Inequalities obtained from considerations of convexity / concavity

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An interesting inequality regarding convex / concave functions is highlighted . By particularizing this inequality to different functions, numerous applications are obtained.

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In [1] the following statement was proposed :
If a , b , c ; x , y , z *are strictly positive real numbers and* $n > 1$, *then :*

$$(ax+by+cz)^n+(ay+bz+cx)^n+(az+bx+cy)^n \le (a+b+c)^n \cdot (x^n+y^n+z^n) \ . \qquad (1)$$

In solving inequality (1), as in the following Lemma, an essential role was played by *Jensen's weighted inequality* , which we recall in the statement :

1. *Theorem* (***Jensen's weighted inequality***)

If $f : \mathbf{I} \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a *convex function* , I an interval , then ,

$$\sum_{k=1}^{n} w_k f(x_k) \ge f\left(\sum_{k=1}^{n} w_k x_k\right) \ , \qquad (\mathrm{J})$$

where , $n \in \mathbb{N}^*$, $w_k > 0$, $x_k \in \mathbf{I}$, $\sum_{k=1}^{n} w_k x_k \in \mathbf{I}$, $\sum_{k=1}^{n} w_k = 1$.

If f is a o *concave function* on $\mathbf{I}$, the inequality sign in (J) is reversed .
The equality in (J) occurs if and only if $x_1 = x_2 = \cdots = x_n$, or when the function f is a *linear (affine)* function .
Starting from this famous inequality, we will obtain a new inequality for *convex functions* , respectively *concave functions* .

2. *Lemma*

If $f : \mathbf{I} \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a *convex function* and $p_k > 0$, $\sum_{k=1}^{n} p_k = 1$, then ,

$$f(p_1x_1+p_2x_2+\ldots+p_nx_n)+f(p_1x_2+p_2x_3+\ldots+p_nx_1)+\ldots+f(p_1x_n+p_2x_1+\ldots+p_nx_{n-1}) \le$$
$$\le f(x_1)+f(x_2)+\ldots+f(x_n) \ . \qquad (2)$$

If the function f is *concave* on $\mathbf{I}$, then in inequality (2) the inequality sign is inverted .

Proof

Using *Jensen's weighted inequality* for convex function , we have :

$$f\left(p_1x_1+p_2x_2+\ldots+p_nx_n\right)\le p_1f\left(x_1\right)+p_2f\left(x_2\right)+\ldots+p_nf\left(x_n\right) \qquad (3_1)$$

$$f\left(p_1x_2+p_2x_3+\ldots+p_nx_1\right)\le p_1f\left(x_2\right)+p_2f\left(x_3\right)+\ldots+p_nf\left(x_1\right) \qquad (3_2)$$

$$\vdots \qquad\qquad \vdots$$

$$f\left(p_1x_n+p_2x_1+\ldots+p_nx_{n-1}\right)\le p_1f\left(x_n\right)+p_2f\left(x_1\right)+\ldots+p_nf\left(x_{n-1}\right) \cdot \qquad (3_n)$$

Adding the relations (3_1) , (3_2), . . . , (3_n), grouping and using the condition relationship , $\sum_{k=1}^{n} p_k = 1$ we obtain the inequality from *Lemma*'s statement .

Starting from this *Lemma* , by particularizations of the function f , numerous other inequalities are obtained .

3. *Proposition*

If $a_1 , a_2 , \cdots , a_n$; $x_1 , x_2 , \cdots , x_n$ *are strictly positive real numbers , and* $m > 1$, $n \in \mathbb{N}^*$, then the following inequality occurs :

$$\left(a_1x_1+a_2x_2+\ldots+a_nx_n\right)^m+\left(a_1x_2+a_2x_3+\ldots+a_nx_1\right)^m+\ldots+\left(a_1x_n+a_2x_1+\ldots+a_nx_{n-1}\right)^m\le$$
$$\le\left(a_1+a_2+\ldots+a_n\right)^m\cdot\left(x_1^m+x_2^m+\ldots+x_n^m\right) \cdot \qquad \mathbf{(4)}$$

If $0 < m < 1$ the inequality in relation (4) is inverted .

