quad (3)$$

In the similar way, we easily find that $\angle FEK = \angle KAM$; (4)

Since $EFDA$ *is a parallelogram, we have*

$$FD = EA = [\angle KEA = 90^\circ; \angle EAK = 30^\circ] = \frac{KA\sqrt{3}}{2}$$

Also $DM = [\angle MDA = 90^\circ; \angle MAD = 60^\circ] = \frac{MA\sqrt{3}}{2}$. *Therefore,* $\frac{FD}{DM} = \frac{KA}{MA}$, *so, taking into account (3), we see that the triangles* FDM *and* KAM *are similar, hence* $\angle MFD = \angle MKA$, $\angle FMD = \angle KMA$. *Thus,*

$$\angle KMF = \angle KMA + \angle AMF = \angle FMD + \angle AMF = \angle AMD = 30^\circ$$

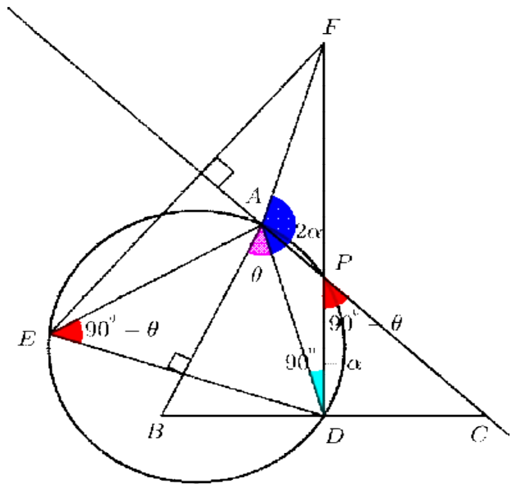
In the same way, we get $\Delta KEF \sim \Delta KAM$, *since*

$$\frac{KE}{EF} = \frac{KE}{AD} = \frac{1}{2}KA : \frac{1}{2}MA = \frac{KA}{MA}, \angle KEF = \angle KAM.$$

So $\angle EKF = \angle AKM$, *and then* $\angle FKM = \angle FKA + \angle AKM = \angle FKA + \angle EKF = \angle EKA = 60^\circ$. *Thus,* $\angle KFM = 180^\circ - \angle KMF - \angle FKM = 180^\circ - 30^\circ - 60^\circ = 90^\circ$. *In the same way we can consider the case when the points* A *and* B, C *lie in the different half-plane with respect to the line* LN.

3.141 Let ABC be a triangle and D a point on the side BC. Point E is the symmetric of D with respect to AB. Point F is the symmetric of E with respect to AC. Point P is the intersection of line DF with line AC. Prove that the quadrilateral $AEDP$ is cyclic.

Solution:



Let $\alpha = \angle BAC$ and $\theta = \angle BAD$. Because E is the symmetric of D with respect to AB, we have $AD = AE$ and DE is perpendicular to AB. We deduce that $\angle EAD = 2\theta$ and $\angle DEA = 90^\circ - \theta$. Because F is symmetric of E with respect to AC, we have $AE = AF$ and EF is perpendicular to AC. We deduce that

$$\angle CAF = \angle EAC = \angle EAB + \angle BAC = \theta + \alpha$$

and

$$\angle PDA = 90^\circ - \frac{1}{2}\angle DAF = 90^\circ - \frac{1}{2}(\angle DAC + \angle CAF) =$$

$$= 90^\circ - \frac{1}{2}\big((\alpha - \theta) + (\alpha + \theta)\big) = 90^\circ - \alpha$$

We deduce that $\angle DPC = \angle PDA + \angle DAC = \big(90^\circ - \alpha\big) + (\alpha - \theta) = 90^\circ - \theta = \angle DEA$. This proves that quadrilateral; $AEDP$ is cyclic.

