MATH PHENOMENON RELOADED

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PROBLEMS

ALGEBRA

PROBLEM A.001.

If $x, y, z \in (0,1); x^6+y^6+z^6=\frac{1}{9}$ then:

$$\left(\frac{2}{1-x^2}\right)^6+\left(\frac{2}{1-y^2}\right)^6+\left(\frac{2}{1-z^2}\right)^6 \geq 3^7$$

PROBLEM A.002.

If $x, y, z > 0; \frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}=1$ then:

$$\sum_{cyc(x,y,z)} \frac{y^3+z^3+1}{x^3} \geq 3xyz$$

PROBLEM A.003.

If $x=\frac{(a+b+c+d)^4}{256abcd}; y=\frac{(a+b+c)^3}{27abc}; z=\frac{(a+b)^2}{4ab}; a,b,c,d \geq 1$ then:

$$ab(1+c+cd)(x+y+z) \leq 3(abcdx+abcdy+abz)$$

PROBLEM A.004.

If $A, B \in M_n(\mathbb{R}); n \geq 2; p \geq 1; n, p \in \mathbb{N}$

$$A^{2p+1}+B^{2p}=I_n; A^{4p+1}=A^{2p} \text{ then: } \det(I_n+A^{2p}+B^{2p}) \geq 0.$$

PROBLEM A.005.

If $x, y, z > 0; xyz=9$ then:

$$\sqrt{z}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2+\sqrt{x}\left(\frac{y+z}{\sqrt{y}+\sqrt{z}}\right)^2+\sqrt{y}\left(\frac{z+x}{\sqrt{z}+\sqrt{x}}\right)^2 \geq 9$$

PROBLEM A.006.

If $x, y, z > 0; x^4+y^4+z^4=x^2y^2z^2$ then:

$$\left(\frac{zx^2+zy^2}{x^4+y^4}\right)^2+\left(\frac{xy^2+xz^2}{y^4+z^4}\right)^2+\left(\frac{yz^2+zx^2}{z^4+x^4}\right)^2 \leq 1$$

PROBLEM A.007.

If $a, b, c, d \geq 0; p \geq q \geq r \geq 0$

$$x=\frac{a+b+c+d}{4}-\sqrt[4]{abcd}; y=\frac{a+b+c}{3}-\sqrt[3]{abc}; z=\frac{a+b}{2}-\sqrt{ab} \text{ then:}$$

$$3(4px+3qy+2rz) \geq (4x+3y+2z)(p+q+r)$$

PROBLEM A.008.

If $a,b,c>0; a+b+c\leq 1$ then:

$$\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}+1-a-b-c \geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1-a-b-c\right)^2$$

PROBLEM A.009.

If $a\geq b\geq c\geq 0$ then:

$$\sqrt{a^2-b^2}+\sqrt{b^2-c^2}+\sqrt{a^2-c^2}+\sqrt{2}(a+b+c) \geq$$
$$\geq \sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{a^2+c^2}$$

PROBLEM A.010.

If $a,b,c\in(0,1); 2(a+b+c)=3$ then:

$$\sum (3+(\log_a c)^4)\left(3+\frac{1}{(a+b)^4}\right) \geq 48$$

PROBLEM A.011. If $x,y,z,t>0$ then:

$$\sum \frac{yzt}{\left(\sqrt[3]{ztx}+\sqrt[3]{txy}+\sqrt[3]{yxz}\right)^3} \geq \frac{4}{27}$$

PROBLEM A.012.

If $1\leq x<y$ then:

$$\frac{(y^5-x^5)(y^7-x^7)(y^9-x^9)}{(y^6-x^6)(y^8-x^8)(y^{10}-x^{10})} < \frac{21}{32}$$

PROBLEM A.013.

If $x,y,z,t\geq 1$ then:

$$(\ln xy)(\ln^2 x+\ln^2 y-\ln x\ln y-\ln z\ln t) \geq$$
$$\geq (\ln zt)(\ln x\ln y+\ln z\ln t-\ln^2 z-\ln^2 t)$$

PROBLEM A.014.

If $a,b,c>1; ab+bc+ca=abc$ then:

$$abc^c+bca^a+cab^b \geq a^2b^2c^2$$

PROBLEM A.015.

Let be the sequence: $3,8,13,18,23,28,\dots$

Find three different terms of the sequence with same sum of digits.

PROBLEM A.016.

Solve for real numbers:

$$\frac{1}{1+m^{3x}}+\frac{1}{1+n^{3x}}+\frac{1}{1+p^{3x}}=\frac{1}{1+(mnp)^{x}}$$

where $m,n,p\in\mathbb{N}; m,n,p\geq 3$ are different in pairs.

PROBLEM A.017.

If $a,b,c>0$ then:

$$(a+b+c)\left(\frac{a}{b^{10}}+\frac{b}{c^{10}}+\frac{c}{a^{10}}\right)\geq\left(\frac{a}{b^{5}}+\frac{b}{c^{5}}+\frac{c}{a^{5}}\right)^{2}$$

PROBLEM A.018.

If $a,b,c,d>0$ are different in pairs then:

$$\sum_{cyc(a,b,c,d)}\frac{a^3}{(b+c+d)(a-b)(a-c)(a-d)}<\frac{(a+b+c+d)^3}{81abcd}$$

PROBLEM A.019.

If $a,b,c>0$ then:

$$2(a^2+b^2+c^2+a^3+b^3+c^3)\leq\sqrt{2}\sum_{cyc(a,b,c)}\sqrt{a^6+b^6}+\sqrt[3]{4}\sum_{cyc(a,b,c)}\sqrt[3]{a^6+b^6}$$

PROBLEM A.020.

If $x,y\geq 0$ then:

$$(e^x+1)\sqrt{e^y}+(e^y+1)\sqrt{e^x}\leq(e^x+1)(e^y+1)$$

PROBLEM A.021.

If $A\in M_5(\mathbb{R}); \det(A^5+I_5)\neq 0; A^{20}-I_5=A^5(A^5+I_5)$ then: $\sqrt[4]{\det A}\in\mathbb{R}$

PROBLEM A.022.

If $x,y,z\in(0,1)$ then:

$$\frac{(x^x\cdot y^y\cdot z^z)^2}{xyz}\geq\frac{((1-x)^x\cdot(1-y)^y\cdot(1-z)^z)^2}{(1-x)(1-y)(1-z)}$$

PROBLEM A.023.

Find:

$$\Omega=\det\left[\begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix}^{100}+\begin{pmatrix}25 & -5\\ -5 & 1\end{pmatrix}^{100}\right]$$

PROBLEM A.024.

If $a,b,c>0$ then:

$$\sum_{cyc(a,b,c)}\left(\frac{1}{a^2b^2}-\frac{1}{ab}\right)+2\sum_{cyc(a,b,c)}\frac{bc^2(ab+1)}{a(b^2c^2+1)}\ge 6$$

PROBLEM A.025.

If $a,b,c,d,e>0;\ c+d+e=1$ then:

$$\left(a+\frac{b}{c}\right)^4+\left(a+\frac{b}{d}\right)^4+\left(a+\frac{b}{e}\right)^4\ge 3(a+3b)^4$$

PROBLEM A.026.

If $a,b>0$ then:

$$\left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{\sqrt{2(a^2+b^2)}}{a+b}}+\left(\frac{\sqrt{2(a^2+b^2)}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}}\ge 1+\sqrt{\frac{a^2+b^2}{2ab}}$$

PROBLEM A.027.

If $0\le x,y,z\le a$ then:

$$\sqrt{x^2-xz+z^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}\le a\left(1+\sqrt{2}+\sqrt{3}\right)$$

PROBLEM A.028.

If $x,y,z\ge e$ then:

$$\frac{1}{\ln x}+\frac{1}{\ln y}+\frac{1}{\ln z}+\ln x\ln y\ln z\ge\frac{1}{\ln x\ln y\ln z}+\ln(xyz)$$

PROBLEM A.029.

If $0<a,b,c\le 1$ then:

$$\frac{1}{a+a^a}+\frac{1}{b+b^b}+\frac{1}{c+c^c}\ge\frac{9}{3+a^2+b^2+c^2}$$

PROBLEM A.030.

f $a,b>0$ then:

$$\left(\frac{2ab}{a^2+b^2}\right)^{\frac{\sqrt{2(a^2+b^2)}}{a+b}}+\left(\frac{\sqrt{2(a^2+b^2)}}{a+b}\right)^{\frac{2ab}{a^2+b^2}}\le 1+\frac{2\sqrt{2ab}}{(a+b)\sqrt{a^2+b^2}}$$

PROBLEM A.031.

If $a,b,c\ge 1$ then:

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)} \le \frac{15}{2}$$

PROBLEM A.032.

If $a,b,c \ge 0$ then:

$$3\sqrt{3}(a+b)(b+c)(c+a) \le 8\sqrt{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}$$

PROBLEM A.033.

If $a,b,c \ge 0$ then:

$$3\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\right) \ge 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+\sqrt[4]{ab}+\sqrt[4]{bc}+\sqrt[4]{ca}$$

PROBLEM A.034.

If $A \in M_5(\mathbb{R}); A^2 = O_5$ then $\det(A^2 - I_5) \le 0$

PROBLEM A.035.

If $x,y,z > 1; xyz = 2\sqrt{2}$ then:

$$x^y + y^z + z^x + y^x + z^y + x^z > 9$$

PROBLEM A.036.

Solve for real numbers:

$$\frac{1}{(x+1)^3}+\frac{1}{(x+1)^2}+\frac{1}{x+1} = 156 + \log_5(x+1)$$

PROBLEM A.037.

In ΔABC the following relationship holds:

$$(2+a^3)(3+a^4)+3(3+b^8)+2\left(2+\frac{c^3}{a^3}\right)(3+c^4a^4) \ge 432r^2$$

PROBLEM A.038.

$$\Omega(x) = \begin{vmatrix} \sin x & \frac{-\sqrt{3}\cos x}{2} & \frac{-\cos x}{2} \\ \frac{\sqrt{3}\cos x}{2} & \sin x & -1 \\ \frac{\cos x}{2} & 1 & \sin x \end{vmatrix}$$

If $x,y,z \ge 0$ then: $\Omega(x)\Omega(y)+\Omega(y)\Omega(z)+\Omega(z)\Omega(x) \le 4(x^2+y^2+z^2)$

PROBLEM A.039.

If $a,b > 0, a^2+b^2 = 1$ then: $\frac{1}{a}+\frac{1}{b} \ge 2\sqrt{2}$

PROBLEM A.040.

If $a,b,c>0; a^4+b^4+c^4=1$ then:

$$\frac{a+b+c}{abc}\geq 3\sqrt{3}$$

PROBLEM A.041.

If $a,b,c,d>0; a^3+b^3+c^3+d^3=1$ then:

$$\frac{a+b+c+d}{abcd}\geq 16$$

PROBLEM A.042.

If $a,b,c>0; a^3+b^3+c^3=1$ then:

$$\frac{a+b+c}{abc}\geq 3\sqrt[3]{9}$$

PROBLEM A.043.

If $a,b,c,d>0; a^2+b^2+c^2+d^2=1$ then:

$$\frac{a+b+c+d}{abcd}\geq 32$$

PROBLEM A.044.

If $a,b,c>0; a^2+b^2+c^2=1$ then:

$$\frac{a+b+c}{abc}=9$$

PROBLEM A.045.

If $x,y,z,t>0$ then:

$$(xy+yz+zt+tx)\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}+\frac{1}{t^4}\right)\geq\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)^2$$

PROBLEM A.046.

If $a,b,c,d>0$ then:

$$(ab+bc+cd+da)(a^4+b^4+c^4+d^4)\geq abc(a+b+c+d)^2$$

PROBLEM A.047. Find $x,y,z>0$ such that:

$$\begin{cases}\log_2 x+\log_2 y+\log_2 z=3\\ 3^x+3^y+3^z=27\end{cases}$$

PROBLEM A.048.

If $a_1,a_2,\dots,a_8\geq 1$ then:

$$a_1^4+a_2^4+\cdots+a_8^4\le(a_1a_2\dots a_8)^4+7$$

PROBLEM A.049.

If $0<a\le b\le c$ then:

$$\frac{1}{(a-b+c)^6}+\frac{1}{b^6}\le\frac{1}{a^6}+\frac{1}{c^6}$$

PROBLEM A.050.

If $a,b,c>0$ then:

$$\left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{a^2}}+e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}\right)\ge\left(e^{\frac{a}{b}}+e^{\frac{b}{c}}+e^{\frac{c}{a}}\right)^2$$

PROBLEM A.051.

If $a,b,c\ge 0$ then:

$$(a+b)\sqrt{a^2+b^2}+(b+c)\sqrt{b^2+c^2}+(c+a)\sqrt{c^2+a^2}\ge\left(2\sqrt{3}-1\right)(ab+bc+ca)$$

PROBLEM A.052.

If $a,b,c>0$ then:

$$(a+b)\sqrt{a^2+b^2-ab}+(b+c)\sqrt{b^2+c^2-bc}+(c+a)\sqrt{c^2+a^2-ca}\ge$$
$$\ge 2(ab+bc+ca)$$

PROBLEM A.053.

If $a,b\ge 0$ then:

$$\left(a+b+\sqrt{a^2+b^2}\right)^2\ge 6\sqrt{3}ab$$

PROBLEM A.054.

If $a,b,c\ge 0$ then:

$$4(a+b+c)\le\left(3\sqrt{3}-2\right)\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$$

PROBLEM A.055.

If $a,b\ge 0$ then:

$$\begin{cases}4ab\le\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)\\ 4ab\sqrt{a^2+b^2}\le(a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)\end{cases}$$

PROBLEM A.056.

If $a,b>0$ then:

$$a^3+b^3+\left(\sqrt{a^2+b^2}\right)^3+\frac{4a^2b^2}{a+b+\sqrt{a^2+b^2}}>4ab\sqrt{a^2+b^2}$$

PROBLEM A.057.

If $a,b,c\geq 0$ then:

$$2(a+b+c)+\sum_{cyc}\sqrt{a^2+b^2-ab}\geq 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

PROBLEM A.058.

If $a,b\geq 0$ then:

$$\left(a+b+\sqrt{a^2+b^2}\right)^2\geq 18a^2b^2$$

PROBLEM A.059.

If $a,b>0;\, x,y,z\in\mathbb{R}$ then:

$$\frac{(a+b)y^2}{a}+\frac{(a+b)z^2}{b}\geq x(2y+2z-x)$$

PROBLEM A.060.

If $a,b,c>0;\, x,y,z,t\in\mathbb{R}$ then:

$$\frac{(a+b+c)y^2}{a}+\frac{(a+b+c)z^2}{b}+\frac{(a+b+c)t^2}{c}\geq x(2y+2z+2t-x)$$

PROBLEM A.061.

If $x,y,z,t>1$ then:

$$(\log_{xzt}x)^2+\left(\log_{xyt}y\right)^2+\left(\log_{xyz}z\right)^2+\left(\log_{yzt}t\right)^2>\frac{1}{4}$$

PROBLEM A.062.

If $x,y,z,t>1$ then:

$$(\log_{xzt}x)\left(\log_{xyt}y\right)\left(\log_{xyz}z\right)\left(\log_{yzt}t\right)<\frac{1}{16}$$

PROBLEM A.063.

Solve for real numbers:

$$\begin{cases}2x\sqrt{1-y^2}+2y\sqrt{1-x^2}=\sqrt{3}\\2y\sqrt{1-z^2}+2z\sqrt{1-y^2}=\sqrt{3}\\2z\sqrt{1-x^2}+2x\sqrt{1-z^2}=\sqrt{3}\end{cases}$$

PROBLEM A.064.

Find $x, y, z \geq 0$ such that:

$$\frac{2x^2+4}{z^2+2y+3}+\frac{2y^2+4}{x^2+2z+3}+\frac{2z^2+4}{y^2+2x+3}=3$$

PROBLEM A.065.

If $a, b, c > 0$ then:

$$\left(\frac{a^4}{4}+\frac{b^8}{8}+\frac{5\sqrt[5]{c^8}}{8}\right)\left(\frac{5\sqrt[5]{c^8}}{8}+\frac{b^8}{8}+\frac{c^4}{4}\right) \geq \frac{27(abc)^4}{(ab+bc+ca)^3}$$

PROBLEM A.066.

If $x, y, z > 0$; $\sqrt{x}+\sqrt{y}+\sqrt{z}=3\sqrt{xyz}$ then:

$$\frac{(x^2+1)(y^2+1)}{(x^3+1)(y^3+1)}+\frac{(y^2+1)(z^2+1)}{(y^3+1)(z^3+1)}+\frac{(z^2+1)(x^2+1)}{(z^3+1)(x^3+1)} \leq 3$$

PROBLEM A.067.

If $x, y, z \in \mathbb{R}$; $\begin{cases} x^2+xy+xz+yz=2 \\ yx+y^2+yz+zx=3 \\ zx+zy+z^2+xy=6 \end{cases}$ then find $\Omega = \max|x+y+z|$.

PROBLEM A.068.

Solve for real numbers:

$$\begin{cases} \sum_{cyc} \frac{x}{x+2\sqrt{yz}} = \sum_{cyc} \frac{y}{x+\sqrt{2(y^2+z^2)}} \\ 2x+\log_2 y+2^z=9 \end{cases}$$

PROBLEM A.069.

If $a, b, c > 0$ are different in pairs then:

$$\left(\frac{a}{cb}\left(\frac{1}{c}-\frac{1}{b}\right)+\frac{b}{ac}\left(\frac{1}{a}-\frac{1}{c}\right)+\frac{c}{ba}\left(\frac{1}{b}-\frac{1}{a}\right)\right)^2 <$$

$$< (a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)$$

PROBLEM A.070.

If $a, b, c \in \mathbb{R}^*$ then:

$$a^2+\frac{1}{b^2}+\frac{1}{c^4}+1 \geq \left|a+\frac{a}{b}+\frac{1}{bc^2}+\frac{1}{c^2}\right|$$

PROBLEM A.071.

If $a, b, c > 0$; $a^2+b^2+c^2=9$ then:

$$\frac{a^3b^2}{(a^2+1)^2}+\frac{b^3c^2}{(b^2+1)^2}+\frac{c^3a^2}{(c^2+1)^2}<\frac{27\sqrt{3}}{16}$$

PROBLEM A.072.

If $a,b,c\in\mathbb{R}; a^2+b^2+c^2=3$ then:

$$|a+(a+c)b+c|\le 4$$

PROBLEM A.073.

If $x,y,z>0; xyz=1$ then:

$$z\left(\frac{x}{y}\right)^{x-y}+x\left(\frac{y}{z}\right)^{y-z}+y\left(\frac{z}{x}\right)^{z-x}\ge 3$$

PROBLEM A.074.

If $a,b,c>0; a^3+b^3+c^3=3$ then:

$$\left(\frac{a^2+1}{a+1}\right)^3+\left(\frac{b^2+1}{b+1}\right)^3+\left(\frac{c^2+1}{c+1}\right)^3\ge 3$$

PROBLEM A.075.

Solve for real numbers:

$$\begin{cases}\frac{x^4y^4}{z^4}+\frac{y^4z^4}{x^4}+\frac{z^4x^4}{y^4}=xyz\sqrt[4]{27(x^4+y^4+z^4)}\\ x^4-4y^3+6z^2-4y+1=0\end{cases}$$

PROBLEM A.076.

Find $x,y,z\ge 1$ such that:

$$\begin{cases}\log_2(2x^3)=\log_{8x}z^{16}\\ \log_2(2y^3)=\log_{8y}x^{16}\\ \log_2(2z^3)=\log_{8z}y^{16}\end{cases}$$

PROBLEM A.077.

If $a,b,c>0; a+b+c=3$ then:

$$a^6+b^6+c^6+\frac{1}{32}((3-a)^6+(3-b)^6+(3-c)^6)\ge 9$$

PROBLEM A.078.

If $A,B,C\in M_2(\mathbb{R}); \det A, \det B, \det C>0; \det(ABC)=8$ then:

$$\det(A^2+B^2+C^2)+\det(A^2+B^2-C^2)+\det(A^2-B^2+C^2)+$$
$$+\det(-A^2+B^2+C^2)\ge 48$$

PROBLEM A.079.

If $x,y,z>0;\frac{x+y}{2x+y}+\frac{y+z}{2y+z}+\frac{z+x}{2z+x}=2$ then:

$$\frac{3x^2+xy+2y^2}{2x^2+y^2}+\frac{3y^2+yz+2z^2}{2y^2+z^2}+\frac{3z^2+zx+2x^2}{2z^2+x^2}\le 6$$

PROBLEM A.080.

If $a,b,c>0$ then:

$$a^a\cdot b^b\cdot c^c\cdot(4a+4b+4c)^{a+b+c}\ge 3^{a+b+c}\cdot(a+b)^{a+b}\cdot(b+c)^{b+c}\cdot(c+a)^{c+a}$$

PROBLEM A.081.

If $a,b,c,x,y,z>0, a^3x+b^3y+c^3z=xyz$ then:

$$x+y+z\ge(a+b+c)\sqrt{a+b+c}$$

PROBLEM A.082.

If $a,b,c,d\in\mathbb{R}$ then: $ac+bd+|ad-bc|\le\sqrt{2(a^2+b^2)(c^2+d^2)}$

PROBLEM A.083.

If $a,b,c,d\in\mathbb{R}$ then:

$$2|ad-bc|(ac+bd)+(ac+bd)^2\le(ad-bc)^2+(a^2+b^2)(c^2+d^2)\sqrt{2}$$

PROBLEM A.084.

If $x,y,z>0;xyz(3x+2y+36z)=6$ then:

$$\left(\frac{x^2y^2}{36}+1\right)(4y^2z^2+1)(9z^2x^2+1)\ge 64x^4y^4z^4$$

PROBLEM A.085.

If $x,y\in\mathbb{R}$ then: $(x^3+2y^3-3xy^2)^2\le(x^2+2y^2)^3$

PROBLEM A.086.

If $a,b>0$ then:

$$\frac{\left((ab)^6+\left(\frac{a+b}{2}\right)^{12}\right)\left(ab+\left(\frac{a+b}{2}\right)^2\right)}{\left((ab)^3\sqrt{ab}+\left(\frac{a+b}{2}\right)^7\right)^2}\le\frac{(a^5+b^5)^2}{4(ab)^5}$$

PROBLEM A.087.

If $a,b,c>0;a^{b^2}\cdot b^{c^2}\cdot c^{a^2}=1$ then:

$$b^2\left(\sqrt[3]{a^{a+b+c}}-a^{\sqrt[3]{abc}}\right)+c^2\left(\sqrt[3]{b^{a+b+c}}-b^{\sqrt[3]{abc}}\right)+a^2\left(\sqrt[3]{c^{a+b+c}}-c^{\sqrt[3]{abc}}\right)\geq 0$$

PROBLEM A.088.

If $x, y, z \geq 0$ then:

$$\frac{7^{\sqrt{xy}}}{5^{\sqrt{xy}}+3^{\sqrt{xy}}}+\frac{7^{\sqrt{yz}}}{5^{\sqrt{yz}}+3^{\sqrt{yz}}}+\frac{7^{\sqrt{zx}}}{5^{\sqrt{zx}}+3^{\sqrt{zx}}}\leq\frac{\sqrt{7^{x+y}}}{\sqrt{5^{x+y}}+\sqrt{3^{x+y}}}+\frac{\sqrt{7^{y+z}}}{\sqrt{5^{y+z}}+\sqrt{3^{y+z}}}+\frac{\sqrt{7^{z+x}}}{\sqrt{5^{z+x}}+\sqrt{3^{z+x}}}$$

PROBLEM A.089.

If $x, y, z, t \in \mathbb{R}; x^2+y^2=z^2+t^2=10$ then:

$$(10-x-3y)(10-xz-yt)(10-z-3t)\leq 10125$$

PROBLEM A.090.

If $a, b, c > 0$ then:

$$\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}}+\left(\frac{2\sqrt{bc}}{b+c}\right)^{\frac{b+c}{2\sqrt{bc}}}+\left(\frac{2\sqrt{ca}}{c+a}\right)^{\frac{c+a}{2\sqrt{ca}}}\geq 6-\left(\frac{a+b}{2\sqrt{ab}}+\frac{b+c}{2\sqrt{bc}}+\frac{c+a}{2\sqrt{ca}}\right)$$

PROBLEM A.091.

If $a, b, c, d \in \left(0, \frac{1}{3}\right)$ then:

$$a^{b+c+d}+b^{c+d+a}+c^{d+a+b}+d^{a+b+c}>1$$

PROBLEM A.092.

If $x_1, x_2, \ldots, x_n \in \left(0, \frac{1}{n}\right); n \in \mathbb{N}; n \geq 2$ then:

$$x_1^{x_2+x_3+\cdots+x_n}+x_2^{x_1+x_3+\cdots+x_n}+\cdots+x_n^{x_1+x_2+\cdots+x_{n-1}}>1$$

PROBLEMS

GEOMETRY

PROBLEM G.001.

If in $\Delta\, ABC: 3a + m_a = 3b + m_b = 3c + m_c$ then find:

$$\Omega = \frac{m_a m_b m_c}{h_a h_b h_c} + \frac{r_a r_b r_c}{s_a s_b s_c}$$

PROBLEM G.002.

In $\Delta\, ABC$ the following relationship holds:

$$\prod \cos\frac{3A}{2} = 0 \Rightarrow \sum a \sin^2 A \geq \frac{9\sqrt{3}r}{2}$$

PROBLEM G.003.

If in $\Delta\, ABC; 2b = a + c$ then:

$$\sin\frac{B}{2} < \frac{s}{3\sqrt{ac}}$$

PROBLEM G.004.

If in $\Delta\, ABC, m_a \leq m_b \leq m_c$ then:

$$\left(\sqrt{a-c} + \sqrt{b-c}\right)\left(\sqrt{a^2-c^2} + \sqrt{b^2-c^2}\right)\left(\sqrt{a^3-c^3} + \sqrt{b^3-c^3}\right) \leq \frac{a^3b^3}{c^3}$$

PROBLEM G.005.

If T is area of pedal's triangle of G – centroid in $\Delta\, ABC$ then:

$$T \geq \frac{3\sqrt{3}r^4}{R^2}$$

PROBLEM G.006.

In acute $\Delta\, ABC$ the following relationship holds:

$$2m_a^{\frac{a}{a+b+c}} \cdot m_b^{\frac{a}{a+b+c}} \cdot m_c^{\frac{a}{a+b+c}} \leq 3R$$

PROBLEM G.007.

Let P be any point in same plane with $\Delta\, ABC$. Prove that:

$$\frac{AP}{BP} + \frac{BP}{CP} + \frac{CP}{AP} \geq \frac{108r^2}{a^2+b^2+c^2}$$

PROBLEM G.008.

If in $\Delta\ ABC, \Omega$ is first Brocard point then:

$$\frac{A\Omega}{B\Omega}+\frac{B\Omega}{C\Omega}+\frac{C\Omega}{A\Omega}\ge\frac{4s^2}{ab+bc+ca}$$

PROBLEM G.009.

Solve for real numbers:

$$4\sin^2(x+y)=1+4\cos^2 x+4\cos^2 y$$

PROBLEM G.010.

In $\Delta\ ABC$ the following relationship holds:

$$a^2m_a+b^2m_b+c^2m_c\ge bch_a+cah_b+ah_c$$

PROBLEM G.011.

If in $\Delta ABC, c_a, c_b, c_c$ are Gergonne's cevians then:

$$c_ac_bc_c\ge 8R^3\sin^2 A\sin^2 B\sin^2 C$$

PROBLEM G.012.

If Ω is area of pedal's triangle of G – centroid in $\Delta\ ABC$ then:

$$\Omega\ge\frac{3\sqrt{3}r^4}{R^2}$$

PROBLEM G.013.

In ΔABC the following relationshiop holds:

$$(m_a\sin A)^2+(m_b\sin B)^2+(m_c\sin C)^2\le\frac{3}{4}\left(h_a^2+h_b^2+h_c^2\right)$$

PROBLEM G.014.

In ΔABC the following relationship holds:

$$3\sqrt[3]{w_a^2w_b^2w_c^2}\le s^2\le\sqrt{3(m_a^4+m_b^4+m_c^4)}$$

PROBLEM G.015.

In ΔABC the following relationship holds:

$$\sqrt[3]{\left(\frac{a^3+b^3+c^3}{3}\right)^2}+\sqrt[5]{\left(\frac{a^5+b^5+c^5}{3}\right)^2}+\sqrt[7]{\left(\frac{a^7+b^7+c^7}{3}\right)^2}\ge 4\sqrt{3}S$$

PROBLEM G.016.

In Δ ABC the following relationship holds:

$$\sum\left(\sqrt{a^2+c^2+4S}+\sqrt{b^2+c^2+4S}-\sqrt{a^2+b^2+4S}\right)^2\geq 2(a^2+b^2+c^2)+12S$$

PROBLEM G.017.

If in Δ ABC, N – ninepoint center, I_a, I_b, I_c – excenters then:

$$8NI_a\cdot NI_b\cdot NI_c\geq\left(2r_a+\sqrt{\frac{abc}{2s}}\right)\left(2r_b+\sqrt{\frac{abc}{2s}}\right)\left(2r_c+\sqrt{\frac{abc}{2s}}\right)$$

PROBLEM G.018.

In ΔABC the following relationship holds:

$$\frac{\sin^8\frac{\pi}{16}}{r_a}+\frac{\sin^8\frac{3\pi}{16}}{r_b}+\frac{\sin^8\frac{5\pi}{16}}{r_c}+\frac{\sin^8\frac{7\pi}{16}}{r}>\frac{9}{20R}$$

PROBLEM G.019.

If in ΔABC; $A>B>C$ then:

$$\left(\frac{a}{b}+\frac{a}{c}+\frac{b}{c}\right)^2>27\left(\frac{a+s}{c+s}\right)^2$$

PROBLEM G.020.

In Δ ABC the following relationship holds:

$$a(2s-a)\cos\frac{A}{2}+b(2s-b)\cos\frac{B}{2}+c(2s-c)\cos\frac{C}{2}\geq 36\sqrt{3}r^2$$

PROBLEM G.021.

In acute ΔABC the following relatioinship holds:

$$AI+BI+CI\leq\sqrt{6R(h_a+h_b+h_c-6r)}$$

PROBLEM G.022.

In ΔABC the following relationship holds:

$$2\sqrt[3]{abc}\leq\sqrt{3}(3R-2r)$$

PROBLEM G.023.

If $x,y,z\in\left(0,\frac{\pi}{2}\right)$; $x+y+z=\pi$ then:

$$\frac{xy(\tan x+\sin x)}{x^2+\sin x\tan x}+\frac{yz(\tan y+\sin y)}{y^2+\sin y\tan y}+\frac{zx(\tan z+\sin z)}{z^2+\sin z\tan z}>\pi$$

PROBLEM G.024.

If $x, y, z \in \left(0, \frac{\pi}{2}\right)$; $\sin x + \sin y + \sin z = 1$ then:

$$\cos^2 x \cos^2 y \cos^2 z \geq 512 \sin^2 x \sin^2 y \sin^2 z$$

PROBLEM G.025.

In acute ΔABC the following relationship holds:

$$\sum_{cyc(a,b,c)} \frac{1}{b\cos B + c\cos C - a\cos A} = \frac{R}{4R\cos A\cos B\cos C} + \frac{2R^2}{abc}$$

PROBLEM G.026.

In acute triangle ABC the following relationship holds:

$$\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} > A^2 + B^2 + C^2 + \cos A + \cos B + \cos C$$

PROBLEM G.027.

In ΔABC the following relationship holds:

$$\frac{a(s-a)}{b+c} + \frac{b(s-b)}{c+a} + \frac{c(s-c)}{a+b} \leq \frac{3\sqrt{3}R}{4}$$

PROBLEM G.028.

If in ΔABC; I – incenter then:

$$\left(\frac{AI+BI}{CI}\right)^5 + \left(\frac{BI+CI}{AI}\right)^5 + \left(\frac{CI+AI}{BI}\right)^5 > \left(\frac{BC}{AI}\right)^5 + \left(\frac{CA}{BI}\right)^5 + \left(\frac{AB}{CI}\right)^5$$

PROBLEM G.029.

In ΔABC the following relationship holds:

$$\left(\frac{2m_a+2m_b}{m_c}\right)^7 + \left(\frac{2m_b+2m_c}{m_a}\right)^7 + \left(\frac{2m_c+2m_a}{m_b}\right)^7 > \left(\frac{3a}{m_a}\right)^7 + \left(\frac{3b}{m_b}\right)^7 + \left(\frac{3c}{m_c}\right)^7$$

PROBLEM G.030.

If $x, y, z \in (0, 1)$ then:

$$\sum_{cyc(x,y,z)} \frac{y(\sin^{-1} x + \tan^{-1} x)}{x^2 + \tan^{-1} x \cdot \sin^{-1} x} > 3$$

PROBLEM G.031.

Solve for $x, y, z \in \left(0, \frac{\pi}{2}\right)$:

$$\begin{cases} \frac{3\cos^2 x}{\cos^2 y} = 1 + \frac{2\sin^2 x}{\sin^2 y} \\ \frac{3\cos^2 y}{\cos^2 z} = 1 + \frac{2\sin^2 y}{\sin^2 z} \\ x + 2^y + \log_2 z = 3 \end{cases}$$

PROBLEM G.032.

In acute ΔABC the following relationship holds:

$$\sum_{cyc(a,b,c)} \left(\frac{b\cos B}{c\cos C} + \frac{c\cos C}{b\cos B}\right)\cos 2A = -3$$

PROBLEM G.033.

If $0 \le x \le \frac{\sqrt{6}}{3}$ then:

$$\sqrt{2x^2 + (\sqrt{2} - \sqrt{6})x + 2} + \sqrt{2x^2 - (\sqrt{2} + \sqrt{6})x + 2} \ge 2$$

PROBLEM G.034.

Solve for real numbers:

$$\begin{cases} \tan x \le -3 \\ 13\tan 3x \ge -9 \\ 5\sqrt{10}\cos 3x + 13\sin 3y - 13 = 0 \end{cases}$$

PROBLEM G.035.

In acute ΔABC the following relationship holds:

$$2\sum a^2\cos^2 A\,(b\cos B + c\cos C)^2 \le \left(\sum a\cos A\right)\prod(b\cos B + c\cos C)$$

PROBLEM G.036.

In $\Delta ABC, I$ – incenter, $\Delta A'B'C'$ - circumcevian triangle of I. Prove that:

$$IA' + IB' + IC' \ge 48\sqrt{3}r^3\left(\frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{1}{(a+b)^2}\right)$$

PROBLEM G.037.

If $A, B \in M_2(\mathbb{C}); \det A \ne 0; \det B \ne 0$ then:

$$\det(A\det B + B\det A) + \det\left(\frac{A}{\det A} + \frac{B}{\det B}\right) = \det(A+B)\left(\det(AB) + \frac{1}{\det(AB)}\right)$$

PROBLEM G.038.

In acute ΔABC the following relationship holds:

$$\sum a^3\cos^3 A+\frac{3abc(a^2+b^2+c^2-8R^2)}{8R^2}\ge 2\sum ba^2\cos B\cos^2 A$$

PROBLEM G.039.

If in $\Delta ABC, \mu(A)=\frac{\pi}{3}$ then:

$$3\sqrt{3}R+a\ge\frac{4bc}{a}$$

PROBLEM G.040.

If in $\Delta ABC; a\ge b\ge c$ then:

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\le\frac{a^2b+b^2c+c^2a}{2S}$$

PROBLEM G.041.

If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$\left(\sin^2 x\sin^2 y\sin^2 z+\cos^2 x\cos^2 y\cos^2 z\right)\left(\frac{1}{\sin^2 x\cos^2 y}+\frac{1}{\sin^2 y\cos^2 z}+\frac{1}{\sin^2 z\cos^2 x}\right)\ge 3$$

PROBLEM G.042.

If in $\Delta ABC, N$ is Nagel's point then:

$$\sum\frac{a^2\cdot AN^2}{5(b^2\cdot BN^2+c^2\cdot CN^2)-a^2\cdot AN^2}\ge\frac{1}{3}$$

PROBLEM G.043.

In $\Delta\ ABC$ the following relationship holds:

$$\frac{s}{ab+bc+ca}+\frac{8Rr}{(2s-a)(2s-b)(2s-c)}\ge\frac{2\sqrt{3}}{9R}$$

PROBLEM G.044.

If $M\in Int\ (\Delta ABC); AM=x; BM=y; CM=z$ then:

$$\frac{ax}{ax+by+98cz}+\frac{by}{bx+cy+98az}+\frac{cz}{cz+ay+98bz}\ge\frac{3}{100}$$

PROBLEM G.045.

If $x,y\in\left(0,\frac{\pi}{2}\right)$ then:

$$\left(1+\frac{1}{\sin^2 x}\right)\left(1+\frac{1}{\sin^2 y\cdot\cos^2 x}\right)\left(1+\frac{1}{\cos^2 x\cdot\cos^2 y}\right)\ge 64$$

PROBLEM G.046.

If $0 < a, b, c < 1$ then in $\Delta\ ABC$ the following relationship holds:

$$\left(a^b + b^a\right)\left(b^c + c^b\right)\left(c^a + a^c\right) > 32Rrs$$

PROBLEM G.047.

In $\Delta\ ABC$ the following relationship holds:

$$\sin A\,|\cos B\cos C| + \sin B\,|\cos C\cos A| + \sin C\,|\cos A\cos B| \geq \frac{rs}{2R^2}$$

PROBLEM G.048.

In $\Delta\ ABC$ the following relationship holds:

$$a^2 m_b m_c + b^2 m_c m_a + c^2 m_a m_b \geq \frac{(a^2 + b^2 + c^2)^2}{4}$$

PROBLEM G.049.

In $\Delta\ ABC$ the following relationship holds:

$$\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}} + \frac{\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}} \geq \frac{2s}{r}$$

PROBLEM G.050.

In $\Delta\ ABC$ the following relationship holds:

$$4(am_b m_c + bm_c m_a + cm_a m_b) \geq 9abc$$

PROBLEM G.051.

If $x > 0$ then ΔABC the following relationship holds:

$$ba^{x+1} + cb^{x+1} + ac^{c+1} \geq 3\left(2\sqrt{3}r\right)^{x+2}$$

PROBLEM G.052.

If in ΔABC; $a^2 + b^2 + c^2 = 1$ then: $18Rr \leq 1$

PROBLEM G.053.

In acute ΔABC the following relationship holds:

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} + \frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} > 6\sqrt{2}$$

PROBLEM G.054.

If $x, y > 0$ then in ΔABC the following relationship holds:

$$\frac{1}{3r_a^2+7r_ar_b}+\frac{1}{3r_b^2+7r_br_c}+\frac{1}{3r_c^2+7r_cr_a}\ge\frac{2}{15R^2}$$

PROBLEM G.055.

If $\Delta ABC\sim\Delta A'B'C'$ then:

$$\sum\frac{(a'+b')(a'+c')}{b'c'}+3\ge\frac{15(b+c)(c'+a')(a'+b')}{8ab'c'}$$

PROBLEM G.056.

If $x\in\left(0,\frac{\pi}{2}\right)$ then:

$$2(\sin x)^{1-\sin x}(1-\sin x)^{\sin x}\le 1$$

PROBLEM G.057.

If $x,y\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{(\sin^2 x+\sin^2 y)^{\sin^2 x+\sin^2 y}\cdot(\cos^2 x+\cos^2 y)^{\cos^2 x+\cos^2 y}}{(\sin x)^{2\sin^2 x}\cdot(\sin y)^{2\sin^2 y}\cdot(\cos x)^{2\cos^2 x}\cdot(\cos y)^{2\cos^2 y}}\le 4$$

PROBLEM G.058.

If $\Delta\ ABC\sim\Delta A'B'C'$ then:

$$4(m_am_{a'}+m_bm_{b'}+m_cm_{c'})+aa'+bb'+cc'\ge$$
$$\ge 4\left(\sqrt{aa'bb'}+\sqrt{bb'cc'}+\sqrt{cc'aa'}\right)$$

PROBLEM G.059.

In ΔABC the following relationship holds:

$$\frac{h_ah_b}{m_am_b}\le 24\sqrt{3}R^2r\cdot\sum\left(\frac{c}{(b^2+c^2)(a^2+c^2)}\right)$$

PROBLEM G.060.

In acute ΔABC the following relationship holds:

$$\frac{\log_{\sin A}\sin B}{\tan\frac{A}{2}}+\frac{\log_{\sin B}\sin C}{\tan\frac{B}{2}}+\frac{\log_{\sin C}\sin A}{\tan\frac{C}{2}}\ge 3\sqrt{3}$$

PROBLEM G.061.

In ΔABC the following relationship holds:

$$\sum_{cyc(a,b,c)}a\sqrt{((b-c)^2+4r^2)((c-a)^2+4r^2)}\ge abc$$

PROBLEM G.062.

In acute ΔABC the following relationship holds:

$$2^{\sin A}+2^{\sin B}+2^{\sin C}+2^{\cos A}+2^{\cos B}+2^{\cos C}>9$$

PROBLEM G.063.

In acute $\Delta\ ABC$ the following relationship holds:

$$\frac{a\cos A}{b\cos B}+\frac{b\cos B}{c\cos C}+\frac{c\cos C}{a\cos A}\leq\frac{3}{8\cos A\cos B\cos C}$$

PROBLEM G.064.