Proof

We consider the function $f:(0,\infty)\longrightarrow(0,\infty)$, $f(x)= x^m$, $m > 1$, - obviously *convex* on $(0,\infty)$ and the weights :

$$p_1=\frac{a_1}{a_1+a_2+\ldots+a_n} \;,\; p_2=\frac{a_2}{a_1+a_2+\ldots+a_n} \;,\cdots,\; p_n=\frac{a_n}{a_1+a_2+\ldots+a_n} \;,$$

for which we obvious have $p_1 + p_2 + \ldots + p_n = 1$.

With these in *Lemma*'s inequality (2) , we get :

$$\left(\frac{a_1}{a_1+a_2+\ldots+a_n}\cdot x_1+\frac{a_2}{a_1+a_2+\ldots+a_n}\cdot x_2+\ldots+\frac{a_n}{a_1+a_2+\ldots+a_n}\cdot x_n\right)^m+$$
$$+\left(\frac{a_1}{a_1+a_2+\ldots+a_n}\cdot x_2+\frac{a_2}{a_1+a_2+\ldots+a_n}\cdot x_3+\ldots+\frac{a_n}{a_1+a_2+\ldots+a_n}\cdot x_1\right)^m+$$
$$+$$
$$\vdots$$
$$+\left(\frac{a_1}{a_1+a_2+\ldots+a_n}\cdot x_n+\frac{a_2}{a_1+a_2+\ldots+a_n}\cdot x_1+\ldots+\frac{a_n}{a_1+a_2+\ldots+a_n}\cdot x_{n-1}\right)^m\le$$
$$\le x_1^m+x_2^m+\ldots+x_n^m \;,$$

where does the inequality in the statement come from.

Equality occurs when $x_1 = x_2 = \cdots = x_n$.

If $0 < m < 1$, the function f is concave and we will apply inequality (2) with the opposite sense .

For example, if in (4) we operate the substitution $m \to 1/m$, we obtain :

$$\sqrt[m]{a_1x_1+a_2x_2+\ldots+a_nx_n}+\sqrt[m]{a_1x_2+a_2x_3+\ldots+a_nx_1}+\ldots+\sqrt[m]{a_1x_n+a_2x_1+\ldots+a_nx_{n-1}} \geq$$

$$\geq \sqrt[m]{a_1+a_2+\ldots+a_n}\cdot\left(\sqrt[m]{x_1}+\sqrt[m]{x_2}+\ldots+\sqrt[m]{x_n}\right). \qquad (4')$$

4. *Remark* Inequality (4) is a generalization of inequality (1) .

For positive real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ we will use for their *arithmetic mean* the notation

$$\mathbf{A}_n[\alpha_1,\alpha_2,\ldots,\alpha_n] := \frac{\alpha_1+\alpha_2+\ldots+\alpha_n}{n}. \qquad (5)$$

With this notation , we can reformulate the result from *Proposition* 3 in the language of means :

5. *Corollary*

If $a_1, a_2, \cdots, a_n$; $x_1, x_2, \cdots, x_n$ *are strictly positive real numbers , and* $m > 1$, $n \in \mathbb{N}^*$, then the following inequality occurs:

$$\left(\mathbf{A}_n[a_1x_1,a_2x_2,\cdots,a_nx_n]\right)^m+\left(\mathbf{A}_n[a_1x_2,a_2x_3,\cdots,a_nx_1]\right)^m+\ldots+\left(\mathbf{A}_n[a_1x_n,a_2x_1,\cdots,a_nx_{n-1}]\right)^m \leq$$

$$\leq \left(\mathbf{A}_n[a_1,a_2,\cdots,a_n]\right)^m\cdot\left(x_1^m+x_2^m+\ldots+x_n^m\right). \qquad (6)$$

Proof

Everything results from relation (4) , by dividing by n^m and recognizing the respective *arithmetic means* , in accordance with the notation (5) .