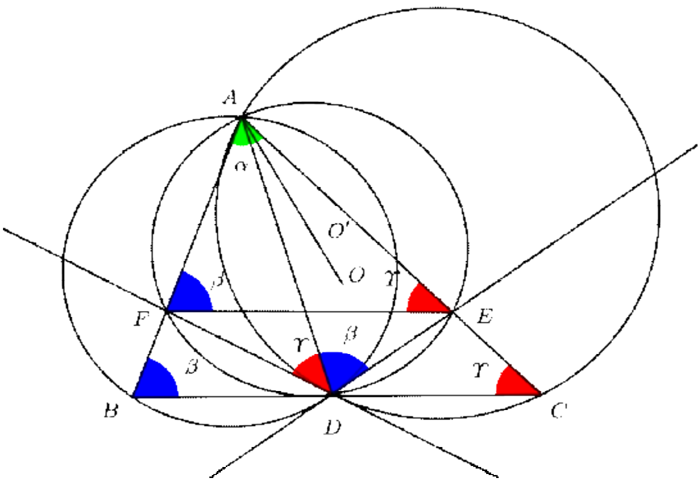
3.142 Let ABC be a triangle and D a point on the side BC. The tangent line to the circumcircle of the triangle ABD at the point D intersect the side AC at E. The tangent line to the circumcircle of the triangle ACD at the point D intersect the side AB at F. Prove that the point A and the circumcenters of the triangles ABC and DEF are collinear.

Malik Talbi-Saudi Arabia NMO-2017

Solution:

Let $\alpha = \angle BAC, \beta = \angle CBA, \gamma = \angle ACB$ and $\theta = \angle BAD$.Because line DE is tangent to the circumcircle of triangle ABD, we have $\angle EDA = \angle DBA = \beta$. Because line DF is tangent to the circumcircle of triangle ADC, we have

$$\angle ADF = \angle ACD = \gamma.$$



Therefore $\angle EDF + \angle FAE = \angle EDA + \angle ADF + \angle BAC = \alpha + \beta + \gamma = 180^\circ$

This proves that quadrilateral $AFDE$ is cyclic and hence $\angle EFA = \angle EDA = \beta$ and $\angle AEF = \angle ADF = \gamma$. This proves that sides EF and BC are parallel. Let O and O' be circumcenters of triangle ABC and AFE, respectively. We have:

$$\angle FAO' = 90^\circ - \angle AEF = 90^\circ - \gamma = \angle BAO.$$

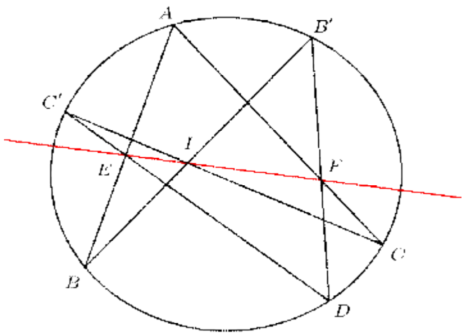
This proves that A, O and O' are collinear.

3.143 Let ABC be a triangle, I its incenter, and D a point on the arc $\widehat{BC}$ of the circumcircle of ABC not containing A. The bisector of the angle $\angle ADB$ intersects the segment AB at E. The bisector of the angle $\angle CDA$ intersects the segment AC at F. Prove that the points E, F, I are collinear.

Malik Talbi-Saudi Arabia NMO-2015

Solution:

Let B' and C' be the midpoints of the arcs $\widehat{CA}$ and $\widehat{AB}$, respectively. Because D and DC' are the bisectors of the angles $\angle ADB$ and $\angle CDA$, respectively, point E is the intersection of AB and DC', point I is the intersection of BB' and CC' and point F is the intersection of $B'D$ and CA. We deduce from Pascal's theorem applied to the cyclic hexagon $ABB'DC'C$ that the three points E, I and F are collinear.