In acute $\Delta\ ABC$ the following relationship holds:

$$\frac{2\sqrt{3}}{R}\leq\frac{1}{a\cos A}+\frac{1}{b\cos B}+\frac{1}{c\cos C}\leq\frac{\sqrt{3}}{4R\cos A\cos B\cos C}$$

PROBLEM G.065.

If $x\geq 0$ then:

$$\sin x\,(16\sin^4 x+5)\leq 5x(4x^2+1)$$

PROBLEM G.066.

In $\Delta\ ABC$ the following relationship holds:

$$\prod_{cyc}(a|\cos A|+b|\cos B|-c|\cos C|)\leq|abc\cdot\cos A\cos B\cos C|$$

PROBLEM G.067.

If $a,b,c\in(0,1);\ x,y\in\mathbb{R}$ then:

$$(a+b+c)\left(\tan^{-1}x-\tan^{-1}\left(\sqrt{x^2+y^2}\right)\right)\leq 3\left(\tan^{-1}\left(\sqrt{x^2+y^2}\right)-\tan^{-1}y\right)$$

PROBLEM G.068.

If in ΔABC: $a\leq b\leq c$ then:

$$\frac{bm_c}{cm_b}+\frac{am_b}{bm_a}+\frac{cm_a}{am_c}\geq\frac{cm_b}{bm_c}+\frac{bm_a}{am_b}+\frac{am_c}{cm_a}$$

PROBLEM G.069.

In $\Delta\ ABC$ the following relationship holds:

$$\sqrt[3]{\frac{\sin A}{\sin B}}+\sqrt[3]{\frac{\sin B}{\sin C}}+\sqrt[3]{\frac{\sin C}{\sin A}}-\sqrt[3]{\frac{\sin A}{\sin C}}-\sqrt[3]{\frac{\sin B}{\sin A}}-\sqrt[3]{\frac{\sin C}{\sin B}}<1$$

PROBLEM G.070.

In acute ΔABC the following relationship holds:

$$\frac{b^2+c^2+2bc}{b^2+c^2-a^2}+\frac{c^2+a^2+2ca}{c^2+a^2-b^2}+\frac{a^2+b^2+2ab}{a^2+b^2-c^2}>9$$

PROBLEM G.071.

In acute ΔABC the following relationship holds:

$$\sum_{cyc} a^4(b^2+c^2-a^2)\geq 32RS^2\sqrt{2(a^2+b^2+c^2)\cos A\cos B\cos C}$$

PROBLEM G.072.

In ΔABC the following relationship holds:

$$\frac{\left(\frac{2}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)^2}{\frac{1}{ab}+\frac{2}{bc}+\frac{1}{ca}}+\frac{\left(\frac{2}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)^2}{\frac{1}{bc}+\frac{2}{ca}+\frac{1}{ab}}+\frac{\left(\frac{2}{ca}+\frac{1}{ab}+\frac{1}{bc}\right)^2}{\frac{1}{ca}+\frac{2}{ab}+\frac{1}{bc}}\geq\frac{4}{R^2}$$

PROBLEM G.073.

In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{\left(\frac{1}{ab\sin\frac{A}{2}\sin\frac{B}{2}}\right)^7}{\left(\frac{1}{bc\sin\frac{B}{2}\sin\frac{C}{2}}\right)^6+\left(\frac{1}{ca\sin\frac{C}{2}\sin\frac{A}{2}}\right)^6}\right)\geq\frac{1}{2r^2}$$

PROBLEM G.074.

If $0<a\leq b<\frac{\pi}{2}$ then:

$$(a+b)\left(\sin(\sqrt{ab})-\cos(\sqrt{ab})\right)\leq 2\sqrt{ab}\left(\sin\left(\frac{a+b}{2}\right)-\cos\left(\frac{a+b}{2}\right)\right)$$

PROBLEM G.075.

If $x,y,z\geq 0$ then:

$$\sqrt{x^2+z^2+xz\sqrt{2}}+\sqrt{y^2+z^2+yz\sqrt{2}}\geq\sqrt{x^2+y^2}$$

PROBLEM G.076.

If $x,y,z\geq 0$ then:

$$\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2 \geq 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$$

PROBLEM G.077.

If $x, y \in \mathbb{R}$ then:

$$(1+\cos(x-y)-\sin y-\sin x\cos y-\cos x)^2 < 12$$

PROBLEM G.078.

If $x, y, z > 0$ then:

$$(x+y+z)\left(\frac{\sqrt{3}}{3}+\tan 20^\circ\right) > 4\sum_{cyc}\frac{xy}{x\cot 50^\circ + y\cot 10^\circ}$$

PROBLEM G.079.

If $0 < x, y, z, t < \frac{\pi}{2}$ then:

$$\sum_{cyc(x,y,z,t)}(\sin^2 x+\csc^2 x)^3 + \sum_{cyc(x,y,z,t)}(\cos^2 x+\sec^2 x)^3 \geq 125$$

PROBLEM G.080.

If $x, y > 0$; $xy \geq \frac{1}{8}$ then:

$$\frac{x^2}{\sin\frac{3\pi}{11}}+\frac{y^2}{\sin\frac{4\pi}{11}} > \frac{1}{\left(\cos\frac{2\pi}{11}+\sin\frac{5\pi}{11}\right)^2}$$

PROBLEM G.081.

If $m \geq 0$; $x \in \left(0, \frac{\pi}{2}\right)$ then:

$$\left(\sin^2 x+\frac{1}{\sin^2 x}\right)^{m+1}+\left(\cos^2 x+\frac{1}{\cos^2 x}\right)^{m+1} \geq \frac{5^{m+1}}{2^m}$$

PROBLEM G.082.

If $2\sin^2 x + 2\sin^2 y = 1$; $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$2\tan x\tan y + 2\tan x + 2\tan y < 3$$

PROBLEM G.083.

Find: $x, y, z \in \left(0, \frac{\pi}{2}\right]$ such that:

$$\frac{\sin^2 x}{1+\sin^2 x}+\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{\sin^2 z}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}+\frac{1}{8\sin x\sin y\sin z} \leq 1$$

PROBLEM G.084.

If $x, y \in \mathbb{R}$ then:

$$|\sin x + \sin y \cos x + \cos y| \leq 2$$

PROBLEM G.085.

If $a, b, c > 0; ab + bc + ca = 3$ then:

$$4(\tan^{-1} 2)(\tan^{-1}(\sqrt[3]{abc})) \leq \pi \tan^{-1}(1 + \sqrt[3]{abc})$$

PROBLEM G.086.

If $x, y, z \in \left(0, \frac{\pi}{2}\right)$; $\cos x \cos y \cos z = \frac{\sqrt{2}}{2}$ then:

$$15(\cos 2x + \cos 2y + \cos 2z) + 6(\cos 4x + \cos 4y + \cos 4z) +$$
$$+ \cos 6x + \cos 6y + \cos 6z \geq 18$$

PROBLEM G.087.

Solve for real numbers:

$$\begin{cases} 3(\cos 2x + \cos 2y + \cos 2z) + 4(\sin x \sin y + \sin y \sin z + \sin z \sin x) = 1 \\ \sin x + \sin z = 2 + \sin y \end{cases}$$

PROBLEM G.088.

In ΔABC; $\Delta A'B'C'$ the following relationship holds:

$$(a + a')(b + b')(c + c') \geq 64rr'\sqrt{ss'} + 4\left(\sqrt{Rrs} - \sqrt{R'r's'}\right)^2$$

PROBLEM G.089.

Find $x, y, z \in \left(0, \frac{\pi}{2}\right)$ such that:

$$\begin{cases} 2\sin^2 x = \tan y \\ \dfrac{3\sin^3 y \cos y}{\sin^4 y + \cos^2 y} = \tan z \\ \dfrac{4\sin^4 z + \cos^2 z}{\sin^4 z + \cos^4 z} = \tan x \end{cases}$$

PROBLEM G.090.

If $x \in \left[0, \frac{\pi}{14}\right)$ then: $\cos^{413} x \geq (\cos 3x)^{21} \cdot (\cos 5x)^7 \cdot \cos 7x$

PROBLEM G.091.

In ΔABC the following relationship holds:

$$\frac{aw_a^2}{h_a}+\frac{bw_b^2}{h_b}+\frac{cw_c^2}{h_c}\geq 2r^2\sqrt{\frac{486r}{R}}$$

PROBLEM G.092.

In ΔABC the following relationship holds:

$$\frac{am_a}{h_a}+\frac{bm_b}{h_b}+\frac{cm_c}{h_c}\geq 2\sqrt{3\sqrt{3}S}$$

PROBLEM G.093.

In ΔABC the following relationship holds:

$$\frac{ar_a}{h_a}+\frac{br_b}{h_b}+\frac{cr_c}{h_c}\geq a+b+c$$

PROBLEM G.094.

If $a,b,c>1$ then:

$$\frac{\sin\left(\frac{2}{a+b}\right)\sin\left(\frac{2}{b+c}\right)\sin\left(\frac{2}{c+a}\right)}{\sin\left(\frac{1}{\sqrt{ab}}\right)\sin\left(\frac{1}{\sqrt{bc}}\right)\sin\left(\frac{1}{\sqrt{ca}}\right)}\geq\left(\frac{8abc}{(a+b)(b+c)(c+a)}\right)^2$$

PROBLEM G.095.

In ΔABC the following relationship holds:

$$\left(m_b(h_c-h_a)+m_a(h_b-h_c)+m_c(h_a-h_b)\right)^2<\frac{9}{4}(a^2+b^2+c^2)\left(h_a^2+h_b^2+h_c^2\right)$$

PROBLEM G.096.

If $x,y,z,t\in\mathbb{R};\alpha\in\left[0,\frac{\pi}{2}\right]$ then:

$$(x+z)\sin\alpha+(1-\sin\alpha)(y+t)\leq\sqrt{2(x^2+y^2+z^2+t^2)}$$

PROBLEM G.097.

If in $\Delta ABC, a\geq b\geq c$ then the following relationship holds:

$$\sqrt[5]{\frac{m_a}{m_b}}+\sqrt[5]{\frac{m_b}{m_c}}+\sqrt[5]{\frac{m_c}{m_a}}-\sqrt[5]{\frac{m_a}{m_c}}-\sqrt[5]{\frac{m_b}{m_a}}-\sqrt[5]{\frac{m_c}{m_b}}<1$$

PROBLEM G.098.

In ΔABC the following relationship holds:

$$\frac{am_a^5+bm_b^5+cm_c^5}{(am_a+bm_b+cm_c)^5}\geq\frac{1}{729R^4}$$

PROBLEM G.099.

If in $\Delta ABC, a \le b \le c$ then:

$$h_a^{20} - h_b^{20} + h_c^{20} \ge (h_a - h_b + h_c)^{20}$$

PROBLEM G.100.

In ΔABC the following relationship holds:

$$4\left(\sum_{cyc} m_a(h_b - h_c)\right)^2 < 9\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} h_a^2\right)$$

PROBLEM G.101.

In ΔABC the following relationship holds:

$$\sqrt[5]{\frac{2(s-a)}{c}} + \sqrt[5]{\frac{2(s-b)}{a}} + \sqrt[5]{\frac{2(s-c)}{b}} \le 3$$

PROBLEM G.102.

In Δ ABC the following relationship holds:

$$\frac{a^8}{r_b r_c} + \frac{b^8}{r_c r_a} + \frac{c^8}{r_a r_b} \ge 6912 r^6$$

PROBLEM G.103.

In ΔABC the following relationship holds:

$$\frac{1}{64}\left(\frac{a^8 r_b}{b^2} + \frac{b^8 r_c}{c^2} + \frac{c^8 r_a}{a^2}\right) \ge \frac{729 r^8}{R + r}$$

PROBLEM G.104.

In ΔABC the following relationship holds:

$$\sqrt{r_a}\cos A + \sqrt{r_b}\cos B + \sqrt{r_c}\cos C \le \frac{3\sqrt{3}R}{4\sqrt{r}}$$

PROBLEM G.105.

In ΔABC the following relationship holds:

$$\frac{b^2 + c^2 - a^2}{\sqrt{r_b r_c}} + \frac{c^2 + a^2 - b^2}{\sqrt{r_c r_a}} + \frac{a^2 + b^2 - c^2}{\sqrt{r_a r_b}} \le 4(R + r)$$

PROBLEM G.106.

In ΔABC the following relationship holds:

$$\frac{1}{b^2\sin\frac{A}{2}}+\frac{1}{c^2\sin\frac{B}{2}}+\frac{1}{a^2\sin\frac{C}{2}}\geq\frac{4s}{abc}$$

PROBLEM G.107.

In ΔABC the following relationship holds:

$$\frac{m_a}{\sqrt{b}}+\frac{m_b}{\sqrt{c}}+\frac{m_c}{\sqrt{a}}\geq\frac{h_a}{\sqrt[4]{bc}}+\frac{h_b}{\sqrt[4]{ca}}+\frac{h_c}{\sqrt[4]{ab}}$$

PROBLEM G.108.

If $x,y,z\geq 0$ then in ΔABC the following relationship holds:

$$\frac{x}{\sin\frac{A}{2}}+\frac{y}{\sin\frac{B}{2}}+\frac{z}{\sin\frac{C}{2}}\geq 2\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)$$

PROBLEM G.109.

In ΔABC the following relationship holds:

$$\frac{\sqrt{bc}\cos A}{m_a}+\frac{\sqrt{ca}\cos B}{m_b}+\frac{\sqrt{ab}\cos C}{m_c}\leq\frac{(4R+r)m_am_bm_c}{rs^3}$$

PROBLEM G.110.

In ΔABC the following relationship holds:

$$s^3\geq\frac{3\sqrt{3}r^2(4R+r)^3}{(2R-r)(2R+5r)}$$

PROBLEM G.111.

If $0<x,y,z<\frac{\pi}{6}$ then:

$$(\sin^2 x)^{\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right)}+(\sin^2 y)^{\sin\left(\frac{z+x}{2}\right)\cos\left(\frac{z-x}{2}\right)}+(\sin^2 z)^{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}>1$$

PROBLEMS

ANALYSIS

PROBLEM AN.001.

If $x_1 = \frac{1}{2}; 6x_{n+1} = 3\sin x_n + 2\cos x_n ; n \geq 1$ then find:

$$\Omega = \lim_{n\to\infty}(x_1 \cdot x_2 \cdot \ldots \cdot x_n)$$

PROBLEM AN.002.

Find:

$$\Omega = \lim_{n\to\infty} \int_0^{2018} (nx - [nx])\, dx;$$

PROBLEM AN.003.

Find:

$$\Omega = \lim_{n\to\infty} \int_0^{2018} \left|\left[nx + \frac{1}{2}\right] - nx\right| dx\,;$$

$[*]$ - great integer function.

PROBLEM AN.004.

If $0 < a \leq b < \frac{\pi}{2}$ then:

$$4\int_a^b ((\sin^2 x + \csc^2 x)^3 + (\cos^2 x + \sec^2 x)^3)\, dx \geq 125(b-a)$$

PROBLEM AN.005.

Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=0}^{n}\left(\frac{3^{n-k-1}(4n-4k-1)}{n-k+1} \cdot \binom{n+1}{k+1}\right)\right)$$

PROBLEM AN.006.

Find:

$$\Omega = \lim_{n\to\infty} n\left(\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)^{\sum_{k=1}^{n}\frac{1}{k^2}} - \left(\frac{\pi^2}{6}\right)^{\frac{\pi^2}{6}}\right)$$

PROBLEM AN.007.

If $si\,(x) = -\int_x^\infty \left(\frac{\sin t}{t}\right) dt; x > 0$ then:

$$\int_\gamma^e \left(\frac{1}{x}\left(si\,(e^2x) - si\,(\pi x)\right)\right) dx = \int_\pi^{e^2} \left(\frac{1}{x}\left(si\,(ex) - si\,(\gamma x)\right)\right) dx$$

PROBLEM AN.008.

If $0 \le a, b, c, d, e, f, x, y, z \le 1$ then:

$$108 \int_0^1 \int_0^1 \int_0^1 \frac{(3abcdefxyz - 1)dxdydz}{(a^2 + b^2 + x^2 + 3)(c^2 + d^2 + y^2 + 3)(e^2 + f^2 + z^2 + 3)} \le 1$$

PROBLEM AN.009.

If $a, b, c \ge 0$ then:

$$e^{2\sqrt{3}(a+b+c)} \ge \left((a^2 + a + 1)(b^2 + b + 1)(c^2 + c + 1)\right)^3$$

PROBLEM AN.010.

Find:

$$\Omega = \int \frac{(\sin x - \cos x)(1 + x) + 3 - 2\cos x}{(\sin x + \cos x + 4)^2} dx; x \in \left(0, \frac{\pi}{2}\right)$$

PROBLEM AN.011.

Find:

$$\Omega = \int \frac{(\sin x - \cos x)(1 + x) + 3 - 2\cos x}{(x - \sin x)(\sin x + \cos x + 4)} dx; x \in \left(0, \frac{\pi}{2}\right)$$

PROBLEM AN.012.

Prove that:

$$\left| \frac{1}{e^{2x^2}} - \int_0^1 \frac{dy}{e^{2y^2}} \right| \le \frac{2\sqrt{e}}{e}, x \in (0, 1)$$

PROBLEM AN.013.

Find:

$$\Omega = \int \frac{x^4 e^x dx}{(x^4 + 4x^3 + 12x^2 + 24x + 24 + 72e^x)^2}; x > 0$$

PROBLEM AN.014.

Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$

$$f(x)+f(y)+xy(x+y)=f(x+y);(\forall)x,y\in\mathbb{R}$$

PROBLEM AN.015.

If $a,b,c>0$ **and** $\Omega(a)=\int_a^{2a}\int_a^{2a}\frac{(x+y)^2+1}{xy+(x+y)\sqrt{3}}dxdy$ **then:**

$$\Omega(a)+\Omega(b)+\Omega(c)\geq ab+bc+ca$$

PROBLEM AN.016.

If $a,b,c>0;a+b+c=2$ **then:**

$$b^3\Omega(a)+c^3\Omega(b)+a^3\Omega(c)\geq\frac{8}{25}$$

Where $\Omega(a)=\lim_{n\to\infty}n\int_0^1\frac{x^n}{(x+a)^2}dx$

PROBLEM AN.017.

If $0<a\leq b\leq c;abc=1$ **then:**

$$\frac{\log^2(a^2+1)}{\log\left(\frac{b+1}{b}\right)\log\left(\frac{c+1}{c}\right)}\leq\frac{\log\left(\frac{b+1}{b}\right)\log\left(\frac{c+1}{c}\right)}{\log(b^2+1)\log(c^2+1)}$$

PROBLEM AN.018.

Find:

$$\Omega=\int\frac{x(\tan x+2\tan 2x+4\tan 4x)}{\cot x-8\cot 8x}dx$$

PROBLEM AN.019. Find:

$$\Omega=\lim_{n\to\infty}\left(1-\frac{\pi^2}{6}+\sum_{k=1}^{n}\frac{1}{k^2}\right)^n$$

PROBLEM AN.020.

Find:

$$\Omega=\lim_{n\to\infty}\left(1-\log 2+\sum_{k=1}^{n}\sin\frac{1}{n+k}\right)^n$$

PROBLEM AN.021.

If $f:[0,a]\to[0,\infty);a\geq 0;f$ **continuous then:**

$$\int_0^a\int_0^a \sqrt{f^2(x)+f^2(y)}\,dx\,dy+\int_0^a\int_0^a \sqrt{2f(x)f(y)}\,dx\,dy\le 2a\sqrt{2}\int_0^a f(x)\,dx$$

PROBLEM AN.022.

Find:

$$\Omega=\lim_{n\to\infty} n^{10}\int_0^{\frac{1}{n^7}}\frac{1+x^2}{1+x^2+x^4}\,dx$$

PROBLEM AN.023.

If $a,b,c\le 0$ then:

$$2\left(\frac{1}{11a+1}+\frac{1}{11b+1}+\frac{1}{11c+1}\right)\le 3+\frac{1}{10a+9b+1}+\frac{1}{10b+9c+1}+\frac{1}{10c+9a+1}$$

PROBLEM AN.024.

If $a,b,c\in\mathbb{N}^*;\ \Omega(a)=\lim_{n\to\infty} n\int_0^1\frac{x^n}{(x+\pi)^a}\,dx$ then:

$$(1+\pi)^b\Omega(a)+(1+\pi)^c\Omega(b)+(1+\pi)^a\Omega(c)\ge 3$$

PROBLEM AN.025.

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{6}{\pi^2}\sum_{k=1}^{n}\frac{1}{n^2}\right)^n$$

PROBLEM AN.026.

Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[7]{\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)(\log n)^6}-\log n\right)$$

PROBLEM AN.027.

If $\frac{\sqrt{3}}{3}\le a,b,c\le 1$ then:

$$\sqrt[3]{abc}\cdot\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)\le\sqrt{\frac{ab+bc+ca}{3}}\cdot\tan^{-1}\left(\sqrt[3]{abc}\right)$$

PROBLEM AN.028.

In $\Delta\, ABC$ the following relationship holds:

$$3\sum_{cyc(A,B,C)} A^2+\frac{9}{2}\le 3\sum_{cyc(A,B,C)} \sin A\tan A+\pi^2$$

PROBLEM AN.029.

If $f:[1,a]\to[1,\infty); a\ge 1; f$ continuous then:

$$3(a-1)^2\int_1^a\left(\frac{1}{f(x)}-f(x)\right)dx\ge\left(\int_1^a\frac{dx}{f(x)}\right)^3-\left(\int_1^a f(x)dx\right)^3$$

PROBLEM AN.030.

Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{\frac{\sqrt{1}}{0!}+\frac{\sqrt{2}}{1!}+\cdots+\frac{\sqrt{k+1}}{k!}}{(k+1)\sqrt{k+2}+(k+2)\sqrt{k+1}}\right)$$

PROBLEM AN.031.

Let be $x_n=\frac{3\pi}{2}-\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\right); n\ge 1$.Find:

$$\Omega=\lim_{n\to\infty}(1+x_n)^n$$

PROBLEM AN.032.

Find:

$$\Omega=\lim_{n\to\infty}\left(2\log 2-1+\sum_{k=1}^{n}\frac{1}{k(2k+1)}\right)^n$$

PROBLEM AN.033.

Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{\prod_{k=1}^{n}(e+k)}\right)\left(\frac{\pi^2}{8}-\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\right)$$

PROBLEM AN.034.

If $\omega(x)=\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\tanh\left(\frac{x}{2^n}\right)\right); x>0$ then find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to 0\\ x>0}}\left(x^n\omega(x)\omega(2x)\cdot\ldots\cdot\omega(nx)\right)}$$

PROBLEM AN.035.

If $0<a\le b\le c$ then:

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^{b}}\le\frac{1}{1+e^{a}}+\frac{1}{1+e^{c}}$$

PROBLEM AN.036.

If $1 \le x, y, z \le e$ then:

$$e^3 \le x\log x + y\log y + z\log z + e^{x+y+z} \le 3e + e^3$$

PROBLEM AN.037.

Find:

$$\Omega = \int \frac{242(x+2)^5 - (x+1)^5 - (x+3)^5}{26(x+2)^3 - (x+1)^3 - (x+3)^3} dx; x > 0$$

PROBLEM AN.038.

If $0 < a, b, c \le 1$;

$$\Omega(a) = \frac{8}{\pi}\int_0^a \frac{x\cos(10\cos^{-1}x)\,dx}{(x^2+a^2)(\cos(11\cos^{-1}x) + \cos(9\cos^{-1}x))}$$

Prove that:

$$\left(\Omega(a) + \Omega(b) + \Omega(c)\right)\left(6 + \frac{1}{\Omega^3(a)} + \frac{1}{\Omega^3(b)} + \frac{1}{\Omega^3(c)}\right) \ge 27$$

PROBLEM AN.039.

If $a, b > 1$ then:

$$\int_1^a\int_1^a \frac{dx\,dy}{\sqrt{x^2y + xy^2}} < \log\sqrt[4]{ab(a-1)(b-1)} + \left(\log\sqrt{a}\right)\left(\log\sqrt{b}\right)$$

PROBLEM AN.040.

If $a, b > 1; a < b$ then:

$$\int_a^b\int_a^b\int_a^b \left(\sqrt{\log_{yz} x} + \sqrt{\log_{zx} y} + \sqrt{\log_{xy} z}\right) \ge 2(b-a)^3$$

PROBLEM AN.041.

$\Omega_n = \binom{n}{7} + 2\binom{n-1}{7} + 3\binom{n-2}{7} + \cdots + (n-6)\binom{7}{7}; n \ge 7$**.Find:**

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\Omega_n}$$

PROBLEM AN.042.

If $0 < a \le b \le c$ then: $e^{(a-b+c)^2} + e^{b^2} \le e^{a^2} + e^{c^2}$

PROBLEM AN.043.

If $1 < a \le b$ then:

$$4\int_a^b\int_a^b (x^y + y^x)\,dx\,dy \ge (b-a)^2(4+(b-a)^2)$$

PROBLEM AN.044.

If $x, y > 0$;

$$\Omega(x,y) = \sum_{n=1}^{\infty} \frac{2n^2 + (2x+2y+5)n + 2xy + 6x - y}{3^n(n+y)(n+y+1)(n+y+2)}$$

then:

$$\Omega(x,y)\cdot\Omega(y,x) \le \frac{1}{9\sqrt[3]{xy}}$$

PROBLEM AN.045.

Let be:

$$\Omega = \sum_{n=1}^{\infty} \frac{an^2 + bn + c}{n!}; a,b,c > 0$$

Prove that:

$$\Omega(a,b,c) + \Omega(b,c,a) + \Omega(c,a,b) \ge 3(4e-1)\sqrt[3]{abc}$$

PROBLEM AN.046.

If $a,b,c > 0; 2e(a+b+c) = 3e+2; \Omega = \lim_{n\to\infty} n\left(\left(\frac{n+1}{n}\right)^{n+a} - e\right)$ then:

$$\Omega(a)\Omega(b)\Omega(c) \le \frac{1}{27}$$

PROBLEM AN.047.

If $a_n > 0; n \in \mathbb{N}; n \ge 2; \lim_{n\to\infty} \frac{a_n\cdot a_{n+2}^6\cdot a_{n+4}}{a_{n+1}^4\cdot a_{n+3}^4} = 10$. Find: $\Omega = \lim_{n\to\infty} \sqrt[n^4]{a_n}$.

PROBLEM AN.048.

If $a,b \in \mathbb{N}; a,b \ge 2$ then:

$$(2a-1)(3a-1)\cdot\ldots\cdot(a^2-1) + (2b-1)(3b-1)\cdot\ldots\cdot(b^2-1) > 2\sqrt{\frac{a!\cdot b!\cdot a^a\cdot b^b}{ab\cdot\sqrt[ab]{a^b\cdot b^a}}}$$

PROBLEM AN.049.

If $0 \le a < b$ then:

$$(1+ab-a^2)e^{a^2} < \frac{1}{b-a}\int_a^b e^{x^2}\,dx < (1-ab+b^2)e^{b^2}$$

PROBLEM AN.050.

Find all functions $f:\mathbb{R}\to[0,\infty)$ such that:

$$\sum f(x)\,f^2(y)+4\sum f(x)=4\sum f(x)\,f(y),(\forall)x,y,z\in\mathbb{R}$$

PROBLEM AN.051.

$$\Omega_n=\int_0^{\frac{\pi}{4}}(4x^2-\pi x+4n^2)\ln(1+\tan x)\,dx;n\in\mathbb{N}^*$$

Find: $\Omega=\lim_{n\to\infty}\frac{\Omega_n}{1+2+3+\cdots+n}$

PROBLEM AN.052.

If $a,b,c\geq 1$ then:

$$e^{ab}+e^{bc}+e^{ca}>3+\frac{c}{a}+\frac{b}{c}+\frac{a}{b}$$

PROBLEM AN.053.

If $a,b,c\in\mathbb{N}^*$ then:

$$\frac{1}{3}\sum\int_0^1 \sin^{-1}\left(x^a(1-x)^b\right)dx\geq\sqrt[3]{\prod\frac{(a!)^2}{(b+a+1)!}}$$

PROBLEM AN.054.

Find:

$$\Omega=\lim_{n\to\infty}\left(n^7\cdot\int_0^{\frac{1}{n^7}}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}\,dx\right)$$

PROBLEM AN.055.

If $0<a<b<1$ then:

$$\frac{2(\ln b-\ln a)}{b-a}+\frac{1}{b-a}\int_a^b\frac{dx}{\ln(1-x)}<1+\frac{1}{\sqrt{ab}}$$

PROBLEM AN.056.

Find:

$$\Omega = \lim_{n\to\infty} n^8 \int_0^{\frac{1}{n^5}} \frac{x^2+1}{x^4+x^2+1}\,dx$$

PROBLEM AN.057.

Find:

$$\Omega = \lim_{n\to\infty} n^2\left(\sqrt[n+5]{7} - \sqrt[n+8]{7}\right)$$

PROBLEM AN.058.

If $0 < a \le b$ then:

$$\int_a^b \left(\frac{\cos^2 x + \tan x}{\cos^4 x + \tan^2 x}\right)^2 dx \le \log\left|\frac{\tan b}{\tan a}\right|$$

PROBLEM AN.059.

Let be $f, g, h: \mathbb{R} \to (0,\infty)$ continuous functions and $a, b, c > 0$. Prove that:

$$\int_0^a\int_0^b\int_0^c \left(\sum \frac{f(x)}{\sqrt{f(y)\left(f(x)+f(y)\right)}}\right) dz\,dy\,dx \ge \frac{81\sqrt{2}}{2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}$$

PROBLEM AN.060.

Find:

$$\Omega = \lim_{n\to\infty} n^2\left(\sqrt[n+3]{\log 5} - \sqrt[n+4]{\log 5}\right)$$

PROBLEM AN.061.

If $p, q, r, s: \mathbb{R} \to (0,\infty)$ are continuous functions; $a \ge 0$ then:

$$\int_0^a s(x)\,dx \ge 4\int_0^a \sqrt[4]{p(x)q(x)r(x)s(x)}\,dx - 3\int_0^a \sqrt[3]{p(x)q(x)r(x)}\,dx$$

PROBLEM AN.062.

If $\frac{\pi}{4} < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{\sqrt[4]{1+\sin 2x} - \sqrt[4]{1-\sin 2x}}{\sqrt[4]{1+\sin 2x} + \sqrt[4]{1-\sin 2x}} \le \log\left|\frac{\cos a}{\cos b}\right|$$

PROBLEM AN.063.

If $f: \mathbb{R} \to \left[0, \frac{\pi}{2}\right]$ f continuous and $\int_0^a \sin^2 f(x)\,dx = \frac{\pi}{2}$; $a > 0$ then:

$$\int_0^a \cos(\sin f(x))\,dx < \frac{1}{2}$$

PROBLEM AN.064.

If $a, b, c \geq \frac{\pi}{4}$ then:

$$\big(1 + 2\Omega(a)\big)b^2 + \big(1 + 2\Omega(b)\big)c^2 + \big(1 + 2\Omega(c)\big)a^2 \leq a^4 + b^4 + c^4$$

where:

$$\Omega(a) = \int_{\frac{\pi}{4}}^{a} \left(\frac{x^2 + 1 + \tan^{-1} x}{x^4 + 2x^2 + 1 + (\tan^{-1} x)^2}\right) dx$$

PROBLEM AN.065.

If $0 < a < b < 1$ then:

$$\frac{1}{2} + \frac{2}{a+b} < \frac{2(\ln b - \ln a)}{b-a} + \frac{1}{b-a}\int_a^b \frac{dx}{\ln(1-x)}$$

PROBLEM AN.066.

Prove that:

$$\int_0^1 (1 + 2x + 3x^2 + 4x^3)e^{x^2}\,dx \geq 4\int_0^1 \sqrt[16]{e^{\left(x+\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}\right)^2}}\,dx$$

PROBLEM AN.067.

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\Omega_n(a) - (a+1)!}\,; n \in \mathbb{N};$$

$$\Omega_n(a) = \sum_{k=0}^{n} (k^2 - a^2 + 1)(a+k)!\,; n \in \mathbb{N}$$

PROBLEM AN.068.

If $x < a$ then:

$$(1 + ax - x^2)e^{x^2} < \frac{1}{a-x}\int_x^a e^{x^2}\,dx < (a^2 - ax)e^{a^2} + e^{x^2}$$

PROBLEM AN.069.

Let be

$$\Omega(a)=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{3^k}\sin^3(3^k\sin a)\right)$$

Prove that if $a,b,c\in\left[0,\frac{\pi}{2}\right]$ then: $4\big(b\Omega(a)+c\Omega(b)+a\Omega(c)\big)\le 3(a^2+b^2+c^2)$

PROBLEM AN.070.

Let be:

$$\Omega(x)=\lim_{n\to\infty}\sum_{k=1}^{n}\left(2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right)\right)$$

Prove that: $\Omega(A)+\Omega(B)+\Omega(C)>ABC-\pi$

PROBLEM AN.071.

Find:

$$\Omega=\lim_{p\to\infty}\sqrt[p]{\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k^2+kp-1}{(p+k+1)!}\right)}$$

PROBLEM AN.072.

If $a,b,c>0;a+b+c=3;0\le x\le 1$ then:

$$a\left(\frac{b}{a}\right)^x+b\left(\frac{c}{b}\right)^x+c\left(\frac{a}{c}\right)^x+b\left(\frac{a}{b}\right)^x+c\left(\frac{b}{c}\right)^x+a\left(\frac{c}{a}\right)^x\le 6$$

PROBLEM AN.073.

Find:

$$\Omega=\sum_{k=1}^{\infty}\left(\sum_{\substack{n=1\\ n\neq k}}^{\infty}\left(\frac{1}{n^2-k^2}\right)\right)$$

PROBLEM AN.074.

Find:

$$\Omega=\int\left(\sum_{n=1}^{\infty}\left(3^n\sinh^3\left(\frac{x}{3^n}\right)\right)\right)dx$$

PROBLEM AN.075. If $1<a\le b$ then:

$$\int_a^b\int_a^b\int_a^b\frac{dx\,dy\,dz}{1+\sqrt[3]{xyz}}\le\log\left(\sqrt[3]{\frac{b+1}{a+1}}\right)^{(b-a)^2}$$

PROBLEM AN.076.

If $0 < a \le b$ then:

$$\frac{e^{4b^2} - e^{4a^2}}{8be^{b^2}} \le \int_a^b e^{4x^2}\,dx \le \frac{e^{4b^2} - e^{4a^2}}{8ae^{a^2}}$$

PROBLEM AN.077.

If $0 < a \le b < \frac{\pi}{2}$ then:

$$\frac{\cos(\cos b) - \cos(\cos a)}{\sin a} \le \int_a^b \sin(\cos x)\,dx \le \frac{\cos(\cos b) - \cos(\cos a)}{\sin b}$$

PROBLEM AN.078.

If $0 < a \le c \le b$ then:

$$\frac{(b^{30} - a^{30})(b^{30} - c^{30})}{36b^{10}} \le \frac{(b^{25} - a^{25})(b^{25} - c^{25})}{25} \le \frac{(b^{30} - a^{30})(b^{30} - c^{30})}{36(ac)^5}$$

PROBLEM AN.079.

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{6 - 2\sum_{i=2}^{n} \frac{1}{i+1}\binom{2i}{i} + 3\sum_{i=2}^{n}\binom{2i}{i}}$$

PROBLEM PROBLEM AN.080.

If $1 \le a \le b$ then:

$$\int_a^b\int_a^b\int_a^b \left(\frac{1}{1+\sqrt{xy}} + \frac{1}{1+\sqrt{yz}} + \frac{1}{1+\sqrt{zx}}\right) dx\,dy\,dz \ge 3(b-a)^2 \log\left(\frac{b+1}{a+1}\right)$$

PROBLEM AN.081.

If $f: [a,b] \to [1,\infty)$; $0 < a \le b$; f integrable then:

$$\int_a^b\int_a^b\int_a^b \frac{3 + f(x) + f(y) + f(z)}{f(x)f(y) + f(y)f(z) + f(z)f(y)}\,dx\,dy\,dz \le (b-a)^3 + \left(\int_a^b \frac{dx}{f(x)}\right)^3$$

PROBLEM AN.082.

If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ then find: $\Omega = e^A \cdot (e^B)^{-1}$; e^A – exponential matrix

PROBLEM AN.083.

If $a,b,c\in\left(0,\frac{\pi}{2}\right)$;

$$\Omega(a,b)=\int_{-\sin a}^{\sin a}\frac{dx}{x^5+\sin b+\sqrt{x^{10}+\sin^2 b}}$$

then: $\Omega(a,b)+\Omega(b,c)+\Omega(c,a)\ge 3$

PROBLEM AN.084.

If $\alpha,\beta>1; 3\alpha-2\beta>1$ then:

$$\zeta(3\alpha-2\beta)\big(\zeta(\beta)\big)^2\ge\big(\zeta(\alpha)\big)^3;\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

PROBLEM AN.085.

If $\Omega_n=\sum_{k=1}^{n}\left(\int_{-\frac{1}{k}}^{\frac{1}{k}}(2x^8+3x^6+1)\cdot\cos^{-1}(kx)\,dx\right)$ then find:

$$\Omega=\lim_{n\to\infty}(\Omega_n-\pi\cdot H_n)$$

PROBLEM AN.086.

Find:

$$\Omega=\lim_{n\to\infty}\left(H_n^2\left(\frac{\pi}{2}-\tan^{-1}(H_n)\right)-H_n\right)$$

where $H_n=\sum_k^n\frac{1}{k}$ – harmonic number

PROBLEM AN.087.

If $\Omega(k)=\int_0^1\frac{x^k-kx^{k-1}-x^{k-1}+1}{x^{2k+1}+2x^{k+1}+x^k+x+1}dx; k\in\mathbb{N}; k\ge 1$ then find:

$$\Omega=\lim_{n\to\infty}\left(\log 2\cdot\log n-n\cdot\log\left(\frac{3}{2}\right)+\sum_{k=1}^{n}\Omega(k)\right)$$

PROBLEM AN.088.

If $x,y,z>0$ then:

$$\tan^{-1}\left(\frac{x^4+y^4}{(x^2+y^2)(x^2-xy+y^2)}\right)+\tan^{-1}\left(\frac{y^4+z^4}{(y^2+z^2)(y^2-yz+z^2)}\right)+$$
$$+\tan^{-1}\left(\frac{z^4+x^4}{(z^2+x^2)(z^2-zx+x^2)}\right)\ge\frac{3\pi}{4}$$

PROBLEM AN.089.

If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \left(e^{\sin^2 x + \frac{\sin^4 x}{2\cos^2 x}}\right) dx \ge \tan b - \tan a$$

PROBLEM AN.090.

If $f:[a,b] \to (0,\infty)$, f continuous; $0 < a \le b$ then:

$$\int_a^b \int_a^b \left(\frac{f(x)}{f(y)} + \frac{f(y)}{f(x)}\right) \cdot \frac{dxdy}{f(x)+f(y)} \ge (b-a)\int_a^b \frac{dx}{f(x)}$$

PROBLEM AN.091.

If $a, b, c > 0$ then:

$$\tan^{-1}\left(\frac{(2a+b)(a+2b)}{9ab}\right) + \tan^{-1}\left(\frac{(2b+c)(b+2c)}{9bc}\right) +$$

$$+\tan^{-1}\left(\frac{(2c+a)(c+2a)}{9ca}\right) \ge \frac{3\pi}{4}$$

PROBLEM AN.092.

If $0 < a \le b$ then:

$$\frac{3}{2}\int_a^b \int_a^b \left(\frac{x^2+y^2}{x^4+x^2y^2+y^4}\right) dxdy \le \left(\log\frac{b}{a}\right)^2$$

PROBLEM AN.093.

If $f:[a,b] \to (0,\infty)$; $0 < a \le b$; f continuous then:

$$\int_a^b \int_a^b \int_a^b \Omega(x,y,z)\, dxdydz \ge 3(b-a)^2 \int_a^b f^3(x)\, dx$$

$$\Omega(x,y,z) = \left(\frac{f^2(x)+f^2(y)}{f(x)+f(y)}\right)^3 + \left(\frac{f^2(y)+f^2(z)}{f(y)+f(z)}\right)^3 + \left(\frac{f^2(z)+f^2(x)}{f(z)+f(x)}\right)^3$$

PROBLEM AN.094.

If $f:[a,b] \to (0,\infty)$; $0 < a \le b$; f continuous then:

$$\int_a^b \int_a^b \frac{\left(f^2(x)+f(y)\right)\left(f^2(y)+f(x)\right)}{\left(1+f(x)\right)\left(1+f(y)\right)} dxdy \ge \left(\int_a^b f(x)\, dx\right)^2$$

PROBLEM AN.095.

If $x, y, z > 0$ then:

$$\frac{x^2}{2}\left(\frac{x}{3}+y+z\right)+\frac{y^3}{3}\left(x+\frac{y}{4}+z\right)+\frac{z^6}{6}\left(x+y+\frac{z}{7}\right)>\frac{xyz(x+y+z)}{2}$$

PROBLEM AN.096.

If $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}; s>1$ then:

$$\frac{1}{2}\zeta(4)+\frac{1}{3}\zeta(9)+\frac{1}{6}\zeta(30)>\zeta(10)$$

PROBLEM AN.097.