If we also consider the *power-mean* (or *generalized mean* , or *Hölder mean*) of positive real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$, noted and defined as follows ,

$$\mathbf{M}_n^{(m)}[x_1,x_2,\cdots,x_n] := \left(\frac{x_1^m+x_2^m+\ldots+x_n^m}{n}\right)^{\frac{1}{m}}, \qquad (7)$$

then we will have another reformulation in the language of means :

6. *Corollary*

If $a_1, a_2, \cdots, a_n$; $x_1, x_2, \cdots, x_n$ *are strictly positive real numbers , and* $m > 1$, $n \in \mathbb{N}^*$, then the following inequality holds :

$$\mathbf{M}_n^{(m)}\left[\mathbf{A}_n[a_1x_1,a_2x_2,\cdots,a_nx_n],\mathbf{A}_n[a_1x_2,a_2x_3,\cdots,a_nx_1],\ldots,\mathbf{A}_n[a_1x_n,a_2x_1,\cdots,a_nx_{n-1}]\right] \leq$$

$$\leq \mathbf{A}_n[a_1,a_2,\cdots,a_n]\cdot\mathbf{M}_n^{(m)}[x_1,x_2,\cdots,x_n]. \qquad (8)$$

Proof

The inequality results from relation (6), by dividing by n and raising to to the power $1/m$.

7. *Proposition* , [3]

If $a_1 , a_2 , \cdots , a_n ; x_1 , x_2 , \cdots , x_n$ *are strictly positive real numbers* , then the following inequality holds :

$$\frac{1}{a_1x_1+a_2x_2+\ldots+a_nx_n}+\frac{1}{a_1x_2+a_2x_3+\ldots+a_nx_1}+\ldots+\frac{1}{a_1x_n+a_2x_1+\ldots+a_nx_{n-1}}\le$$
$$\le\frac{1}{a_1+a_2+\ldots+a_n}\cdot\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right). \qquad \textbf{(9)}$$

Proof

The function $f:(0,\infty)\longrightarrow(0,\infty)$, $f(x)=\frac{1}{x}$, is a convex function pe $(0,\infty)$, so applying the inequality from *Lemma* , in the form ,

$$\frac{1}{p_1x_1+p_2x_2+\ldots+p_nx_n}+\frac{1}{p_1x_2+p_2x_3+\ldots+p_nx_1}+\ldots+\frac{1}{p_1x_n+p_2x_1+\ldots+p_nx_{n-1}}\le$$
$$\le\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}.$$

with the weights ,

$$p_1=\frac{a_1}{a_1+a_2+\ldots+a_n}\ ,\ p_2=\frac{a_2}{a_1+a_2+\ldots+a_n}\ ,\cdots,\ p_n=\frac{a_n}{a_1+a_2+\ldots+a_n}\ ,$$

(for which we obviously have $\sum_{k=1}^{n}p_k=1$) , we immediately obtain the inequality from the statement.

Also, here we can give a description in the language of means, now using the notation :

$$\mathbf{H}_n[x_1,x_2,\ldots,x_n]:=\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}}\ , \qquad \textbf{(10)}$$

for the *harmonic mean* of positive real numbers $x_1 , x_2 , \cdots , x_n$.