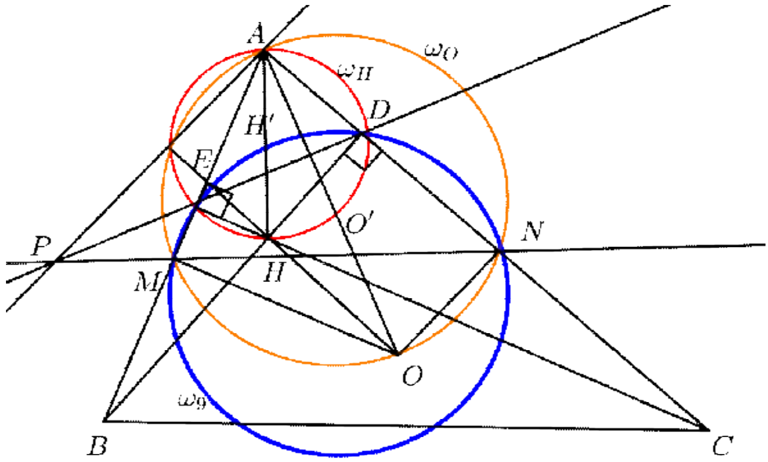


3.144 Let BD and CE be altitudes of an arbitrary scalene triangle ABC with orthocenter H and circumcenter O. Let M and N be the midpoints of sides AB, respectively AC, and P the intersection point of lines MN and DE. Prove that lines AP and OH are perpendicular.

Liana Topan-Saudi Arabia NMO-2015

Solution: *Because $\angle ADH = \angle HEA = 90^\circ$, quadrilateral $AEHD$ is cyclic with AH a diameter of its circumcircle ω_H. Because $\angle ANO = \angle OMA = 90^\circ$, quadrilateral $AMON$ is cyclic with AO a diameter of its circumcircle ω_O.*

Let ω_9 be the nine point circle of the triangle ABC. It is the circumcircle of quadrilateral $DEMN$.



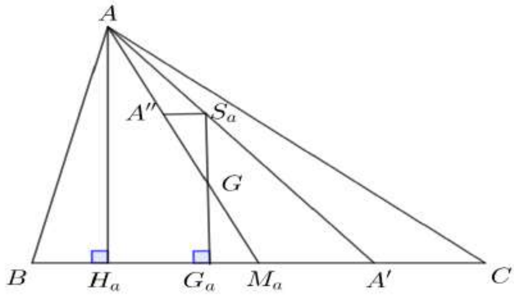
Because the circles ω_H and ω_9 intersect at D and E, the line DE is radical axis of ω_H and ω_9. We deduce that P is the radical center of the circles ω_H, ω_O and ω_9, and therefore, the line AP is the radical axis of the circles ω_H , which is perpendicular to the line $H'O'$, where H' is the center of ω_H and O' the center of ω_O. But H' is the midpoint of AH and O' is the midpoint of AO. We deduce that AP is perpendicular to HO.

3.145 Let Abc be a triangle and G centroid. Let G_a, G_b and G_c be the orthogonal projections of G on sides BC, CA, respectively AB. If S_a, S_b and S_c are the symmetrical points of G_a, G_b respectively G_c with respect to G, prove that AS_a, BS_b and CS_c are concurent.

Liana Topan-Saudi Arabia NMO-2015

Solution:

Let H_a be the foot of altitude from A, M_a the midpoint of side BC, A' the intersection point of line AS_a and side BC and A'' the intersection point of the parallel line to BC passing through S_a with AM_a.



Because GG_a and AH_a are parallel, we have

$$\frac{H_aM_a}{G_aM_a} = \frac{AM_a}{GM_a} = 3$$

Because S_aA'' and BC are parallel, we have

$$\frac{G_aM_a}{S_aA''} = \frac{G_aG}{s_aG} = 1$$

We also have

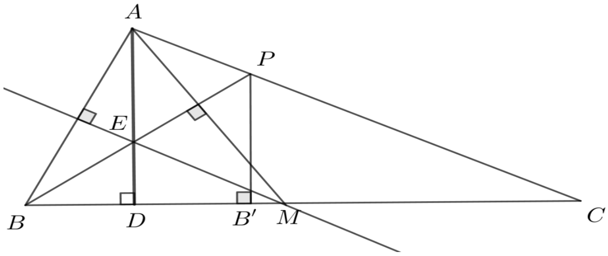
$$\frac{S_aA''}{A'M_a} = \frac{AA''}{AM_a} = 1 - \frac{M_aA''}{AM_a} = 1 - 2\frac{M_aG}{AM_a} = \frac{1}{3}$$

Multiplying these three relations we deduce that $A'M_a = M_aH_a$, which means that AS_a and AH_a are isotomic conjugate. Similarly, BS_b and BH_b are isotomic conjugate and CS_c and CH_c are isotomic conjugate. Since the altitudes of a triangle are concurrent, so are AS_a, BS_b and CS_c.