Find:

$$\Omega=\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\frac{n^2+n+k^2}{n^2+k^2}\right)\right)$$

PROBLEM AN.098.

If $f, g, h: [0,1] \to (0,\infty); f, g, h$ continuous then:

$$27e^{\int_0^1 \log(f(x)\cdot g(x)\cdot h(x))dx}\le\left(\int_0^1\big(f(x)+g(x)+h(x)\big)\,dx\right)^3$$

PROBLEM AN.099.

If $0<a<b$ then:

$$\frac{\int_a^b(\tan^{-1}x)\,dx}{\int_a^{\sqrt{ab}}(\tan^{-1}x)\,dx}>1+\sqrt{\frac{b}{a}}$$

PROBLEM AN.100.

If $0<a<b$ then:

$$\frac{\int_a^{\frac{a+b}{2}}(\tan^{-1}t)\,dt}{\int_a^b(\tan^{-1}t)\,dt}<\frac{1}{2}$$

PROBLEM AN.101.

If $1<a<b$ then:

$$\frac{\int_a^b(\log t)\,dt}{\int_{\sqrt{ab}}^b(\log t)\,dt}<1+\sqrt{\frac{a}{b}}$$

PROBLEM AN.102.

Prove that if $n \in \mathbb{N}; n \geq 1$ then:

$$\left|\int_0^1 \log(1+x^2)\,dx - \frac{1}{n}\sum_{k=1}^{n} \log\left(1+\frac{k^2}{n^2}\right)\right| \leq \frac{1}{2n}$$

PROBLEM AN.103.

If $n \in \mathbb{N}; n \geq 1$ then:

$$\left|\sum_{k=1}^{n} \tan^{-1}\left(\frac{k+1}{\sqrt{3}}\right) - \int_0^n \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) dx\right| \leq \frac{n\sqrt{3}}{18}$$

PROBLEM AN.104.

If $a, b \in \mathbb{R}; a \leq b$ then:

$$\left(\int_a^b \left(\frac{x^2+1}{x^4+1}\right) dx\right)\left(\int_a^b \left(\frac{x^4+1}{x^6+1}\right) dx\right)\left(\int_a^b \left(\frac{x^6+1}{x^2+1}\right) dx\right) \geq (b-a)^3$$

PROBLEM AN.105.

If $a \leq b; a, b \in \mathbb{R}; f_1, f_2, f_3: [a, b] \to (0, \infty)$ then:

$$\left(\int_a^b \frac{f_1(x)}{f_2(x)} dx\right)\left(\int_a^b \frac{f_2(x)}{f_3(x)} dx\right)\left(\int_a^b \frac{f_3(x)}{f_1(x)} dx\right) \geq (b-a)^3$$

PROBLEM AN.106.

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt{3}}\prod_{k=1}^{n}\left(\frac{4k-1}{4k+1}\right)\right)$$

PROBLEM AN.107.

Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{2^n}\prod_{k=1}^{n}\left(2-\frac{3}{3k-1}\right)$$

PROBLEM AN.108.

Find:

$$\Omega = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{(25k^2+5k-6)(n-k+1)^2}\right)$$

PROBLEM AN.109.

Find:

$$\Omega = \lim_{m\to\infty}\sqrt[m]{\sum_{n=1}^{\infty}\left(\frac{2n^2+2nm+n-1}{(2n+2m+2)!!}\right)}$$

PROBLEM AN.110.

If $f:[1,2]\to(0,\infty)$; f continuous; $\int_1^2 f(x)\,dx=1$ then:

$$\int_1^2 f^2(x)\,dx+\int_1^2\frac{dx}{f(x)}\geq 2$$

PROBLEM AN.111.

If $f\in C^1([1,2])$; $f(1)=1$; $f(2)=2$ then:

$$\int_1^2\left(f'(x)\right)^2dx+\int_1^2\frac{dx}{f'(x)}\geq 2$$

PROBLEM AN.112.

If $f\in C^1([1,2])$; $f(1)=1$; $f(2)=2$ then:

$$\int_1^2\int_1^2\int_1^2\frac{\left(\left(f'(x)\right)^2+\left(f'(y)\right)^2\right)\left(\left(f'(y)\right)^2+\left(f'(z)\right)^2\right)\left(\left(f'(z)\right)^2+\left(f'(x)\right)^2\right)}{\left(f'(x)+f'(y)\right)\left(f'(y)+f'(z)\right)\left(f'(z)+f'(x)\right)}dx\,dy\,dz\geq 1$$

PROBLEM AN.113.

If $x_n\subset(0,\infty)$; $n\in\mathbb{N}$; $n\geq 1$ and $\lim_{n\to\infty}\frac{x_{n+4}x_{n+2}^6x_n}{x_{n+3}^4x_{n+1}^4}=e^{24}$ then find: $\Omega=\lim_{n\to\infty}\sqrt[n^4]{x_n}$

PROBLEM AN.114.

If $x_n=\sum_{k=0}^{n}\frac{2^k}{5^n}\binom{n}{2n-k}$ then find: $\Omega=\lim_{n\to\infty}\left(\frac{1}{x_n}-\log(x_n+1)^{\frac{1}{x_n^2}}\right)$

PROBLEM AN.115.

If $0<a\leq b$; $f:[a,b]\to(0,1]$, f continuous then:

$$\int_a^b \int_a^b \frac{f(x)+f(y)}{\left(f(x)\right)^{f(y)}+\left(f(y)\right)^{f(x)}}\,dx\,dy+\left(\int_a^b f(x)\,dx\right)^2 \le 2(b-a)\int_a^b f(x)\,dx$$

PROBLEM AN.116.

If $0<a\le b$ then:

$$\int_a^b\left(\int_a^b\left(\sqrt{\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}}\right)dx\right)dy \ge \sqrt{3}\log\left(\frac{b}{a}\right)^{b-a}$$

PROBLEM AN.117.

If $f:[a,b]\to(0,\infty)$; $0<a\le b$; f derivable; f' continuous then:

$$\int_a^b \frac{f'(x)\sqrt{f(x)}}{f^3(x)+1}\,dx \le \tan^{-1}\left(\frac{f(b)-f(a)}{1+f(a)f(b)}\right)$$

PROBLEM AN.118.

If $0<a\le b$ then:

$$\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}\,dx \le \frac{1}{2}\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right)$$

PROBLEM AN.119.

Find:

$$\Omega=\int \frac{(x-1)\cos x-(x+1)\sin x}{x^2+\sin 2x+1}\,dx;\ x\in\mathbb{R}$$

PROBLEM AN.120.

Find:

$$\Omega=\lim_{n\to\infty}\frac{\tan\left(\left(\frac{H_n}{n}\right)^{\frac{n^2}{H_n^2}}\right)}{\left(\tan\left(\frac{H_n}{n}\right)\right)^{\frac{n^2}{H_n^2}}};\ H_n=1+\frac{1}{2}+\cdots+\frac{1}{n};\ n\in\mathbb{N}^*$$

PROBLEM AN.121.

If $0<a\le b<\frac{\pi}{2}$ then:

$$\left(\int_0^{\sqrt{ab}}\left(\sqrt[3]{x}\sin x\right)dx\right)\left(\int_0^{\frac{a+b}{2}}\left(\sqrt[3]{x}\cos x\right)dx\right)\le\left(\int_0^{\sqrt{ab}}\left(\sqrt[3]{x}\cos x\right)dx\right)\left(\int_0^{\frac{a+b}{2}}\left(\sqrt[3]{x}\sin x\right)dx\right)$$

PROBLEM AN.122.

Find:

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n]{1+\sin x}+\sqrt[n]{1-\sin x}-2}{\sqrt[n+1]{1+\sin x}+\sqrt[n+1]{1-\sin x}-2}$$

PROBLEM AN.123.

If $0<a\le b$; $f:[a,b]\to(0,\infty)$; f derivable in $[a,b]$ then:

$$\int_a^b\int_a^b\int_a^b\left(\frac{f(x)f'(y)f'(z)}{f^2(x)+f(y)f(z)}\right)dx\,dy\,dz\le\log\left(\sqrt{\frac{f(b)}{f(a)}}\right)^{(b-a)(f(b)-f(a))}$$

PROBLEM AN.124.

Find:

$$\Omega=\lim_{n\to\infty}\left(1-\frac{\pi^4}{90}+\sum_{k=1}^{n}\frac{1}{k^4}\right)^n$$

PROBLEM AN.125.

If $H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$; $n\ge1$ then find:

$$\Omega=\lim_{n\to\infty}H_n^2\left[\left(\frac{1+H_n}{H_n}\right)^{H_n}-\log\left(\frac{1+H_n}{H_n}\right)^{eH_n}\right]$$

PROBLEM AN.126.

If $0<a\le b<\frac{\pi}{2}$ then:

$$\int_a^b\sin(e^x)\,dx\ge\log\left(\frac{e^b+\sqrt{1+e^{2b}}}{e^a+\sqrt{1+e^{2a}}}\right)$$

PROBLEM AN.127.

Find:

$$\Omega=\lim_{n\to\infty}\left(4-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)^n$$

PROBLEM AN.128.

If $a, b, c > 1$;

$$\Omega(a,b) = \lim_{x\to\infty} \frac{x\left(a^{\frac{1}{x}} - 1\right) - \ln a}{x\left(b^{\frac{1}{x}} - 1\right) - \ln b}$$

then: $\ln a \cdot \Omega(a,b) + \ln b \cdot \Omega(b,c) + \ln c \cdot \Omega(c,a) \geq \ln(abc)$

PROBLEM AN.129.

Find:

$$\Omega = \lim_{n\to\infty}\left(\log(2n+1) - \sum_{k=1}^{n}\left(\frac{1}{k[\sqrt{k}]} \cdot \left[\frac{\sqrt{k} + \sqrt{k+1} + \sqrt{k+2}}{3}\right]\right)\right)$$

$[*]$ **- great integer function.**

PROBLEM AN.130.

Let be:

$$\omega(n) = \sum_{k=1}^{n}\left[\frac{i^2 + i + 1}{i^2 - i + 1}\right]; [*] - great\ integer\ function$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\log(3n+1) - \sum_{k=1}^{n}\frac{1}{\omega(k)}\right)$$

PROBLEM AN.131.

If $x_n = \sum_{i=1}^{n}\left[\frac{i-\sqrt{i}}{i-\sqrt{i+\sqrt{i}}}\right]$; $[*]$ - great integer function then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1 + x_n^2 \log\left(\frac{1+x_n}{x_n}\right)}{x_n}\right)^{x_n}$$

PROBLEM AN.132.

If $a, b \in \mathbb{R}$ then:

$$6\int_a^b (\tan^{-1} x)\, dx \geq 3\log\left(\frac{1+b^2}{1+a^2}\right) + a^3 - b^3$$

PROBLEM AN.133.

Let be:

$$\omega=\sum_{n=1}^{\infty}\frac{1}{\left[\sqrt[3]{(n^3+2n+1)}\right]^2}$$

Find:

$$\Omega=\lim_{n\to\infty}n\left(\omega-\sum_{k=1}^{n}\frac{1}{k^2}\right)$$

PROBLEM AN.134.

If $n\geq 1$ then:

$$\frac{1}{\log 2}\left(\frac{2^n-1}{n}\right)^{2n+1}\leq\frac{1\cdot 3\cdot 7\cdot\ldots\cdot(2^{2n}-1)}{(2n)!}$$

PROBLEM AN.135.

If $x_1=2; x_2=4; x_3=10;$

$$x_{n+3}-5x_{n+2}+7x_{n+1}-3x_n=0; n\in\mathbb{N}; n\geq 1$$

then find:

$$\Omega=\lim_{n\to\infty}\left(x_n^2\left(3^{\frac{1}{x_n}}-1\right)-x_n\log 3\right)$$

PROBLEM AN.136.

Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{p=1}^{n}\left(\frac{1}{\sum_{k=1}^{p}\left\{\frac{kp}{p+1}\right\}}\right)-2\log(2n+1)\right)$$

PROBLEM AN.137.

$$\omega(n)=n\prod_{i=2}^{n}\left(\frac{i^3+1}{i^3-1}\right)$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\omega^2(n)\left(1+\frac{1}{\omega(n)}\right)^{\frac{1}{\omega(n)}}-\omega^2(n)\cos\left(\frac{1}{\omega^2(n)}\right)\right)$$

PROBLEM AN.138.

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{2(k+1)^2}\right)\tan^{-1}\left(\frac{2k^2+4k+1}{2(k+1)}\right)\right)$$

PROBLEM AN.139.

Let be $f:\mathbb{R}\to(0,\infty)$ continuous such that for $a,b,c>0$, fixed values

$$a^3f(x)+b^3f(y)+c^3f(z)=f(x)f(y)f(z),(\forall)\ x,y,z\in\mathbb{R}$$

Prove that:

$$\int_{\alpha}^{\beta} f(x)\,dx \geq \frac{(\beta-\alpha)(a+b+c)\sqrt{a+b+c}}{3};(\forall)0<\alpha\leq\beta$$

PROBLEM AN.140.

Find:

$$\Omega = \int e^x\left(4\cot^3 x+\cot^2 x+\cot x-2\right)dx; x\in\left(0,\frac{\pi}{2}\right)$$

PROBLEM AN.141.

Let be $x_n>0; n\geq 1; \lim_{n\to\infty}\left(n(n+1)(x_{n+1}-x_n)\right)=a, \lim_{n\to\infty}x_n=b; a,b\in\mathbb{R}$. Find in terms of a,b:

$$\Omega = \lim_{n\to\infty}\left(n\left(x_n^b-b^{x_n}\right)\right)$$

PROBLEM AN.142.

If $0<a\leq b$ then:

$$\int_a^b\int_a^b\int_a^b\int_a^b \frac{(x+y+z+t)dxdydzdt}{\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}} \leq \frac{(b+a)^2(b-a)^4}{4ab}$$

PROBLEM AN.143.

If $0\leq a\leq b\leq c\leq d$ then:

$$\tan^{-1}d-\tan^{-1}a \leq \frac{b-a}{1+a^2}+\frac{c-b}{1+b^2}+\frac{d-c}{1+c^2}$$

PROBLEM AN.144.

If $H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}; n\geq 1$ then find:

$$\Omega = \lim_{n\to\infty}H_n\left(\left(\pi+\frac{1}{H_n}\right)^{\pi^{\left(\pi+\frac{1}{H_n}\right)}}-\pi^{\left(\pi+\frac{1}{H_n}\right)^{\pi}}\right)$$

PROBLEM AN.145.

If $f:(0,\infty)\to(1,\infty)$; f continuos; $0<a\le b$ then:

$$4(b-a)^3+6(b-a)^2\int_a^b \log(f(x))\,dx\le 3(b-a)^2\int_a^b f(x)\,dx+\left(\int_a^b f(x)\,dx\right)^3$$

PROBLEM AN.146.

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{H_n}\sum_{k=1}^{n}\left(\frac{1\cdot\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[k]{k!}}{(k+1)!}\right)\right)$$

where $H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$; $n\ge 1$ (harmonic number)

PROBLEM AN.147.

If $f:[a,b]\to(0,\infty)$; $a<b$; f continuous then:

$$3(b-a)\int_a^b f^2(x)\,dx+(b-a)^2\ge 2\int_a^b f(x)\,dx+2\left(\int_a^b f(x)dx\right)^2$$

PROBLEM AN.148.

If $x_1=14; x_2=81; x_3=564$

$$x_{n+3}-14x_{n+2}+65x_{n+1}-100x_n=0;\ n\ge 1;\ x_n\in\mathbb{R}$$

then find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\frac{x_{n+1}}{x_n}}$$

PROBLEM AN.149.

If $a,b,c\in\left(0,\frac{\pi}{2}\right)$ then:

$$\left(\frac{a+b+c}{ab+bc+ca}\sin\left(\frac{ab+bc+ca}{a+b+c}\right)\right)^{a+b+c}\ge\left(\frac{\sin b}{b}\right)^a\left(\frac{\sin c}{c}\right)^b\left(\frac{\sin a}{a}\right)^c$$

PROBLEM AN.150.

If $0<z<y<x<\frac{\pi}{2}$ then:

$$\frac{\sin x}{\sin y}+\frac{\sin x+\sin y}{\sin z}>\frac{6}{\pi}\sqrt[3]{\left(\frac{x}{z}\right)^2}$$

PROBLEM AN.151.

Let be the sequence: $x_1 = 10, x_2 = 64, x_3 = 352, x_4 = 1702$

$$x_{n+4} - 10x_{n+3} + 36x_{n+2} - 54x_{n+1} + 27x_n = 0; n \in \mathbb{N}; n \geq 1$$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{x_n \cdot x_{n+3}}{x_{n+1} \cdot x_{n+2}}}$$

PROBLEM AN.152.

If $0 < a \leq b < \frac{\pi}{2}$ **then:**

$$\int_a^b \left((\sin x)^{2\cos^2 x} + (\cos x)^{2\sin^2 x}\right) dx \geq \sqrt{2}\left(\tan^{-1}\left(\frac{\tan b}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{\tan a}{\sqrt{2}}\right)\right)$$

PROBLEM AN.153.

If $0 < a \leq b < \frac{\pi}{4}$ **then:**

$$\int_a^b \left((\tan x)^{\cot x} + (\cot x)^{\tan x}\right) dx \geq b - a$$

SOLUTIONS

ALGEBRA

SOLUTION A.001.

$$x \in (0,1) \Rightarrow x\sqrt{3} \in \left(0, \sqrt{3}\right)$$

$$\left(x\sqrt{3}-1\right)^2\left(x\sqrt{3}+2\right) \geq 0$$

$$\left(3x^2-2\sqrt{3}x+1\right)\left(x\sqrt{3}+2\right) \geq 0$$

$$3\sqrt{3}x^3+6x^2-6x^2-4\sqrt{3}x+x\sqrt{3}+2 \geq 0$$

$$3\sqrt{3}x^3-3\sqrt{3}x+2 \geq 0$$

$$2 \geq 3\sqrt{3}x-3\sqrt{3}x^2$$

$$2 \geq 3\sqrt{3}x(1-x^2)$$

$$\frac{2}{3(1-x^2)} \geq \sqrt{3}x$$

$$\left(\frac{2}{3(1-x^2)}\right)^6 \geq \left(\sqrt{3}x\right)^6 = 3^3 \cdot x^6$$

$$\left(\frac{2}{1-x^2}\right)^6 \geq 3^9 \cdot x^6$$

$$\sum_{cyc(x,y,z)} \left(\frac{2}{1-x^2}\right)^6 \geq 3^9 \cdot \sum_{cyc(x,y,z)} x^6 = 3^9 \cdot \frac{1}{9} = 3^7$$

Equality holds for $x^6 = y^6 = z^6 = \frac{1}{27}$ ***or*** $x = y = z = \frac{1}{\sqrt{3}}$.

SOLUTION A.002.

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3} = 1 \Rightarrow x^3y^3+y^3z^3+z^3x^3 = x^3y^3z^3$$

$$x^3y^3(z^3-1) = z^3(y^3+x^3)$$

$$z^3-1 = z^3 \cdot \frac{y^3+x^3}{x^3y^3}$$

$$z^3-1 = \frac{z^3}{x^3}+\frac{z^3}{y^3} \quad (1)$$

Analogous: $x^3-1=\frac{x^3}{y^3}+\frac{x^3}{z^3}$ *(2);* $y^3-1=\frac{y^3}{z^3}+\frac{y^3}{x^3}$ *(3)*

By adding (1); (2); (3):

$$\sum_{cyc(x,y,z)} x^3-3=\sum\frac{y^3+z^3}{x^3}$$

$$\sum_{cyc(x,y,z)} x^3-3\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)\le\sum\frac{y^3+z^3}{x^3}$$

$$\sum_{cyc(x,y,z)}\frac{y^3+z^3+3}{x^3}\ge\sum_{cyc(x,y,z)}x^3\overset{AM-GM}{\ge}3xyz$$

Equality holds for: $x=y=z=\sqrt[3]{3}$.

SOLUTION A.003.

$$\frac{a+b}{2}\overset{AM-GM}{\ge}\sqrt{ab}\Rightarrow a+b\ge2\sqrt{ab}\Rightarrow(a+b)^2\ge4ab\Rightarrow\frac{(a+b)^2}{4ab}\ge1\Rightarrow z\ge1$$

$$\frac{a+b+c}{3}\overset{AM-GM}{\ge}\sqrt[3]{abc}\Rightarrow\frac{(a+b+c)^3}{27}\ge abc\Rightarrow\frac{(a+b+c)^3}{27abc}\ge1\Rightarrow y\ge1$$

$$\frac{a+b+c+d}{4}\overset{AM-GM}{\ge}\sqrt[4]{abcd}\Rightarrow\frac{(a+b+c+d)^4}{256}\ge abcd\Rightarrow\frac{(a+b+c+d)^4}{256abcd}\ge1\Rightarrow x\ge1$$

We prove that: $z\le y\le x$ *(1)*

$$z\le y\Leftrightarrow\frac{(a+b)^2}{4ab}\le\frac{(a+b+c)^3}{27abc}\Leftrightarrow$$

$$\Leftrightarrow 27c(a+b)^2\le4(a+b+c)^3 \quad \textit{(to prove) (2)}$$

$$2c(a+b)^2=2c(a+b)(a+b)\overset{AM-GM}{\le}\left(\frac{2c+a+b+a+b}{3}\right)^3=\frac{8(a+b+c)^3}{27}$$

$$27\cdot2c(a+b)^2\le8(a+b+c)^3\Rightarrow \textit{(2)}$$

$$y\le x\Leftrightarrow\frac{(a+b+c)^3}{27abc}\le\frac{(a+b+c+d)^4}{256abcd}\Leftrightarrow$$

$$\Leftrightarrow256d(a+b+c)^3\le27(a+b+c)^4 \quad \textit{(to prove) (3)}$$

$$3d(a+b+c)^3=3d(a+b+c)(a+b+c)(a+b+c)\overset{AM-GM}{\le}$$

$$\le\left(\frac{3d+a+b+c+a+b+c+a+b+c}{4}\right)^4=\frac{81(a+b+c)^4}{256}$$

$$3d(a+b+c)^3\cdot256\ge81(a+b+c+d)^4\Rightarrow \textit{(3)}$$

By $a,b,c,d\ge1\Rightarrow ab\le abc\le abcd$

Systems (x, y, z); $(abcd, abc, ab)$ are same orientation. By Cebyshev's inequality:

$$(abcdx + abcy + abz) \geq \frac{1}{3}(abcd + abc + ab)(x + y + z)$$

$$3(abcdx + abcy + abz) \geq (abcd + abc + ab)(x + y + z)$$

SOLUTION A.004.

$$A^{2p}B^{2p} = A^{2p}(I_n - A^{2p+1}) = A^{2p} - A^{4p+1} = A^{2p} - A^{2p} = O_n$$

$$B^{2p}A^{2p} = (I_n - A^{2p+1})A^{2p} = A^{2p} - A^{4p+1} = O_n$$

$$(I_n + A^{2p})(I_n + B^{2p}) = I_n + A^{2p} + B^{2p} + A^{2p}B^{2p} = I_n + A^{2p} + B^{2p} + O_n =$$

$$= I_n + A^{2p} + B^{2p}$$

$$\det(I_n + A^{2p} + B^{2p}) = \det\Big((I_n + A^{2p})(I_n + B^{2p})\Big) = \det(I_n + A^{2p}) \cdot \det(I_n + B^{2p}) =$$

$$= \det(I_n + (A^p)^2) \cdot \det(I_n + (B^p)^2) \geq 0$$

SOLUTION A.005.

First, we prove: $\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2 \geq \sqrt{xy} \Leftrightarrow (x+y)^2 \geq \sqrt{xy}\left(\sqrt{x}+\sqrt{y}\right)^2 \Leftrightarrow$

$$\Leftrightarrow x^2 + 2xy + y^2 \geq \sqrt{xy}\left(x + y + 2\sqrt{xy}\right)$$

$$x^2 + 2xy + y^2 \geq x\sqrt{xy} + y\sqrt{xy} + 2xy, x^2 + y^2 \geq x\sqrt{xy} + y\sqrt{xy}$$

$$x^2 - x\sqrt{xy} + y^2 - y\sqrt{xy} \geq 0,\ x\sqrt{x}\left(\sqrt{x} - \sqrt{y}\right) - y\sqrt{y}\left(\sqrt{x} - \sqrt{y}\right) \geq 0$$

$$\left(\sqrt{x} - \sqrt{y}\right)\left(x\sqrt{x} - y\sqrt{y}\right) \geq 0,\ \left(\sqrt{x} - \sqrt{y}\right)\left[\left(\sqrt{x}\right)^3 - \left(\sqrt{y}\right)^3\right] \geq 0$$

$$\left(\sqrt{x} - \sqrt{y}\right)^2\left(x + y + \sqrt{xy}\right) \geq 0$$

$$\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2 \geq \sqrt{xy} \Rightarrow \sqrt{z}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2 \geq \sqrt{xyz} \Rightarrow$$

$$\Rightarrow \sum_{cyc(x,y,z)} \sqrt{z}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2 \geq 3\sqrt{xyz} = 3\sqrt{9} = 9$$

Equality holds for $x = y = z = \sqrt[3]{9}$.

SOLUTION A.006.

First, we prove that: $\left(\frac{x^2+y^2}{x^4+y^4}\right)^2 \leq \frac{1}{x^2y^2}$

$$(x^2 + y^2)^2 \cdot x^2y^2 \leq (x^4 + y^4)^2,\ x^2y^2(x^4 + y^4 + 2x^2y^2) \leq x^8 + y^8 + 2x^4y^4$$

$$x^6y^2 + x^2y^6 + 2x^4y^4 \leq x^8 + y^8 + 2x^4y^4$$

$$x^8+y^8-x^2y^6-x^6y^2\ge 0,\ x^6(x^2-y^2)-y^6(x^2-y^2)\ge 0$$

$$(x^2-y^2)(x^6-y^6)\ge 0,\ (x^2-y^2)^2(x^4+x^2y^2+y^4)\ge 0$$

$$\left(\frac{x^2+y^2}{x^4+y^4}\right)^2\le\frac{1}{x^2y^2}\Rightarrow z^2\left(\frac{x^2+y^2}{x^4+y^4}\right)^2\le\frac{z^2}{x^2y^2},\ \left(\frac{zx^2+zy^2}{x^4+y^4}\right)^2\le\frac{z^2}{x^2y^2}$$

$$\sum_{cyc(x,y,z)}\left(\frac{zx^2+zy^2}{x^4+y^4}\right)^2\le\sum_{cyc(x,y,z)}\frac{z^2}{x^2y^2}=\frac{z^2}{x^2y^2}+\frac{x^2}{y^2z^2}+\frac{y^2}{z^2x^2}=\frac{x^4+y^4+z^4}{x^2y^2z^2}=\frac{x^2y^2z^2}{x^2y^2z^2}=1$$

Equality holds for $x=y=z=\sqrt{3}$.

SOLUTION A.007.

$$4x=a+b+c+d-4\sqrt[4]{abcd}\ge 3y=a+b+c-3\sqrt[3]{abc}$$

$$\Leftrightarrow d-4\sqrt[4]{abcd}\ge -3\sqrt[3]{abc},\ d+3\sqrt[3]{abc}\ge 4\sqrt[4]{abcd}$$

$$d+3\sqrt[3]{abc}=d+\sqrt[3]{abc}+\sqrt[3]{abc}+\sqrt[3]{abc}\ge$$

$$\overset{AM-GM}{\ge} d+3\sqrt[3]{\left(\sqrt[3]{abc}\right)^3}=d+3\sqrt[3]{abc}$$

Hence: $4x\ge 3y$ *(1)*

$$3y=a+b+c-3\sqrt[3]{abc}\ge a+b-2\sqrt{ab}=2z\Leftrightarrow c-3\sqrt[3]{abc}\ge -2\sqrt{ab}$$

$$c+2\sqrt{ab}\ge 3\sqrt[3]{abc}$$

$$c+2\sqrt{ab}=c+\sqrt{ab}+\sqrt{ab}\overset{AM-GM}{\ge}3\sqrt[3]{c\cdot\sqrt{ab}\cdot\sqrt{ab}}=3\sqrt[3]{abc}$$

Hence: $3y\ge 2z$ *(2)*

By (1); (2)$\Rightarrow 4x\ge 3y\ge 2z\ge 0$

$$4x\ge 3y\ge 2z;\ p\ge q\ge r\ge 0$$

By Cebyshev's inequality:

$$4xp+3yq+2zr\ge\frac{1}{3}(4x+3y+2z)(p+q+r)$$

$$3(4xp+3yq+2zr)\ge(4x+3y+2z)(p+q+r)$$

SOLUTION A.008.

Let be the random variable:

$$X=\begin{pmatrix}\frac{1}{b} & \frac{1}{c} & \frac{1}{a} & 1\\ a & b & c & 1-a-b-c\end{pmatrix},\ X^2=\begin{pmatrix}\frac{1}{b^2} & \frac{1}{c^2} & \frac{1}{a^2} & 1\\ a & b & c & 1-a-b-c\end{pmatrix}$$

$$M(X)=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1-a-b-c,\ M(X^2)=\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}+1-a-b-c$$

$$D^2(X) \geq 0 \Rightarrow M(X^2) - \left(M(X)\right)^2 \geq 0 \Rightarrow M(X^2) \geq \left(M(X)\right)^2$$

$$\frac{a}{b^2} + \frac{b}{c^2} + \frac{c}{a^2} + 1 - a - b - c \geq \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1 - a - b - c\right)^2$$

SOLUTION A.009.

$$\left(\sqrt{a^2 - b^2} + b\sqrt{2}\right)^2 = a^2 - b^2 + 2b^2 + 2\sqrt{2}b\sqrt{a^2 - b^2} =$$

$$= a^2 + b^2 + 2b\sqrt{2(a^2 - b^2)} \geq a^2 + b^2$$

$\sqrt{a^2 - b^2} + b\sqrt{2} \geq \sqrt{a^2 + b^2}$ *(1). Analogous:* $\sqrt{b^2 - c^2} + c\sqrt{2} \geq \sqrt{b^2 + c^2}$ *(2)*

$$\sqrt{a^2 - c^2} + c\sqrt{2} \geq \sqrt{a^2 + c^2}$$

But $a \geq c \Rightarrow \sqrt{a^2 - c^2} + a\sqrt{2} \geq \sqrt{a^2 - c^2} + c\sqrt{2} \geq \sqrt{a^2 + c^2}$ *(3)*

Adding (1); (2); (3):

$$\sqrt{a^2 - b^2} + \sqrt{b^2 - c^2} + \sqrt{a^2 - c^2} + \sqrt{2}(a + b + c) \geq \sqrt{a^2 + b^2} + \sqrt{b^2 + c^2} + \sqrt{a^2 + c^2}$$

SOLUTION A.010.

$$3 + (\log_a c)^4 = 1 + 1 + 1 + (\log_a c)^4 \overset{AM-GM}{\geq} 4\sqrt[4]{1 \cdot 1 \cdot 1 \cdot 1(\log_a c)^4} = 4\log_a c \quad (1)$$

$$3 + \frac{1}{(a+b)^4} = 1 + 1 + 1 + \frac{1}{(a+b)^4} \overset{AM-GM}{\geq} 4\sqrt[4]{1 \cdot 1 \cdot 1 \cdot \frac{1}{(a+b)^4}} = \frac{4}{a+b} \quad (2)$$

$$\sum (3 + (\log_a c)^4)\left(3 + \frac{1}{(a+b)^4}\right) \overset{(1);(2)}{\geq} \sum (4\log_a c)\left(\frac{4}{a+b}\right) = 16\sum \frac{\log_a c}{a+b} \overset{AM-GM}{\geq}$$

$$\geq 16 \cdot 3\sqrt[3]{\frac{\log_a c \cdot \log_b a \cdot \log_c b}{(a+b)(b+c)(c+a)}} = \frac{48}{\sqrt[3]{(a+b)(b+c)(c+a)}} \overset{AM-GM}{\geq}$$

$$\geq \frac{48}{\frac{(a+b)+(b+c)+(c+a)}{3}} = \frac{48}{\frac{2(b+c+a)}{3}} = \frac{48}{\frac{3}{3}} = 48$$

SOLUTION A.011.

Denote: $\sqrt[3]{yzt} = u;\ \sqrt[3]{ztx} = v;\ \sqrt[3]{txy} = w;\ \sqrt[3]{xyz} = t$

$$\sum \frac{yzt}{\left(\sqrt[3]{ztx} + \sqrt[3]{txy} + \sqrt[3]{yxz}\right)^3} \geq \frac{4}{27} \Leftrightarrow\Leftrightarrow \sum \frac{u^3}{(v+w+t)^3} \geq \frac{4}{27} \quad (1)$$

Denote: $S = u + v + w + t$

$$(1) \Leftrightarrow \sum \left(\frac{u}{S-u}\right)^3 \geq \frac{4}{27}$$

Let be $f: (0, \infty) \to (0, \infty);\ f(x) = \left(\frac{x}{S-x}\right)^3;\ f'(x) = 3 \cdot \frac{S}{(S-x)^2} \cdot \left(\frac{x}{S-x}\right)^2$

$$f''(x)=3\cdot\frac{S}{(S-x)^3}\cdot\left(\frac{x}{S-x}\right)^2+3\cdot\left(\frac{S}{(S-x)^2}\right)^2\cdot\left(\frac{x}{S-x}\right)>0$$

$$\sum\left(\frac{u}{S-u}\right)^3=\sum f(u)\overset{JENSEN}{\geq}4f\left(\frac{u+v+w+t}{4}\right)=4f\left(\frac{S}{4}\right)=4\left(\frac{\frac{S}{4}}{S-\frac{S}{4}}\right)^3=4\left(\frac{1}{3}\right)^3=\frac{4}{27}$$

SOLUTION A.012.

$$Let\ be\ f,g:[x,y]\to\mathbb{R};n,m\in\mathbb{N};n<m$$

$$f(x)=x^n;g(x)=x^m$$

$$By\ Cauchy's\ theorem\ (\exists)x\in(x,y)$$

$$\frac{f(y)-f(x)}{g(y)-g(x)}=\frac{f'(c)}{g'(c)}\Rightarrow\frac{y^n-x^n}{y^m-x^m}=\frac{nc^{n-1}}{mc^{m-1}}=\frac{n}{m}\cdot\frac{1}{c^{m-n}}<\frac{n}{m}\ because:$$

$$c>x\geq 1\Rightarrow c^{m-n}>1\Rightarrow\frac{1}{c^{m-n}}<1$$

$$m=5;n=6\Rightarrow\frac{y^5-x^5}{y^6-x^6}<\frac{5}{6}\quad(1)$$

$$m=7;n=8\Rightarrow\frac{y^7-x^7}{y^8-x^8}<\frac{7}{8}\quad(2)$$

$$m=9;n=10\Rightarrow\frac{y^9-x^9}{y^{10}-x^{10}}<\frac{9}{10}\quad(3)$$

$$By\ multiplying\ (1);(2);(3):\frac{y^5-x^5}{y^6-x^6}\cdot\frac{y^7-x^7}{y^8-x^8}\cdot\frac{y^9-x^9}{y^{10}-x^{10}}<\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{9}{10}=\frac{21}{32}$$

SOLUTION A.013.

$$Let\ be\ a,b,c,d\geq 0$$

$$a^3+b^3+c^3\overset{AM-GM}{\geq}3\sqrt[3]{a^3b^3c^3}=3abc\quad(1)$$

$$b^3+c^3+d^3\geq 3\sqrt[3]{b^3c^3d^3}=3bcd\quad(2)$$

$$c^3+d^3+a^3\geq 3\sqrt[3]{c^3d^3a^3}=3cda\quad(3)$$

$$d^3+a^3+b^3\geq\sqrt[3]{d^3a^3b^3}=3dab\quad(4)$$

$$By\ adding\ (1);(2);(3);(4):3(a^3+b^3+c^3+d^3)\geq 3(abc+bcd+cda+dab)$$

$$a^3+b^3+c^3+d^3\geq abc+bcd+cda+dab\quad(5)$$

$$Take\ a=\ln x;b=\ln y;c=\ln z;d=\ln t$$

$$(a+b)(a^2-ab+b^2)-cd(a+b)\geq ab(c+d)-(c+d)(c^2-cd+d^2)$$

$$(a+b)(a^2+b^2-ab-cd)\geq(c+d)(ab+cd-c^2-d^2)$$

$$(\ln x+\ln y)(\ln^2 x+\ln^2 y-\ln x\ln y-\ln z\ln t)\geq$$

$$\geq (\ln z + \ln t)(\ln x \ln y + \ln z \ln t - \ln^2 z - \ln^2 t)$$

$$(\ln xy)(\ln^2 x + \ln^2 y - \ln x \ln y - \ln z \ln t) \geq (\ln zt)(\ln x \ln y + \ln z \ln t - \ln^2 z - \ln^2 t)$$

SOLUTION A.014.

$$\frac{(ab)\cdot c^c + (bc)\cdot a^a + (ca)\cdot b^b}{ab+bc+ca} \overset{AM-GM}{\geq} \left((c^c)^{ab}\cdot (a^a)^{bc}\cdot \left(b^b\right)^{ca}\right)^{\frac{1}{ab+bc+ca}} =$$

$$= \left(c^{abc}\cdot a^{abc}\cdot b^{abc}\right)^{\frac{1}{abc}} = abc$$

$$(ab)\cdot c^c + (bc)\cdot a^2 + (ca)\cdot b^b \geq (ab+bc+ca)\cdot abc = abc\cdot abc = (abc)^2 = a^2b^2c^2$$

SOLUTION A.015.

The sequence is an arithmetical progression:

$$a_1 = 3; r = 5.$$

$$a_n = a_1 + (n-1)r$$

$$a_{101} = 3 + 100\cdot 5 = 503$$

$$S(a_{101}) = 5 + 0 + 3 = 8$$

$$a_{1001} = 3 + 1000\cdot 5 = 5003$$

$$S(a_{1001}) = 5 + 0 + 0 + 3 = 8$$

$$a_{10001} = 3 + 10000\cdot 5 = 50003$$

$$S(a_{10001}) = 5 + 0 + 0 + 0 + 3 = 8$$

$$S(a_{101}) = S(a_{1001}) = S(a_{10001})$$

SOLUTION A.016.

Let be $f: [0,\infty) \to \mathbb{R}; f(x) = \frac{1}{1+e^x};$

$$f'(x) = \frac{-e^x}{(1+e^x)^2}; f''(x) = \frac{e^x(e^x-1)}{(1+e^x)^3} \geq 0, f \text{ – convexe}$$

If $u, v, w \geq 0$ ***then by Jensen's inequality:***

$$f\left(\frac{u+v+w}{3}\right) \leq \frac{1}{3}\left(f(u) + f(v) + f(w)\right)$$

$$\frac{1}{1+e^{\frac{u+v+w}{3}}} \leq \frac{1}{3}\left(\frac{1}{1+e^u} + \frac{1}{1+e^v} + \frac{1}{1+e^w}\right)$$

Denote $a = e^u;\ b = e^v; c = e^w$

$$\frac{1}{1+\sqrt[3]{abc}} \leq \frac{1}{3}\left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right)$$

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\le\frac{3}{1+\sqrt[3]{abc}}$$

Equality holds if $a=b=c$.

Denote $a=m^{3x}; b=n^{3x}; c=p^{3x}$

$$\frac{1}{1+m^{3x}}+\frac{1}{1+n^{3x}}+\frac{1}{1+p^{3x}}\le\frac{3}{1+\sqrt[3]{m^{3x}\cdot n^{3x}\cdot p^{3x}}}=\frac{3}{1+(mnp)^x}$$

Equality holds for:

$$m^{3x}=n^{3x}=p^{3x}\Rightarrow x=0$$

SOLUTION A.017.

Let be the random variable:

$$X=\begin{pmatrix}\frac{1}{b^5} & \frac{1}{c^5} & \frac{1}{a^5}\\ \frac{a}{a+b+c} & \frac{b}{a+b+c} & \frac{c}{a+b+c}\end{pmatrix}$$

$$M(X)=\frac{a}{b^5(a+b+c)}+\frac{b}{c^5(a+b+c)}+\frac{c}{a^5(a+b+c)}$$

$$X^2=\begin{pmatrix}\frac{1}{b^{10}} & \frac{1}{c^{10}} & \frac{1}{a^{10}}\\ \frac{a}{a+b+c} & \frac{b}{a+b+c} & \frac{c}{a+b+c}\end{pmatrix}$$

$$M(X^2)=\frac{a}{b^{10}(a+b+c)}+\frac{b}{c^{10}(a+b+c)}+\frac{c}{a^{10}(a+b+c)}$$

$$D^2(X)\ge 0\Rightarrow M(X^2)-\left(M(X)\right)^2\ge 0\Rightarrow M(X^2)\ge\left(M(X)\right)^2$$

$$\frac{1}{a+b+c}\left(\frac{a}{b^5}+\frac{b}{c^5}+\frac{c}{a^5}\right)\ge\frac{1}{(a+b+c)^2}\left(\frac{a}{b^{10}}+\frac{b}{c^{10}}+\frac{c}{a^{10}}\right)^2$$

$$(a+b+c)\left(\frac{a}{b^5}+\frac{b}{c^5}+\frac{c}{a^5}\right)\ge\left(\frac{a}{b^{10}}+\frac{b}{c^{10}}+\frac{c}{a^{10}}\right)^2$$

SOLUTION A.018.

$$\sum_{cyc(a,b,c,d)}\frac{a^3}{(b+c+d)(a-b)(a-c)(a-d)}=$$

$$=\frac{(a+b+c+d)^3}{(b+c+d)(c+d+a)(d+a+b)(a+b+c)}\overset{AM-GM}{<}$$

$$< \frac{(a+b+c+d)^3}{3\sqrt[3]{bcd}\cdot 3\sqrt[3]{cda}\cdot 3\sqrt[3]{dab}\cdot 3\sqrt[3]{abc}} = \frac{(a+b+c+d)^3}{81abcd}$$

SOLUTION A.019.