8. *Corollary*

If $a_1 , a_2 , \cdots , a_n$; $x_1 , x_2 , \cdots , x_n$ *are strictly positive real numbers* , then the following inequalities hold :

a)

$$\frac{1}{\mathbf{A}_n[a_1x_1,a_2x_2,\ldots,a_nx_n]}+\frac{1}{\mathbf{A}_n[a_1x_2,a_2x_3,\ldots,a_nx_1]}+\ldots+\frac{1}{\mathbf{A}_n[a_1x_n,a_2x_1,\ldots,a_nx_{n-1}]}\le$$
$$\le n\cdot\frac{1}{\mathbf{A}_n[a_1,a_2,\ldots,a_n]}\cdot\frac{1}{\mathbf{H}_n[x_1,x_2,\ldots,x_n]}\ , \qquad \textbf{(11)}$$

b)

$$\mathbf{H}_n\big[\mathbf{A}_n[a_1x_1,a_2x_2,\ldots,a_nx_n],\mathbf{A}_n[a_1x_2,a_2x_3,\ldots,a_nx_1],\ldots,\mathbf{A}_n[a_1x_n,a_2x_1,\ldots,a_nx_{n-1}]\big]\ge$$
$$\ge\mathbf{A}_n[a_1,a_2,\ldots,a_n]\cdot\mathbf{H}_n[x_1,x_2,\ldots,x_n]. \qquad \textbf{(12)}$$

Proof

a) results from *Proposition* 7 in accordance with notations (5) and (10).

b) Rewriting ***a)*** in the form

$$\frac{n}{\mathbf{H}_n\left[\mathbf{A}_n[a_1x_1, a_2x_2, \ldots, a_nx_n], \mathbf{A}_n[a_1x_2, a_2x_3, \ldots, a_nx_1], \ldots, \mathbf{A}_n\left[a_1x_n, a_2x_1, \ldots, a_nx_{n-1}\right]\right]} \leq$$

$$\leq n \cdot \frac{1}{\mathbf{A}_n[a_1, a_2, \ldots, a_n]} \cdot \frac{1}{\mathbf{H}_n[x_1, x_2, \ldots, x_n]} ,$$

immediately yields the result from the statement .

9. *Proposition*

If $a_1, a_2, \cdots, a_n$; $x_1, x_2, \cdots, x_n$ *are strictly positive real numbers* , then the following inequality holds :

$$(a_1x_1 + a_2x_2 + \ldots + a_nx_n) \cdot (a_1x_2 + a_2x_3 + \ldots + a_nx_1) \cdot \ldots \cdot (a_1x_n + a_2x_1 + \ldots + a_nx_{n-1}) \geq$$

$$\geq (a_1 + a_2 + \ldots + a_n)^n \cdot x_1x_2 \ldots x_n . \qquad \textbf{(13)}$$

Proof

Function $f : (0, \infty) \longrightarrow (0, \infty)$, $f(x) = \ln x$ is a *concave* function on $(0, \infty)$, so applying the inequality from *Lemma* , but with the inequality sign reversed, and with the *weights* : $p_1 = \frac{a_1}{\sum_{k=1}^{n} a_k}$, $p_2 = \frac{a_2}{\sum_{k=1}^{n} a_k}$, $\cdots$, $p_n = \frac{a_n}{\sum_{k=1}^{n} a_k}$, we will obtain :

$$\ln\left(\frac{a_1}{\sum_{k=1}^{n} a_k} x_1 + \frac{a_2}{\sum_{k=1}^{n} a_k} x_2 + \ldots + \frac{a_n}{\sum_{k=1}^{n} a_k} x_n\right) + \ln\left(\frac{a_1}{\sum_{k=1}^{n} a_k} x_2 + \frac{a_2}{\sum_{k=1}^{n} a_k} x_3 + \ldots + \frac{a_n}{\sum_{k=1}^{n} a_k} x_1\right) + \ldots +$$

$$+ \ln\left(\frac{a_1}{\sum_{k=1}^{n} a_k} x_n + \frac{a_2}{\sum_{k=1}^{n} a_k} x_1 + \ldots + \frac{a_n}{\sum_{k=1}^{n} a_k} x_{n-1}\right) \geq \ln x_1 + \ln x_2 + \ldots + \ln x_n \Leftrightarrow$$