3.146 Let ABC be a triangle, with $AB < AC$, D the foot of the altitude from A, M the midpoint of BC, and $B'$the symmetric of B with respect to D. The perpendicular line to BC at B' intersects AC at point P. Prove that if BP and AM are perpendicular then triangle ABC is right-angled.

Liana Topan-Saudi Arabia NMO-2015

Solution*: Let E be the intersect point of AD and BP. Because AD is perpendicular to BM and BP is perpendicular to AM, the point E is the orthocentre of triangle ABM and therefore ME is perpendicular to AB.*



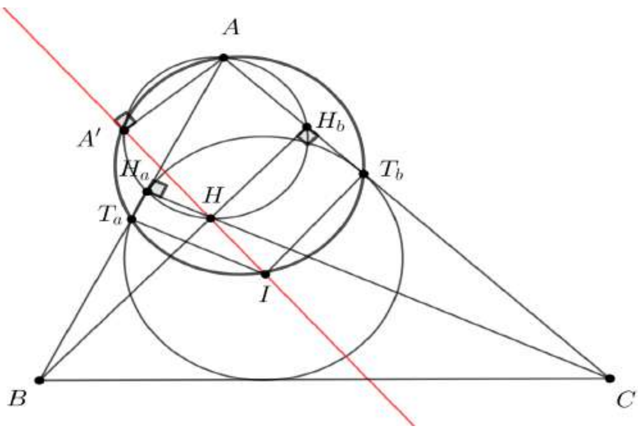
Because point D is the midpoint of the segment BB' and DE and $B'P$ are parallel, we deduce that point E is the midpoint of segment BP. But M is the midpoint of segment BC. We deduce that lines ME and BC are parallel.

It follows that AB and AC are perpendicular and triangle ABC is right-angled.

3.147 Let ABC be a triangle, H_a, H_b and H_c the feet of its altitudes from A, B and C, respectively, T_a, T_b, T_c its intouch points on the sides BC, CA and AB respectively. The circumcircles of triangles AH_bH_c and AT_bT_c intersect again at A'. The circumcircles of triangles BH_cH_a and BT_cT_a intersect again at B'. The circumcircles of triangles CH_aH_b and CT_aT_b intersect again at C'. Prove that the points A', B', C' are collinear.

Malik Talbi-Saudi Arabia TST-2015

Solution:



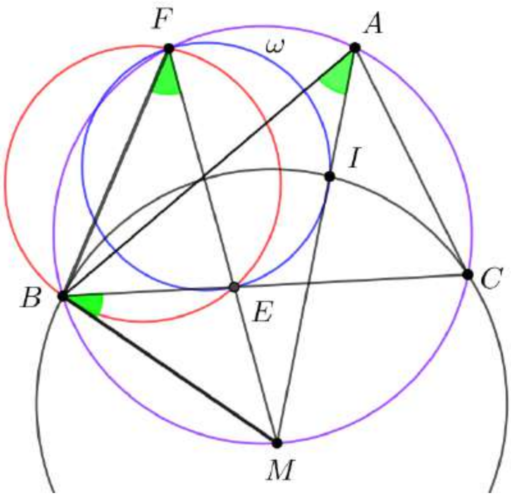
Let H and I be the orthocentre and the incenter of triangle ABC, respectively. Because $\angle AH_bH = \angle HH_cA = 90^\circ$, AH is a diameter of the circumcircle of AH_cHH_b, and therefore $\angle AA'H = 90^\circ$. Because $\angle AT_bI = \angle IT_cA = 90^\circ$, AI is a diameter of the circumcircle of AT_cIT_b and therefore $\angle AA'I = 90^\circ$. We deduce that point A' is on the line HI. Similarly, we prove that the points B' and C' are on the line HI, which prove that the points A', B', C' are collinear.