First, we prove that: $\sqrt{x}+\sqrt{y}\le\sqrt{2(x+y)}$ ***(1)***

$$\textit{By squaring: } x+y+2\sqrt{xy}\le 2x+2y \Leftrightarrow \left(\sqrt{x}-\sqrt{y}\right)^2\ge 0$$

$$\textit{Analogous: } \sqrt[3]{x}+\sqrt[3]{y}\le\sqrt[3]{4(x+y)} \quad (2)$$

$$x+y+3\sqrt[3]{xy}\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\le 4x+4y$$

$$3\sqrt[3]{x^2y}+3\sqrt[3]{xy^2}\le 3x+3y$$

$$x+y-\sqrt[3]{x^2y}-\sqrt[3]{xy^2}\ge 0$$

$$\sqrt[3]{x^2}\left(\sqrt[3]{x}-\sqrt[3]{y}\right)-\sqrt[3]{y^2}\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\ge 0$$

$$\left(\sqrt[3]{x}-\sqrt[3]{y}\right)^2\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\ge 0$$

Replace $x=a^6; y=b^6$ ***in (1); (2):*** $\sqrt{a^6}+\sqrt{b^6}\le\sqrt{2(a^6+b^6)}\Rightarrow a^3+b^3\le\sqrt{2(a^6+b^6)}$

$$\sqrt[3]{a^6}+\sqrt[3]{b^6}\le\sqrt[3]{4(a^6+b^6)}\Rightarrow a^2+b^2\le\sqrt[3]{4(a^6+b^6)}$$

$$\sum_{cyc(a,b,c)}(a^3+b^3)+\sum_{cyc(a,b,c)}(a^2+b^2)\le\sum_{cyc(a,b,c)}\sqrt{2(a^6+b^6)}+\sum_{cyc(a,b,c)}\sqrt[3]{4(a^6+b^6)}$$

$$2(a^2+b^2+c^2+a^3+b^3+c^3)\le\sqrt{2}\cdot\sum_{cyc(a,b,c)}\sqrt{a^6+b^6}+\sqrt[3]{4}\cdot\sum_{cyc(a,b,c)}\sqrt[3]{a^6+b^6}$$

SOLUTION A.020.

$$\textit{With } \sqrt{e^x}=a; \sqrt{e^y}=b \textit{ the inequality can be written:}$$

$$(a^2+1)\,b+(b^2+1)a\le(a^2+1)(b^2+1)$$

$$(a^2+1)b+(b^2+1)a\le\frac{(a^2+1)(b^2+1)}{2}+\frac{(a^2+1)(b^2+1)}{2}$$

$$(a^2+1)\left(b-\frac{b^2+1}{2}\right)+(b^2+1)\left(a-\frac{a^2+1}{2}\right)\le 0$$

$$\frac{(a^2+1)(2b-b^2-1)}{2}+\frac{(b^2+1)(2a-a^2-1)}{2}\le 0$$

$$\frac{(a^2+1)(b-1)^2}{2}+\frac{(b^2+1)(a-1)^2}{2}\ge 0$$

SOLUTION A.021.

$$A^{20}=A^{10}+A^5+I_5\Rightarrow A^{20}+A^5=A^{10}+2A^5+I_5$$

$$A^{20}+A^5=\left(A^5+I_5\right)^2,\ \left(A^{15}+I_5\right)A^5=\left(A^5+I_5\right)^2$$

$$\left(\left(A^5\right)^3+I_5^3\right)A^5=\left(A^5+I_5\right)^2,\ \ \left(A^5+I_5\right)\left(A^{10}-A^5+I_5\right)A^5=\left(A^5+I_5\right)^2$$

$$\textit{Multiplying}\ \left(A^{10}-A^5+I_5\right)A^5=A^5+I_5,\ A^{15}-A^{10}+A^5=A^5+I_5$$

$$A^{15}-A^{10}=I_5\Rightarrow A^{10}\left(A^5-I_5\right)=I_5$$

$$(\det A)^{10}\det\left(A^5-I_5\right)=\det I_5=1>0\Rightarrow \det\left(A^5-I_5\right)>0\ \ (1)$$

$$A^{20}-I_5=A^{10}+A^5,\ \left(A^{10}+I_5\right)\left(A^{10}-I_5\right)=A^5\left(A^5+I_2\right)$$

$$\left(A^{10}+I_5\right)\left(A^5-I_5\right)\left(A^5+I_5\right)=A^5\left(A^5+I_5\right)$$

$$\textit{Multiplying by}\ \left(A^5+I_5\right)^{-1}:$$

$$\left(A^{10}+I_5\right)\left(A^5-I_5\right)=A^5$$

$$\det\left(A^5\right)=\det\left(A^{10}+I_5\right)\cdot\left(A^5-I_5\right)=\det\left(A^{10}+I_5\right)\cdot\det\left(A^5-I_3\right)\geq 0$$

$$(\det A)^5\geq 0\Rightarrow \det A\geq 0\Rightarrow \sqrt[4]{\det A}\geq 0$$

SOLUTION A.022.

$$\textit{If}\ x\in\left(0,\tfrac{1}{2}\right)\ \textit{then}\ x<\tfrac{1}{2}\Rightarrow 2x-1<0\ \ (1)$$

$$x<1-x\Rightarrow \tfrac{x}{1-x}<1\Rightarrow \log\left(\tfrac{x}{1-x}\right)<0\ \ (2)$$

$$\textit{By (1); (2)}\Rightarrow (2x-1)\log\left(\tfrac{x}{1-x}\right)>0\ \ (3)$$

$$\textit{If}\ x\in\left(\tfrac{1}{2},1\right)\ \textit{then}\ x>\tfrac{1}{2}\Rightarrow 2x-1>0\ \ \ (4)$$

$$x>1-x\Rightarrow \tfrac{x}{1-x}>1\Rightarrow \log\left(\tfrac{x}{1-x}\right)>0\ \ \ (5)$$

$$\textit{By (4); (5)}\Rightarrow (2x-1)\log\left(\tfrac{x}{1-x}\right)>0\ \ \ (6)$$

$$\textit{By (3); (6)}\Rightarrow (2x-1)\log\left(\tfrac{x}{1-x}\right)\geq 0,(\forall)x\in(0,1)\ \ \ (7)$$

$$\textit{Equality holds if}\ x=\tfrac{1}{2}.$$

$$(2x-1)\log\left(\frac{x}{1-x}\right)\geq 0\Rightarrow \log\left(\frac{x}{1-x}\right)^{2x-1}\geq \log 1$$

$$\left(\frac{x}{1-x}\right)^{2x-1}\geq 1\Rightarrow x^{2x-1}\geq (1-x)^{2x-1}$$

$$\frac{(x^x)^2}{x} \ge \frac{((1-x)^x)^2}{1-x}; (\forall)x \in (0,1)$$

$$\textit{Analogous: } \frac{(y^y)^2}{y} \ge \frac{((1-y)^y)^2}{1-y}; \frac{(z^z)^2}{z} \ge \frac{((1-z)^z)^2}{1-z}$$

$$\textit{By multiplying: } \frac{(x^x)^2\cdot(y^y)^2\cdot(z^z)^2}{xyz} \ge \frac{((1-x)^x\cdot(1-y)^y\cdot(1-z)^z)^2}{(1-x)(1-y)(1-z)}; \forall\, x \in (0,1)$$

$$\textit{Equality holds if: } x = y = z = \frac{1}{2}$$

SOLUTION A.023.

$$\textit{Let } x = \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}; Y = \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}$$

$$XY = \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 25-25 & -5+5 \\ 125-125 & -25+25 \end{pmatrix} = O_2$$

$$X^iY^j = O_2; (\forall)i,j \in \overline{1,100}$$

$$(X+Y)^{100} = \sum_{k=0}^{100}\binom{100}{k}X^{100-k}Y^k = X^{100} + Y^{100}$$

$$\Omega = \det(X^{100} + Y^{100}) = \det[(X+Y)^{100}] = [\det(X+Y)]^{100} =$$

$$= \left(\det\left[\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} + \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}\right]\right)^{100} = \left(\begin{vmatrix} 26 & 0 \\ 0 & 26 \end{vmatrix}\right)^{100} = 26^{200}$$

SOLUTION A.024.

$$2\sum\frac{bc^2(ab+1)}{a(b^2c^2+1)} = 2\sum\frac{b^2c^2(ab+1)}{ab(b^2c^2+1)} =$$

$$= 2\sum\left(\frac{b^2c^2}{b^2c^2+1}\cdot\frac{ab+1}{ab}\right) = 2\sum\left(\frac{b^2c^2+1-1}{b^2c^2+1}\cdot\left(1+\frac{1}{ab}\right)\right) =$$

$$= 2\sum\left(1-\frac{1}{b^2c^2+1}\right)\left(1+\frac{1}{ab}\right) =$$

$$= 2\sum\left(1+\frac{1}{ab}-\frac{1}{b^2c^2+1}-\frac{1}{ab(b^2c^2+1)}\right) \overset{AM-GM}{\ge} 6 + 2\sum\frac{1}{ab} - \sum\frac{1}{bc} - \frac{1}{abc}\sum\frac{1}{a}$$

$$LHS \ge \sum\frac{1}{a^2b^2} - \sum\frac{1}{ab} + 6 + \sum\frac{1}{ab} - \frac{1}{abc}\sum\frac{1}{a} \ge 6 = RHS$$

$$\textit{Remains to prove: } \sum\frac{1}{a^2b^2} \ge \frac{1}{abc}\sum\frac{1}{a} \Leftrightarrow a^2+b^2+c^2 \ge ab+bc+ca$$

$$\Leftrightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 \ge 0$$

SOLUTION A.025.

$$c + d + e = 1 \Rightarrow \sqrt[3]{cde} \le \frac{c+d+e}{3} = \frac{1}{3} \Rightarrow$$

$$\frac{1}{\sqrt[3]{cde}} \ge 3 \Rightarrow \frac{b}{\sqrt[3]{cde}} \ge 3b \Rightarrow a + \frac{b}{\sqrt[3]{cde}} \ge a + 3b \quad (1)$$

$$\left(a + \frac{b}{c}\right)^4 + \left(a + \frac{b}{d}\right)^4 + \left(a + \frac{b}{e}\right)^4 \overset{AM-GM}{\ge} 3\left(\sqrt[3]{\left(a + \frac{b}{c}\right)\left(a + \frac{b}{d}\right)\left(a + \frac{b}{c}\right)}\right)^4 \ge$$

$$\overset{HOLDER}{\ge} 3\left(\sqrt[3]{a \cdot a \cdot a} + \sqrt[3]{\frac{b}{c} \cdot \frac{b}{d} \cdot \frac{b}{e}}\right)^4 = 3\left(a + \frac{b}{\sqrt[3]{cde}}\right)^4 \overset{(1)}{\ge} (a + 3b)^4$$

Equality holds for $c = d = e = \frac{1}{3}$.

SOLUTION A.026.

Lemma:

$$\text{If } x \ge 1; y \ge 1 \text{ then: } x^y + y^x \ge 1 + xy$$

Proof:

$$x^y = \left(1 + (x - 1)\right)^y \overset{Bernoulli}{\ge} 1 + (x - 1)y \quad (1)$$

$$y^x = \left(1 + (y - 1)\right)^x \overset{Bernoulli}{\ge} 1 + (y - 1)x \quad (2)$$

By adding (1); (2): $x^y + y^x \ge 1 + (x - 1)y + 1(y - 1)x = 1 + xy - x - y + 1 + xy =$

$$= 1 + xy + (x - 1)(y - 1) \ge 1 + xy$$

Hence: $x^y + y^x \ge 1 + xy \quad (3)$

Denote: $m_a = \frac{a+b}{2}; m_g = \sqrt{ab}; m_q = \sqrt{\frac{a^2+b^2}{2}}$

$$m_q \ge m_a \ge m_g \Rightarrow \frac{m_a}{m_g} \ge 1; \frac{m_q}{m_a} \ge 1$$

In (3) we take $x = \frac{m_a}{m_g} \ge 1; y = \frac{m_q}{m_a} \ge 1$

$$\left(\frac{m_a}{m_g}\right)^{\frac{m_q}{m_a}} + \left(\frac{m_q}{m_a}\right)^{\frac{m_a}{m_g}} \ge 1 + \frac{m_a}{m_g} \cdot \frac{m_q}{m_a}$$

$$\left(\frac{a + b}{2\sqrt{ab}}\right)^{\frac{\sqrt{2(a^2+b^2)}}{a+b}} + \left(\frac{\sqrt{2(a^2 + b^2)}}{a^2 + b^2}\right)^{\frac{a+b}{2\sqrt{ab}}} \ge 1 + \sqrt{\frac{a^2 + b^2}{2ab}}$$

Equality holds for $a = b$.

SOLUTION A.027.

If $x \le z$ *then* $x - z \le 0$

$$\sqrt{x^2 - xz + z^2} = \sqrt{z^2 + x(x-z)} \le \sqrt{z^2} = z \le a$$

If $x \ge z$ *then* $z - x \le 0$

$$\sqrt{x^2 - xz + z^2} = \sqrt{x^2 + z(z-x)} \le \sqrt{x^2} = x \le a$$

Hence: $\sqrt{x^2 - xz + z^2} \le 1$ *(1)*

$$\sqrt{y^2 + z^2} = \sqrt{2} \cdot \sqrt{\frac{y^2 + z^2}{2}} \overset{QM-AM}{\le} \sqrt{2} \cdot \frac{y+z}{2} \le$$

$$\le \sqrt{2} \cdot \frac{a+a}{2} = a\sqrt{2} \quad (2)$$

$$\sqrt{x^2 + xy + y^2} \le \sqrt{a^2 + a \cdot a + a^2} = a\sqrt{3} \quad (3)$$

By adding: (1); (2); (3):

$$\sqrt{x^2 - xz + z^2} + \sqrt{y^2 + z^2} + \sqrt{x^2 + xy + y^2} \le a + a\sqrt{2} + a\sqrt{3} = a\left(1 + \sqrt{2} + \sqrt{3}\right)$$

Equality holds for $x = y = z = a$.

SOLUTION A.028.

Lemma: If $a_1, a_2, \dots, a_n \ge 1; n \in \mathbb{N}; n \ge 1$ *then:*

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + a_1 a_2 \dots a_n \ge a_1 + a_2 + \dots + a_n + \frac{1}{a_1 a_2 \cdot \dots \cdot a_n}$$

Proof:

For $n = 1$: $\frac{1}{a_1} + a_1 = a_1 + \frac{1}{a_1}$ *(true)*

$$P(n): \sum_{k=1}^{n} \frac{1}{a_k} + a_1 a_2 \dots a_n \ge \sum_{k=1}^{n} a_k + \frac{1}{a_1 a_2 \cdot \dots \cdot a_n} \quad \text{(true)}$$

$$P(n+1): \sum_{k=1}^{n} \frac{1}{a_k} + a_1 a_2 \dots a_n a_{n+1} \ge \sum_{k=1}^{n} a_k + \frac{1}{a_1 a_2 \dots a_n a_{n+1}} \quad \text{(to prove)}$$

Denote $u = a_1 a_2 \dots a_n \ge 1$

$$\sum_{k=1}^{n+1} \frac{1}{a_k} + a_1 a_2 \dots a_n a_{n+1} = \sum_{k=1}^{n} \frac{1}{a_k} + \frac{1}{a_{n+1}} + u a_{n+1} =$$

$$= \sum_{k=1}^{n} \frac{1}{a_k} + u + \frac{1}{a_{n+1}} + u a_{n+1} - u \overset{P(n)}{\ge}$$

$$\geq \sum_{k=1}^{n} a_k + \frac{1}{u} + \frac{1}{a_{n+1}} + ua_{n+1} - u\sum_{k=1}^{n+1} a_k + \frac{1}{ua_{n+1}}$$

$$\frac{1}{u} + \frac{1}{a_{n+1}} + ua_{n+1} - u \geq a_{n+1} + \frac{1}{ua_{n+1}}$$

$$a_{n+1} + u + u^2a_{n+1}^2 - u^2a_{n+1} \geq a_{n+1}^2u + 1$$

$$a_{n+1} + u + u^2a_{n+1}^2 - u^2a_{n+1} - ua_{n+1}^2 - 1 \geq 0$$

$$u^2a_{n+1}(a_{n+1} - 1) + (a_{n+1} - 1) - u(a_{n+1} - 1)(a_{n+1} + 1) \geq 0$$

$$(a_{n+1} - 1)(u^2a_{n+1} + 1 - ua_{n+1} - u) \geq 0$$

$$(a_{n+1} - 1)[ua_{n+1}(u - 1) - (u - 1)] \geq 0$$

$$(a_{n+1} - 1)(u - 1)(ua_{n+1} - 1) \geq 0$$

Which is true because: $a_{n+1} \geq 1; u \geq 1; ua_{n+1} \geq 1$

$$P(n) \to P(n+1)$$

Back to the problem:

In lemma: $n = 3; a_1 = \ln x\,; a_2 = \ln y\,; a_3 = \ln z$

$$\frac{1}{\ln x} + \frac{1}{\ln y} + \frac{1}{\ln z} + \ln x \ln y \ln z \geq \ln x + \ln y + \ln z + \frac{1}{\ln x \ln y \ln z}$$

$$\frac{1}{\ln x} + \frac{1}{\ln y} + \frac{1}{\ln z} + \ln x \ln y \ln z \geq \ln(xyz) + \frac{1}{\ln x \ln y \ln z}$$

Equality holds for $x = y = z = e.$

SOLUTION A.029.

$$a^a = (1 + a - 1)^a \overset{BERNOULLI}{\leq} 1 + a(a - 1)$$

$$a + a^a \leq 1 + a(a - 1) + a = 1 + a^2 - a + a = 1 + a^2$$

$$\frac{1}{a + a^a} \geq \frac{1}{1 + a^2} \ (1)$$

Analogous: $\frac{1}{b + b^b} \geq \frac{1}{1 + b^2}$ *(2);* $\frac{1}{c + c^c} \geq \frac{1}{1 + c^2}$ *(3)*

By adding (1); (2); (3): $\frac{1}{a + a^a} + \frac{1}{b + b^b} + \frac{1}{c + c^c} \geq \frac{1}{1 + a^2} + \frac{1}{1 + b^2} + \frac{1}{1 + c^2} \geq$

$$\overset{BERGSTROM}{\geq} \frac{(1 + 1 + 1)^2}{(1 + a^2) + (1 + b^2) + (1 + c^2)} = \frac{9}{3 + a^2 + b^2 + c^2}$$

SOLUTION A.030.

$$0 < \frac{2ab}{a^2 + b^2} \leq 1 \text{ because } 2ab \leq a^2 + b^2 \Leftrightarrow (a - b)^2 \geq 0$$

$$\frac{\sqrt{2(a^2+b^2)}}{a+b} \geq 1 \text{ because } \sqrt{2(a^2+b^2)} \geq a+b \Leftrightarrow$$

$$\Leftrightarrow 2a^2+2b^2 \geq a^2+2ab+b^2 \Leftrightarrow a^2-2ab+b^2 \geq 0 \Rightarrow (a-b)^2 \geq 0$$

$$0 < x \leq 1; y \geq 1$$

$$y^x = \left(1+(y-1)\right)^x \overset{Bernoulli}{\leq} 1+x(y-1) \quad (1)$$

$$x \in [0,1] \Rightarrow x^y \leq x^1 \quad (2)$$

$$\text{By adding (1); (2): } x^y + y^x \leq 1 + x(y-1) + x = 1 + xy$$

$$\text{For } x = \frac{2ab}{a^2+b^2}; y = \frac{\sqrt{2(a^2+b^2)}}{a+b}$$

$$\left(\frac{2ab}{a^2+b^2}\right)^{\frac{\sqrt{2(a^2+b^2)}}{a+b}} + \left(\frac{\sqrt{2(a^2+b^2)}}{a+b}\right)^{\frac{2ab}{a^2+b^2}} \leq 1 + \frac{2ab}{a^2+b^2} \cdot \frac{\sqrt{2(a^2+b^2)}}{a+b} =$$

$$= 1 + \frac{2ab \cdot \sqrt{2(a^2+b^2)}}{(a+b) \cdot \left(\sqrt{a^2+b^2}\right)^2} = 1 + \frac{2\sqrt{2}ab}{(a+b)\sqrt{a^2+b^2}}$$

SOLUTION A.031.

$$\frac{1+a+a^2}{1+a^2} - \frac{3}{2} = \frac{2+2a+2a^2-3-3a^2}{2(1+a^2)} = \frac{-a^2+2a-1}{2(1+a^2)} =$$

$$= -\frac{(a-1)^2}{2(1+a^2)} \leq 0 \Rightarrow \frac{1+a+a^2}{1+a^2} \leq \frac{3}{2} \quad (1)$$

$$\frac{1+b+b^2+b^3}{1+b^3} - \frac{4}{2} = \frac{2+2b+2b^2+2b^3-4-4b^3}{2(1+b^3)} =$$

$$= \frac{-2b^3+2b^2+2b-2}{2(1+b^3)} = \frac{-2b^2(b-1)+2(b-1)}{2(1+b^3)} =$$

$$= \frac{-(b-1)(b^2-1)}{1+b^3} = \frac{-(b-1)^2(b+1)}{1+b^3} \leq 0 \Rightarrow$$

$$\Rightarrow \frac{1+b+b^2+b^3}{1+b^3} \leq \frac{4}{2} \quad (2)$$

$$\frac{1+c+c^2+c^3+c^4}{1+c^4} - \frac{5}{2} = \frac{2+2c+2c^2+2c^4-5-5c^4}{2(1+c^4)} =$$

$$= \frac{-3c^4+2c^3+2c^2+2c-3}{2(1+c^4)} = \frac{-3c^4+3c^3-c^3+c^2-c+3c-3}{2(1+c^4)} =$$

$$= \frac{-3c^3(c-1)-c^2(c-1)+c(c-1)+3(c-1)}{2(1+c^4)} =$$

$$=\frac{(c-1)(-3c^3-c^2+c+3)}{2(1+c^4)}=\frac{(c-1)[-3(c-1)(c^2+c+1)-c(c-1)]}{2(1+c^4)}=$$

$$=\frac{(c-1)^2(-3c^2-3c-3+c)}{2(1+c^4)}=\frac{-(c-1)^2(3c^2+2c+3)}{2(1+c^4)}\leq 0\Rightarrow$$

$$\frac{1+c+c^2+c^3+c^4}{1+c^4}\leq\frac{5}{2}\quad (3)$$

By multiplying (1); (2); (3): $\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)}\leq\frac{3}{2}\cdot\frac{4}{2}\cdot\frac{5}{2}=\frac{15}{2}$

SOLUTION A.032.

$$a^2+ab+b^2=\frac{4(a^2+ab+b^2)}{4}=\frac{3a^2+6ab+3b^2+a^2-2ab+b^2}{4}=$$

$$=\frac{3(a+b)^2+(a-b)^2}{4}\geq\frac{3(a+b)^2}{4}$$

$$\sqrt{a^2+ab+b^2}\geq\frac{(a+b)\sqrt{3}}{2}$$

$$\sqrt{3}(a+b)\leq 2\sqrt{a^2+ab+b^2}$$

$$\prod\sqrt{3}(a+b)\leq\prod\left(2\sqrt{a^2+ab+b^2}\right)$$

$$3\sqrt{3}(a+b)(b+c)(c+a)\leq 8\sqrt{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}$$

SOLUTION A.033.

$$\sqrt{\frac{a+b}{2}}=\sqrt{\frac{(\sqrt{a})^2+(\sqrt{b})^2}{2}}\overset{QM-AM}{\geq}\frac{\sqrt{a}+\sqrt{b}}{2}\overset{AM-GM}{\geq}\sqrt[4]{ab}\quad (1)$$

$$3\sum\sqrt{\frac{a+b}{2}}=2\sum\sqrt{\frac{a+b}{2}}+\sum\sqrt{\frac{a+b}{2}}\overset{(1)}{\geq}2\sum\frac{\sqrt{a}+\sqrt{b}}{2}+\sum\sqrt[4]{ab}=$$

$$=2\sum\sqrt{a}+\sum\sqrt[4]{ab}=$$

$$=2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+\sqrt[4]{ab}+\sqrt[4]{bc}+\sqrt[4]{ca}$$

SOLUTION A.034. ***Lemma:***

If $X,Y\in M_n(\mathbb{R})$ ***then*** $\det(X^2+Y^2)\geq 0$

Proof:

$$X=(a_{ij})_{\substack{1\leq i\leq n\\1\leq j\leq n}};\ Y=(b_{ij})_{\substack{1\leq i\leq n\\1\leq j\leq n}}$$

$$X+iY=(a_{ij}+ib_{ij})_{\substack{1\leq i\leq n\\1\leq j\leq n}}=(z_{ij})_{\substack{1\leq i\leq n\\1\leq j\leq n}}$$

$$z_{ij} = a_{ij} + b_{ij}$$

$$X - iY = \left(a_{ij} - ib_{ij}\right)_{\substack{1\le i\le n\\1\le j\le n}} = \left(\overline{z_{ij}}\right)_{\substack{1\le i\le n\\1\le j\le n}}$$

$$\det(X^2 + Y^2) = \det(X^2 - i^2Y^2) = \det(X - iY)(X + iY) = \det(X + iY)\det(X - iY) =$$

$$= z \cdot \bar{z} = |z|^2 \ge 0$$

Back to the problem:

$$\det(A + I_5) = \det(O_5 + A + I_5) =$$

$$= \det(A^2 + A + I_5) = \det\left(\left(A + \frac{1}{2}I_5\right)^2 + \left(\frac{\sqrt{3}}{2}I_5\right)^2\right) \overset{Lemma}{\ge} 0$$

$$\det(A - I_5) = \det(O_5 + A - I_5) = \det(-A^2 + A - I_5) = (-1)^5\det(A^2 - A + I_5) =$$

$$= -\det\left(\left(A - \frac{1}{2}I_5\right)^2 + \left(\frac{\sqrt{3}}{2}I_5\right)^2\right) \overset{Lemma}{\le} 0$$

$$\det(A^2 - I_5) = \det(A - I_5)(A + I_5) = \det(A - I_5) \cdot \det(A + I_5) \le 0$$

SOLUTION A.035.

$$x^y = \left(1 + (x - 1)\right)^y \overset{Bernoulli}{\ge} 1 + (x - 1)y \quad (1)$$

$$y^x = \left(1 + (y - 1)\right)^x \overset{Bernoulli}{\ge} 1 + (y - 1)x \quad (2)$$

By adding (1); (2): $x^y + y^x \ge 1 + (x - 1)y + 1(y - 1)x = 1 + xy - x - y + 1 + xy =$

$$= 1 + xy + (x - 1)(y - 1) \ge 1 + xy$$

$$x^y + y^x > 1 + xy\ (1)$$

Analogous: $y^z + z^y > 1 + yz\ \ (2);\ z^x + x^z > 1 + zx\ \ (3)$

By adding (1); (2); (3):

$$x^y + y^z + z^x + y^x + z^y + x^z > 1 + 1 + 1 + xy + yz + zx \overset{AM-GM}{\ge} 3 + 3\sqrt[3]{(xyz)^2} =$$

$$= 3 + 3\left(\sqrt[3]{2\sqrt{2}}\right)^2 = 3 + 3\left(\sqrt{2}\right)^2 = 3 + 3 \cdot 2 = 9$$

SOLUTION A.036.

$$Put\ \frac{1}{x+1} = t,\ \ (t^3 + t^2 + t) - 156 = -\log_5 t$$

$$Let\ f(t) = t^3 + t^2 + t - 156$$

$$f'(t) = 3t^2 + 2t + 1$$

$$\Delta < 0\ \ f'(t) > 0\ \forall t \in \mathbb{R} \Rightarrow f(t)\ is\ increasing\ and\ -\log_5(t) = g(t)\quad t > 0$$

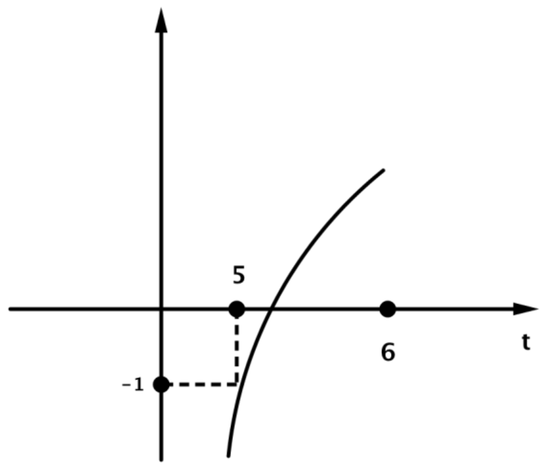
$$g'(t) = -\frac{1}{t \ln 5} \quad g'(t) < 0; \forall t > 0$$

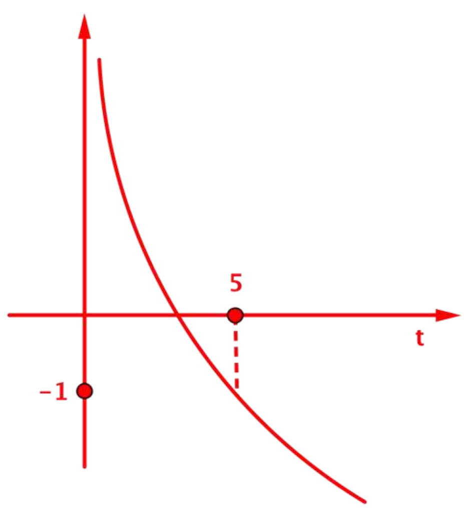
Hence decreasing

***Hence possible number of solutions is* 1.**

$$f(0) = -156, f(5) = -1, f(6) = 102$$

$f(5)f(6) < 0$ ***Hence one root between* 5 *and* 6**





$g(t) = -\log_5(t)$. ***Clearly at*** $t = 5$

$$g(5) = -1$$

Hence $t = 5$ ***is the only solution*** $\frac{1}{x+1} = 5; x = \frac{1}{5} - 1$

$x = -\frac{4}{5}$ ***(Answer)***

SOLUTION A.037.

$$2 + a^3 = 1 + 1 + a^3 \overset{AM-GM}{\geq} 3\sqrt[3]{1 \cdot 1 \cdot a^3} = 3a \quad (1)$$

$$3+a^4=1+1+1+a^4\overset{AM-GM}{\geq}4\sqrt[4]{1\cdot 1\cdot 1\cdot a^4}=4a \quad (2)$$

$$3+b^8=1+1+1+b^8\overset{AM-GM}{\geq}4\sqrt[4]{1\cdot 1\cdot 1\cdot b^8}=4b^2 \quad (3)$$

$$2+\frac{c^3}{a^3}=1+1+\left(\frac{c}{a}\right)^3\overset{AM-GM}{\geq}3\sqrt[3]{1\cdot 1\cdot\left(\frac{c}{a}\right)^3}=\frac{3c}{a} \quad (4)$$

$$3+c^4a^4=1+1+1+c^4a^4\overset{AM-GM}{\geq}4\sqrt[4]{1\cdot 1\cdot 1\cdot c^4a^4}=4ac \quad (5)$$

$$(2+a^3)(3+a^4)+3(3+b^8)+\left(2+\frac{c^3}{a^3}\right)(3+c^4a^4)\overset{(1)-(5)}{\geq}$$

$$\geq 3a\cdot 4a+3\cdot 4b^2+\frac{3c}{a}\cdot 4ac=12a^2+12b^2+12c^2=$$

$$=12(a^2+b^2+c^2)\overset{IONESCU-WEITZENBOCK}{\geq}$$

$$\geq 12\cdot 4\sqrt{3}S=48\sqrt{3}S=48\sqrt{3}rs\overset{MITRINOVIC}{\geq}48\sqrt{3}r\cdot 3\sqrt{3}r=48\cdot 9r^2=432r^2$$

SOLUTION A.038.

$$\textit{Denote } a=\sin x\,;\, b=\frac{\sqrt{3}\cos x}{2}\,;\, c=\frac{\cos x}{2}$$

$$\Omega(x)=\begin{vmatrix} a & -b & -c \\ b & a & -1 \\ c & 1 & a \end{vmatrix}=a^3+bc-bc+ac^2+ab^2+a=$$

$$=a(a^2+b^2+c^2+1)=\sin x\left(\sin^2 x+\frac{3\cos^2 x}{4}+\frac{\cos^2 x}{4}+1\right)=$$

$$=\sin x\,(\sin^2 x+\cos^2 x+1)=2\sin x$$

$$\Omega(x)\Omega(y)+\Omega(y)\Omega(z)+\Omega(z)\Omega(x)=$$

$$=4(\sin x\sin y+\sin y\sin z+\sin z\sin x)\leq 4(xy+yz+zx)\leq 4(x^2+y^2+z^2)$$

Equality holds for $x=y=z=0$.

SOLUTION A.039.

$$1=a^2+b^2\geq 2\sqrt{a^2b^2}=2ab\Rightarrow ab\leq\frac{1}{2}\Rightarrow\frac{1}{ab}\geq 2 \quad (1)$$

$$\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}\overset{AM-GM}{\geq}\frac{2\sqrt{ab}}{ab}=\frac{2}{\sqrt{ab}}=2\left(\frac{1}{ab}\right)^{\frac{1}{2}}\geq 2\cdot 2^{\frac{1}{2}}=2\sqrt{2}$$

Equality holds if $a=b=\frac{\sqrt{2}}{2}$

SOLUTION A.040.

$$1=a^4+b^4+c^4\geq 3\sqrt[3]{(abc)^4}\Rightarrow\frac{1}{3}\geq\sqrt[3]{(abc)^4}$$

$$\Rightarrow (abc)^4 \le \frac{1}{27} \Rightarrow abc \le \frac{1}{\sqrt[4]{27}} \Rightarrow \frac{1}{abc} \ge \sqrt[4]{27} \quad (1)$$

$$\frac{a+b+c}{abc} \ge \frac{3\sqrt[3]{abc}}{abc} = \frac{3\sqrt[3]{abc}}{\sqrt[3]{(abc)^3}} = \frac{3}{\sqrt[3]{(abc)^2}} =$$

$$= 3 \cdot \left(\frac{1}{abc}\right)^{\frac{2}{3}} \overset{(1)}{\ge} 3 \cdot \left(\sqrt[4]{27}\right)^{\frac{2}{3}} = 3 \cdot 3^{\frac{3}{4}\cdot\frac{2}{3}} = 3 \cdot 3^{\frac{1}{2}} = 3\sqrt{3}$$

Equality holds for $a = b = c = \frac{1}{\sqrt[4]{3}}$.

SOLUTION A.041.

$$1 = a^3 + b^3 + c^3 + d^3 \ge 4\sqrt[4]{a^3b^3c^3d^3} = 4\sqrt[4]{(abcd)^3}$$

$$\sqrt[4]{(abcd)^3} \le \frac{1}{4} \Rightarrow (abcd)^3 \le \frac{1}{4^4} \Rightarrow abcd \le \frac{1}{\sqrt[3]{2^8}} \Rightarrow \frac{1}{abcd} \ge \sqrt[3]{2^8} \quad (1)$$

$$\frac{a+b+c+d}{abcd} \overset{AM-GM}{\ge} \frac{4\sqrt[4]{abcd}}{abcd} = \frac{4\sqrt[4]{abcd}}{\sqrt[4]{(abcd)^4}} = 4 \cdot \frac{1}{\sqrt[4]{(abcd)^3}} =$$

$$= 4 \cdot \left(\frac{1}{abcd}\right)^{\frac{3}{4}} \overset{(1)}{\ge} 4 \cdot \left(\sqrt[3]{2^8}\right)^{\frac{3}{4}} = 2^2 \cdot 2^{\frac{8}{3}\cdot\frac{3}{4}} = 2^2 \cdot 2^2 = 16$$

Equality holds for: $a = b = c = d = \frac{1}{\sqrt[3]{4}}$.

SOLUTION A.042.

$$1 = a^3 + b^3 + c^3 \ge 3\sqrt[3]{a^3b^3c^3} = 3abc \Rightarrow abc \le \frac{1}{3}$$

$$\Rightarrow \frac{1}{abc} \ge 3 \quad (1)$$

$$\frac{a+b+c}{abc} \overset{AM-GM}{\ge} \frac{3\sqrt[3]{abc}}{abc} = \frac{3\sqrt[3]{abc}}{\sqrt[3]{(abc)^3}} = 3 \cdot \frac{1}{\sqrt[3]{(abc)^2}} = 3 \cdot \left(\frac{1}{abc}\right)^{\frac{2}{3}} \overset{(1)}{\ge} 3 \cdot 3^{\frac{2}{3}} = 3\sqrt[3]{9}$$

Equality holds for: $a = b = c = \frac{1}{\sqrt[3]{3}}$.

SOLUTION A.043.

$$1 = a^2 + b^2 + c^2 + d^2 \overset{AM-GM}{\ge} 4 \cdot \sqrt[4]{a^2b^2c^2d^2} = 4\sqrt{abcd} \Rightarrow \sqrt{abcd} \le \frac{1}{4} \Rightarrow abcd \le \frac{1}{16}$$

$$\Rightarrow \frac{1}{abcd} \ge 16 \quad (1)$$

$$\frac{a+b+c+d}{abcd} \overset{AM-GM}{\ge} \frac{4\sqrt[4]{abcd}}{abcd} = \frac{4\sqrt[4]{abcd}}{\sqrt[4]{(abcd)^4}} = \frac{4}{\sqrt[4]{(abcd)^3}} = 4 \cdot \left(\frac{1}{abcd}\right)^{\frac{3}{4}} \overset{(1)}{\ge} 4 \cdot 16^{\frac{3}{4}} =$$

$$= 4 \cdot (2^4)^{\frac{3}{4}} = 4 \cdot 2^3 = 32$$

Equality holds for: $a = b = c = d = \frac{1}{2}$

SOLUTION A.044.

$$1 = a^2 + b^2 + c^2 \overset{AM-GM}{\geq} 3\sqrt[3]{a^2b^2c^2} = 3\sqrt[3]{(abc)^2}$$

$$1 \geq 3\sqrt[3]{(abc)^2} \Rightarrow \frac{1}{27} \geq (abc)^2 \Rightarrow abc \leq \frac{1}{3\sqrt{3}} \quad (1)$$

$$\frac{a+b+c}{abc} \overset{AM-GM}{\geq} \frac{3\sqrt[3]{abc}}{abc} = \frac{3\sqrt[3]{abc}}{\sqrt[3]{(abc)^3}} = 3 \cdot \frac{1}{\sqrt[3]{(abc)^2}} = 3 \cdot \left(\frac{1}{abc}\right)^{\frac{2}{3}} \overset{(1)}{\geq} 3 \cdot \left(3\sqrt{3}\right)^{\frac{2}{3}} =$$

$$= 3 \cdot \left(\left(\sqrt{3}\right)^3\right)^{\frac{2}{3}} = 3 \cdot \left(\sqrt{3}\right)^2 = 3 \cdot 3 = 9. \text{ Equality holds for } a = b = c = \frac{1}{\sqrt{3}}.$$

SOLUTION A.045.

Denote $a = \frac{1}{x}; b = \frac{1}{y}; c = \frac{1}{z}; d = \frac{1}{t}$*. Inequality can be written:*

$$\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{da}\right)(a^4 + b^4 + c^4 + d^4) \geq (a+b+c+d)^2$$

$$\frac{1}{abcd}(ab + bc + cd + da)(a^4 + b^4 + c^4 + d^4) \geq (a+b+c+d)^2$$

$$(ab + bc + cd + da)(a^4 + b^4 + c^4 + d^4) \geq abcd(a+b+c+d)^2$$

$$(ab + bc + cd + da)(a^4 + b^4 + c^4 + d^4) \overset{AM-GM}{\geq} 4\sqrt[4]{a^2b^2c^2d^2}\left(\frac{a^4}{1} + \frac{b^4}{1} + \frac{c^4}{1} + \frac{d^4}{1}\right) \overset{RADON}{\geq}$$

$$\geq 4\sqrt{abcd} \cdot \frac{(a+b+c+d)^4}{(1+1+1+1)^3} = 4\sqrt{abcd} \cdot \frac{1}{64}(a+b+c+d)^2(a+b+c+d)^2 \geq$$

$$\overset{AM-GM}{\geq} \frac{1}{16}\sqrt{abcd} \cdot \left(4\sqrt[4]{abcd}\right)^2 \cdot (a+b+c+d)^2 =$$

$$= \frac{1}{16}abcd \cdot 16\sqrt{abcd} \cdot (a+b+c+d)^2 = abcd(a+b+c+d)^2$$

SOLUTION A.046.

$$(ab + bc + cd + da)(a^4 + b^4 + c^4 + d^4) \overset{AM-GM}{\geq}$$

$$\geq 4\sqrt[4]{a^2b^2c^2d^2}\left(\frac{a^4}{1} + \frac{b^4}{1} + \frac{c^4}{1} + \frac{d^4}{1}\right) \overset{RADON}{\geq} 4\sqrt{abcd} \cdot \frac{(a+b+c+d)^4}{(1+1+1+1)^3} =$$

$$= \frac{1}{16}\sqrt{abcd} \cdot (a+b+c+d)^2 \cdot (a+b+c+d)^2 \geq$$

$$\overset{AM-GM}{\geq} \frac{1}{16}\sqrt{abcd}\left(4\sqrt[4]{abcd}\right)^2(a+b+c+d)^2 =$$

$$= \frac{1}{16}\sqrt{abcd}\cdot 16\sqrt{abcd}\cdot(a+b+c+d)^2 = abcd(a+b+c+d)^2$$

Equality holds for $a = b = c = d$.