$$\Leftrightarrow \quad \ln\left(\frac{a_1x_1 + a_2x_2 + \ldots + a_nx_n}{\sum_{k=1}^{n} a_k} \cdot \frac{a_1x_2 + a_2x_3 + \ldots + a_nx_1}{\sum_{k=1}^{n} a_k} \cdot \ldots \cdot \frac{a_1x_n + a_2x_1 + \ldots + a_nx_{n-1}}{\sum_{k=1}^{n} a_k}\right) \geq$$

$$\geq \ln(x_1x_2 \ldots x_n) ,$$

from which the inequality from the statement results .

Here too we can convert the result from the statement by reformulating it in the language of *means* .

For this, let us also recall the *geometric mean* of positive real numbers. $\alpha_1, \alpha_2, \cdots, \alpha_n$ with notation and definition , $\mathbf{G}_n[\alpha_1, \alpha_2, \ldots, \alpha_n] := \sqrt[n]{\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n}$. **(14)**

10. *Corollary*

If $a_1, a_2, \cdots, a_n$; $x_1, x_2, \cdots, x_n$ are strictly positive real numbers , then the following inequalities hold :

a)

$$\mathbf{G}_n\left[a_1x_1+a_2x_2+\ldots+a_nx_n,\ a_1x_2+a_2x_3+\ldots+a_nx_1,\ldots,\ a_1x_n+a_2x_1+\ldots+a_nx_{n-1}\right]\geq$$

$$\geq n\cdot\mathbf{A}_n\left[a_1,\ a_2,\ \ldots,a_n\right]\cdot\mathbf{G}_n\left[x_1,\ x_2,\ \ldots,x_n\right]\cdot \quad \mathbf{(15)}$$

b)

$$\mathbf{G}_n\left[\mathbf{A}_n\left[a_1x_1,a_2x_2,\ldots,a_nx_n\right],\mathbf{A}_n\left[a_1x_2,a_2x_3,\ldots,a_nx_1\right],\ldots,\mathbf{A}_n\left[a_1x_n,a_2x_1,\ldots,a_nx_{n-1}\right]\right]\geq$$

$$\geq \mathbf{A}_n\left[a_1,\ a_2,\ \ldots,a_n\right]\cdot\mathbf{G}_n\left[x_1,\ x_2,\ \ldots,x_n\right]\cdot \quad \mathbf{(16)}$$

a*) , *b*)** The ***Proof

follows from Proposition 9 in agreement with notations (5) and (14).

11. *Remark*

As a matter of fact , we also have ,

$$\mathbf{G}_n\left[a_1x_1+a_2x_2+\ldots+a_nx_n,\ a_1x_2+a_2x_3+\ldots+a_nx_1,\ldots,\ a_1x_n+a_2x_1+\ldots+a_nx_{n-1}\right]\leq$$

$$\leq\mathbf{A}_n\left[a_1x_1+a_2x_2+\ldots+a_nx_n,\ a_1x_2+a_2x_3+\ldots+a_nx_1,\ldots,\ a_1x_n+a_2x_1+\ldots+a_nx_{n-1}\right]=$$

$$=\frac{\left(a_1x_1+a_2x_2+\ldots+a_nx_n\right)+\left(a_1x_2+a_2x_3+\ldots+a_nx_1\right)+\ldots+\left(a_1x_n+a_2x_1+\ldots+a_nx_{n-1}\right)}{n}=$$

$$=\frac{\left(a_1+a_2+\ldots+a_n\right)\cdot\left(x_1+x_2+\ldots+x_n\right)}{n}=n\cdot\mathbf{A}_n\left[a_1,\ a_2,\ \ldots,a_n\right]\cdot\mathbf{A}_n\left[x_1,\ x_2,\ \ldots,x_n\right],$$

it turns out that we have even the very next *refinement* of the inequality of means ,