3.148 Let ABC be a triangle, Γ its circumcircle, I its incenter, and ω a tangent circle to the line AI at I and to the side BC. Prove that the circles Γ and ω are tangent.

Malik Talbi-Saudi Arabia TST-2015

Solution:

Let M be the midpoint of arc BC not containing A, E the tangent point of BC to ω, F the second intersection point of EM with ω. Remember that M is the circumcenter of triangle BCI and therefore $MI = MB$.



Because MI is tangent to ω, we have from the power of the point M with respect to ω: $ME \cdot MF = MI^2 = BM^2$. We deduce that the line BM is tangent to the circumcircle of triangle BEF. Therefore,

$$\angle BFM = \angle MBE = \angle MAC = \angle BAM.$$

Because the tangent line to Γ at M is parallel to the tangent line to ω at E and F, E, M are collinear and F is an intersection point of Γ and ω. F is the center of the homothety of the circles ω and Γ and therefore, they are tangent.

3.149 The picture shows the path of a rabbit that was being chased by a wolf in the fog. The rabbit first ran to the east, then it turned right, after a while, it turned left, and soon after, it turned left again. After the last turn, the rabbit again ran to the east. How many degrees does the angle at the second turn measure?

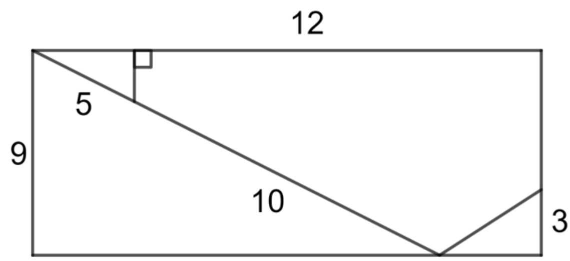
$(\boldsymbol{A})\mathbf{98}^\circ$ $(\boldsymbol{B})\mathbf{96}^\circ$ $(\boldsymbol{C})\mathbf{88}^\circ$ $(\boldsymbol{D})\mathbf{90}^\circ$ $(\boldsymbol{E})\mathbf{92}^\circ$

Slovenia NMO-2013

Solution:

The first and the last section of the rabbit's path are parallel. If we add another line passing through the second turn, we get $\alpha = 44^\circ$ and $\beta = 180^\circ - 132^\circ = 48^\circ$. The angle at the second turn is therefore equal to $\alpha + \beta = 92^\circ$. The correct answer is E.

3.150 A rectangle has been divided into four parts by three line segments, as shown in the picture. After that, the four shapes obtained have been rearranged to from a square. What is the perimeter of this square?

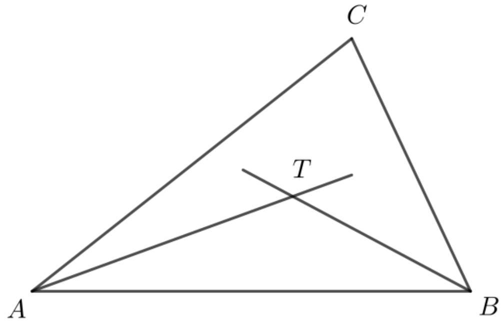


Slovenia NMO-2013

Solution: *Let us the notation suggested in the figure. By Pythagora's theorem we have $y = \sqrt{15^2 - 9^2} = \sqrt{144} = 12$. The two right triangles on the left side are similar, so $\frac{x}{5} = \frac{y}{15}$, or $x = \frac{y}{3} = 4$. Hence, the sides of the rectangle measure 9 and 16, and its area is 144. The area of the square is therefore also equal to 144, so its side has the length 12, and its perimeter is 48.*

3.151 In the triangle ABC, the bisectors of the angles $\angle BAC$ and $\angle CBA$ intersect at the point T. Let γ denote the size of the angle $\angle ACB$. What is the size of the angle $\angle ATB$?