SOLUTION A.047.

$$\log_2 x + \log_2 y + \log_2 z = 3 \Rightarrow \log_2(xyz) = \log_2 8 \Rightarrow xyz = 8$$

$$27 = 3^x + 3^y + 3^z \overset{AM-GM}{\geq} 3\cdot\sqrt[3]{3^x\cdot 3^y\cdot 3^z} =$$

$$= 3\sqrt[3]{3^{x+y+z}} \overset{AM-GM}{\geq} 3\sqrt[3]{3^{3\sqrt[3]{xyz}}} = 3\sqrt[3]{3^{3\sqrt[3]{8}}} = 3\sqrt[3]{3^6} = 3\cdot 3^2 = 27$$

$$3^x = 3^y = 3^z \Rightarrow x = y = z$$

$$xyz = 8 \Rightarrow x^3 = 8 \Rightarrow x = 2; y = 2; z = 2$$

SOLUTION A.048.

$$\left.\begin{matrix} a_1 \geq 1 \Rightarrow a_1^4 \geq 1 \Rightarrow a_1^4 - 1 \geq 0 \\ a_2 \geq 1 \Rightarrow a_2^4 \geq 1 \Rightarrow a_2^4 - 1 \geq 0 \end{matrix}\right\} \Rightarrow \left(a_1^4 - 1\right)\left(a_2^4 - 1\right) \geq 0$$

$$a_1^4a_2^4 - \left(a_1^4 + a_2^4\right) + 1 \geq 0 \Rightarrow a_1^4 + a_2^4 \leq (a_1a_2)^4 + 1 \quad (1)$$

$$\textit{Analogous: } a_3^4 + a_4^4 \leq (a_3a_4)^4 + 1 \quad (2)$$

$$\textit{By (1); (2): } a_1^4 + a_2^4 + a_3^4 + a_4^4 \leq (a_1a_2)^4\cdot(a_3a_4)^4 + 1$$

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 \leq (a_1a_2a_3a_4)^4 + 1 \quad (3)$$

$$\textit{Analogous: } a_5^4 + a_6^4 + a_7^4 + a_8^4 \leq (a_5a_6a_7a_8)^4 + 1 \quad (4)$$

By (3); (4):

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4 + a_6^4 + a_7^4 + a_8^4 \leq (a_1a_2a_3a_3)^4\cdot(a_5a_6a_7a_8)^4 + 1 =$$

$$= (a_1a_2a_3a_4a_5a_6a_7a_8)^4 + 1$$

SOLUTION A.049.

$$\textit{Let be } f:(0,\infty)\to\mathbb{R}; f(x) = \frac{1}{x^6} = x^{-6};$$

$$f'(x) = -6x^{-5}; f''(x) = +30x^{-4} = \frac{30}{x^4} > 0;\ f \textit{ convexe}$$

Suppose that a, b, c are different in pairs.

$$bc - ab = ac - ab - ac + bc$$

$$(c-a)b = (c-b)a + (c-a-c+b)c$$

$$b = \frac{c-b}{c-a}a + \left(1 - \frac{c-b}{c-a}\right)c$$

$$f(b)=f\left(\frac{c-b}{c-a}a+\left(1-\frac{c}{c-a}\right)c\right)\overset{f\ convexe}{\leq}\frac{c-b}{c-a}f(a)+\frac{b-a}{c-a}f(c)\quad (1)$$

$$a-b+c=a-\left(\frac{c-b}{c-a}a+\frac{b-a}{c-a}c\right)+c=\frac{b-a}{c-a}a+\frac{c-b}{c-a}c$$

$$f(a-b+c)=f\left(\frac{b-a}{c-a}a+\frac{c-b}{c-a}c\right)\overset{f\ convexe}{\leq}\frac{b-a}{c-a}f(a)+\frac{c-b}{c-a}f(c)\quad (2)$$

By adding (1); (2):

$$f(a-b+c)+f(b)\leq\left(\frac{b-a}{c-a}+\frac{c-b}{c-a}\right)f(a)+\left(\frac{c-b}{c-a}+\frac{b-a}{c-a}\right)f(c)$$

$$f(a-b+c)+f(b)\leq f(a)+f(c)$$

$$\frac{1}{(a-b+c)^6}+\frac{1}{b^6}\leq\frac{1}{a^6}+\frac{1}{b^6}.\ \textit{Equality holds for } a=b=c.$$

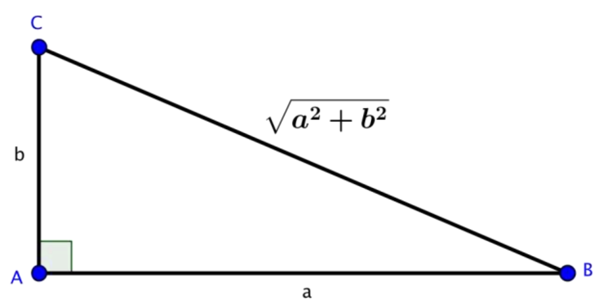
SOLUTION A.050.

$$\left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{a^2}}+e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}\right)\overset{CBS}{\geq}\left(\sqrt{e^{a^2+\frac{1}{b^2}}}+\sqrt{e^{b^2+\frac{1}{c^2}}}+\sqrt{e^{c^2+\frac{1}{a^2}}}\right)^2=$$

$$=\left(e^{\frac{1}{2}\left(a^2+\frac{1}{b^2}\right)}+e^{\frac{1}{2}\left(b^2+\frac{1}{c^2}\right)}+e^{\frac{1}{2}\left(c^2+\frac{1}{a^2}\right)}\right)^2\overset{AM-GM}{\geq}\left(e^{\frac{1}{2}\cdot\frac{2a}{b}}+e^{\frac{1}{2}\cdot\frac{2b}{c}}+e^{\frac{1}{2}\cdot\frac{2c}{a}}\right)^2=\left(e^{\frac{a}{b}}+e^{\frac{b}{c}}+e^{\frac{c}{a}}\right)^2$$

SOLUTION A.051.

$$\textit{Let be } \Delta\, ABC \textit{ with: } AB=a;AC=b;\mu(\hat{A})=90^\circ;BC=\sqrt{a^2+b^2}$$



$$\textit{By Gordon's inequality: } AB\cdot AC+AB\cdot BC+AC\cdot BC>4\sqrt{3}S[ABC]$$

$$a\cdot b+a\cdot\sqrt{a^2+b^2}+b\cdot\sqrt{a^2+b^2}>4\sqrt{3}\cdot\frac{ab}{2}$$

$$(a+b)\sqrt{a^2+b^2}>ab\left(2\sqrt{3}-1\right)\quad (1)$$

Analogous:

$$(b+c)\sqrt{b^2+c^2}>bc\left(2\sqrt{3}-1\right)\quad (2)$$

$$(c+a)\sqrt{c^2+a^2}>ca\left(2\sqrt{3}-1\right)\quad (3)$$

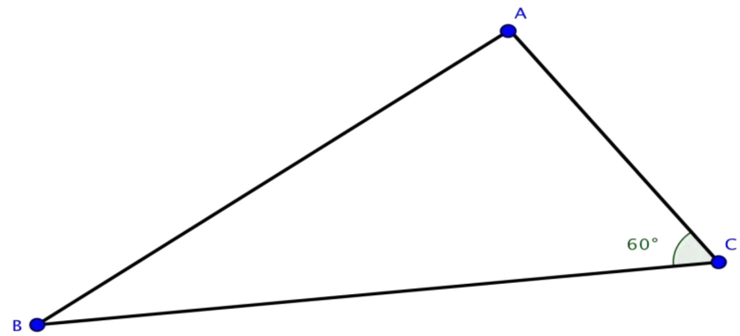
By (1); (2); (3):

$$(a+b)\sqrt{a^2+b^2}+(b+c)\sqrt{b^2+c^2}+(c+a)\sqrt{c^2+a^2}>\left(2\sqrt{3}-1\right)(ab+bc+ca)$$

Equality holds if $a=b=c=0$.

SOLUTION A.052.

Let be ΔABC with $BC=a; AC=b; \mu(\widehat{C})=60^\circ$



By cosine law: $AB=\sqrt{a^2+b^2-2ab\cos 60^\circ}=\sqrt{a^2+b^2-ab}$

By Gordon's inequality: $AB\cdot BC+BC\cdot CA+CA\cdot AB\ge 4\sqrt{3}S[ABC]$

$$a\cdot b+a\sqrt{a^2+b^2-ab}+b\sqrt{a^2+b^2-ab}\ge 4\sqrt{3}\cdot\frac{AC\cdot BC\cdot\sin C}{2}$$

$$ab+(a+b)\sqrt{a^2+b^2-ab}\ge 2\sqrt{3}\cdot ab\cdot\frac{\sqrt{3}}{2}$$

$$ab+(a+b)\sqrt{a^2+b^2-ab}\ge 3ab$$

$$(a+b)\sqrt{a^2+b^2-ab}\ge 2ab \quad (1)$$

Analogous: $(b+c)\sqrt{b^2+c^2-bc}\ge 2bc$ ***(2);*** $(c+a)\sqrt{c^2+a^2-ca}\ge 2ca$ ***(3)***

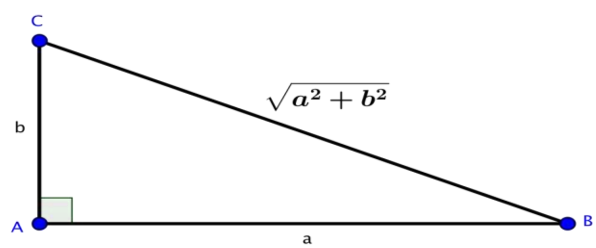
By adding (1); (2); (3):

$$(a+b)\sqrt{a^2+b^2-ab}+(b+c)\sqrt{b^2+c^2-bc}+(c+a)\sqrt{c^2+a^2-ca}\ge$$

$\ge 2(ab+bc+ca)$. ***Equality holds for $a=b=c$.***

SOLUTION A.053.

Let be ΔABC with: $AB=a; AC=b; \mu(\widehat{A})=90^\circ$



$$R=\frac{\sqrt{a^2+b^2}}{2}\quad \textit{(circumradii);}$$

$$r=\frac{S[ABC]}{\frac{AB+AC+BC}{2}}=\frac{\frac{ab}{2}}{\frac{a+b+\sqrt{a^2+b^2}}{2}}=\frac{ab}{a+b+\sqrt{a^2+b^2}}\quad \textit{(inradii)}$$

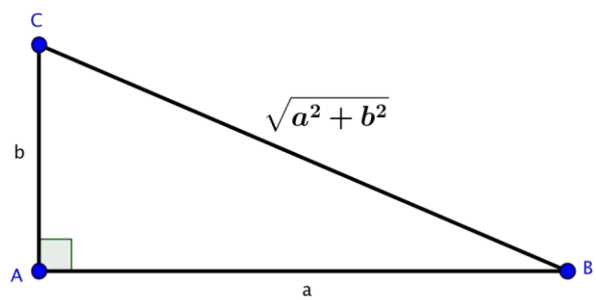
By Mitrinovic's inequality: $s\geq 3\sqrt{3}r$ *(s – semiperimeter)*

$$\frac{AB+BC+CA}{2}\geq 3\sqrt{3}\cdot\frac{ab}{a+b+\sqrt{a^2+b^2}}$$

$$\frac{a+b+\sqrt{a^2+b^2}}{2}\geq\frac{3\sqrt{3}ab}{a+b+\sqrt{a^2+b^2}}$$

$$\left(a+b+\sqrt{a^2+b^2}\right)^2\geq 6\sqrt{3}ab.\ \textit{Equality holds if } a=b=0.$$

SOLUTION A.054.



$$R=\frac{\sqrt{a^2+b^2}}{2}\quad \textit{(circumradii)}$$

$$s=\frac{AB+AC+BC}{2}=\frac{a+b+\sqrt{a^2+b^2}}{2}\quad \textit{(semiperimeter)}$$

By Mitrinovic's inequality: $s\leq\frac{3\sqrt{3}R}{2}$

$$\frac{a+b+\sqrt{a^2+b^2}}{2}<\frac{3\sqrt{3}}{2}\cdot\frac{\sqrt{a^2+b^2}}{2}$$

$$a+b+\sqrt{a^2+b^2}<3\sqrt{3}\cdot\frac{\sqrt{a^2+b^2}}{2}$$

$$2(a+b)<\left(3\sqrt{3}-2\right)\sqrt{a^2+b^2}\quad (1)$$

$$\textit{Analogous: } 2(b+c)<\left(3\sqrt{3}-2\right)\sqrt{b^2+c^2}\quad (2)$$

$$2(c+a)<\left(3\sqrt{3}-2\right)\sqrt{c^2+a^2}\quad (3)$$

$$4(a+b+c)<\left(3\sqrt{3}-2\right)\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$$

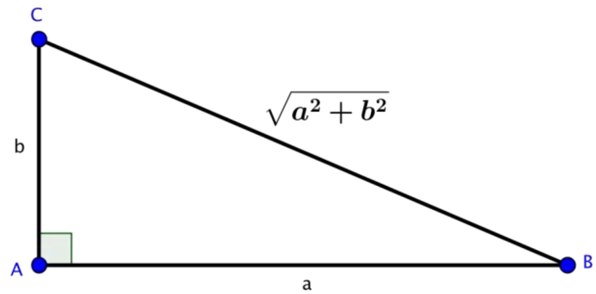
Equality holds for $a=b=c=0.$

$$4(a+b+c) \leq \left(3\sqrt{3}-2\right)\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$$

SOLUTION A.055.

If $a=b=0$ *or* $a=0; b\neq 0$ *or* $a\neq 0; b=0$ *the relationships are obvious.*

Suppose that $a\neq 0; b\neq 0$*. Let be* $\Delta\, ABC$ *with:* $AB=a; AC=b; \mu(\widehat{A})=90°$.



$$R=\frac{\sqrt{a^2+b^2}}{2}\ \textit{(circumradii)}$$

$$r=\frac{S[ABC]}{\frac{AB+BC+CA}{2}}=\frac{\frac{ab}{2}}{\frac{a+b+\sqrt{a^2+b^2}}{2}}=\frac{ab}{a+b+\sqrt{a^2+b^2}}\ \textit{(inradii)}$$

By Murray – Klamkin's inequality (1967):

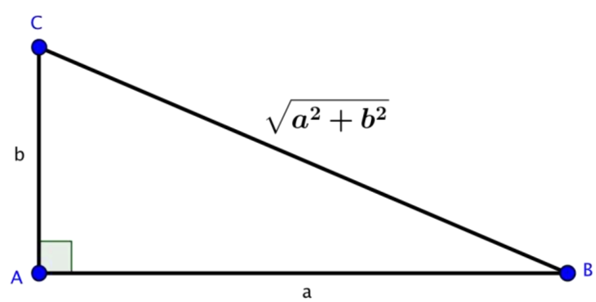
$$4r^2 \leq \frac{ab\sqrt{a^2+b^2}}{a+b+\sqrt{a^2+b^2}} \leq \frac{a^2+b^2}{4}=R^2$$

$$\frac{4a^2b^2}{\left(a+b+\sqrt{a^2+b^2}\right)^2} \leq \frac{ab\sqrt{a^2+b^2}}{a+b+\sqrt{a^2+b^2}} \leq \frac{a^2+b^2}{4}$$

$$\begin{cases} 4ab \leq \sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right) \\ 4ab\sqrt{a^2+b^2} \leq (a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right) \end{cases}$$

SOLUTION A.056.

Let be ΔABC *with* $AB=a; AC=b; \mu(\widehat{A})=90°$.



$$R=\frac{\sqrt{a^2+b^2}}{2}\ \textit{(circumradii)}$$

$$r=\frac{S[ABC]}{\frac{AB+BC+CA}{2}}=\frac{\frac{ab}{2}}{\frac{a+b+\sqrt{a^2+b^2}}{2}}=\frac{ab}{a+b+\sqrt{a^2+b^2}}\ \textit{(inradii)}$$

By Guba's inequality: $AB^3+BC^3+CA^3>8S[ABC](2R-r)$

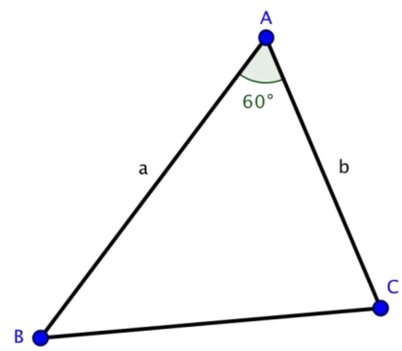
$$a^3+b^3+\left(\sqrt{a^2+b^2}\right)^3>8\cdot\frac{ab}{2}\left(2\cdot\frac{\sqrt{a^2+b^2}}{2}-\frac{ab}{a+b+\sqrt{a^2+b^2}}\right)$$

$$a^3+b^3+\left(\sqrt{a^2+b^2}\right)^3>4ab\left(\sqrt{a^2+b^2}-\frac{ab}{a+b+\sqrt{a^2+b^2}}\right)$$

$$a^3+b^3+\left(\sqrt{a^2+b^2}\right)^3+\frac{4a^2b^2}{a+b+\sqrt{a^2+b^2}}>4ab\sqrt{a^2+b^2}$$

SOLUTION A.057.

Let be $\Delta\ ABC$ *with* $AB=a; AC=b; m(\widehat{A})=60°$.



By cosine law: $BC^2=a^2+b^2-2ab\cos A$

$$BC=a^2+b^2-2ab\cdot\frac{1}{2}=a^2+b^2-ab$$

$$s=\frac{a+b+\sqrt{a^2+b^2-ab}}{2}$$

$$r=\frac{S}{s}=\frac{\frac{1}{2}\cdot a\cdot b\cdot\sin A}{\frac{a+b+\sqrt{a^2+b^2-ab}}{2}}=\frac{\frac{ab\sqrt{3}}{4}}{\frac{a+b+\sqrt{a^2+b^2-ab}}{2}}=\frac{ab\sqrt{3}}{2\left(a+b+\sqrt{a^2+b^2-ab}\right)}$$

By Mitrinovic's inequality: $s\geq 3\sqrt{3}r$

$$\frac{a+b+\sqrt{a^2+b^2-ab}}{2}\geq\frac{9ab}{2\left(a+b+\sqrt{a^2+b^2-ab}\right)}$$

$$\left(a+b+\sqrt{a^2+b^2-ab}\right)^2\geq 9ab\Rightarrow a+b+\sqrt{a^2+b^2-ab}\geq 3\sqrt{ab}\quad\textit{(1)}$$

Analogous: $b+c+\sqrt{b^2+c^2-bc}\geq 3\sqrt{bc}$ *(2)*

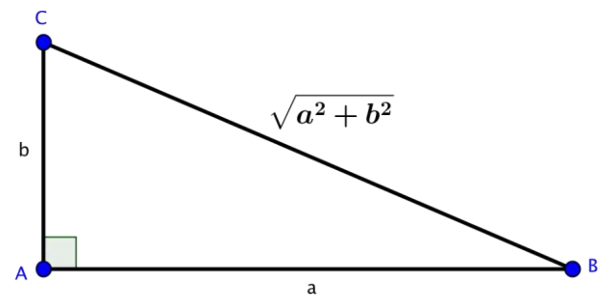
$$c+a+\sqrt{c^2+a^2-ca}\ge 3\sqrt{ca}\quad (3)$$

By adding (1); (2); (3):

$$2(a+b+c)+\sum_{cyc}\sqrt{a^2+b^2-ab}\ge 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

SOLUTION A.058.

Let be $\Delta\, ABC$ *with:* $AB=a;\ AC=b;\ \mu(\widehat{A})=90^\circ$



$$r=\frac{S[ABC]}{\frac{AB+AC+BC}{2}}=\frac{\frac{ab}{2}}{\frac{a+b+\sqrt{a^2+b^2}}{2}}=\frac{ab}{a+b+\sqrt{a^2+b^2}}\quad (inradii)$$

By Neuberg's inequality: $a^2+b^2+\left(\sqrt{a^2+b^2}\right)^2\ge 36r^2$

$$2(a^2+b^2)\ge 36\cdot\frac{a^2b^2}{\left(a+b+\sqrt{a^2+b^2}\right)^2}$$

$$(a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2\ge 18a^2b^2$$

SOLUTION A.059.

$$x^2+\frac{(a+b)y^2}{a}+\frac{(a+b)z^2}{b}=x^2+\frac{y^2}{\frac{a}{a+b}}+\frac{z^2}{\frac{b}{a+b}}\overset{CBS}{\ge}x^2+\frac{(y+z)^2}{\frac{a}{a+b}+\frac{b}{a+b}}=$$

$$=x^2+(y+z)^2\ge 2\sqrt{x^2(y+z)^2}=2x(y+z)$$

$$x^2+\frac{(a+b)y^2}{a}+\frac{(a+b)z^2}{b}\ge 2x(y+z)$$

$$\frac{(a+b)y^2}{a}+\frac{(a+b)z^2}{b}\ge 2x(y+z)-x^2=x(2y+2z-x)$$

SOLUTION A.060.

$$x^2+\frac{(a+b+c)y^2}{a}+\frac{(a+b+c)z^2}{b}+\frac{(a+b+c)t^2}{c}=$$

$$= x^2 + \frac{y^2}{\frac{a}{a+b+c}} + \frac{z^2}{\frac{b}{a+b+c}} + \frac{t^2}{\frac{c}{a+b+c}} \overset{CBS}{\geq}$$

$$\geq x^2 + \frac{(y+z+t)^2}{\frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c}} = x^2 + \frac{(y+z+t)^2}{\frac{a+b+c}{a+b+c}} =$$

$$= x^2 + (y+z+t)^2 \overset{AM-GM}{\geq} 2\sqrt{x^2(y+z+t)^2} = 2x(y+z+t)$$

$$\frac{(a+b+c)y^2}{a} + \frac{(a+b+c)z^2}{b} + \frac{(a+b+c)t^2}{c} \geq$$

$$\geq 2x(y+z+t) - x^2 = x(2y+2z+2t-x)$$

SOLUTION A.061.

$$\frac{\lg x}{\lg x + \lg z + \lg t} > \frac{\lg x}{\lg x + \lg y + \lg y + \lg t} \quad (1)$$

$$\frac{\lg y}{\lg x + \lg y + \lg t} > \frac{\lg y}{\lg x + \lg y + \lg z + \lg t} \quad (2)$$

$$\frac{\lg z}{\lg x + \lg y + \lg z} > \frac{\lg z}{\lg x + \lg y + \lg z + \lg t} \quad (3)$$

$$\frac{\lg t}{\lg y + \lg z + \lg t} > \frac{\lg t}{\lg x + \lg y + \lg z + \lg t} \quad (4)$$

By adding (1); (2); (3); (4): $\frac{\lg x}{\lg(xzt)} + \frac{\lg y}{\lg(xyz)} + \frac{\lg z}{\lg(xyz)} + \frac{\lg t}{\lg(yzt)} > \frac{\lg x + \lg y + \lg z + \lg t}{\lg x + \lg y + \lg z + \lg t} = 1$

$$\log_{xzt} x + \log_{xyz} y + \log_{xyz} z + \log_{yzt} t > 1 \quad (5)$$

By AM-QM: $\log_{xzt} x + \log_{xyz} y + \log_{xyz} z + \log_{yzt} t <$

$$< 2\sqrt{(\log_{xzt} x)^2 + (\log_{xyz} y)^2 + (\log_{xyz} z)^2 + (\log_{yzt} t)^2} \quad (6)$$

By (5); (6): $2\sqrt{(\log_{xzt} x)^2 + (\log_{xyz} y)^2 + (\log_{xyz} z)^2 + (\log_{yzt} t)^2} > 1$

$$(\log_{xzt} x)^2 + (\log_{xyz} y)^2 + (\log_{xyz} z)^2 + (\log_{yzt} t)^2 > \frac{1}{4}$$

SOLUTION A.062.

$$\frac{\lg x}{\lg x + \lg z + \lg t} < \frac{\lg x}{\lg x + \lg z} \quad (1)$$

$$\frac{\lg y}{\lg x + \lg y + \lg t} < \frac{\lg y}{\lg y + \lg t} \quad (2)$$

$$\frac{\lg z}{\lg x + \lg y + \lg z} < \frac{\lg z}{\lg x + \lg z} \quad (3)$$

$$\frac{\lg t}{\lg y + \lg z + \lg t} < \frac{\lg t}{\lg y + \lg t} \quad (4)$$

By adding (1); (2); (3); (4):

$$\frac{\lg x}{\lg(xzt)}+\frac{\lg y}{\lg(xyt)}+\frac{\lg z}{\lg(xyz)}+\frac{\lg t}{\lg(yzt)}<\frac{\lg x+\lg z}{\lg x+\lg z}+\frac{\lg y+\lg t}{\lg y+\lg t}=2$$

$$2>\log_{xzt}x+\log_{xyt}y+\log_{xyz}z+\log_{yzt}t\geq$$

$$\overset{AM-GM}{\geq}4\sqrt[4]{(\log_{xzt}x)(\log_{xyt}y)(\log_{xyz}z)(\log_{yzt}t)}$$

$$16>256(\log_{xzt}x)(\log_{xyt}y)(\log_{xyz}z)(\log_{yzt}t)$$

$$(\log_{xzt}x)(\log_{xyt}y)(\log_{xyz}z)(\log_{yzt}t)<\frac{1}{16}$$

SOLUTION A.063.

From: $x\sqrt{1-y^2}+y\sqrt{1-x^2}=\frac{\sqrt{3}}{2}\Rightarrow$

$$\sin^{-1}x+\sin^{-1}y\in\left\{\frac{\pi}{3},\frac{2\pi}{3}\right\}\quad (1)$$

Analogous: $\sin^{-1}y+\sin^{-1}z\in\left\{\frac{\pi}{3},\frac{2\pi}{3}\right\}\quad (2)$

$$\sin^{-1}x+\sin^{-1}z\in\left\{\frac{\pi}{3},\frac{2\pi}{3}\right\}\quad (3)$$

By adding (1); (2); (3): $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z\in\left\{\frac{\pi}{2},\pi\right\}\quad (4)$

If $x=y=z\Rightarrow\sin^{-1}x=\sin^{-1}y=\sin^{-1}z\in\left\{\frac{\pi}{6},\frac{\pi}{3}\right\}$

Solutions: $\left(\frac{1}{2};\frac{1}{2};\frac{1}{2}\right);\left(\frac{\sqrt{3}}{2};\frac{\sqrt{3}}{2};\frac{\sqrt{3}}{2}\right)$

If $x=y\neq z\Rightarrow z=0; x=y=\frac{\sqrt{3}}{2}$

$$z=1; x=y=\frac{1}{2}$$

If $x=z\neq y\Rightarrow y=0; x=z=\frac{\sqrt{3}}{2}$

$$y=1;\ x=z=\frac{1}{2}$$

If $y=z\neq x\Rightarrow x=0; y=z=\frac{\sqrt{3}}{2}$

$$x=1;\ y=z=\frac{1}{2}$$

Solutions: $\left(\frac{\sqrt{3}}{2};\frac{\sqrt{3}}{2};0\right);\left(\frac{1}{2},\frac{1}{2},1\right);\left(\frac{\sqrt{3}}{2},0,\frac{\sqrt{3}}{2}\right)$

$$\left(\frac{1}{2},1,\frac{1}{2}\right);\left(0,\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right);\left(1;\frac{1}{2};\frac{1}{2}\right)$$

SOLUTION A.064.

$$\frac{2x^2+4}{z^2+2y+3}\overset{AM-GM}{\leq}\frac{2x^2+4}{z^2+(y^2+1)+3}=\frac{2(x^2+2)}{(z^2+2)+(y^2+2)}\quad (1)$$

Analogous: $\frac{2y^2+4}{x^2+2z+3}\leq\frac{2(y^2+2)}{(x^2+2)+(z^2+2)}$ *(2);* $\frac{2z^2+4}{y^2+2x+3}\leq\frac{2(z^2+2)}{(y^2+2)+(x^2+2)}$ *(3)*

By adding (1); (2); (3):

$$\sum_{cyc}\frac{2x^2+4}{z^2+2y+3}\leq 2\sum_{cyc}\frac{x^2+2}{(z^2+2)+(y^2+2)}\overset{NESBIT}{\leq}2\cdot\frac{3}{2}=3$$

Equality holds if: $x^2+2=y^2+2=z^2+2\Rightarrow x=y=z\Rightarrow$

$$3\cdot\frac{2x^2+4}{x^2+2x+3}=3\Rightarrow 2x^2+4=x^2+2x+3\Rightarrow$$

$$\Rightarrow x^2-2x+1=0\Rightarrow (x-1)^2=0\Rightarrow x=y=z=1.$$

SOLUTION A.065.

$\frac{1}{4}+\frac{1}{8}+\frac{5}{8}=\frac{1}{4}+\frac{6}{8}=\frac{1}{4}+\frac{3}{4}=1$. *By Young's inequality:*

$$\frac{a^4}{4}+\frac{b^8}{8}+\frac{c^{\frac{8}{5}}}{\frac{8}{5}}\geq abc\quad (1);\ \frac{a^{\frac{8}{5}}}{8}+\frac{b^8}{8}+\frac{c^4}{4}\geq abc\quad (2)$$

By multiplying (1); (2): $\left(\frac{a^4}{4}+\frac{b^8}{8}+\frac{c^{\frac{8}{5}}}{\frac{8}{5}}\right)\left(\frac{a^{\frac{8}{5}}}{\frac{8}{5}}+\frac{b^8}{8}+\frac{c^4}{4}\right)\geq abc\cdot abc\geq$

$$\overset{GM-HM}{\geq}abc\cdot\left(\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\right)^3=\frac{27abc}{\frac{(ab+bc+ca)^3}{(abc)^3}}=\frac{27(abc)^4}{(ab+bc+ca)^3}$$

SOLUTION A.066.

First, we prove that:

$$\frac{x^2+1}{x^3+1}\leq\frac{1}{\sqrt{x}}\quad (1)$$

$$(x^2+1)^2\cdot x\leq(x^3+1)^2;\ x^5+2x^3+x\leq x^6+2x^3+1$$

$$x^6-x^5-x+1\geq 0;\ x^5(x-1)-(x-1)\geq 0$$

$$(x-1)(x^5-1)\geq 0;\ (x-1)^2(x^4+x^3+x^2+x+1)\geq 0$$

Analogous: $\frac{y^2+1}{y^3+1}\leq\frac{1}{\sqrt{y}}$ *(2)*

By multiplying (1); (2): $\frac{(x^2+1)(y^2+1)}{(x^3+1)(y^3+1)} \le \frac{1}{\sqrt{xy}}$ *(3)*

Analogous: $\frac{(y^2+1)(z^2+1)}{(y^3+1)(z^3+1)} \le \frac{1}{\sqrt{yz}}; \frac{(z^2+1)(x^2+1)}{(z^3+1)(x^3+1)} \le \frac{1}{\sqrt{zx}}$ *(4)*

By (3); (4): $\sum_{cyc} \frac{(x^2+1)(y^2+1)}{(x^3+1)(y^3+1)} \le \sum_{cyc} \frac{1}{\sqrt{xy}}$ *(5)*

$\sqrt{x} + \sqrt{y} + \sqrt{z} = 3\sqrt{xyz} \Rightarrow \sum_{cyc} \frac{1}{\sqrt{xy}} = 3$ *(6)*

By (5); (6): $\sum_{cyc} \frac{(x^2+1)(y^2+1)}{(x^3+1)(y^3+1)} \le 3$

SOLUTION A.067.

$$\begin{cases} x(x+y) + z(x+y) = 2 \\ y(x+y) + z(y+x) = 3 \\ x(z+y) + z(z+y) = 6 \end{cases} \Rightarrow \begin{cases} (x+y)(x+z) = 2 & (1) \\ (x+y)(y+z) = 3 & (2) \\ (z+y)(x+z) = 6 & (3) \end{cases}$$

By multiplying (1); (2); (3): $(x+y)^2(y+z)^2(z+x)^2 = 36$

$$(x+y)(y+z)(z+x) = \pm 6$$

By (1): $y + z = \pm 3$ *(4)*

By (2): $z + x = \pm 2$ *(5)*

By (3): $x + y = \pm 1$ *(6)*

By adding (4); (5); (6): $2(x+y+z) = \pm(1+2+3)$

$$\Omega = |x+y+z| = \left|\pm\frac{6}{2}\right| = 3$$

SOLUTION A.068.

$$\sum_{cyc} \frac{x}{x + 2\sqrt{yz}} \overset{AM-GM}{\ge} \sum \frac{x}{x+y+z} = 1 = \sum \frac{y}{x+y+z} \overset{QM-AM}{\le}$$

$$\le \sum_{cyc} \frac{y}{x + 2\sqrt{\frac{y^2+z^2}{z}}} = \sum_{cyc} \frac{y}{x + \sqrt{2(y^2+z^2)}}$$

Hence: $x = y = z$*. Be second equation:* $2x + \log_2 x + 2^x = 9$

Let be $f: (0, \infty) \to \mathbb{R}; f(x) = 2x + \log_2 x + 2^x$

$f'(x) = 2 + \frac{1}{x \log 2} + 2^x \log 2 > 0$; f *increasing*

$$f(2) = 9 \Rightarrow x = y = z = 2$$

SOLUTION A.069.

$$\text{Let be } A=\begin{pmatrix} a & b & c\\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}\\ \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2}\end{pmatrix}.$$

$$\det A=\frac{a}{cb}\left(\frac{1}{c}-\frac{1}{b}\right)+\frac{b}{ac}\left(\frac{1}{a}-\frac{1}{c}\right)+\frac{c}{bc}\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$\text{By Hadamard's inequality: } \left(\frac{a}{cb}\left(\frac{1}{c}-\frac{1}{b}\right)+\frac{b}{ac}\left(\frac{1}{a}-\frac{1}{c}\right)+\frac{c}{ba}\left(\frac{1}{b}-\frac{1}{a}\right)\right)^2<$$

$$<(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)$$

SOLUTION A.070.

$$\text{Let be } A=\begin{pmatrix} a & \frac{1}{b} & \frac{1}{c^2} & 1\\ 1 & a & \frac{1}{b} & \frac{1}{c^2}\end{pmatrix}\in M_{2,4}(\mathbb{R})$$

$$A^T=\begin{pmatrix} a & 1\\ \frac{1}{b} & a\\ \frac{1}{c^2} & \frac{1}{b}\\ 1 & \frac{1}{c^2}\end{pmatrix}\in M_{4,2}(\mathbb{R})$$

$$A\cdot A^T=\begin{pmatrix} a & \frac{1}{b} & \frac{1}{c^2} & 1\\ 1 & a & \frac{1}{b} & \frac{1}{c^2}\end{pmatrix}\begin{pmatrix} a & 1\\ \frac{1}{b} & a\\ \frac{1}{c^2} & \frac{1}{b}\\ 1 & \frac{1}{c^2}\end{pmatrix}$$

$$A\cdot A^T=\begin{pmatrix} a^2+\frac{1}{b^2}+\frac{1}{c^4}+1 & a+\frac{a}{b}+\frac{1}{bc^2}+\frac{1}{c^2}\\ a+\frac{a}{b}+\frac{1}{bc^2}+\frac{1}{c^2} & a^2+\frac{1}{b^2}+\frac{1}{c^4}+1\end{pmatrix}$$

$$\text{By Binet-Cauchy's theorem: } \det(A\cdot A^T)\geq 0$$

$$\left(a^1+\frac{1}{b^2}+\frac{1}{c^4}+1\right)^2\geq\left(a+\frac{a}{b}+\frac{1}{bc^2}+\frac{1}{c^2}\right)^2$$

$$a^2+\frac{1}{b^2}+\frac{1}{c^4}+1\geq\left|a+\frac{a}{b}+\frac{1}{bc^2}+\frac{1}{c^2}\right|$$

Equality holds for $a=b=c=1$.