12. ***Corollary*** **(*O rafinare a inegalitatii GM–AM*)**

If $a_1, a_2, \cdots, a_n\ ; x_1, x_2, \cdots, x_n$ are strictly positive real numbers , then the following inequality holds :

$$\mathbf{G}_n\left[x_1,\ x_2,\ \ldots,x_n\right]\leq$$

$$\leq\frac{\mathbf{G}_n\left[a_1x_1+a_2x_2+\ldots+a_nx_n,\ a_1x_2+a_2x_3+\ldots+a_nx_1,\ldots,\ a_1x_n+a_2x_1+\ldots+a_nx_{n-1}\right]}{n\cdot\mathbf{A}_n\left[a_1,\ a_2,\ \ldots,a_n\right]}\leq$$

$$\leq\mathbf{A}_n\left[x_1,\ x_2,\ \ldots,x_n\right]\cdot \quad \mathbf{(17)}$$

13. *Proposition*

If $a_1, a_2, \cdots, a_n\ ;\ x_1, x_2, \cdots, x_n$ are strictly positive real numbers , with the notations : $\mathbf{A}:=a_1+a_2+\cdots+a_n$ and $\mathbf{X}:=x_1+x_2+\cdots+x_n$,then the following inequality holds :

$$\left(a_1x_1+a_2x_2+\ldots+a_nx_n\right)^{a_1x_1+a_2x_2+\ldots+a_nx_n}\cdot\left(a_1x_2+a_2x_3+\ldots+a_nx_1\right)^{a_1x_2+a_2x_3+\ldots+a_nx_1}\cdot\ldots$$

$$\ldots\cdot\left(a_1x_n+a_2x_1+\ldots+a_nx_{n-1}\right)^{a_1x_n+a_2x_1+\ldots+a_nx_{n-1}}\leq\mathbf{A}^{\mathbf{AX}}\cdot\left(x_1^{x_1}x_2^{x_2}\ldots x_n^{x_n}\right)^{\mathbf{A}}\cdot \quad \mathbf{(18)}$$

Proof

The function $f:(0,\infty)\longrightarrow(0,\infty),\ f(x)=x\cdot\ln x$ is a convex function on $(0,\infty)$, so applying the inequality in *Lemma* , we obtain in a first instance:

$$\left(p_1x_1+p_2x_2+\ldots+p_nx_n\right)^{p_1x_1+p_2x_2+\ldots+p_nx_n}\cdot\left(p_1x_2+p_2x_3+\ldots+p_nx_1\right)^{p_1x_2+p_2x_3+\ldots+p_nx_1}\cdot\ldots$$
$$\ldots\cdot\left(p_1x_n+p_2x_1+\ldots+p_nx_{n-1}\right)^{p_1x_n+p_2x_1+\ldots+p_nx_{n-1}}\le x_1^{x_1}x_2^{x_2}\ldots x_n^{x_n}\,. \qquad \mathbf{(19)}$$

Then taking the weights : $p_1=\dfrac{a_1}{\sum_{k=1}^{n}a_k}=\dfrac{a_1}{A}$, $p_2=\dfrac{a_2}{\sum_{k=1}^{n}a_k}=\dfrac{a_2}{A}$ $,\cdots,$ $p_n=\dfrac{a_n}{\sum_{k=1}^{n}a_k}=\dfrac{a_n}{A}$,

after some routine calculations , the relationship in the statement is obtained .