Slovenia NMO-2013

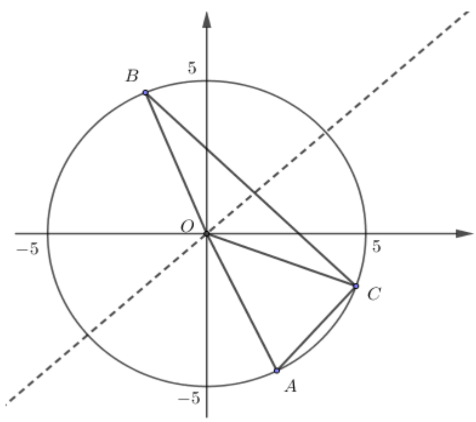


Solution: *We have* $\angle ATB = 180^\circ - \angle BAT - \angle TBA = 180^\circ - \frac{\alpha}{2} - \frac{\beta}{2} = 180^\circ - \frac{1}{2}(\alpha + \beta)$. *Since* $\alpha + \beta + \gamma = 180^\circ$, *we get* $\alpha + \beta = 180^\circ - \gamma$. *So,* $\angle ATB = 180^\circ - \frac{1}{2}(180^\circ - \gamma) = 90^\circ + \frac{\gamma}{2}$. *The correct answer is* E.

3.152 Let O be the origin of the coordinate system. The point $A\left(\frac{5}{2}, -\frac{5\sqrt{3}}{2}\right)$ is rotated around O by 2013π into the point B. The point B ius reflected across the bisector of the odd quadrats into the point C. Find the size of the angle $\angle AOC$.

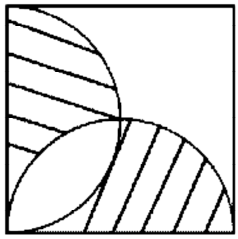
Slovenia NMO-2013

Solution:
The coordinates of the point B are $\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$ *and the coordinates of C are* $\left(\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right)$. *All three points lie on circle with the centre at O and radius 5. From the values of the trigonometric functions* $sin(-30^\circ) = -\frac{1}{2}$ *and* $cos(-30^\circ) = \frac{\sqrt{3}}{2}$ *we conclude*



that the angle between OC and the positive ray of the x –axisis is 30°.
Similarly, $sin(-60^\circ) = -\frac{\sqrt{3}}{2}$ *and* $cos(-60^\circ) = \frac{1}{2}$ *imply that the angle between* OA *and the positive ray of the* x *–axis is* 60°. *From here we conclude that* $\angle AOC = 30^\circ$.

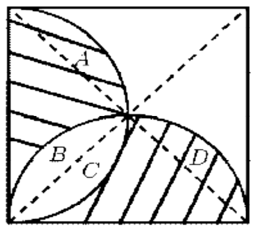
3.153 Into a square with the side of length 2 we draw two semicircles whose diameters are the sides of the square as shown in the figure. What is the area of the unshaded part of the square?



$$(A)\ \frac{\pi}{2}\quad (B)\ 2\quad (C)\ \frac{3}{2}+\frac{\pi}{4}\quad (D)\ \frac{3\pi}{4}-\frac{1}{2}\quad (E)\ \frac{3\pi}{4}$$

Slovenia NMO-2013

Solution: *Adding both diagonals onto the figure we notice that the parts* A, B, C *and* D *have equal area. The area of the unshaded part is therefore equal to one half of the area of the square, i.e.* $\frac{4}{2} = 2$. *The correct answer is* B.



3.154 Let $ABCDEF$ be a regular hexagon, let P be the midpoint of the side AB and let R the midpoint of the side EF, as shown in the figure. What is the ratio of the area of the quadrilateral $APRF$ to the area of the quadrilateral $BCDP$?

Slovenia NMO-2013

Solution: *Denote the length of the side of the regular hexagon by* a. *The quadrilateral* $APRF$ *is a trapezium with the altitude* $\frac{1}{2}\cdot\frac{a\sqrt{3}}{2} = \frac{a\sqrt{3}}{4}$, *so its area* *equals* $P_{APRF} = \frac{a\sqrt{3}}{4} \cdot \frac{|PR|+|AF|}{2} = \frac{a\sqrt{3}\left(\frac{3}{2}a+a\right)}{8} = \frac{5\sqrt{3}a^2}{16}$. *The triangle* PBD *has a side of length* $|BD| = \frac{a}{2}$, *so its area equals*