SOLUTION A.071.

$$\textit{Let be } f:(0,\infty)\to\mathbb{R};\ f(x)=\frac{x^3}{\left(x^2+1\right)^2}$$

$$f'(x)=\frac{x^2(3-x^2)}{(x^2+1)^3}$$

$$\min f(x)=f\left(\sqrt{3}\right)=\frac{3\sqrt{3}}{(3+1)^2}=\frac{3\sqrt{3}}{16}\Rightarrow f(a)\le\frac{3\sqrt{3}}{16}$$

$$\frac{a^3}{\left(a^2+1\right)^2}\le\frac{3\sqrt{3}}{16}\quad (1)$$

$$\textit{By multiplying in (1): }\frac{a^3b^2}{\left(a^2+1\right)^2}\le\frac{3\sqrt{3}}{16}b^2\quad (2)$$

$$\textit{Analogous: }\frac{b^3c^2}{\left(b^2+1\right)^2}\le\frac{3\sqrt{3}}{16}c^2\quad (3)$$

$$\frac{c^3a^2}{\left(c^2+1\right)^2}\le\frac{3\sqrt{3}}{16}a^2\quad (4)$$

$$\textit{By adding (2); (3); (4): }\frac{a^3b^2}{\left(a^2+1\right)^2}+\frac{b^3c^2}{\left(b^2+1\right)^2}+\frac{c^3a^2}{\left(c^2+1\right)^2}\le\frac{3\sqrt{3}}{16}(a^2+b^2+c^2)=\frac{3\sqrt{3}}{16}\cdot 9=\frac{27\sqrt{3}}{16}$$

SOLUTION A.072.

$$\textit{Let be } A=\begin{pmatrix} a & b & c & 1\\ 1 & a & b & c\end{pmatrix}\in M_{2,4}(\mathbb{R})$$

$$A^T=\begin{pmatrix} a & 1\\ b & a\\ c & b\\ 1 & c\end{pmatrix}\in M_{4,2}(\mathbb{R})$$

$$A\cdot A^T=\begin{pmatrix} a & b & c & 1\\ 1 & a & b & c\end{pmatrix}\begin{pmatrix} a & 1\\ b & a\\ c & b\\ 1 & c\end{pmatrix}=\begin{pmatrix} a^2+b^2+c^2+1 & a+ab+bc+c\\ a+ab+bc+c & a^2+b^2+c^2+1\end{pmatrix}=$$

$$=\begin{pmatrix} 4 & a+ab+bc+c\\ a+ab+bc+c & 4\end{pmatrix}$$

$$\textit{By Cauchy – Binet's theorem: }\det(A\cdot A^T)\ge 0$$

$$\begin{vmatrix} 4 & a+ab+bc+c\\ a+ab+bc+c & 4\end{vmatrix}\ge 0$$

$$16-(a+ab+bc+c)^2\ge 0$$

$$(a+ab+bc+c)^2\le 16,\ \ |a+b(a+c)+c|\le 4$$

$$\textit{Equality holds for } a=b=c=1.$$

SOLUTION A.073.

$$If\ x \geq y \Rightarrow x - y \geq 0$$

$$\frac{x}{y} \geq 1 \Rightarrow \left(\frac{x}{y}\right)^{x-y} \geq 1^{x-y} = 1$$

$$If\ x \leq y \Rightarrow x - y \leq 0 \Rightarrow y - x \geq 0$$

$$\frac{x}{y} \leq 1 \Rightarrow \frac{y}{x} \geq 1 \Rightarrow \left(\frac{y}{x}\right)^{y-x} \geq 1 \Rightarrow \left(\frac{x}{y}\right)^{x-y} \geq 1$$

$$\left(\frac{x}{y}\right)^{x-y} \geq 1 \Rightarrow z\left(\frac{x}{y}\right)^{x-y} \geq z \quad (1)$$

$$Analogous:\ x\left(\frac{y}{z}\right)^{y-z} \geq x \quad (2)$$

$$y\left(\frac{z}{x}\right)^{z-x} \geq y \quad (3)$$

By adding (1); (2); (3):

$$z\left(\frac{x}{y}\right)^{x-y} + x\left(\frac{y}{z}\right)^{y-z} + y\left(\frac{z}{x}\right)^{z-x} \geq x + y + z \overset{AM-GM}{\geq} 3\sqrt[3]{xyz} = 3 \cdot \sqrt[3]{1} = 3$$

SOLUTION A.074.

$$(x-1)^4(x^2+x+1) \geq 0$$

$$(x-1)^2[x^3(x-1)-(x-1)] \geq 0$$

$$(x-1)^2(x^4-x^3-x+1) \geq 0$$

$$x^4(x-1)^2 - x^3(x-1)^2 - x(x-1)^2 + (x-1)^2 \geq 0$$

$$x^6 - 2x^5 + x^4 - x^5 + 2x^4 - x^3 - x^3 + 2x^2 - x + x^2 - 2x + 1 \geq 0$$

$$x^6 - 3x^5 + 3x^4 - 2x^3 + 3x^2 - 3x + 1 \geq 0$$

$$x^6 + 3x^5 + 3x^4 + x^3 + x^3 + 3x^2 + 3x + 1 \leq 2x^6 + 6x^4 + 6x^2 + 2$$

$$(x^3+1)(x^3+3x^2+3x+1) \leq 2(x^6+3x^4+3x^2+1)$$

$$(x^3+1)(x+1)^3 \leq 2(x^2+1)^3$$

$$\left(\frac{x^2+1}{x+1}\right)^3 \geq \frac{x^3+1}{2} \quad (1)$$

Replacing in (1) $x = a; x = b; x = c$*:*

$$\left(\frac{a^2+1}{a+1}\right)^3 \geq \frac{a^3+1}{2} \quad (2)$$

$$\left(\frac{b^2+1}{b+1}\right)^3 \geq \frac{b^3+1}{2} \quad (3)$$

$$\left(\frac{c^2+1}{c+1}\right)^3 \ge \frac{c^3+1}{2} \quad (4)$$

By adding (2); (3); (4): $\left(\frac{a^2+1}{a+1}\right)^3 + \left(\frac{b^2+1}{b+1}\right)^3 + \left(\frac{c^2+1}{c+1}\right)^3 \ge \frac{a^3+b^3+c^3+3}{2} = \frac{3+3}{2} = \frac{6}{2} = 3$

SOLUTION A.075.

$$\text{Denote } x^4 = a; y^4 = b; z^4 = c$$

$$\frac{x^4y^4}{z^4} + \frac{y^4z^4}{x^4} + \frac{z^4x^4}{y^4} = \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}$$

We prove that: $\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \ge a + b + c \quad (1)$

$$a^2b^2 + b^2c^2 + a^2c^2 \ge abc(a+b+c), 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - 2abc(a+b+c) \ge 0$$

$$(ab-bc)^2 + (bc-ca)^2 + (ca-ab)^2 \ge 0$$

We prove that: $a + b + c \ge \sqrt{3(ab+bc+ca)} \quad (2)$

$$a^2 + b^2 + c^2 + 2(ab+bc+ca) \ge 3(ab+bc+ca),\ a^2+b^2+c^2 \ge ab+bc+ca$$

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \ge 0$$

We prove that: $\sqrt{3(ab+bc+ca)} \ge 3\sqrt[4]{\frac{abc(a+b+c)}{3}} \quad (3)$

$$9(ab+bc+ca)^2 \ge 81 \cdot \frac{abc(a+b+c)}{3}$$

$$a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a+b+c) \ge 3abc(a+b+c)$$

$$a^2b^2 + b^2c^2 + c^2a^2 - abc(a+b+c) \ge 0$$

$$(ab-bc)^2 + (bc-ca)^2 + (ca-ab)^2 \ge 0$$

By (1); (2); (3): $\frac{x^4y^4}{z^4} + \frac{y^4z^4}{x^4} + \frac{z^4x^4}{y^4} \ge 3\sqrt[4]{\frac{abc(a+b+c)}{3}} = \sqrt[4]{27abc(a+b+c)} =$

$$= \sqrt[4]{27x^4y^4z^4(x^4+y^4+z^4)} = xyz\sqrt[4]{27(x^4+y^4+z^4)}$$

Equality holds for $x = y = z$*. Replace* $x = y = z$ *in second equation:*

$$x^4 - 4x^3 + 6x^2 - 4x + 1 = 0,\ (x-1)^2 = 0 \Rightarrow x = 1$$

Answer $x = y = z = 1$*.*

SOLUTION A.076.

$$\log_2(2x^3) = \log_{8x} z^{16} \Rightarrow \log_2 2 + \log_2 x^3 = 16\log_{8x} z$$

$$1 + 3\log_2 x = \frac{16\log_2 z}{\log_2(8x)}$$

$$(1 + 3\log_2 x)(\log_2 8 + \log_2 x) = 16\log_2 z$$

$$(1+3\log_2 x)(3+\log_2 x)=16\log_2 z$$

$$x,y,z\ge 1\Rightarrow \log_2 x\,;\log_2 y\,;\log_2 z\ge 0$$

$$\textit{Denote } a=\log_2 x\,;b=\log_2 y\,;c=\log_2 z,\ \ a,b,c>0$$

$$16c=(1+3a)(3+a)=(1+a+a+a)(1+1+1+a)\ge$$

$$\overset{AM-GM}{\ge} 4\cdot\sqrt[4]{1\cdot a\cdot a\cdot a}\cdot 4\cdot\sqrt[4]{1\cdot a\cdot a\cdot a}=16a$$

$$16c\ge 16a\Rightarrow c\ge a\ \ (1)$$

$$\textit{Analogous: } a\ge b; b\ge c\ \ (2)$$

$$\textit{By (1); (2):} c\ge a\ge b\ge c\Rightarrow a=b=c.$$

$$(1+3a)(3+a)=16a, 3+a+9a+3a^2-16a=0$$

$$3a^2-6a+3=0$$

$$a^2-2a+1=0\Rightarrow (a-1)^2=0\Rightarrow a=1\Rightarrow a=b=c=1$$

$$\log_2 x=\log_2 y=\log_2 z=1\Rightarrow x=y=z=2$$

SOLUTION A.077.

$$\frac{a^6}{1^5}+\frac{b^6}{1^5}+\frac{c^6}{1^5}+\frac{1}{32}\left(\frac{(3-a)^6}{1^5}+\frac{(3-b)^6}{1^5}+\frac{(3-c)^6}{1^5}\right)\ge$$

$$\overset{RADON}{\ge}\frac{(a+b+c)^6}{(1+1+1)^5}+\frac{1}{32}\cdot\frac{\big((3-a)+(3-b)+(3-c)\big)^6}{(1+1+1)^5}=$$

$$=\frac{3^6}{3^5}+\frac{1}{32}\cdot\frac{(9-3)^6}{3^5}=3+\frac{1}{32}\cdot\frac{6^6}{3^5}=3+\frac{2^6\cdot 3^6}{2^5\cdot 3^5}=3+2\cdot 3=9$$

SOLUTION A.078.

$$\det\Big((A^2+B^2)+C^2\Big)+\det\Big((A^2+B^2)-C^2\Big)=2(\det(A^2+B^2)+\det C^2)\ \ \textit{(1)}$$

$$\det(A^2-B^2+C^2)+\det(-A^2+B^2+C^2)=2(\det C^2+\det(A^2-B^2))\ \ \textit{(2)}$$

By adding (1); (2):

$$\det(A^2+B^2+C^2)+\det(A^2+B^2-C^2)+\det(A^2-B^2+C^2)+\det(-A^2+B^2+C^2)=$$

$$=2(2\det C^2+\det(A^2+B^2)+\det(A^2-B^2))=$$

$$=2(2\det C^2+2\det A^2+2\det B^2)=4(2\det C+2\det A+2\det B)=$$

$$=8(\det A+\det B+\det C)\ge 8\cdot 3\sqrt[3]{\det A\cdot\det B\cdot\det C}\ge 24\sqrt[3]{\det(ABC)}=$$

$$=24\sqrt[3]{8}=24\cdot\sqrt[3]{2^3}=48$$

$$\textit{Equality holds for } A=B=C\in\left\{\begin{pmatrix}-\sqrt{2} & 0\\ 0 & -\sqrt{2}\end{pmatrix};\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}\right\}$$

SOLUTION A.079.

$$y(y-x)^2 \geq 0,\ y^3 - 2xy^2 + x^2y \geq 0$$

$$6x^3 + 3x^2y + 2x^2y + xy^2 + 4xy^2 + 2y^3 \leq 6x^3 + 3xy^2 + 6x^2y + 3y^3$$

$$3x^2(2x+y) + xy(2x+y) + 2y^2(2x+y) \leq 3x(2x^2+y^2) + 3y(2x^2+y^2)$$

$$(3x^2+xy+2y^2)(2x+y) \leq (3x+3y)(2x^2+y^2)$$

$$\frac{3x^2+xy+2y^2}{2x^2+y^2} \leq \frac{3(x+y)}{2x+y} \quad (1)$$

Analogous: $\frac{3y^2+yz+2z^2}{2y^2+z^2} \leq \frac{3(y+z)}{2y+z}$ *(2);* $\frac{3z^2+zx+2x^2}{2z^2+x^2} \leq \frac{3(z+x)}{2z+x}$ *(3)*

By adding (1); (2); (3): $\frac{3x^2+xy+2y^2}{2x^2+y^2} + \frac{3y^2+yz+2z^2}{2y^2+z^2} + \frac{3z^2+zx+2x^2}{2z^2+x^2} \leq$

$$\leq 3\left(\frac{x+y}{2x+y} + \frac{y+z}{2y+z} + \frac{z+x}{2z+x}\right) = 3 \cdot 2 = 6$$

SOLUTION A.080.

Let be $f: (0,\infty) \to \mathbb{R};\ f(x) = x \log x$

$$f'(x) = \log x + x \cdot \frac{1}{x} = 1 + \log x;\ f''(x) = \frac{1}{x} > 0 \Rightarrow f \text{ concave}$$

By Popoviciu's inequality:

$$f(a) + f(b) + f(c) + 3f\left(\frac{a+b+c}{3}\right) \geq 2\left[f\left(\frac{b+c}{2}\right) + f\left(\frac{c+a}{2}\right) + f\left(\frac{a+b}{2}\right)\right]$$

$$a\log a + b\log b + c\log c + 3 \cdot \frac{a+b+c}{3} \cdot \log\left(\frac{a+b+c}{3}\right) \geq$$

$$\geq 2\left(\frac{b+c}{2}\log\left(\frac{b+c}{2}\right) + \frac{c+a}{2} + \log\frac{c+a}{2} + \frac{a+b}{2} + \log\frac{a+b}{2}\right)$$

$$\log a^a + \log b^b + \log c^c + (a+b+c)\log\left(\frac{a+b+c}{3}\right) \geq$$

$$\geq \log\left(\frac{b+c}{2}\right)^{b+c} + \log\left(\frac{c+a}{2}\right)^{c+a} + \log\left(\frac{a+b}{2}\right)^{a+b}$$

$$\log\left(a^a \cdot b^b \cdot c^c\left(\frac{a+b+c}{3}\right)^{a+b+c}\right) \geq \log\left(\left(\frac{b+c}{2}\right)^{b+c} \cdot \left(\frac{c+a}{2}\right)^{c+a} \cdot \left(\frac{a+b}{2}\right)^{a+b}\right)$$

$$a^a \cdot b^b \cdot c^c \cdot \frac{(a+b+c)^{a+b+c}}{3^{a+b+c}} \geq \frac{(b+c)^{b+c}(c+a)^{c+a}(a+b)^{a+b}}{2^{2a+2b+2c}}$$

$$a^a \cdot b^b \cdot c^c \cdot (a+b+c)^{a+b+c} \cdot 4^{a+b+c} \geq 3^{a+b+c}(b+c)^{b+c}(c+a)^{c+a}(a+b)^{a+b}$$

$$a^a \cdot b^b \cdot c^c \cdot (4a+4b+4c)^{a+b+c} \geq 3^{a+b+c} \cdot (b+c)^{b+c}(c+a)^{c+a}(a+b)^{a+b}$$

SOLUTION A.081.

$$xyz = a^3x + b^3y + c^3z \Rightarrow$$

$$\Rightarrow 1 = \frac{a^3}{yz} + \frac{b^3}{zx} + \frac{c^3}{xy} \overset{AM-GM}{\geq} \frac{a^3}{\left(\frac{y+z}{2}\right)^2} + \frac{b^3}{\left(\frac{z+x}{2}\right)^2} + \frac{c^3}{\left(\frac{x+y}{2}\right)^2} \overset{RADON}{\geq} \frac{(a+b+c)^3}{(x+y+z)^2}$$

$$1 \geq \frac{(a+b+c)^3}{(x+y+z)^2} \Rightarrow (x+y+z)^2 \geq (a+b+c)^3$$

$$x + y + z \geq (a+b+c)\sqrt{a+b+c}$$

SOLUTION A.082.

$$\textit{Let be } \vec{u} = a\vec{i} + b\vec{j}; \vec{v} = c\vec{j} + d\vec{j}$$

$$|\vec{u}| = \sqrt{a^2+b^2}; |\vec{v}| = \sqrt{c^2+d^2}$$

$$\cos\left(\widehat{\vec{u},\vec{v}}\right) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|} = \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}} \quad (1)$$

$$\sin^2\left(\widehat{\vec{u},\vec{v}}\right) = 1 - \cos^2\left(\widehat{\vec{u},\vec{v}}\right) = 1 - \frac{(ac+bd)^2}{(a^2+b^2)(c^2+d^2)} =$$

$$= \frac{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 - a^2c^2 - b^2d^2 - 2abcd}{(a^2+b^2)(c^2+d^2)} =$$

$$= \frac{a^2d^2 - 2abcd + b^2c^2}{(a^2+b^2)(c^2+d^2)} = \frac{(ad-bc)^2}{(a^2+b^2)(c^2+d^2)}$$

$$\sin\left(\widehat{\vec{u},\vec{v}}\right) = \frac{|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}} \quad (2)$$

$$\textit{If } x \in \mathbb{R} \textit{ then: } \sin x + \cos x = \sin x + \tan\frac{\pi}{4}\cdot\cos x =$$

$$= \sin x + \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}\cdot\cos x = \frac{1}{\cos\frac{\pi}{4}}\left(\sin x\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x\right) = \frac{1}{\cos\frac{\pi}{4}}\cdot\sin\left(x+\frac{\pi}{4}\right) \leq \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$$

$$\sin x + \cos x \leq \sqrt{2}; (\forall)x \in \mathbb{R}$$

$$\sin\left(\widehat{\vec{u},\vec{v}}\right) + \cos\left(\widehat{\vec{u},\vec{v}}\right) \leq \sqrt{2}$$

$$\textit{By (1); (2): } \frac{|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}} + \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}} \leq \sqrt{2}$$

$$|ad-bc| + ac + bd \leq \sqrt{2(a^2+b^2)(c^2+d^2)}$$

SOLUTION A.083.

$$\textit{Let be } \vec{u} = a\vec{i} + b\vec{j}; \vec{v} = c\vec{i} + d\vec{j}$$

$$\cos(\widehat{\vec{u},\vec{v}})=\frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}$$

$$\sin^2(\widehat{\vec{u},\vec{v}})=1-\cos^2(\widehat{\vec{u},\vec{v}})=1-\frac{(ac+bd)^2}{(a^2+b^2)(c^2+d^2)}=$$

$$=\frac{a^2c^2+a^2d^2+b^2c^2+b^2d^2-a^2c^2-b^2d^2-2abcd}{(a^2+b^2)(c^2+d^2)}=$$

$$=\frac{a^2d^2-2abcd+b^2c^2}{(a^2+b^2)(c^2+d^2)}=\frac{(ad-bc)^2}{(a^2+b^2)(c^2+d^2)}$$

$$\sin(\widehat{\vec{u},\vec{v}})=\frac{|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

$$\sin 2(\widehat{\vec{u},\vec{v}})=2\sin(\widehat{\vec{u},\vec{v}})\cos(\widehat{\vec{u},\vec{v}})=2\cdot\frac{|ad-bc|(ac+bd)}{(a^2+b^2)(c^2+d^2)}$$

$$\cos 2(\widehat{\vec{u},\vec{v}})=2\cos^2(\widehat{\vec{u},\vec{v}})-1=2\cdot\frac{(ac+bd)^2}{(a^2+b^2)(c^2+d^2)}-1=$$

$$=\frac{2(a^2c^2+b^2c^2+2abcd)-(a^2+b^2)(c^2+d^2)}{(a^2+b^2)(c^2+d^2)}=$$

$$=\frac{2a^2c^2+2b^2d^2+4abcd-a^2c^2-a^2d^2-b^2c^2-b^2d^2}{(a^2+b^2)(c^2+d^2)}=$$

$$=\frac{a^2c^2+b^2d^2+4abcd-a^2d^2-b^2c^2}{(a^2+b^2)(c^2+d^2)}=\frac{(ac+bd)^2-(ad-bc)^2}{(a^2+b^2)(c^2+d^2)}$$

$$\sin 2x+\cos 2x=\sin 2x+\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}\cos 2x=$$

$$=\frac{\sin 2x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos 2x}{\frac{\sqrt{2}}{2}}=\frac{\sin\left(2x+\frac{\pi}{4}\right)}{\frac{1}{\sqrt{2}}}=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)\le\sqrt{2}$$

$$\sin 2x+\cos 2x\le\sqrt{2}$$

$$\sin 2(\widehat{\vec{u},\vec{v}})+\cos 2(\widehat{\vec{u},\vec{v}})\le\sqrt{2}$$

$$\frac{2|ad-bc|(ac+bd)}{(a^2+b^2)(c^2+d^2)}+\frac{(ac+bd)^2-(ad-bc)^2}{(a^2+b^2)(c^2+d^2)}\le\sqrt{2}$$

$$2(ad-bc)(ac+bd)+(ac+bd)^2-(ad+bc)^2\le\sqrt{2}(a^2+b^2)(c^2+d^2)$$

$$2(ad-bc)(ac+bd)+(ac+bd)^2\le(ad-bc)^2+(a^2+b^2)(c^2+d^2)\sqrt{2}$$

SOLUTION A.084.

Denote $a = \frac{x}{2}; b = \frac{y}{3}; c = 6z$

$$xyz(3x + 2y + 36z) = 6 \Rightarrow \frac{x}{2} \cdot \frac{y}{3} \cdot 6z\left(\frac{x}{2} + \frac{y}{3} + 6z\right) = 1$$

$$\Rightarrow abc(a + b + c) = 1 \Rightarrow a^2bc + ab^2c + abc^2 = 1$$

$$a^2b^2 + 1 = a^2b^2 + a^2bc + ab^2c + abc^2 =$$

$$= ab(ab + ac + bc + c^2) = ab\big(a(b + c) + c(b + c)\big) =$$

$$= ab \cdot (b + c)(a + c) \overset{AM-GM}{\geq} ab \cdot 2\sqrt{bc} \cdot 2\sqrt{ac} = 4abc\sqrt{ab} \quad (1)$$

Analogous: $b^2c^2 + 1 \geq 4abc\sqrt{bc}$ *(2);* $c^2a^2 + 1 \geq 4abc\sqrt{ca}$ *(3)*

By multiplying (1); (2); (3):

$$(a^2b^2 + 1)(b^2c^2 + 1)(c^2a^2 + 1) \geq 64(abc)\sqrt[3]{a^2b^2c^2} = 64a^4b^4c^4 \quad (4)$$

Replace in (4): $a = \frac{x}{2}; b = \frac{y}{3}; c = 6z$

$$\left(\frac{x^2}{4} \cdot \frac{y^2}{9} + 1\right)\left(\frac{y^2}{9} \cdot 36z^2 + 1\right)\left(36z^2 \cdot \frac{x^2}{4} + 1\right) \geq$$

$$\geq 64 \cdot \frac{x^4}{2^4} \cdot \frac{y^4}{3^4} \cdot 36^4z^4 = \frac{64x^4y^4z^4}{36^4} \cdot 36^5 = 64x^4y^4z^4$$

Equality holds for $x = \frac{2}{\sqrt[3]{4}}; y = \frac{3}{\sqrt[3]{4}}; z = 6\sqrt[3]{4}$

SOLUTION A.085.

$$\text{Let be } A = \begin{pmatrix} x & y & y \\ y & y & x \\ y & x & y \end{pmatrix} \in M_3(\mathbb{R})$$

$$\det A = \begin{vmatrix} x & y & y \\ y & y & x \\ y & x & y \end{vmatrix} = xy^2 + xy^2 + xy^2 - y^3 - y^3 - x^3$$

$$\det A = 3xy^2 - 2y^3 - x^3$$

By Hadamards' inequality: $(\det A)^2 \leq (x^2 + y^2 + y^2)(y^2 + y^2 + x^2)(y^2 + x^2 + y^2)$

$$(3xy^2 - 2y^2 - 2y^3 - x^3)^2 \leq (x^2 + 2y^2)^3, (x^3 + 2y^3 - 3xy^2)^2 \leq (x^2 + 2y^2)^3$$

SOLUTION A.086.

Denote $\frac{a+b}{2} = x; \sqrt{ab} = y \Rightarrow a + b = 2x$

$$\frac{\left(a^5+b^5\right)^2}{4(ab)^5} \overset{CEBYSHEV}{\geq} \frac{\left(\frac{1}{2}(a^2+b^2)(a^3+b^3)\right)^2}{4(y^2)^5} \overset{CEBYSHEV}{\geq}$$

$$\geq \frac{\left(\frac{1}{4}(a+b)^2(a^3+b^3)\right)^2}{4y^{10}} \overset{MUIRHEAD}{\geq} \frac{\left(\frac{1}{4}(a+b)^2ab(a+b)^2\right)^2}{4y^{10}} = \frac{\left(\frac{1}{4}\cdot 4x^2\cdot y^2\cdot 2x\right)^2}{4y^{10}} =$$

$$= \frac{x^4\cdot y^4\cdot 4x^2}{4y^{10}} = \left(\frac{x}{y}\right)^6$$

Remains to prove that:

$$\frac{\left((ab)^6+\left(\frac{a+b}{2}\right)^{12}\right)\left(ab+\left(\frac{a+b}{2}\right)^2\right)}{\left((ab)^3\sqrt{ab}+\left(\frac{a+b}{2}\right)^7\right)^2} = \frac{(x^{12}+y^{12})(x^2+y^2)}{(x^7+y^7)^2} \leq \left(\frac{x}{y}\right)^6$$

$$x^6(x^7+y^7)^2 \geq y^6(x^{12}+y^{12})(x^2+y^2)$$

$$x^6(x^{14}+y^{14}+2x^7y^7) \geq y^6(x^{14}+x^{12}y^2+x^2y^2+y^{14})$$

$$x^{20}+x^6y^{14}+2x^{13}y^7-x^{14}y^6-x^{12}y^6-x^2y^{18}-y^{20} \geq 0$$

$$\text{Denote } t = \frac{x}{y} = \frac{\frac{a+b}{2}}{\sqrt{ab}} \overset{AM-GM}{\geq} 1 \Rightarrow t-1 \geq 0$$

$$\frac{x^{20}}{y^{20}}+\frac{x^6}{y^6}+\frac{2x^{13}}{y^{13}}-\frac{x^{14}}{y^{14}}-\frac{x^{12}}{y^{12}}-\frac{x^2}{y^2}-1 \geq 0$$

$$t^{20}+t^6+2t^{13}-t^{14}-t^{12}-t^2-1 \geq 0$$

$$t^{20}-t^{14}+t^{13}+t^{12}+t^{13}-t^2+t^6-1 \geq 0$$

$$t^{14}(t^6-1)+t^{13}(t-1)+t^2(t^{11}-1)+(t^6-1) \geq 0$$

Which is true because:

$$t \geq 1 \Rightarrow t^6 \geq 1; t^{11} \geq 1 \Rightarrow t-1 \geq 0; t^6-1 \geq 0; t^{11}-1 \geq 0$$

SOLUTION A.087.

$$\sum\left(\frac{b^2}{a^2+b^2+c^2}\cdot a^{\frac{a+b+c}{3}}\right) = \sum\left(\frac{b^2}{a^2+b^2+c^2}\cdot a^{\frac{a+b+c}{3}-\sqrt[3]{abc}+\sqrt[3]{abc}}\right) \geq$$

$$\overset{CEBYSHEV}{\geq} \frac{1}{(a^2+b^2+c^2)^2}\left(\sum b^2\cdot a^{\sqrt[3]{abc}}\right)\left(\sum b^2\cdot a^{\frac{a+b+c}{3}-\sqrt[3]{abc}}\right)$$

$$\sum\left(b^2\cdot a^{\frac{a+b+c}{3}}\right) \geq \frac{1}{a^2+b^2+c^2}\left(\sum b^2\cdot a^{\sqrt[3]{abc}}\right)\left(\sum b^2\, a^{\frac{a+b+c}{3}-\sqrt[3]{abc}}\right) \geq$$

$$\overset{AM-GM}{\geq}\left(\sum b^2\cdot a^{\sqrt[3]{abc}}\right)\cdot\sqrt[a^2+b^2+c^2]{\left(a^{b^2}\cdot b^{c^2}\cdot c^{a^2}\right)^{\frac{a+b+c}{3}-\sqrt[3]{abc}}}=$$

$$=\left(\sum b^2\cdot a^{\sqrt[3]{abc}}\right)\cdot\sqrt[a^2+b^2+c^2]{1^{\frac{a+b+c}{3}-\sqrt[3]{abc}}}=\sum\left(b^2\cdot a^{\sqrt[3]{abc}}\right)$$

$$\sum\left(b^2\cdot a^{\frac{a+b+c}{3}}\right)\geq\sum\left(b^2\cdot a^{\sqrt[3]{abc}}\right),\ \ \sum b^2\left(a^{\frac{a+b+c}{3}}-a^{\sqrt[3]{abc}}\right)\geq 0$$

$$b^2\left(\sqrt[3]{a^{a+b+c}}-a^{\sqrt[3]{abc}}\right)+c^2\left(\sqrt[3]{b^{a+b+c}}-b^{\sqrt[3]{abc}}\right)+a^2\left(\sqrt[3]{c^{a+b+c}}-c^{\sqrt[3]{abc}}\right)\geq 0$$

SOLUTION A.088.

$$\textit{Let be } f:[0,\infty)\to\mathbb{R};\ f(x)=\frac{7^x}{5^x+3^x}$$

$$f'(x)=\frac{7^x\log 7\,(5^x+3^x)-7^x(5^x\log 5+3^x\log 3)}{(5^x+3^x)^2}$$

$$f'(x)=\frac{7^x\cdot 5^x(\log 7-\log 5)+7^x\cdot 3^x(\log 7-\log 3)}{(5^x+3^x)^2}>0$$

$$f\textit{ increasing; }\sqrt{xy}\leq\frac{x+y}{2}\Rightarrow f\left(\sqrt{xy}\right)\leq f\left(\frac{x+y}{2}\right)$$

$$\frac{7^{\sqrt{xy}}}{5^{\sqrt{xy}}+3^{\sqrt{xy}}}\leq\frac{7^{\frac{x+y}{2}}}{5^{\frac{x+y}{2}}+3^{\frac{x+y}{2}}}\Rightarrow\frac{7^{\sqrt{xy}}}{5^{\sqrt{xy}}+3^{\sqrt{xy}}}\leq\frac{\sqrt{7^{x+y}}}{\sqrt{5^{x+y}}+\sqrt{3^{x+y}}}\quad(1)$$

$$\textit{Analogous: }\frac{7^{\sqrt{yz}}}{5^{\sqrt{yz}}+3^{\sqrt{yz}}}\leq\frac{\sqrt{7^{y+z}}}{\sqrt{5^{y+z}}+\sqrt{3^{y+z}}}\quad(2)$$

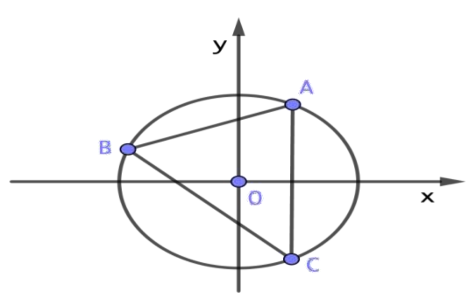
$$\frac{7^{\sqrt{zx}}}{5^{\sqrt{zx}}+3^{\sqrt{zx}}}\leq\frac{\sqrt{7^{z+x}}}{\sqrt{5^{z+x}}+\sqrt{3^{z+x}}}\quad(3)$$

By adding (1); (2); (3):

$$\frac{7^{\sqrt{xy}}}{5^{\sqrt{xy}}+3^{\sqrt{xy}}}+\frac{7^{\sqrt{yz}}}{5^{\sqrt{yz}}+3^{\sqrt{yz}}}+\frac{7^{\sqrt{zx}}}{5^{\sqrt{zx}}+3^{\sqrt{zx}}}\leq\frac{\sqrt{7^{x+y}}}{\sqrt{5^{x+y}}+\sqrt{3^{x+y}}}+\frac{\sqrt{7^{y+z}}}{\sqrt{5^{y+z}}+\sqrt{3^{y+z}}}+\frac{\sqrt{7^{z+x}}}{\sqrt{5^{z+x}}+\sqrt{3^{z+x}}}$$

SOLUTION A.089.

Let be $A(1,3);B(x,y);C(z,t)$



$$R=OA=\sqrt{1^2+3^2}=\sqrt{10}\ \ (1)$$

$$A, B, C \in \mathcal{C}(O, R);\ \mathcal{C}: x^2 + y^2 = 10$$

$$AB = \sqrt{(x-1)^2 + (y-3)^2} = \sqrt{x^2 - 2x + 1 + y^2 - 6y + 9} =$$

$$= \sqrt{10 - 2x - 6y + 10} = \sqrt{20 - 2x - 6y}$$

$$AC = \sqrt{(z-1)^2 + (t-3)^2} = \sqrt{z^2 - 2z + 1 + t^2 - 6t + 9} =$$

$$= \sqrt{10 - 2z - 6t + 10} = \sqrt{20 - 2z - 6t}$$

$$BC = \sqrt{(x-z)^2 + (y-t)^2} = \sqrt{x^2 - 2xz + z^2 + y^2 - 2yt + t^2} = \sqrt{20 - 2xz - 2yt}$$

The maximum of area of ΔABC is obtained when ΔABC is an equilateral one.

The side AB can be obtained by: $\frac{2}{3} \cdot \frac{AB\sqrt{3}}{2} = R \Rightarrow AB = \frac{3R}{\sqrt{3}} = R\sqrt{3} \overset{(1)}{=} \sqrt{30}$

$$S_{\max}[ABC] = \frac{\left(\sqrt{30}\right)^2 \cdot \sqrt{3}}{4} = \frac{30\sqrt{3}}{4} = \frac{15\sqrt{3}}{2}$$

$$\frac{AB \cdot AC \cdot BC}{4 \cdot R} \le \frac{15\sqrt{3}}{2}$$

$$AB \cdot AC \cdot BC \le \frac{15\sqrt{3} \cdot 4 \cdot \sqrt{30}}{2} = 30\sqrt{90} = 90\sqrt{10}$$

$$\sqrt{20 - 2x - 6y} \cdot \sqrt{20 - 2z - 6t} \cdot \sqrt{20 - 2xz - 2yt} \le 90\sqrt{10}$$

$$\sqrt{(10 - x - 3y)(10 - xz - yt)(10 - z - 3t)} \le 45\sqrt{5}$$

$$(10 - x - 3y)(10 - xz - yt)(10 - z - 3t) \le \left(45\sqrt{5}\right)^2 = 10125$$

SOLUTION A.090.

By Bernoulli's inequality if $\alpha \ge 1; x > 0$ then $x^\alpha \ge \alpha x - \alpha + 1$

For $\alpha = \frac{a+b}{2\sqrt{ab}} \overset{AM-GM}{\ge} 1$ we have:

$$x^{\frac{a+b}{2\sqrt{ab}}} \ge \frac{a+b}{2\sqrt{ab}} x - \frac{a+b}{2\sqrt{ab}} + 1;\ (\forall) x > 0$$

For $x = \frac{2\sqrt{ab}}{a+b} > 0$ we have:

$$\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}} \ge \frac{a+b}{2\sqrt{ab}} \cdot \frac{2\sqrt{ab}}{a+b} - \frac{a+b}{2\sqrt{ab}} + 1 = 2 - \frac{a+b}{2\sqrt{ab}}$$

By summing:

$$\sum_{cyc}\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}} \geq 6-\sum_{cyc}\left(\frac{a+b}{2\sqrt{ab}}\right)$$

SOLUTION A.091.

By Bernoulli's inequality: $(1+x)^{\alpha} < 1+\alpha x; x > -1; 0 < \alpha < 1$

For $x = p-1; \alpha = 1-q$

$$(1+p-1)^{1-q} < 1+(1-q)(p-1)$$

$$p^{1-q} < 1+p-1-pq+q = p+q-pq < p+q$$

$$p^{q-q} > \frac{1}{p+q} \Rightarrow p^{q} > \frac{p}{p+q} \quad (1)$$

By (1): $a^{b+c+d} > \frac{a}{a+b+c+d}$ *(2),* $b^{c+d+a} > \frac{b}{a+b+c+d}$ *(3),* $c^{d+a+b} > \frac{c}{a+b+c+d}$ *(4)*

$d^{a+b+c} > \frac{d}{a+b+c+d}$ *(5). By adding (2); (3); (4); (5):*

$$a^{b+c+d}+b^{c+d+a}+c^{d+a+b}+d^{a+b+c} > \frac{a+b+c+d}{a+b+c+d} = 1$$

SOLUTION A.092.

By Bernoulli's inequality: $(1+x)^{\alpha} < 1+\alpha x; x > -1; 0 < \alpha < 1$

For $x = p-1; \alpha = 1-q$

$$(1+p-1)^{1-q} < 1+(1-q)(p-1)$$

$$p^{1-q} < 1+p-1-pq+q = p+q-pq < p+q$$

$$p^{q-1} > \frac{1}{p+q} \Rightarrow p^{q} > \frac{p}{p+q} \quad (a)$$

By (a):

$$x_1^{x_1+x_2+\cdots+x_n} > \frac{x_1}{x_1+x_2+\cdots+x_n} \quad (1)$$

$$x_2^{x_1+x_3+\cdots x_n} > \frac{x_2}{x_1+x_2+\cdots+x_n} \quad (2)$$

$$- - - - - - - - - - - - - - - -$$

$$x_n^{x_1+x_2+\cdots+x_{n-1}} > \frac{x_n}{x_1+x_2+\cdots+x_n} \quad (n)$$

By adding (1); (2); …; (n):

$$x_1^{x_2+x_3+\cdots+x_n}+x_2^{x_1+x_3+\cdots+x_n}+\cdots+x_n^{x_1+x_2+\cdots+x_{n-1}} >$$

$$> \frac{x_1+x_2+\cdots+x_n}{x_1+x_2+\cdots+x_n} = 1$$

SOLUTIONS

GEOMETRY

SOLUTION G.001.

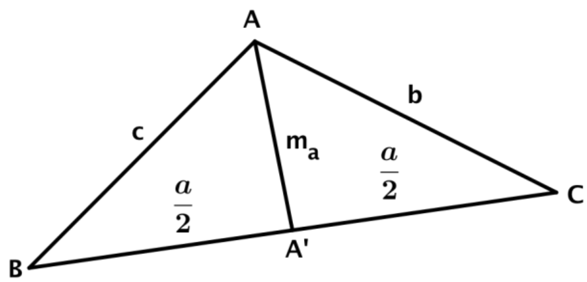
$$3a+m_a=3b+m_b\Rightarrow 3(a-b)=m_b-m_a\Rightarrow 3(a-b)(m_b+m_a)=m_b^2-m_a^2$$

$$3(a-b)(m_b+m_a)=\frac{1}{2}(a^2+c^2)-\frac{1}{4}b^2-\frac{1}{2}(b^2+c^2)+\frac{1}{4}a^2$$

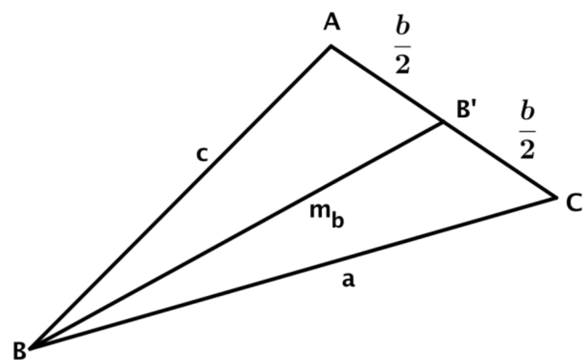
$$3(a-b)(m_a+m_b)=\frac{3}{4}(a^2-b^2),\ (a-b)\left(m_a+m_b-\frac{1}{4}a-\frac{1}{4}b\right)=0$$

$$m_a+\frac{a}{2}>b;\ m_b+\frac{b}{2}>a\Rightarrow m_a+m_b>\frac{a+b}{2}>\frac{a+b}{4}$$

$$\Rightarrow m_a+m_b-\frac{a+b}{4}\neq 0\Rightarrow a-b=0\Rightarrow a=b$$



$$m_a+\frac{a}{2}>b$$



$$m_b+\frac{b}{2}>a$$

Analogous: $b=c;\ c=a\Rightarrow\Delta\ ABC$ ***is an equilateral one*** $\Rightarrow\Omega=2.$

SOLUTION G.002.

$$\prod\cos\frac{3A}{2}=0\Rightarrow\cos\frac{3A}{2}=0\ or\ \cos\frac{3B}{2}=0\ or\ \cos\frac{3C}{2}=0$$

$$WLOG: \cos\frac{3A}{2}=0\Rightarrow\frac{3A}{2}=\frac{\pi}{2}\Rightarrow A=\frac{\pi}{3}$$

$$\cos A=\cos\frac{\pi}{3}=\frac{1}{2}=\frac{b^2+c^2-a^2}{2bc}\Rightarrow b^2+c^2-a^2=bc$$

$$a^2=b^2+c^2-bc$$

$$\sum a\sin^2 A=\sum a\left(\frac{a}{2R}\right)^2=\frac{1}{4R^2}\sum a^3=$$

$$=\frac{1}{4R^2}(a^3+b^3+c^3)=\frac{1}{4R^2}\Big(a^3+(b+c)(b^2-bc+c^2)\Big)=$$

$$=\frac{1}{4R^2}(a^3+(b+c)\cdot a^2)=\frac{a^2}{4R^2}(a+b+c)=$$

$$=\sin^2 A\cdot 2s\overset{MITRINOVIC}{\geq}2\sin^2 A\cdot 3\sqrt{3}r=2\cdot\left(\frac{\sqrt{3}}{2}\right)^2\cdot 3\sqrt{3}r=2\cdot\frac{3}{4}\cdot 3\sqrt{3}r=\frac{9\sqrt{3}r}{2}$$

SOLUTION G.003.

$$2b=a+c\Rightarrow 3b=a+b+c\Rightarrow 3b=2s$$

$$3\cdot 2R\sin B=2s$$

$$6R\cdot 2\sin\frac{B}{2}\cos\frac{B}{2}=a+b+c$$

$$\sin\frac{B}{2}=\frac{a+b+c}{12R\cos\frac{B}{2}}=\frac{2R(\sin A+\sin B+\sin C)}{12R\cos\frac{B}{2}}=\frac{\sin A+\sin B+\sin C}{6\cos\frac{B}{2}}=$$

$$=\frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{6\cos\frac{B}{2}}=\frac{2\cos\frac{A}{2}\cos\frac{C}{2}}{3}=$$

$$=\frac{2}{3}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-c)}{ab}}=\frac{2s}{3b\sqrt{ac}}\sqrt{(s-a)(s-c)}\leq$$

$$\leq\frac{2s}{3b\sqrt{ac}}\cdot\frac{s-a+s-c}{2}=\frac{2s}{3b\sqrt{ac}}\cdot\frac{b}{2}=\frac{s}{3\sqrt{ac}}$$

SOLUTION G.004.

$$m_a\leq m_b\leq m_c\Rightarrow a\geq b\geq c$$

First, we prove that: $\sqrt{a-c}+\sqrt{b-c}\leq\sqrt{\frac{ab}{c}}$ *(1)*

Denote: $a-c=x^2; b-c=y^2$

$$(1) \Leftrightarrow x + y \le \sqrt{\frac{(c+x^2)(c+y^2)}{c}} \Leftrightarrow c(x+y)^2 \le (c+x^2)(c+y^2)$$

$$c(x^2+y^2+2xy) \le c^2 + cy^2 + cx^2 + x^2y^2$$

$$cx^2 + cy^2 + 2cxy \le c^2 + cy^2 + cx^2 + x^2y^2,\ x^2y^2 - 2cxy + c^2 \ge 0 \Leftrightarrow (xy-c)^2 \ge 0$$

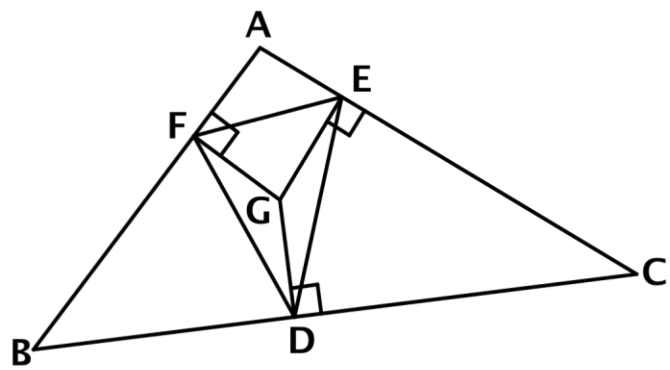
Analogous with (1): $\sqrt{a^2-c^2} + \sqrt{b^2-c^2} \le \sqrt{\frac{a^2b^2}{c^2}} = \frac{ab}{c}$ *(2)*

$$\sqrt{a^3-c^3} + \sqrt{b^3-c^3} \le \sqrt{\frac{a^3b^3}{c^3}} \quad (3)$$

Multiplying (1); (2); (3):

$$\left(\sqrt{a-c}+\sqrt{b-c}\right)\left(\sqrt{a^2-c^2}+\sqrt{b^2-c^2}\right)\left(\sqrt{a^3-c^3}+\sqrt{b^3-c^3}\right) \le \frac{a^3b^3}{c^3}$$

SOLUTION G.005.



$$S[FEG] = \frac{1}{2}\cdot\frac{1}{3}h_b\cdot\frac{1}{3}h_a\cdot\sin A = \frac{1}{18}\cdot\frac{2S}{b}\cdot\frac{2S}{c}\cdot\frac{a}{2R} = \frac{4S^2a}{36Rbc} = \frac{S^2a}{9bcR}$$

$$S[DEF] = S[FEG] + S[EDG] + S[DFG]$$

$$T = \frac{S^2}{9R}\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right) = \frac{S^2}{9R}\cdot\frac{a^2+b^2+c^2}{abc} = \frac{S^2}{9R}\cdot\frac{a^2+b^2+c^2}{4RS} =$$

$$= \frac{(a^2+b^2+c^2)S}{36R^2} \overset{IONESCU-WEITZENBOCK}{\ge} \frac{4\sqrt{3}\cdot S\cdot S}{36R^2} =$$

$$= \frac{\sqrt{3}s^2r^2}{9R^2} \overset{MITRINOVIC}{\ge} \frac{\sqrt{3}\cdot 27r^4}{9R^2} = \frac{3\sqrt{3}r^4}{R^2}$$

SOLUTION G.006.