14. *Proposition*

If $a_1, a_2, \cdots, a_n$; k ; $x_1, x_2, \cdots, x_n$ are strictly positive real numbers , with notation , $A := a_1+a_2+\cdots+a_n$, then the following inequality holds :

$$k^{\frac{a_1x_1+a_2x_2+\ldots+a_nx_n}{A}}+k^{\frac{a_1x_2+a_2x_3+\ldots+a_nx_1}{A}}+\ldots+k^{\frac{a_1x_n+a_2x_1+\ldots+a_nx_{n-1}}{A}}\le k^{x_1}+k^{x_2}+\ldots+k^{x_n}\,. \qquad \mathbf{(20)}$$

Proof

Function $f:(0,\infty)\longrightarrow(0,\infty)$, $f(x)=k^x$, $k>0$, is a convex function on $(0,\infty)$, so applying the inequality from *Lemma* , with the weights :

$$p_1=\frac{a_1}{\sum_{k=1}^{n}a_k}=\frac{a_1}{A}\,,\ p_2=\frac{a_2}{\sum_{k=1}^{n}a_k}=\frac{a_2}{A}\ ,\cdots,\ p_n=\frac{a_n}{\sum_{k=1}^{n}a_k}=\frac{a_n}{A}\ ,$$

after a few simple calculations , the relationship in the statement is obtained .

15. *Proposition*

If $a_1, a_2, \cdots, a_n$; $x_1, x_2, \cdots, x_n$ are strictly positive real numbers , with notation , $A := a_1+a_2+\cdots+a_n$, then the following inequality holds :

$$\sin\frac{a_1x_1+a_2x_2+\ldots+a_nx_n}{A}+\sin\frac{a_1x_2+a_2x_3+\ldots+a_nx_1}{A}+\ldots+\sin\frac{a_1x_n+a_2x_1+\ldots+a_nx_{n-1}}{A}\ge$$
$$\ge\sin x_1+\sin x_2+\ldots+\sin x_n \qquad \mathbf{(21)}$$

Proof

Function $f:[0,\pi]\longrightarrow\mathbb{R}$, $f(x)=\sin x$ is a concave function on $[0,\pi]$, so applying the inequality in *Lemma* , with the weights :

$$p_1=\frac{a_1}{\sum_{k=1}^{n}a_k}=\frac{a_1}{A}\,,\ p_2=\frac{a_2}{\sum_{k=1}^{n}a_k}=\frac{a_2}{A}\ ,\cdots,\ p_n=\frac{a_n}{\sum_{k=1}^{n}a_k}=\frac{a_n}{A}\ ,$$

for which we obviously have $\sum_{k=1}^{n}p_k=1$, the inequality from the statement is obtained .

Let's note that it makes sense to write $\sin\dfrac{a_1x_{\sigma(1)}+a_2x_{\sigma(2)}+\ldots+a_nx_{\sigma(n)}}{A}$, where σ is a a *permutation* of order n . because if $0<x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}<\pi$, then we have too ,

$$0<\frac{a_1x_{\sigma(1)}+a_2x_{\sigma(2)}+\ldots+a_nx_{\sigma(n)}}{A}<\frac{a_1\pi+a_2\pi+\ldots+a_n\pi}{A}=\frac{(a_1+a_2+\ldots+a_n)\pi}{A}=\pi\,.$$

Analogously, if we consider the function $f : [0, \pi/2] \longrightarrow \mathbb{R}$, $f(x) = \cos x$ which is also a *concave function* on $[0, \pi/2]$, is obtained in a similar way ,

15. *Propoziţie*

If $a_1, a_2, \cdots, a_n$; $x_1, x_2, \cdots, x_n$ are strictly positive real numbers , with notation , $A := a_1 + a_2 + \cdots + a_n$, then the following inequality holds :

$$\cos\frac{a_1x_1+a_2x_2+\ldots+a_nx_n}{A}+\cos\frac{a_1x_2+a_2x_3+\ldots+a_nx_1}{A}+\ldots+\cos\frac{a_1x_n+a_2x_1+\ldots+a_nx_{n-1}}{A} \geq$$
$$\geq \cos x_1+\cos x_2+\ldots+\cos x_n \qquad (22)$$

Numerous other inequalities can be obtained by conveniently choosing convex or concave functions – to which *Lemma* 2 is applied.

References :

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