$P_{PBD} = 2 \cdot \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}a^2}{4}$. *The triangle* BCD *has a side of length* $|BD| = \sqrt{3}a$. *The area of the quadrilateral* $BCDP$ *is*

$$P_{BCDP} = P_{PBD} + P_{BCD} = 2 \cdot \frac{\sqrt{3}a^2}{4} = \frac{\sqrt{3}a^2}{2}.$$

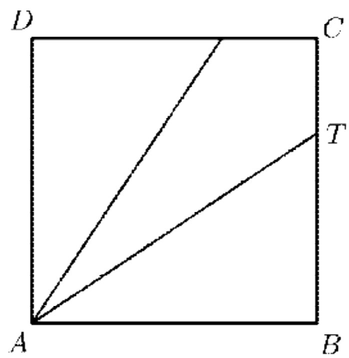
The ratio of the areas is $\frac{P_{APRF}}{P_{BCDP}} = \frac{5}{8}$.

3.155 In the unit square $ABCD$, the two rays from the vertex A divide the right angle in three: the middle angles measures 30° and the other two angles are of the same size. One of the rays intersects the side BC at the point T. Find the length of the line segment BT.

$$(A)\frac{1}{2} \quad (B)\frac{\sqrt{3}}{2} \quad (C)\frac{1}{3} \quad (D)\frac{\sqrt{3}}{3} \quad (E)\frac{\sqrt{2}}{2}$$

Slovenia NMO-2013

Solution:

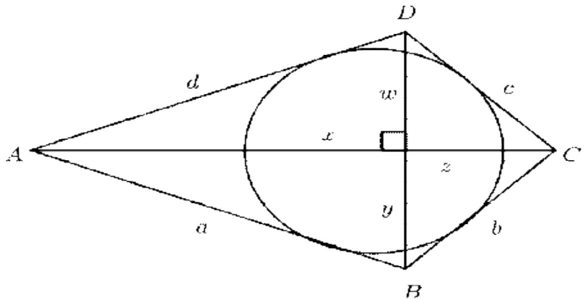


From the given data we determine $\angle BAT = 30^\circ$. *Since* $tan(\angle BAT) = \left|\frac{TB}{AB}\right|$, *we have* $|TB| = |AB|\tan 30^\circ = 1 \cdot \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{3}$. *The correct answer is* D.

3.156 Prove that every tangential quadrilateral whose diagonal intersect at right angles is a deltoid.

Slovenia NMO-2013

Solution:



Let us use the notation from the figure. By Pythagoras' theorem we have

$a^2 = x^2 + y^2,$

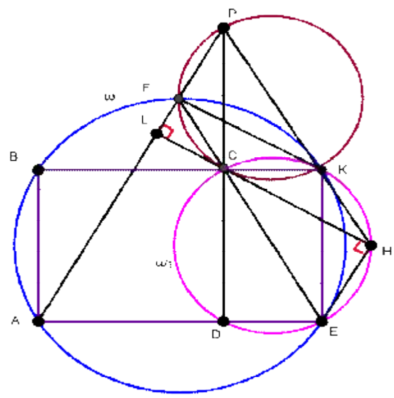
$b^2 = y^2 + z^2, c^2 = z^2 + w^2$ *and* $d^2 = w^2 + x^2$*. This implies* $a^2 + c^2 = b^2 + d^2$*. But the quadrilateral is tangential, so* $a + c = b + d$*. Squaring this equality and using the equalities obtained above we find that* $2ac = 2bd$*, or* $ac = bd$*. Plugging in* $a = b + d - c$ *we get* $c^2 - (b + d)c + bd = 0$*, which can be factored as* $(c - b)(c - d) = 0$*. So,* $c = b$ *and* $a = d$ *or* $c = d$ *and* $a = b$*. In both cases the quadrilateral is a deltoid.*

3.157 A point E lies on the extension of the side AD of the rectangle $ABCD$ over D. The ray EC meets the circumcircle ω of ABE at the point $F \neq E$. The rays DC and AF meet at P. H is the foot of the perpendicular drawn from C to the line l going through E parallel to AF. Prove that the line PH is tangent to ω.