In acute angled $\Delta\, ABC$ *its valid the relationship:*

$$m_a \le R(1+\cos A) \Rightarrow am_a \le aR(1+\cos A)$$

$$\sum am_a \le R\sum a\,(1+\cos A) = R\sum a\left(1+\frac{b^2+c^2-a^2}{2bc}\right) = R\sum\frac{a}{2bc}[(b+c)^2-a^2] =$$

$$=\frac{R}{2}\sum\frac{a}{bc}(b+c-a)(b+c+a)=\frac{R}{2}\cdot 2s\sum\frac{a(b+c-a)}{bc}=Rs\sum\frac{a^2}{abc}(2s-2a)=$$

$$=\frac{Rs}{4RS}\sum 2a^2\,(s-a)=\frac{s\cdot R}{2Rs}\sum(a^2s-a^3)=\frac{1}{2r}[s(a^2+b^2+c^2)-(a^3+b^3+c^3)]=$$

$$=\frac{1}{2r}[s\cdot 2(s^2-r^2-4Rr)-2s(s^2-3r^2-6Rr)]=$$

$$=\frac{1}{2r}\cdot 2s(s^2-r^2-4Rr-s^2+3r^2+6Rr)=\frac{s}{r}(2r^2+2Rr)=2s(R+r)$$

$$R+r\geq\frac{1}{2s}\sum am_a\overset{WEIGHTED\ AM-GM}{\geq}\left(\prod m_a^a\right)^{\frac{1}{2s}}$$

$$m_a^{\frac{a}{2s}}\cdot m_b^{\frac{b}{2s}}\cdot m_c^{\frac{c}{2s}}\leq R+r\overset{Euler}{\leq}\frac{3R}{2}$$

$$2m_a^{\frac{a}{a+b+c}}\cdot m_b^{\frac{b}{a+b+c}}\cdot m_c^{\frac{c}{a+b+c}}\leq 3R$$

SOLUTION G.007.

By AM-GM: $\frac{AP}{BP}+\frac{BP}{CP}+\frac{CP}{AP}\geq 3\sqrt[3]{\frac{AP}{BP}\cdot\frac{BP}{CP}\cdot\frac{CP}{AP}}=3$

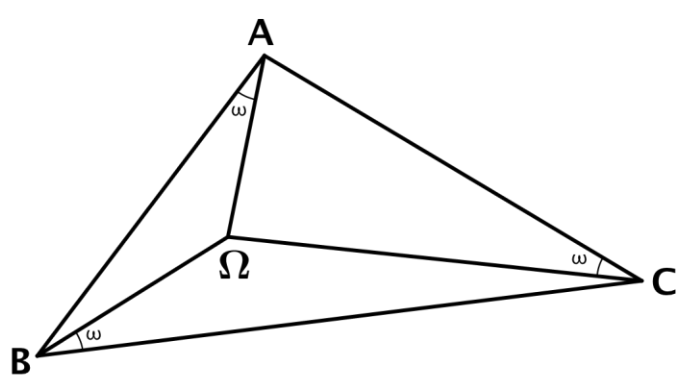
Remains to prove that: $\frac{108r^2}{a^2+b^2+c^2}\leq 3\Leftrightarrow a^2+b^2+c^2\geq 36r^2$

$$a^2+b^2+c^2=2(s^2-r^2-4Rr)\overset{GERRETSEN}{\geq}$$

$$\geq 2(16Rr-5r^2-r^2-4Rr)=2(12Rr-6r^2)\geq$$

$$\overset{EULER}{\geq}2(12\cdot 2r\cdot r-6r^2)=2(24r^2-6r^2)=36r^2$$

SOLUTION G.008.



$$\frac{b}{\sin(180^\circ-A)}=\frac{A\Omega}{\sin\omega}\Rightarrow A\Omega=\frac{b\sin\omega}{\sin A}$$

$$\frac{c}{\sin(180^\circ - B)} = \frac{B\Omega}{\sin\omega} \Rightarrow B\Omega = \frac{c\sin\omega}{\sin B}$$

$$\frac{a}{\sin(180^\circ - C)} = \frac{C\Omega}{\sin\omega} \Rightarrow C\Omega = \frac{a\sin\omega}{\sin C}$$

$$\frac{A\Omega}{B\Omega} + \frac{B\Omega}{C\Omega} + \frac{C\Omega}{A\Omega} = \frac{b\sin\omega}{\sin A}\cdot\frac{\sin B}{c\sin\omega} + \frac{c\sin\omega}{\sin B}\cdot\frac{\sin C}{a\sin\omega} + \frac{a\sin\omega}{\sin C}\cdot\frac{\sin A}{b\sin\omega} =$$

$$= \frac{b\sin B}{c\sin A} + \frac{c\sin C}{a\sin B} + \frac{a\sin A}{b\sin C} = \frac{b\cdot\frac{2R}{b}}{c\cdot\frac{2R}{a}} + \frac{c\cdot\frac{2R}{c}}{a\cdot\frac{2R}{b}} + \frac{a\cdot\frac{2R}{a}}{b\cdot\frac{2R}{c}} =$$

$$= \frac{a}{c} + \frac{b}{a} + \frac{c}{b} = \frac{a^2}{ac} + \frac{b^2}{ab} + \frac{c^2}{bc} \overset{BERGSGRÖM}{\geq} \frac{(a+b+c)^2}{ac+ba+cb} = \frac{4s^2}{ab+bc+ca}$$

SOLUTION G.009.

$$4\cdot\frac{1-\cos(2x+2y)}{2} = 1 + 4\cdot\frac{1+\cos 2x}{2} + 4\cdot\frac{1+\cos 2y}{2}$$

$$2 - 2\cos(2x+2y) = 1 + 2 + 2\cos 2x + 2 + 2\cos 2y$$

$$2\cos 2x + 2\cos(2x+2y) + 2\cos 2y = -3$$

$$2(2\cos^2 x - 1) + 2\cdot 2\cos\frac{2x+2y+2y}{2}\cos\frac{2x+2y-2y}{2} = -3$$

$$4\cos^2 x - 2 + 4\cos(x+2y)\cos x = -3$$

$$4\cos^2 x + 4\cos(x+2y)\cos x = -1$$

$$4\cos^2 x + 4\cos(x+2y)\cos x + \cos^2(x+2y) + 1 - \cos^2(x+2y) = 0$$

$$[2\cos x + \cos(x+2y)]^2 + \sin^2(x+2y) = 0$$

$$\sin^2(x+2y) = 0 \Rightarrow x + 2y = k\pi; k\in\mathbb{Z} \Rightarrow \cos(x+2y) = (-1)^k$$

$$2\cos x + (-1)^k = 0 \Rightarrow \cos x = \frac{(-1)^{k+1}}{2}$$

$$x \in \left\{\pm\arccos\frac{(-1)^{k+1}}{2} + 2k\pi \middle| k\in\mathbb{Z}\right\}$$

$$y = \frac{1}{2}(k\pi - x)$$

$$y \in \frac{1}{2}\left\{k\pi \mp \arccos\frac{(-1)^{k+1}}{2} - 2k\pi \middle| k\in\mathbb{Z}\right\}$$

$$y \in \left\{\frac{1}{2}\left(\mp\arccos\frac{(-1)^{k+1}}{2} - k\pi\right) k\in\mathbb{Z}\right\}$$

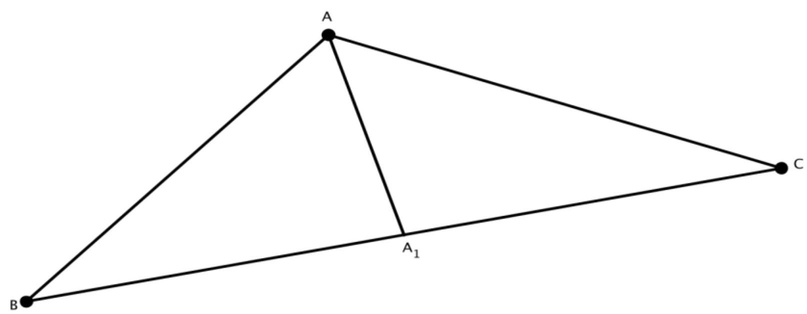
SOLUTION G.010.

$$\sum a^2 m_a \overset{TERESHIN}{\geq} \sum a^2 \cdot \frac{b^2+c^2}{4R} = \frac{1}{4R}\sum a^2(b^2+c^2) =$$

$$= \frac{1}{4R}\left(\sum a^2b^2 + \sum a^2c^2\right) = \frac{1}{4R}\cdot 2\sum a^2b^2 = \frac{1}{2R}\sum a^2b^2 = \sum \frac{a^2b^2}{2R} = \sum \frac{a^2b^2}{2\cdot\frac{abc}{4S}} =$$

$$= \sum \frac{2S\cdot ab}{c} = \sum ab\cdot\frac{2S}{c} = \sum abh_c = abh_c + bch_a + cah_b$$

SOLUTION G.011.



$$S[ABA_1] = \frac{1}{2}AA_1\cdot BA_1 \sin\left(\widehat{AA_1B}\right) \leq \frac{1}{2}AA_1\cdot BA_1 = \frac{1}{2}c_a(s-b) \quad (1)$$

$$S[ACA_1] = \frac{1}{2}AA_1\cdot CA_1 \sin\left(\widehat{AA_1C}\right) \leq \frac{1}{2}AA_1\cdot CA_1 = \frac{1}{2}c_a(s-c) \quad (2)$$

$$S = S[ABC] = S[ABA_1] + S[ACA_1] \overset{(1),(2)}{\leq}$$

$$\leq \frac{1}{2}c_a(s-b) + \frac{1}{2}c_a(s-c) = \frac{1}{2}c_a(s-b+s-c) = \frac{ac_a}{2}$$

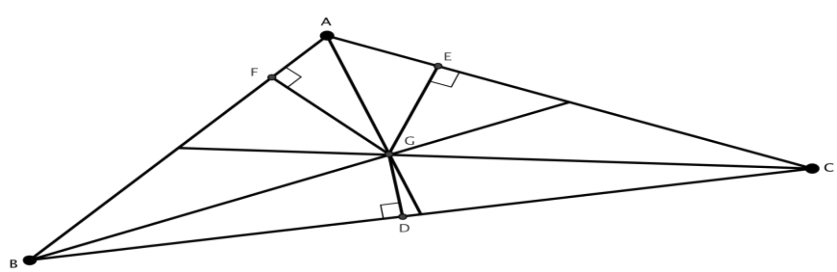
$S \leq \frac{ac_a}{2}$ ***(3). Analogous:*** $S \leq \frac{bc_b}{2}$ ***(4);*** $S = \leq \frac{cc_c}{2}$ ***(5)***

By multiplying (3); (4); (5):

$$S^3 \leq \frac{abc}{8}c_ac_bc_c \Rightarrow c_ac_bc_c \geq \frac{8S^3}{abc} = \frac{8S^3}{4RS} = \frac{2S^2}{R} = \frac{2\cdot 4R^4\sin^2 A\sin^2 B\sin^2 C}{R} =$$

$$= 8R^3\sin^2 A\sin^2 B\sin^2 C$$

SOLUTION G.012.



$$GD = \frac{h_a}{3}; GE = \frac{h_b}{3}; GF = \frac{h_c}{3}$$

$$S[DEF] = S[GDE] + S[GFE] + S[GFD] = \sum \frac{1}{2} \cdot \frac{h_a}{3} \cdot \frac{h_b}{3} \cdot \sin(\widehat{DGE}) = \frac{1}{18} \sum h_a h_b \sin C =$$

$$= \frac{1}{18} \sum \frac{2S}{a} \cdot \frac{2S}{b} \cdot \frac{c}{2R} = \frac{1}{9R} \cdot S^2 \cdot \sum \frac{c}{ab} = \frac{S^2}{9R} \cdot \frac{a^2 + b^2 + c^2}{abc} = \frac{S^2}{9R} \cdot \frac{a^2 + b^2 + c^2}{4RS} =$$

$$= \frac{(a^2 + b^2 + c^2)S}{36R^2} \overset{IONESCU-WEITZENBOCK}{\geq} \frac{4\sqrt{3}S^2}{36R^2} = \frac{\sqrt{3}r^2 s^2}{9R^2} \overset{MITRINOVIC}{\geq} \frac{\sqrt{3}r^2 \cdot 27r^2}{9R^2} =$$

$$= \frac{3\sqrt{3}r^4}{R^2}$$

SOLUTION G.013.

$$\sum (m_a \sin A)^2 = \sum m_a^2 \sin^2 A = \sum \frac{2(b^2 + c^2) - a^2}{4} \cdot \frac{a^2}{4R^2} =$$

$$= \frac{1}{16R^2} \sum a^2 (2b^2 + 2c^2 - a^2) = \frac{1}{16R^2} \sum (2a^2 b^2 + 2b^2 c^2 - a^4) =$$

$$= \frac{1}{16R^2} \left(4 \sum a^2 b^2 - \sum a^4\right) \leq \frac{1}{16R^2} \left(4 \sum a^2 b^2 - \sum a^2 b^2\right) = \frac{3}{16R^2} \sum a^2 b^2 =$$

$$= \frac{3}{4} \sum \frac{a^2 b^2}{4R^2} = \frac{3}{4} \sum \frac{a^2 b^2 c^2}{4R^2 c^2} = \frac{3}{4} \sum \frac{16R^2 S^2}{4R^2 c^2} = \frac{3}{4} \sum \frac{4S^2}{c^2} =$$

$$= \frac{3}{4} \sum \left(\frac{2S}{c}\right)^2 = \frac{3}{4}\left(h_a^2 + h_b^2 + h_c^2\right). \textit{ Equality holds for } a = b = c.$$

SOLUTION G.014.

$$w_a \leq \sqrt{s(s-a)} \Rightarrow w_a^2 \leq s(s-a) \quad (1)$$

$$3\sqrt[3]{\prod w_a^2} \overset{AM-GM}{\leq} w_a^2 + w_b^2 + w_c^2 \overset{(1)}{\leq} s(s-a) + s(s-b) + s(s-c) =$$

$$= s(3s - a - b - c) = s(3s - 2s) = s^2$$

$$m_a \geq \sqrt{s(s-a)} \Rightarrow m_a^2 \geq s(s-a) \quad (2)$$

$$3(m_a^4 + m_b^4 + m_c^4) \geq (m_a^2 + m_b^2 + m_c^2)^2$$

$$\sqrt{3(m_a^4 + m_b^4 + m_c^4)} \geq m_a^2 + m_b^2 + m_c^2 \overset{(2)}{\geq} s(s-a) + s(s-b) + s(s-c) =$$

$$= s(3s - a - b - c) = s(3s - 2s) = s^2$$

SOLUTION G.015.

$$\frac{a^3 + b^3 + c^3}{2} \geq \left(\frac{a + b + c}{3}\right)^3 = \left(\frac{2s}{3}\right)^3$$

$$\sqrt[3]{\left(\frac{a^3+b^3+c^3}{3}\right)^2} \ge \sqrt[3]{\left(\left(\frac{2s}{3}\right)^3\right)^2} = \frac{4s^2}{9} \quad (1)$$

$$\frac{a^5+b^5+c^5}{3} \ge \left(\frac{a+b+c}{3}\right)^5 = \left(\frac{2s}{3}\right)^5$$

$$\sqrt[5]{\left(\frac{a^5+b^5+c^5}{3}\right)^2} \ge \sqrt[5]{\left(\left(\frac{2s}{3}\right)^5\right)^2} = \frac{4s^2}{9} \quad (2)$$

$$\frac{a^7+b^7+c^7}{3} \ge \left(\frac{a+b+c}{3}\right)^7 = \left(\frac{2s}{3}\right)^7$$

$$\sqrt[7]{\left(\frac{a^7+b^7+c^7}{3}\right)^2} \ge \sqrt[7]{\left(\left(\frac{2s}{3}\right)^7\right)^2} = \frac{4s^2}{9} \quad (3)$$

By adding (1); (2); (3): $\sqrt[3]{\left(\frac{a^3+b^3+c^3}{3}\right)^2} + \sqrt[5]{\left(\frac{a^5+b^5+c^5}{3}\right)^2} + \sqrt[7]{\left(\frac{a^7+b^7+c^7}{3}\right)^2} \ge \frac{12s^2}{9} =$

$$= \frac{4}{3}s \cdot s \overset{MITRINOVIC}{\ge} \frac{4}{3}s \cdot 3\sqrt{3}r = 4\sqrt{3}S$$

SOLUTION G.016.

$$\text{If } x, y > 0; \sum x^2 \ge \sum xy \ (1) \Rightarrow 2\sum x^2 \ge 2\sum xy \Rightarrow 3\sum x^2 \ge \sum x^2 + 2\sum xy$$

$$3\sum x^2 - 2\sum xy \ge \sum x^2$$

$$\text{For } x = \sqrt{a^2+c^2+4S}; y = \sqrt{b^2+c^2+4S}; z = \sqrt{a^2+b^2+4S}$$

$$\sum (x+y-z)^2 \ge \sum x^2$$

$$\sum \left(\sqrt{a^2+c^2+4S} + \sqrt{b^2+c^2+4S} - \sqrt{a^2+b^2+4S}\right)^2 \ge$$

$$\ge \left(\sqrt{a^2+b^2+4S}\right)^2 + \left(\sqrt{b^2+c^2+4S}\right)^2 + \left(\sqrt{a^2+c^2+4S}\right)^2 =$$

$$= a^2+b^2+4S+b^2+c^2+4S+a^2+c^2+4S = 2(a^2+b^2+c^2)+12S$$

SOLUTION G.017.

$$\text{By Euler's inequality: } R \ge 2r \Rightarrow R^2 \ge 2Rr \Rightarrow R \ge \sqrt{2Rr}$$

$$2r_a + R \ge 2r_a + \sqrt{2Rr} = 2r_a + \sqrt{\frac{4Rrs}{2s}}$$

$$2\left(r_a+\frac{R}{2}\right)\geq 2r_a+\sqrt{\frac{4RS}{2s}}=2r_a+\sqrt{\frac{abc}{2s}}$$

$$2NI_a\geq 2r_a+\sqrt{\frac{abc}{2s}}$$

Analogous: $2NI_b\geq 2r_b+\sqrt{\frac{abc}{2s}}; 2NI_c\geq 2r_c+\sqrt{\frac{abc}{2s}}$

By multiplying: $8NI_a\cdot NI_b\cdot NI_c\geq\left(2r_a+\sqrt{\frac{abc}{2s}}\right)\left(2r_b+\sqrt{\frac{abc}{2s}}\right)\left(2r_c+\sqrt{\frac{abc}{2s}}\right)$

SOLUTION G.018.

$$\sin^4\frac{\pi}{16}=\left(\sin^2\frac{\pi}{16}\right)^2=\left(\frac{1-\cos\frac{\pi}{8}}{2}\right)^2=\frac{1}{4}\left(1-\cos\frac{\pi}{8}\right)^2=$$

$$=\frac{1}{4}\left(1-2\cos\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right)=\frac{1}{4}\left(1-2\cos\frac{\pi}{8}+\frac{1+\cos\frac{\pi}{4}}{2}\right)=$$

$$=\frac{1}{4}+\frac{1}{8}\left(1+\cos\frac{\pi}{4}\right)-\frac{1}{2}\cos\frac{\pi}{8}\ \ (1)$$

Analogous: $\sin^4\frac{3\pi}{16}=\frac{1}{4}+\frac{1}{8}\left(1+\cos\frac{3\pi}{4}\right)-\frac{1}{2}\cos\frac{3\pi}{8}$ *(2)*

$$\sin^4\frac{5\pi}{16}=\frac{1}{4}+\frac{1}{8}\left(1+\cos\frac{5\pi}{4}\right)-\frac{1}{2}\cos\frac{5\pi}{8}\ \ (3)$$

$$\sin^4\frac{7\pi}{16}=\frac{1}{4}+\frac{1}{8}\left(1+\cos\frac{7\pi}{4}\right)-\frac{1}{2}\cos\frac{7\pi}{8}\ \ (4)$$

By adding (1); (2); (3); (4): $\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4\frac{5\pi}{16}+\sin^4\frac{7\pi}{16}=$

$$=1+\frac{1}{8}\left(4+\cos\frac{\pi}{4}+\cos\frac{3\pi}{4}+\cos\frac{5\pi}{4}+\cos\frac{7\pi}{4}\right)-$$

$$-\frac{1}{2}\left(\cos\frac{\pi}{8}+\cos\frac{3\pi}{8}+\cos\frac{5\pi}{8}+\cos\frac{7\pi}{8}\right)=$$

$$=1+\frac{1}{8}\left(4+\cos\frac{\pi}{4}+\cos\left(\pi-\frac{\pi}{4}\right)+\cos\left(\pi+\frac{\pi}{4}\right)+\cos\left(2\pi-\frac{\pi}{4}\right)\right)-$$

$$-\frac{1}{2}\left(2\cos\frac{\frac{\pi}{8}+\frac{7\pi}{8}}{2}\cos\frac{\frac{7\pi}{8}-\frac{\pi}{8}}{2}+2\cos\frac{\frac{3\pi}{8}+\frac{5\pi}{8}}{2}\cos\frac{\frac{5\pi}{8}-\frac{3\pi}{2}}{2}\right)=$$

$$=1+\frac{1}{8}\left(4+\cos\frac{\pi}{4}-\cos\frac{\pi}{4}-\cos\frac{\pi}{4}+\cos\frac{\pi}{4}\right)-\frac{1}{2}\left(2\cos\frac{\pi}{2}\cos\frac{6\pi}{16}+2\cos\frac{\pi}{2}\cos\frac{2\pi}{16}\right)=$$

$$= 1 + \frac{1}{8} \cdot 4 - \frac{1}{2}\left(0 \cdot \cos\frac{3\pi}{8} + 0 \cdot \cos\frac{\pi}{8}\right) = 1 + \frac{1}{2} = \frac{3}{2} \quad (5)$$

$$\frac{\sin^8\frac{\pi}{16}}{r_a} + \frac{\sin^8\frac{3\pi}{16}}{r_b} + \frac{\sin^8\frac{5\pi}{16}}{r_c} + \frac{\sin^8\frac{7\pi}{16}}{r} \overset{BERGSTROM}{>} \frac{\left(\sin^4\frac{\pi}{16} + \sin^4\frac{3\pi}{16} + \sin^4\frac{5\pi}{16} + \sin^4\frac{7\pi}{16}\right)^2}{r_a + r_b + r_c + r} =$$

$$\overset{(5)}{=} \frac{\left(\frac{3}{2}\right)^2}{4R + r + r} = \frac{\frac{9}{4}}{4R + 2r} \ge \frac{\frac{9}{4}}{4R + 2 \cdot \frac{R}{2}} = \frac{9}{4(4R + r)} = \frac{9}{20R}$$

SOLUTION G.019.

$$A > B > C \Rightarrow a > b > c$$

$$a > b \Rightarrow as > bs \Rightarrow ab + as > ab + bs$$

$$a(b + s) > b(a + s)$$

$\frac{a}{b} > \frac{a+s}{b+s}$ *(1). Analogous:* $\frac{a}{c} > \frac{a+s}{c+s}$ *(2);* $\frac{b}{c} > \frac{b+s}{c+s}$ *(3)*

By adding (1); (2); (3): $\frac{a}{b} + \frac{a}{c} + \frac{b}{c} > \frac{a+s}{b+s} + \frac{b+s}{c+s} + \frac{a+s}{c+s} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{a+s}{b+s} \cdot \frac{b+s}{c+s} \cdot \frac{a+s}{c+s}} = 3\sqrt[3]{\left(\frac{a+s}{b+s}\right)^2}$

$$\left(\frac{a}{b} + \frac{a}{c} + \frac{b}{c}\right)^3 > 27\left(\frac{a+s}{b+s}\right)^2$$

SOLUTION G.020.

$$S = rs\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{s(s-a)}{bc}}\sqrt{(s-b)(s-c)} \cdot \sqrt{bc} =$$

$$= \sqrt{bc} \cdot \sqrt{(s-b)(s-c)} \cdot \cos\frac{A}{2} \overset{AM-GM}{\le} \frac{b+c}{2} \cdot \frac{s-b+s-c}{2} \cdot \cos\frac{A}{2} = \frac{b+c}{2} \cdot \frac{a}{2}\cos\frac{A}{2} =$$

$$= \frac{(2s-a)a}{4}\cos\frac{A}{2}$$

$$3S \le \sum \frac{a(2s-a)}{4}\cos\frac{A}{2}$$

$$\sum a\,(2s-a)\cos\frac{A}{2} \ge 12rs \overset{MITRINOVIC}{\ge} 12r \cdot 3\sqrt{3}r = 36\sqrt{3}r^2$$

SOLUTION G.021.

$$\left(\sum_{cyc(A,B,C)} AI\right)^2 \le 3\sum_{cyc(A,B,C)} AI^2 = 48R^2 \sum_{cyc(A,B,C)} \sin^2\frac{B}{2}\sin^2\frac{C}{2}$$

$$48R^2 \sum_{cyc(A,B,C)} \sin^2\frac{B}{2}\sin^2\frac{C}{2} \le 6R(h_a + h_b + h_c - 6r)$$

$$8R\sum\sin^2\frac{B}{2}\sin^2\frac{C}{2}=8R\cdot\frac{s^2+r^2-8Rr}{16R^2}=\frac{s^2+r^2-8Rr}{2R}=\frac{s^2+r^2+4Rr-12Rr}{2R}=$$

$$=\frac{s^2+r^2+4Rr}{2R}-6r=h_a+h_b+h_c-6r$$

SOLUTION G.022.

$$3R\sqrt{3}=\frac{3R\sqrt{3}}{2}\cdot 2\overset{MITRINOVIC}{\geq}2s=2\sqrt[3]{s^3}=2\sqrt[3]{(s-a+a)(s-b+b)(s-c+c)}\geq$$

$$\overset{MAHLER}{\geq}2\left(\sqrt[3]{(s-a)(s-b)(s-c)}+\sqrt[3]{abc}\right)=2\sqrt[3]{abc}+2\sqrt[3]{\frac{S^2}{s}}=2\sqrt[3]{abc}+2\sqrt[3]{\frac{r^2s^2}{s}}=$$

$$=2\sqrt[3]{abc}+2r\sqrt[3]{\frac{s}{r}}\overset{MITRINOVIC}{\geq}2\sqrt[3]{abc}+2r\sqrt[3]{\frac{3\sqrt{3}r}{r}}=2\sqrt[3]{abc}+2r\sqrt[3]{\left(\sqrt{3}\right)^3}=$$

$$=2\sqrt[3]{abc}+2\sqrt{3}r$$

$$3R\sqrt{3}\geq 2\sqrt[3]{abc}+2\sqrt{3}r$$

$$2\sqrt[3]{abc}\leq 3R\sqrt{3}-2\sqrt{3}r$$

$$2\sqrt[3]{abc}\leq\sqrt{3}(3R-2r)$$

SOLUTION G.023.

$$If\ x\in\left(0,\frac{\pi}{2}\right)\Rightarrow\sin x<x<\tan x$$

$$(x-\sin x)(\tan x-x)>0$$

$$x\tan x-x^2-\sin x\tan x+x\sin x>0$$

$$x(\tan x+\sin x)>x^2+\sin x\tan x$$

$$\frac{x(\tan x+\sin x)}{x^2+\sin x\tan x}>1\Rightarrow\frac{xy(\tan x+\sin x)}{x^2+\sin x\tan x}>y$$

$$\sum_{cyc(x,y,z)}\frac{xy(\tan x+\sin x)}{x^2+\sin x\tan x}>\sum y=\pi$$

SOLUTION G.024.

$$\prod\left(\frac{1}{\sin^2 x}-1\right)=\prod\left(\frac{1-\sin^2 x}{\sin^2 x}\right)=\prod\left(\frac{(1-\sin x)(1+\sin x)}{\sin^2 x}\right)=$$

$$=\frac{1}{\prod\sin^2 x}\prod(\sin x+\sin y+\sin z-\sin x)(\sin x+\sin y+\sin z+\sin x)=$$

$$=\frac{1}{\prod\sin^2 x}\prod(\sin y+\sin z)\cdot\prod(2\sin x+\sin y+\sin z)\geq$$

$$\overset{AM-GM}{\geq} \frac{1}{\prod \sin^2 x} \cdot \prod 2\sqrt{\sin y \sin z} \cdot \prod 4\sqrt[4]{\sin^2 x \sin y \sin z} =$$

$$= \frac{2^3 \cdot 4^3}{\prod \sin^2 x} \cdot \left(\prod \sin x\right) \cdot \prod (\sin x) = \frac{8^3}{(\prod \sin x)^2} \cdot \left(\prod \sin x\right)^2 = 2^9 = 512$$

$$\prod \left(\frac{1}{\sin^2 x} - 1\right) \geq 512$$

$$\prod \left(\frac{1 - \sin^2 x}{\sin^2 x}\right) \geq 512 \Rightarrow \prod \left(\frac{\cos^2 x}{\sin^2 x}\right) \geq 512$$

$$\prod \cos^2 x \geq 512 \prod \sin^2 x$$

Equality holds if $\sin x = \sin y = \sin z = \frac{1}{3} \Rightarrow x = y = z = \arcsin \frac{1}{3}$.

SOLUTION G.025.

First, we prove that in any acute ΔABC: $\sum \frac{1}{b+c-a} = \frac{4R+r}{2rs}$ *(1)*

$$\frac{4R + r}{2rs} = \frac{4R}{2rs} + \frac{r}{2rs} = \frac{2R}{rs} + \frac{1}{2s} = \frac{2R}{\frac{abc}{4R}} + \frac{1}{2s} = \frac{8R^2}{abc} + \frac{1}{2s}$$

With: $a = y + z; b = z + x; c = x + y$

Inequality (1) can be written: $\sum \frac{1}{x+z+x+y-y-z} = \frac{8R^2}{abc} + \frac{1}{2s}$ *(2)*

Now we use: $R = \frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}$;

$$(2) \Leftrightarrow \frac{1}{2}\sum \frac{1}{x} = \frac{8 \cdot \frac{(x+y)^2(y+z)^2(z+x)^2}{16xyz(x+y+z)}}{(x+y)(y+z)(z+x)} + \frac{1}{2(x+y+z)}$$

$$\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{(x + y)(y + z)(z + x)}{2xyz(x + y + z)} + \frac{1}{2(x + y + z)}$$

$$(x + y)(y + z)(z + x) + xyz = (x + y + z)(xy + yz + zx)$$

After calculus we obtain an identity.

Back to the problem: we will apply (1) for the orthic triangle of $\Delta\, ABC$ *with sides:*

$$a_0 = a \cos A\, ; b_0 = b \cos B\, ; c_0 = c \cos C$$

Area: $S_0 = 2S \cos A \cos B \cos C$; *Inradii:* $R_0 = \frac{R}{2}$

$$\sum_{cyc(a_0,b_0,c_0)} \frac{1}{b_0 + c_0 - a_0} = \frac{8R_0^2}{a_0 b_0 c_0} + \frac{1}{2s_0}$$

$$\sum_{cyc(a_0,b_0,c_0)}\frac{1}{b_0+c_0-a_0}=\frac{8\cdot\frac{R^2}{4}}{abc\cos A\cos B\cos C}+\frac{1}{a\cos A+b\cos B\, c\cos C}$$

$$\sum_{cyc(a_0,b_0,c_0)}\frac{1}{b\cos B+c\cos C-a\cos A}=\frac{2R^2}{4RS\cos A\cos B\cos C}+\frac{1}{\frac{abc}{2R^2}}$$

$$\sum_{cyc(a_0,b_0,c_0)}\frac{1}{b\cos B+c\cos C-a\cos A}=\frac{R}{4S\cos A\cos B\cos C}+\frac{2R^2}{abc}$$

SOLUTION G.026.

$$Let\ be\ f:\left[0,\frac{\pi}{2}\right]\to\mathbb{R};f(x)=\sin x\cdot\tan x-x^2$$

$$f'(x)=\sin x+\frac{\sin x}{\cos^2 x}-2x$$

$$f''(x)=\frac{(\cos x-1)(\cos^3 x+\cos^2 x-\cos x-2)}{\cos^3 x}\geq 0$$

$$f''(x)\geq 0\Rightarrow f'(x)\geq f'(0)=0\Rightarrow f(x)\geq f(0)=0$$

$$\sin x\cdot\tan x-x^2\geq 0$$

$$\sin A\cdot\tan A-A^2\geq 0$$

$$\frac{\sin^2 A}{\cos A}-A^2\geq 0\Rightarrow\frac{1-\cos^2 A}{\cos A}\geq A^2$$

$$\frac{1}{\cos A}-\cos A\geq A^2\Rightarrow\frac{1}{\cos A}\geq A^2+\cos A$$

$$\sum_{cyc(A,B,C)}\frac{1}{\cos A}\geq\sum_{cyc(A,B,C)}(A^2+\cos A)$$

$$\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\geq A^2+B^2+C^2+\cos A+\cos B+\cos C$$

SOLUTION G.027.

$$(b+c-2a)^2\geq 0\Rightarrow b^2+c^2+4a^2+2bc-4ab-4ac\geq 0$$

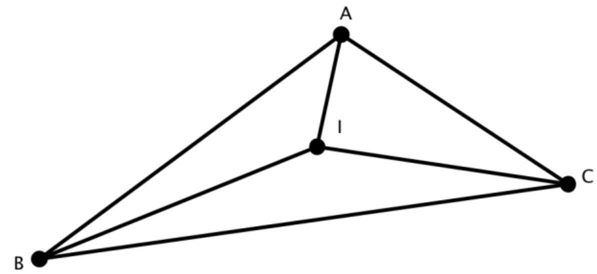
$$4ab+4ac-4a^2\leq b^2+c^2+2bc$$

$$4a(b+c-a)\leq(b+c)^2\Rightarrow\frac{a(b+c-a)}{b+c}\leq\frac{b+c}{4}$$

$$\frac{a(2s-2a)}{b+c}\leq\frac{b+c}{4}\Rightarrow\frac{a(s-a)}{b+c}\leq\frac{b+c}{8}$$

$$\sum_{cyc(a,b,c)}\frac{a(s-a)}{b+c}\le\frac{1}{8}\sum(b+c)=\frac{1}{8}\cdot 4s=\frac{1}{2}s\overset{MITRINOVIC}{\le}\frac{1}{2}\cdot\frac{3\sqrt{3}R}{2}=\frac{3\sqrt{3}R}{4}$$

SOLUTION G.028.



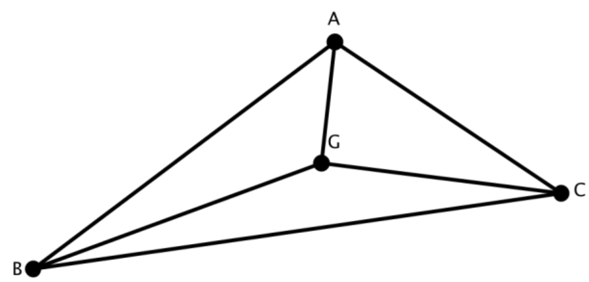
$$In\ \Delta BIC: BI+CI>BC$$

$$\frac{BI+CI}{AI}>\frac{BC}{AI}\Rightarrow\left(\frac{BI+CI}{AI}\right)^5>\left(\frac{BC}{AI}\right)^5$$

Analogous: $\left(\frac{BI+AI}{BI}\right)^5>\left(\frac{CA}{BI}\right)^5;\left(\frac{AI+BI}{CI}\right)^5>\left(\frac{AB}{CI}\right)^5$

By adding: $\left(\frac{AI+BI}{CI}\right)^5+\left(\frac{BI+CI}{AI}\right)^5+\left(\frac{CI+AI}{BI}\right)^5>\left(\frac{BC}{AI}\right)^5+\left(\frac{CA}{BI}\right)^5+\left(\frac{AB}{CI}\right)^5$

SOLUTION G.029.



$$In\ \Delta BGC; G-centroid:$$

$$GB+GC>BC\Rightarrow\frac{2}{3}m_b+\frac{2}{3}m_c>a\Rightarrow 2m_b+2m_c>3a\Rightarrow\frac{2m_b+2m_c}{m_a}>\frac{3a}{m_a}$$

$$\Rightarrow\left(\frac{2m_b+2m_c}{m_a}\right)^7>\left(\frac{3a}{m_a}\right)^7\quad(1)$$

Analogous: $\left(\frac{2m_c+2m_a}{m_b}\right)^7>\left(\frac{3b}{m_b}\right)^7\quad(2)$

$$\left(\frac{2m_a+2m_b}{m_c}\right)^7>\left(\frac{3c}{m_c}\right)^7\quad(3)$$

By adding (1); (2); (3):

$$\left(\frac{2m_a+2m_b}{m_c}\right)^7+\left(\frac{2m_b+2m_c}{m_a}\right)^7+\left(\frac{2m_c+2m_a}{m_b}\right)^7>\left(\frac{3a}{m_a}\right)^7+\left(\frac{3b}{m_b}\right)^7+\left(\frac{3c}{m_c}\right)^7$$

SOLUTION G.030.

$$\tan^{-1}x<x\Rightarrow x-\tan^{-1}x>0$$

$$x<\sin^{-1}x\Rightarrow \sin^{-1}x-x>0$$

$$(x-\tan^{-1}x)(\sin^{-1}x-x)>0$$

$$x\sin^{-1}x-x^2-\tan^{-1}x\sin^{-1}x+x\tan^{-1}x>0$$

$$x(\sin^{-1}x+\tan^{-1}x)>x^2+\tan^{-1}x\sin^{-1}x$$

$$\frac{\sin^{-1}x+\tan^{-1}x}{x^2+\tan^{-1}x\sin^{-1}x}>\frac{1}{x}$$

$$\frac{y(\sin^{-1}x+\tan^{-1}x)}{x^2+\tan^{-1}x\sin^{-1}x}>\frac{y}{x}$$

$$\sum_{cyc(x,y,z)}\frac{y(\sin^{-1}x+\tan^{-1}x)}{x^2+\tan^{-1}x\cdot\sin^{-1}x}>\sum\frac{y}{x}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{y}{x}\cdot\frac{z}{y}\cdot\frac{x}{z}}=3$$

SOLUTION G.031.

$$3\cos^2x\sin^2y=\cos^2y\sin^2y+2\sin^2x\cos^2y$$

$$3\cos^2x\,(1-\cos^2y)=\cos^2y\,(1-\cos^2y)+2(1-\cos^2x)\cos^2y$$

$$3\cos^2x-3\cos^2x\cos^2y=\cos^2y-\cos^4y+2\cos^2y-2\cos^2x\cos^2y$$

$$3\cos^2x-3\cos^2y-\cos^2x\cos^2y+\cos^4y=0$$

$$3(\cos^2x-\cos^2y)-\cos^2y\,(\cos^2x-\cos^2y)=0$$

$$(\cos^2x-\cos^2y)(3-\cos^2y)=0$$

$\cos^2x-\cos^2y=0\Rightarrow\cos^2x=\cos^2y\Rightarrow\cos x=\cos y\Rightarrow x=y$. ***Analogous:***

$$\frac{3\cos^2y}{\cos^2z}=1+\frac{2\sin^2y}{\sin^2z}\Rightarrow y=z\Rightarrow x=y=z$$

$x+2^x+\log_2x=3$. ***Let be*** $f:(0,\infty)\to\mathbb{R}$

$$f(x)=x+2^x+\log_2x\,;f'(x)=2^x\log2+\frac{1}{x\log2}>0$$

f strictly increasing; $f(1)=3$

f injective; $f(1)=f(x)\Rightarrow x=1\Rightarrow x=y=z=1.$

SOLUTION G.032.