A.S.Kuznetsov-Yakuti-Russian NMO-2017

Solution:

Let the lines CH *and* AF *meet at* L*, then* $\angle ALH = 90^\circ$*. Since* $\angle CDE = \angle CHE = 90^\circ$*, the points* C, H, E, K *belong to some circle* ω_1*. Let* K *be the fourth vertex of a rectangle* $EDCK$*, obviously,* K *belongs to* ω *and* ω_1*. Note that* $\angle FKB = \angle BAF = \angle EPC$ *(the first equality is true because the angles subtend a common arc of* ω*, the second because of parallelity). Therefore* $FPKC$ *is cyclic. Then*



$$\angle FPK = 180^\circ - \angle FCK = 90^\circ - \angle CEK = 90^\circ - \angle CHK.$$

We can rewrite this as $\angle LPK = 90^\circ - \angle LHK$. *This means that the quadrilateral* $LPKH$ *is in fact a right triangle, and the points* P, K, H *are collinear. It remains to prove that* PH *is tangent to* ω *at* K, *which is true because* $\angle FKP = \angle FCP = \angle FEK$.

3.158 Point D is chosen on side BC of tha acute triangle ABC so that $AD = AC$. Let P and Q be respectively the feet of the perpendiculars from C and D to side AB. It is known that

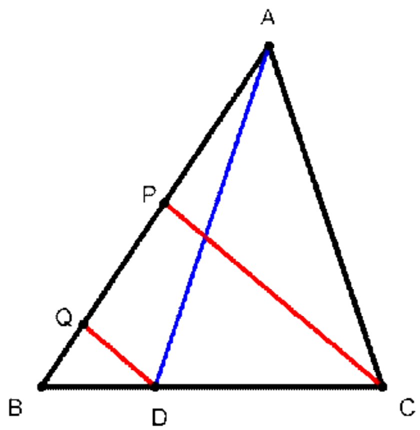
$\boldsymbol{AP^2 + 3BP^2 = AQ^2 + 3BQ^2}$. **Find $\widehat{\mathbf{ABC}}$.**

Argentina NMO-2017

Solution: *Write the condition* $AP^2 + 3BP^2 = AQ^2 + 3BQ^2$ *in the form* $AQ^2 - AP^2 = 3(BP^2 - BQ^2)$. *Express* AQ^2 *and* AP^2 *by Pitagoras theorem for tha right-angled* ADQ *and* ACP:

$AQ^2 = AD^2 - DQ^2, AP^2 = AC^2 - CP^2$. *Since* $AC = AD$, *it follows that* $AQ^2 - AP^2 = CP^2 - DQ^2$. *Likewise the right-angled triangles* BCP *and* BDQ *yield* $BP^2 = BC^2 - CP^2$,

$BQ^2 = BD^2 - DQ^2$. *Hence,*



$BP^2 - BQ^2 = BC^2 - BD^2 - (CP^2 - DQ^2)$.

We showed above that $CP^2 - DQ^2 = AQ^2 - AP^2$; *on the other hand* $AQ^2 - AP^2 = 3(BP^2 - BQ^2)$ *by hypothesis. So the obtained equality can be* *written as* $BP^2 - BQ^2 = BC^2 - BD^2 - 3(BP^2 - BQ^2)$, *which implies* $4(BP^2 - BQ^2) = BC^2 - BD^2$. *Furthermore, we have* $\frac{BC}{BP} = \frac{BD}{BQ} = x$ *with* $x > 0$ *from the similar triangles* BCP *and* BDQ. *Replacing* $BC = xBP$ *and* $BD = xBQ$ *in* $4(BP^2 - BQ^2) = BC^2 - BD^2$ *leads to* $4(BP^2 - BQ^2) = x^2(BP^2 - BQ^2)$. *Because* $BP^2 - BQ^2 \neq 0$ *and* $x > 0$, *it follows that* $x = 2$. *Then* $BD = 2BQ$, *meaning that hypotenuse* BD *of right-angled triangle* BDQ *is twice that its leg* BQ. *Therefore* $\widehat{BDQ} = 30^\circ$ *and so* $\widehat{ABC} = \widehat{DBQ} = 60^\circ$.

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