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos A=\sum\frac{b^2+c^2}{bc}\cdot\frac{b^2+c^2-a^2}{2bc}=$$

$$=\frac{1}{2a^2b^2c^2}\sum a^2(b^2+c^2)(b^2+c^2-a^2)=\frac{6a^2\,b^2c^2}{2a^2b^2c^2}=3$$

Let be a', b', c' - sides of orthic triangle of ΔABC and $A'=\pi-2A, B'=\pi-2B,$

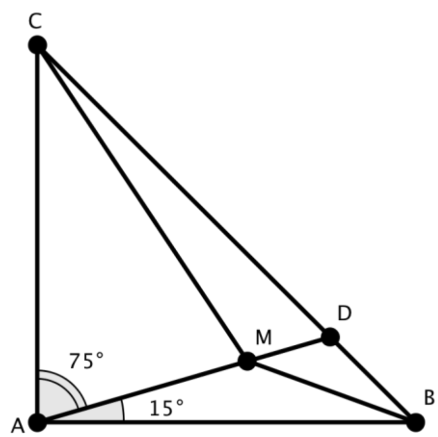
$$C'=\pi-2C$$

$$\sum\left(\frac{b'}{c'}+\frac{c'}{b'}\right)\cos A'=3$$

$$\sum\left(\frac{b\cos B}{c\cos C}+\frac{c\cos C}{b\cos B}\right)\cos(\pi-2A)=3$$

$$\sum\left(\frac{b\cos B}{c\cos C}+\frac{c\cos C}{b\cos B}\right)\cos 2A=-3$$

SOLUTION G.033.



$$AB=AC=1; AB=x$$

$$\cos 15^\circ=\cos(45^\circ-30^\circ)=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$BM^2=AB^2+AM^2-2AB\cdot AM\cos 15^\circ=$$

$$=1+x^2-2x\cdot\frac{\sqrt{6}+\sqrt{2}}{4}=x^2-\frac{x\left(\sqrt{6}+\sqrt{2}\right)}{2}+1$$

$$BM=\frac{1}{\sqrt{2}}\cdot\sqrt{2x^2-x\left(\sqrt{6}+\sqrt{2}\right)+2}\quad (1)$$

$$\cos 75^\circ=\cos(45^\circ+30^\circ)=\cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$CM^2=AC^2+AM^2-2AC\cdot AM\cos 75^\circ=$$

$$= 1 + x^2 - 2x \cdot \frac{\sqrt{6} - \sqrt{2}}{4} = x^2 + \frac{(\sqrt{2} - \sqrt{6})x}{2} + 1$$

$$CM = \frac{1}{\sqrt{2}}\sqrt{2x^2 + (\sqrt{2} - \sqrt{6})x + 2} \quad (2)$$

$$CM + MB \geq BD + DC = BC = \sqrt{2}$$

$$\textit{By (1); (2):} \frac{1}{\sqrt{2}}\left(\sqrt{2x^2 + (\sqrt{2} - \sqrt{6})x + 2} + \sqrt{2x^2 - (\sqrt{2} + \sqrt{6})x + 2}\right) \geq \sqrt{2}$$

$$\sqrt{2x^2 + (\sqrt{2} - \sqrt{6})x + 2} + \sqrt{2x^2 - (\sqrt{2} + \sqrt{6})x + 2} \geq 2$$

SOLUTION G.034.

$$\textit{Let be } f: (-\infty, -3] \to \mathbb{R}; f(\tan x) = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

$$f(\tan x) = \tan 3x$$

$$f(x) = \frac{3x - x^3}{1 - 3x^2}; f'(x) = \frac{3 + 6x^2 + 3x^4}{(1 - 3x^2)^2} > 0$$

f strictly increasing;

$$f(-3) = \frac{-9+27}{1-27} = -\frac{9}{13} \Rightarrow f(x) \leq -\frac{9}{13} \Rightarrow \tan 3x \leq -\frac{9}{13}. \textit{ From hypothesis:}$$

$$\tan 3x \geq -\frac{9}{13} \Rightarrow \tan 3x = -\frac{9}{13} \Rightarrow \cos 3x = \frac{13}{5\sqrt{10}}$$

$$5\sqrt{10} \cdot \frac{13}{5\sqrt{10}} + 13\sin 3y - 13 = 0$$

$$\sin 3y = 0 \Rightarrow 3y = n\pi; n \in \mathbb{Z} \Rightarrow y = \frac{n\pi}{3}; n \in \mathbb{Z}$$

$$\textit{If } \cos 3x = -\frac{13}{5\sqrt{10}} \Rightarrow 5\sqrt{10}\frac{-13}{5\sqrt{10}} + 13\sin 3y - 13 = 0$$

$$13\sin 3y = 26 \Rightarrow \sin 3y = \frac{1}{2}$$

$$3y = (-1)^n \frac{\pi}{6} + n\pi; n \in \mathbb{Z}$$

$$y = (-1)^n \frac{\pi}{18} + \frac{n\pi}{3}; n \in \mathbb{Z}$$

SOLUTION G.035.

In ΔABC:

$$2a^2(b+c)^2 + 2b^2(a+c)^2 + 2c^2(a+b)^2 \leq (a+b+c)(a+b)(b+c)(a+c) \quad (1)$$

$$2a^2b^2 + 4a^2bc + 2a^2c^2 + 2a^2b^2 + 4ab^2c + 2b^2c^2 + 2a^2c^2 + 4abc^2 + 2b^2c^2 \leq$$

$$\le (a^2+ab+ac+ab+b^2+bc)(ab+ac+bc+c^2)$$

$$4a^2b^2+4b^2c^2+4a^2c^2+4abc(a+b+c)\le$$

$$\le a^3b+2a^2b^2+a^2bc+ab^3+ab^2c+a^3c+2a^2bc+a^2c^2+ab^2c+abc^2+$$

$$+a^2bc+2ab^2c+abc^2+b^3c+b^2c^2+a^2b^2+2abc^2+ac^3+b^2c^2+bc^3$$

$$a^3b-2a^2b^2+ab^3+a^3c-2a^2c^2+ac^3+b^3c-2b^2c^2+bc^3\ge 0$$

$$ab(a-b)^2+ac(a-c)^2+bc(b-c)^2\ge 0$$

We apply relationship (1) for orthic triangle with sides: $a_0=a\cos A\,;\,b_0=b\cos B\,;$

$$c=c\cos C$$

$$2\sum a_0^2\,(b_0+c_0)^2\le\left(\sum a_0\right)\cdot\prod(a_0+b_0)$$

$$2\sum a^2\cos^2 A\,(b\cos B+c\cos C)^2\le\left(\sum a\cos A\right)\prod(a\cos A+b\cos B)$$

SOLUTION G.036.

$$\sum IA'=abc\sum\frac{1}{w_a(b+c)}=abc\sum\frac{1}{\frac{2}{b+c}\sqrt{bcs(s-a)}(b+c)}=$$

$$=\frac{abc}{2}\sum\frac{1}{\sqrt{bc}\cdot\sqrt{s(s-a)}}\overset{AM-GM}{\ge}\frac{abc}{2}\sum\frac{1}{\frac{b+c}{2}\cdot\frac{s+s-a}{2}}=$$

$$=2abc\sum\frac{1}{(b+c)^2}=8Rrs\sum\frac{1}{(b+c)^2}\ge$$

$$\overset{EULER}{\ge}16r^2s\sum\frac{1}{(b+c)^2}\overset{MITRINOVIC}{\ge}48\sqrt{3}r^3\sum\frac{1}{(b+c)^2}$$

SOLUTION G.037.

If $x,y\in\mathbb{C}$ *then is known the identity:*

$$\det(xA+yB)=x^2\det A+y^2\det B+xy(\det(A+B)-\det A-\det B)\quad (1)$$

We take in (1): $x=\det B\,;\,y=\det A$*:* $\det(A\det B+B\det A)=$

$$=\det^2 B\cdot\det A+\det^2 A\det B+\det A\cdot\det B\,(\det(A+B)-\det A-\det B)$$

$$\det(A\det B+B\det A)=$$

$$=\det A\det B\,(\det A+\det B)+\det(AB)\,(\det(A+B)-\det A-\det B)=$$

$$=\det(AB)\cdot\det(A+B)\quad (2)$$

We take in (1): $x=\frac{1}{\det A}\,;\,y=\frac{1}{\det B}$

$$\det\left(\frac{A}{\det A}+\frac{B}{\det B}\right)=\frac{1}{\det A}+\frac{1}{\det B}+\frac{1}{\det A\det B}(\det(A+B)-\det A-\det B)=$$

$$=\frac{1}{\det A}+\frac{1}{\det B}+\frac{\det(A+B)}{\det A\cdot\det B}-\frac{1}{\det B}-\frac{1}{\det A}=\frac{\det(A+B)}{\det A\cdot\det B}$$

$$\det\left(\frac{A}{\det A}+\frac{B}{\det B}\right)=\frac{\det(A+B)}{\det A\cdot\det B}\quad (3)$$

By adding (2); (3): $\det(A\det B+B\det A)+\det\left(\frac{A}{\det A}+\frac{B}{\det B}\right)$

SOLUTION G.038.

$$\frac{\sum a^2-8R^2}{8R^2}=\frac{s^2-4Rr-r^2-4R^2}{4R^2}=\cos A\cos B\cos C$$

The inequality can be written:

$$\sum a^3\cos^3 a+3abc\cos A\cos B\cos C\geq 2\sum(a\cos A)^2\,(b\cos B)^2$$

Let a_0,b_0,c_0 *be the sides of the orthic triangle of* ΔABC: $a_0=a\cos A$, $b_0=b\cos B$, $c_0=c\cos C$

$$\sum a_0^3+3a_0b_0c_0\geq 2\sum a_0^2b_0$$

By Schur's inequality:

$$\sum a_0^3+3a_0b_0c_0\geq a_0b_0(a_0+b_0)+b_0c_0(b_0+c_0)+c_0a_0(a_0+c_0)$$

Remains to prove: $\sum a_0b_0\,(a_0+b_0)\geq 2\sum a_0^2b_0$

$a_0b_0(a_0-b_0)+b_0c_0(c_0-b_0)+a_0c_0(a_0-c_0)\geq 0$ *which is true.*

SOLUTION G.039.

By cosine law: $a^2b^2+c^2-2bc\cos A=b^2+c^2-2bc\cdot\frac{1}{2}=b^2+c^2-bc\geq bc$ *because*

$$(b-c)^2\geq 0$$

$$a^2\geq bc\quad (1)$$

$$1+\frac{a}{b}\overset{AM-GM}{\geq}2\sqrt{\frac{a}{b}};1+\frac{a}{c}\overset{AM-GM}{\geq}2\sqrt{\frac{a}{c}}\Rightarrow$$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right)\geq\frac{4a}{\sqrt{bc}}\overset{(1)}{\geq}\frac{4\sqrt{bc}}{\sqrt{bc}}=4\quad (2)$$

By multiplying: (1); (2): $a^2\left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right)\geq 4bc$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right)\geq\frac{4bc}{a^2}$$

$$(b+a)(c+a) \geq \frac{4b^2c^2}{a^2} \quad (3)$$

By AM-GM: $\left(\frac{b+a+c+a}{2}\right)^2 \geq (b+a)(c+a) \quad (4)$

By (3); (4): $\left(\frac{b+a+c+a}{2}\right)^2 \geq \frac{4b^2c^2}{a^2} \Rightarrow \frac{2s+a}{2} \geq \frac{2bc}{a}$

$$2s+a \geq \frac{4bc}{a} \quad (5)$$

By Mitrinovic's inequality: $\frac{3\sqrt{3}R}{2} \geq s$

$$3\sqrt{3}R + a \geq 2s + a \overset{(5)}{\geq} \frac{4bc}{a}$$

$$3\sqrt{3}R + a \geq \frac{4bc}{a}$$

SOLUTION G.040.

$$a \geq b \geq c \Rightarrow b-a \leq 0; c-b \leq 0; a-c \geq 0 \Rightarrow$$

$$(b-a)(c-b)(a-c) \geq 0$$

$$(bc-b^2-ac+ab)(a-c) \geq 0$$

$$abc - bc^2 - ab^2 + b^2c - a^2c + ac^2 + a^2b - abc \geq 0$$

$$a^2b + b^2c + c^2a \geq a^2c + b^2a + c^2b \quad (1)$$

$$a^2c + b^2a + c^2b = ac \cdot a + ab \cdot b + bc \cdot c =$$

$$= \frac{2S}{\sin B} \cdot a + \frac{2S}{\sin C} \cdot b + \frac{2S}{\sin A} \cdot C = \left(\frac{a}{\sin B} + \frac{b}{\sin C} + \frac{c}{\sin A}\right) 2S \quad (2)$$

By (1); (2): $\left(\frac{a}{\sin B} + \frac{b}{\sin C} + \frac{c}{\sin A}\right) 2S \leq a^2b + b^2c + c^2a$

$$\frac{a}{\sin B} + \frac{b}{\sin C} + \frac{c}{\sin B} \leq \frac{a^2b + b^2c + c^2a}{2S}$$

SOLUTION G.041.

$$\frac{\sin^2 z}{\cos^2 y} + \frac{\cos^2 y}{\sin^2 z} \geq 2 \Leftrightarrow \sin^4 z + \cos^4 y \geq 2\sin^2 z \cos^2 y \Leftrightarrow (\sin^2 z - \cos^2 y)^2 \geq 0$$

$$\sum \left(\frac{\sin^2 z}{\cos^2 y} + \frac{\cos^2 y}{\sin^2 z}\right) \geq 6$$

$$\sum \frac{\sin^2 y \sin^2 z + \sin^2 z - \sin^2 y \sin^2 z}{1 - \sin^2 y} + \sum \frac{1 - \sin^2 z}{\sin^2 x} \geq 6$$

$$\sum \left(\frac{\sin^2 y \sin^2 z}{\cos^2 y} + \sin^2 z\right) + \sum \frac{1 - \sin^2 z}{\sin^2 x} \geq 6$$

$$\sum\left(\frac{\sin^2 y\sin^2 z}{\cos^2 y}+\frac{1-\sin^2 z}{\sin^2 x}\right)+\sum\sin^2 z\geq 6$$

$$\sum\frac{\sin^2 y\sin^2 z}{\cos^2 y}+\sum\frac{1}{\sin^2 x}+\sum\sin^2 z-3-\sum\frac{\sin^2 z}{\sin^2 x}\geq 3$$

$$\sum\frac{\sin^2 y\sin^2 z}{\cos^2 y}+\sum\frac{1+\sin^2 x\sin^2 z-\sin^2 x-\sin^2 z}{\sin^2 x}\geq 3$$

$$\sum\frac{\sin^2 y\sin^2 z}{\cos^2 y}+\sum\frac{\cos^2 x\cos^2 z}{\sin^2 x}\geq 3$$

$$\left(\prod\sin^2 x+\prod\cos^2 x\right)\sum\frac{\cos^2 x\cos^2 z}{\sin^2 x}\geq 3$$

$$\left(\prod\sin^2 x+\prod\cos^2 x\right)\sum\frac{1}{\sin^2 x\cos^2 y}\geq 3$$

SOLUTION G.042.

$$\textit{Denote } x=a^2\cdot AN^2;\ y=b^2\cdot BN^2;\ z=c^2\cdot CN^2$$

$$\textit{Inequality can be written: } \sum\frac{x}{5(y+z)-x}\geq\frac{1}{3}$$

$$\sum x(5z+5x-y)(5x+5y-z)\geq\frac{1}{3}\prod(5y+5z-x)$$

$$3\sum(5x+5x^2-xy)(5x+5y-z)\geq\prod(5y+5z-x)$$

$$5(x^3+y^3+z^3)+3xyz\geq 3(x^2y+y^2z+z^2x+y^2x+z^2y+x^2y)$$

$$5\sum x^3+15xyz-12xyz\overset{SCHUR}{\geq}5\sum(x^2y+xy^2)-12xyz\geq 3\sum(x^2y+xy^2)\ \textit{(to prove)}$$

$$5\sum(x^2y+xy^2)-3\sum(x^2y+xy^2)\geq 12xyz$$

$$2\sum(x^2y+xy^2)\geq 12xyz$$

$$\sum(x^2y+xy^2)\geq 6xyz\ \textit{(to prove)}$$

$$\sum(x^2y+xy^2)\overset{AM-GM}{\geq}6\sqrt[6]{x^2y\cdot y^2z\cdot z^2x\cdot y^2x\cdot z^2y\cdot x^2z}=6\sqrt[6]{x^6y^6z^6}=6xyz$$

SOLUTION G.043.

$$\frac{s}{ab+bc+ca}+\frac{8Rr}{(2s-a)(2s-b)(2s-c)}=$$

$$=\frac{s(a+b)(b+c)(c+a)+8Rr(ab+bc+ca)}{(a+b)(b+c)(c+a)(ab+bc+ca)}=$$

$$=\frac{2s^2(s^2+2Rr+r^2)+8Rr(s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)(s^2+4Rr+r^2)}=$$

$$=\frac{s^2(s^2+2Rr+r^2)+4Rr(s^2+4Rr+r^2)}{s(s^2+2Rr+r^2)(s^2+4Rr+r^2)}\geq\frac{1}{s}\Leftrightarrow$$

$$\Leftrightarrow s^4+s^2(6Rr+r^2)+4Rr^2(4R+r)\geq$$

$$\geq s^4+s^2(6Rr+r^2)+r^2(2R+r)(4R+r)$$

$$\Leftrightarrow r^2(4R+r)(2R-r)\geq s^2r^2\Leftrightarrow s^2\leq 8R^2-2Rr-r^2\ \ \textit{(to prove)}$$

$$s^2\overset{GERRETSEN}{\leq}16Rr-5r^2\leq 4R^2+4Rr+3r^2\Leftrightarrow$$

$$\Leftrightarrow 4R^2-12Rr+8r^2\geq 0\Leftrightarrow (R-2r)(R-r)\geq 0\ \ \textit{(true)}$$

$$\frac{s}{ab+bc+ca}+\frac{8Rr}{(a+b)(b+c)(c+a)}\geq\frac{1}{s}\overset{MITRINOVIC}{\leq}\frac{1}{\frac{3\sqrt{3}R}{2}}=\frac{2\sqrt{3}}{9R}$$

SOLUTION G.044.

By Murray-Klamkin's duality principle: ax, by, cz can be sides of a triangle. Denote $ax=a_0, by=b_0, cz=c_0$. We must prove that:

$$\frac{a_0}{a_0+b_0+98c_0}+\frac{b_0}{b_0+c_0+98a_0}+\frac{c_0}{c_0+a_0+98b_0}\geq\frac{3}{100}$$

$$\sum_{cyc(a_0,b_0,c_0)}\frac{a_0}{a_0+b_0+98c_0}=\sum_{cyc(a_0,b_0,c_0)}\frac{a_0^2}{a_0^2+a_0b_0+98a_0c_0}\geq$$

$$\overset{BERGSTROM}{\geq}\frac{(a_0+b_0+c_0)^2}{\sum a_0^2+\sum a_0b_0+98\sum a_0c_0}\geq\frac{3}{100}\Leftrightarrow$$

$$\Leftrightarrow 100(a_0+b_0+c_0)^2\geq 3\sum a_0^2+297\sum a_0b_0$$

$$100\left(\sum a_0^2+2\sum a_0b_0\right)\geq 3\sum a_0^2+297\sum a_0b_0$$

$$97\sum a_0^2\geq 97\sum a_0b_0$$

$$\sum a_0^2\geq\sum a_0b_0$$

$$(a_0-b_0)^2+(b_0-c_0)^2+(c_0-a_0)^2\geq 0$$

SOLUTION G.045.

$$\sin^2 x+\sin^2 y\cos^2 x+\cos^2 x\cos^2 y=$$

$$=\sin^2 x+\cos^2 x(\sin^2 y+\cos^2 y)=\sin^2 x+\cos^2 x=1\quad \textit{(1)}$$

$$1-\frac{1}{\sin^2 x}\overset{(1)}{=}\frac{\sin^2 x+1}{\sin^2 x}=\frac{\sin^2 x+\sin^2 x+\sin^2 y\cos^2 x+\cos^2 x\cos^2 y}{\sin^2 x}\geq$$

$$\overset{AM-GM}{\geq}\frac{1}{\sin^2 x}\cdot 4\cdot\sqrt[4]{\sin^2 x\cdot\sin^2 x\cdot\sin^2 y\cos^2 x\cdot\cos^2 x\cos^2 y}=$$

$$=\frac{4}{\sin^2 x}\cdot\sqrt[4]{\sin^4 x\cos^4 x\sin^2 y\cos^2 y}=\frac{4\cos x}{\sin x}\sqrt{\sin y\cos y}\quad (2)$$

$$1+\frac{1}{\sin^2 y\cos^2 x}\overset{(1)}{=}\frac{\sin^2 y\cos^2 x+\sin^2 x+\sin^2 y\cos^2 x+\cos^2 x\cos^2 y}{\sin^2 y\cos^2 x}\geq$$

$$\overset{AM-GM}{\geq}\frac{1}{\sin^2 y\cos^2 x}\cdot 4\cdot\sqrt[4]{\sin^2 y\cos^2 x\cdot\sin^2 x\cdot\sin^2 y\cos^2 x\cdot\cos^2 x\cos^2 y}=$$

$$=\frac{4}{\sin^2 y\cos^2 x}\cdot\sin y\cdot\cos x\cdot\sqrt{\sin x\cos y\cos x}=\frac{4\sqrt{\sin x\cos y\cos x}}{\sin y\cos x}\quad (3)$$

$$1+\frac{1}{\cos^2 x\cos^2 y}=\frac{\cos^2 x\cos^2 y+\sin^2 x+\sin^2 y\cos^2 x+\cos^2 x\cos^2 y}{\cos^2 x\cos^2 y}\geq$$

$$\overset{AM-GM}{\geq}\frac{4}{\cos^2 x\cos^2 y}\cdot\sqrt[4]{\cos^2 x\cos^2 y\cdot\sin^2 x\cdot\sin^2 y\cos^2 x\cdot\cos^2 x\cos^2 y}=$$

$$=\frac{4}{\cos^2 x\cos^2 y}\cdot\cos x\cos y\sqrt{\sin x\sin y\cos x}=\frac{4\sqrt{\sin x\sin y\cos x}}{\cos x\cos y}\quad (4)$$

By multiplying (2); (3); (4): $\left(1+\frac{1}{\sin^2 x}\right)\left(1+\frac{1}{\sin^2 y\cos^2 x}\right)\left(1+\frac{1}{\cos^2 x\cos^2 y}\right)\geq$

$$\geq\frac{4\cos x}{\sin x}\sqrt{\sin y\cos y}\cdot\frac{4\sqrt{\sin x\cos y\cos x}}{\sin y\cos x}\cdot\frac{4\sqrt{\sin x\sin y\cos x}}{\cos x\cos y}=$$

$$=64\cdot\frac{\cos x\cdot\sin y\cos y\cdot\sin x\cdot\cos x}{\sin y\cos x\cos x\cos y\sin x}=64$$

SOLUTION G.046.

$$\left(\frac{1}{a}\right)^b=\left(1+\left(\frac{1}{a}-1\right)\right)^b<1+b\left(\frac{1}{a}-1\right)=1+\frac{b}{a}-b=\frac{a+b-ab}{a}$$

$$\frac{1}{a^b}<\frac{a+b-ab}{a}\Rightarrow a^b>\frac{a}{a+b-ab}\quad (1)$$

Analogous: $b^a>\frac{b}{a+b-ab}$ *(2)*

By adding (1); (2): $a^b+b^a>\frac{a+b}{a+b-ab}$ *(3)*

$$\frac{a+b}{a+b-ab}>1+ab\Leftrightarrow a+b>a+b-ab+ab(a+b)-a^2b^2\Leftrightarrow$$

$$\Leftrightarrow ab(a+b)-a^2b^2-ab<0\Leftrightarrow ab(a+b-ab-1)<0$$

$$\Leftrightarrow ab\big((a-1)-b(a-1)\big)<0\Leftrightarrow ab(a-1)(1-b)<0$$

$$\Leftrightarrow ab(1-a)(1-b) > 0 \text{ which is true}$$

$$a^b + b^a > \frac{a+b}{a+b-ab} > 1 + ab \Rightarrow a^b + b^a > 1 + ab \quad (4)$$

Analogous: $b^c + c^b > 1 + bc$ *(5);* $c^a + a^c > 1 + ac$ *(6)*

By multiplying (4); (5); (6):

$$\left(a^b + b^a\right)\left(b^c + c^b\right)\left(c^a + a^c\right) > (1+ab)(1+bc)(1+ca) \overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ac} = 8abc = 8 \cdot 4RS = 8 \cdot 4Rrs = 32Rrs$$

SOLUTION G.047.

If H – the orthocenter of triangle ABC is the origin of the system of rectangular axis lets denote x, y, z the affixes of A, B respectively C.

$A(x); B(y); C(z); x, y, z$ *different in pairs*

$$C = AB = |y - x|; b = AC = |z - x|; a = BC = |z - y|$$

$$|x| = AH = 2R|\cos A|; |y| = BH = 2R|\cos B|; z = CH = 2R|\cos C|$$

$$1 = |1| = \left|\frac{xy}{(z-x)(z-y)} + \frac{yz}{(x-y)(x-z)} + \frac{zx}{(y-x)(y-z)}\right| \leq$$

$$\leq \left|\frac{xy}{(z-x)(z-y)}\right| + \left|\frac{yz}{(x-y)(x-z)}\right| + \left|\frac{zx}{(y-x)(y-z)}\right| =$$

$$= \frac{|x| \cdot |y|}{|z-x| \cdot |z-y|} + \frac{|y| \cdot |z|}{|x-y| \cdot |x-z|} + \frac{|z| \cdot |x|}{|y-x| \cdot |y-z|}$$

$$|x| \cdot |y| \cdot |x-y| + |y| \cdot |z| \cdot |y-z| + |z| \cdot |y| \cdot |z-y| \geq |z-x| \cdot |z-y| \cdot |x-y|$$

$$2R|\cos B| \cdot 2R|\cos C| \cdot a + 2R|\cos C| \cdot 2R|\cos A| \cdot b + 2R|\cos A| \cdot 2R|\cos B| \geq abc$$

$$a|\cos B \cos C| + b|\cos C \cos A| + c|\cos A \cos B| \geq \frac{abc}{4R^2} = \frac{4Rrs}{4R^2}$$

$$2R \sin A \cdot |\cos B \cos C| + 2R \sin B \cdot |\cos C \cos A| + 2R \sin C\, |\cos A \cos B| \geq \frac{rs}{R}$$

$$\sin A\, |\cos B \cos C| + \sin B\, |\cos C \cos A| + \sin C\, |\cos A \cos B| \geq \frac{rs}{2R^2}$$

Equality holds for $a = b = c$.

SOLUTION G.048.

If K – Lemoine's point of the triangle ABC is the origin of the system of rectangular axis lets denote x, y, z the affixes of A, B respectively C.

$A(x); B(y); C(z); x, y, z$ *– different in pairs.*

$$c = AB = |y - x|; b = AC = |z - x|; a = BC = |z - y|$$

$$|x| = KA = \frac{2bc\, m_a}{a^2+b^2+c^2};\ |y| = KB = \frac{2cam_b}{a^2+b^2+c^2};\ |z| = KC = \frac{2abm_c}{a^2+b^2+c^2}$$

$$1 = |1| = \left|\frac{xy}{(z-x)(z-y)} + \frac{yz}{(x-y)(x-z)} + \frac{zx}{(y-x)(y-z)}\right| \le$$

$$\le \left|\frac{xy}{(z-x)(z-y)}\right| + \left|\frac{yz}{(x-y)(x-z)}\right| + \left|\frac{zx}{(y-x)(y-z)}\right| =$$

$$= \frac{|x|\cdot|y|}{|z-x|\cdot|z-y|} + \frac{|y|\cdot|z|}{|x-y|\cdot|x-z|} + \frac{|z|\cdot|x|}{|y-x|\cdot|y-z|}$$

$$|x|\cdot|y|\cdot|x-y| + |y|\cdot|z|\cdot|y-z| + |z|\cdot|x|\cdot|z-y| \ge |z-x|\cdot|z-y|\cdot|x-y|$$

$$\sum_{cyc(a,b,c)} \left(a\cdot\frac{2cam_b}{a^2+b^2+c^2}\cdot\frac{2abm_c}{a^2+b^2+c^2}\right) \ge abc$$

$$4abc\sum_{cyc(a,b,c)}\frac{a^2m_bm_c}{(a^2+b^2+c^2)^2} \ge abc$$

$$a^2m_bm_c + b^2m_cm_a + c^2m_am_b \ge \frac{(a^2+b^2+c^2)^2}{4}$$

Equality holds for $a = b = c$.

SOLUTION G.049.

If I – incentre of triangle ABC is the origin of the system of rectangular axis lets denote x, y, z the affixes of A, B respectively C.

$A(x); B(y); C(z); x, y, z$ – different in pairs

$$C = AB = |y-x|;\ b = AC = |z-x|;\ a = BC = |z-y|$$

$$|x| = IA = \frac{r}{\sin\frac{A}{2}};\ |y| = IB = \frac{r}{\sin\frac{B}{2}};\ |z| = IC = \frac{r}{\sin\frac{C}{2}}$$

$$1 = |1| = \left|\frac{xy}{(z-x)(z-y)} + \frac{yz}{(x-y)(x-z)} + \frac{zx}{(y-x)(y-z)}\right| \le$$

$$\le \left|\frac{xy}{(z-x)(z-y)}\right| + \left|\frac{yz}{(x-y)(x-z)}\right| + \left|\frac{zx}{(y-x)(y-z)}\right| =$$

$$= \frac{|x|\cdot|y|}{|z-x|\cdot|z-y|} + \frac{|y|\cdot|z|}{|x-y|\cdot|x-z|} + \frac{|z|\cdot|x|}{|y-x|\cdot|y-z|}$$

$$|x|\cdot|y|\cdot|x-y| + |y|\cdot|z|\cdot|y-z| + |z|\cdot|x|\cdot|z-y| \ge |z-x|\cdot|z-y|\cdot|x-y|$$

$$\sum_{cyc(a,b,c)}\left(a\cdot\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}\right) \ge abc \Rightarrow \sum_{cyc(a,b,c)}\left(\frac{a}{\sin\frac{B}{2}\sin\frac{C}{2}}\right) \ge \frac{abc}{r^2}$$

$$\sum_{cyc(a,b,c)}\left(\frac{2R\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}\right)\geq\frac{4Rrs}{r^2}\Rightarrow\sum_{cyc(a,b,c)}\left(\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}\right)\geq\frac{4Rs}{r\cdot 2R}=\frac{2s}{R}$$

SOLUTION G.050.

If G – the centroid of triangle ABC is the origin of the system of rectangular axis lets denote x, y, z the affixes of A, B respectively C.

$A(x); B(y); C(z); x, y, z$ different in pairs

$$c=AB=|y-x|;b=AC=|z-x|;a=BC=|z-y|$$

$$|x|=GA=\frac{2}{3}m_a;|y|=GB=\frac{2}{3}m_b;|z|=GC=\frac{2}{3}m_c$$

$$1=|1|=\left|\frac{xy}{(z-x)(z-y)}+\frac{yz}{(x-y)(x-z)}+\frac{zx}{(y-x)(y-z)}\right|\leq$$

$$\leq\left|\frac{xy}{(z-x)(z-y)}\right|+\left|\frac{yz}{(x-y)(x-z)}\right|+\left|\frac{zx}{(y-x)(y-z)}\right|=$$

$$=\frac{|x|\cdot|y|}{|z-x|\cdot|z-y|}+\frac{|y|\cdot|z|}{|x-y|\cdot|x-z|}+\frac{|z|\cdot|x|}{|y-x|\cdot|y-z|}$$

$$|x|\cdot|y|\cdot|x-y|+|y|\cdot|z|\cdot|y-z|+|z|\cdot|x|\cdot|z-x|\geq|z-x|\cdot|z-y|\cdot|x-y|$$

$$\frac{2}{3}m_a\cdot\frac{2}{3}m_b\cdot c+\frac{2}{3}m_b\cdot\frac{2}{3}m_c\cdot a+\frac{2}{3}m_c\cdot\frac{2}{3}m_a\cdot b\geq abc$$

$$4(am_bm_c+bm_cm_a+cm_am_b)\geq 9abc$$

Equality holds for $a=b=c$.

SOLUTION G.051.

$$ba^{x+1}+cb^{x+1}+ac^{x+1}=\frac{(ab)^{x+1}}{b^x}+\frac{(bc)^{x+1}}{c^x}+\frac{(ca)^{x+1}}{b^x}\geq$$

$$\overset{RADON}{\geq}\frac{(ab+bc+ca)^{x+1}}{(a+b+c)^x}\overset{GORDON}{\geq}\frac{\left(4\sqrt{3}S\right)^{x+1}}{(2s)^x}=$$

$$=\frac{4^{x+1}\cdot\left(\sqrt{3}\right)^{x+1}\cdot r^{x+1}\cdot s^{x+1}}{2^x\cdot s^x}=2^{2x+2-x}\left(\sqrt{3}\right)^{x+1}\cdot r^{x+1}\cdot s\geq$$

$$\overset{MITRINOVIC}{\geq}2^{x+2}\left(\sqrt{3}\right)^{x+1}\cdot r^{x+1}\cdot 3\sqrt{3}r=2^{x+2}\left(\sqrt{3}\right)^{x+2}\cdot 3\cdot r^{x+2}=3\cdot\left(2\sqrt{3}r\right)^{x+2}$$

Equality holds for $a=b=c$.

SOLUTION G.052.

$$1=a^2+b^2+c^2\overset{AM-GM}{\geq}3\sqrt[3]{a^2b^2c^2}=3\sqrt[3]{(abc)^2}$$

$$\frac{1}{3} \geq \sqrt[3]{(abc)^2} \Rightarrow (abc)^2 \leq \frac{1}{27} \Rightarrow abc \leq \frac{1}{3\sqrt{3}} \Rightarrow \frac{1}{abc} \geq 3\sqrt{3} \quad (1)$$

$$\frac{a+b+c}{abc} \overset{AM-GM}{\geq} \frac{3\sqrt[3]{abc}}{abc} = \frac{3\sqrt[3]{abc}}{\sqrt[3]{(abc)^3}} = \frac{3}{\sqrt[3]{(abc)^2}} =$$

$$= 3 \cdot \left(\frac{1}{abc}\right)^{\frac{2}{3}} \overset{(1)}{\geq} 3 \cdot \left(3\sqrt{3}\right)^{\frac{2}{3}} = 3\left(\left(\sqrt{3}\right)^3\right)^{\frac{2}{3}} = 3\left(\sqrt{3}\right)^2 = 9$$

$$\frac{a+b+c}{abc} \geq 9 \Rightarrow \frac{2s}{4RS} \geq 9 \Rightarrow \frac{2s}{4R \cdot rs} \geq 9$$

$$\frac{1}{2Rr} \geq 9 \Rightarrow 18Rr \leq 1$$

SOLUTION G.053.

$$1 = \sin^2 A + \cos^2 A \overset{AM-GM}{\geq} 2\sqrt{\sin^2 A \cdot \cos^2 A} = 2 \sin A \cos A$$

$$1 \geq 2 \sin A \cos A \Rightarrow \sin A \cos A \leq \frac{1}{2} \Rightarrow \frac{1}{\sin A \cos A} \geq 2 \quad (1)$$

$$\frac{1}{\sin A} + \frac{1}{\cos A} = \frac{\sin A + \cos A}{\sin A \cos A} \overset{AM-GM}{\geq} \frac{2\sqrt{\sin A \cos A}}{\sin A \cos A} =$$

$$= \frac{2}{\sqrt{\sin A \cos A}} = 2 \cdot \left(\frac{1}{\sin A \cos A}\right)^{\frac{1}{2}} \geq 2 \cdot 2^{\frac{1}{2}} = 2\sqrt{2}$$

Equality holds for $A = \frac{\pi}{4}$.

$$\sum_{cyc(A,B,C)} \left(\frac{1}{\sin A} + \frac{1}{\cos A}\right) > 2\sqrt{2} + 2\sqrt{2} + 2\sqrt{2} = 6\sqrt{2}$$

SOLUTION G.054.

$$\sum_{cyc(a,b,c)} \frac{1}{3r_a^2 + 7r_b r_c} \overset{Bergstrom}{\geq} \frac{(1+1+1)^2}{\sum_{cyc(a,b,c)}(3r_a^2 + 7r_b r_c)} =$$

$$= \frac{9}{3\left(r_a^2 + r_b^2 + r_c^2\right) + 7(r_a r_b + r_b r_c + r_c r_a)} =$$

$$= \frac{9}{3((4R+r)^2 - 2s^2) + 7s^2} = \frac{9}{3(4R+r)^2 - 6s^2 + 7s^2} =$$

$$= \frac{9}{3(4R+r)^2 + s^2} \overset{MITRINOVIC}{\geq} \frac{9}{3(4R+r)^2 + 27r^2} \overset{EULER}{\geq}$$

$$\geq \frac{9}{3\left(4R + \frac{R}{2}\right)^2 + 27\left(\frac{R}{2}\right)^2} = \frac{3}{\left(\frac{9R}{2}\right)^2 + 9\left(\frac{R}{2}\right)^2} = \frac{3}{\frac{81R^2 + 9R^2}{3}} = \frac{12R^2}{90} = \frac{2R^2}{15}$$

SOLUTION G.055.

$$\Delta ABC \sim \Delta A'B'C' \Rightarrow \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = k \Rightarrow a' = ka; b' = kb; c' = ck$$

Inequality can be written: $\sum \frac{(ka+kb)(ka+kc)}{kb\cdot kc} + 3 \geq \frac{15(b+c)(kc+ka)(ka+kb)}{8a\cdot kb\cdot kc}$

$$\sum \frac{(a+b)(a+c)}{bc} + 3 \geq \frac{15(b+c)(c+a)(a+b)}{8abc}$$

$$\frac{s^2 - r^2 - Rr}{Rr} + 3 \geq \frac{15}{8} \cdot \frac{2s(s^2 + r^2 + 2Rr)}{4Rrs}$$

$$\frac{s^2 - r^2 - Rr + 3Rr}{Rr} \geq \frac{15(s^2 + r^2 + 2Rr)}{16Rr}$$

$$16(s^2 - r^2 + 2Rr) \geq 15(s^2 + r^2 + 2Rr)$$

$$16s^2 - 16r^2 + 32Rr \geq 15s^2 + 15r^2 + 30Rr$$

$$s^2 \geq 31r^2 - 2Rr \text{ (to prove)}$$

By Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2 \geq 31r^2 - 2Rr \Leftrightarrow 18Rr \geq 36r^2$

$$R \geq 2r \text{ (Euler's inequality)}$$

SOLUTION G.056.

Denote $a = \sin x\, ; b = 1 - \sin x$

$$x \in \left(0, \frac{\pi}{2}\right) \Rightarrow a, b \in (0, 1)$$

$$2(\sin x)^{2-\sin x} \cdot (1 - \sin x)^{\sin x} = 2a^b \cdot b^a \leq$$

$$\overset{AM-GM}{\leq} 2 \cdot \left(\frac{a \cdot b + b \cdot a}{a + b}\right)^{a+b} \overset{AM-GM}{\leq} 2\left(\frac{a+b}{2}\right)^{a+b} =$$

$$= 2\left(\frac{\sin x + 1 - \sin x}{2}\right)^{a+b} = 2\left(\frac{1}{2}\right)^{\sin x + 1 - \sin x} = 2 \cdot \left(\frac{1}{2}\right)^1 = 2 \cdot \frac{1}{2} = 1$$

SOLUTION G.057.

$$\sqrt[a+b]{a^a b^b} \overset{WGM-WHM}{\geq} \frac{a+b}{\frac{a}{a} + \frac{b}{b}} = \frac{a+b}{2}$$

$$a^b b^b \geq \frac{(a+b)^{a+b}}{2^{a+b}} \quad (1)$$

In (1) we take: $a = \sin^2 x\, ; b = \sin^2 y$

$$(\sin^2 x)^{\sin^2 x} \cdot (\sin^2 y)^{\sin^2 y} \geq \frac{(\sin^2 x + \sin^2 y)^{\sin^2 x + \sin^2 y}}{2^{\sin^2 x + \sin^2 y}} \quad (2)$$

In (1) we take: $a = \cos^2 x\, ; b = \cos^2 y$

$$(\cos^2 x)^{\cos^2 x}\cdot(\cos^2 y)^{\cos^2 y}\ge\frac{(\cos^2 x+\cos^2 y)^{\cos^2 x+\cos^2 y}}{2^{\cos^2 x+\cos^2 y}}\quad (3)$$

By multiplying (2); (3): $(\sin^2 x)^{\sin^2 x}\cdot(\sin^2 y)^{\sin^2 y}\cdot(\cos^2 x)^{\cos^2 x}\cdot(\cos^2 y)^{\cos^2 y}\ge$

$$\ge\frac{(\sin^2 x+\sin^2 y)^{\sin^2 x+\sin^2 y}\cdot(\cos^2 x+\cos^2 y)^{\cos^2 x+\cos^2 y}}{2^{\sin^2 x+\sin^2 y+\cos^2 x+\cos^2 y}}=$$

$$=\frac{1}{4}(\sin^2 x+\sin^2 y)^{\sin^2 x+\sin^2 y}\cdot(\cos^2 x+\cos^2 y)^{\cos^2 x+\cos^2 y}$$

$$\frac{(\sin^2 x+\sin^2 y)^{\sin^2 x+\sin^2 y}\cdot(\cos^2 x+\cos^2 y)^{\cos^2 x+\cos^2 y}}{(\sin^2 x)^{\sin^2 x}\cdot(\sin^2 y)^{\sin^2 y}\cdot(\cos^2 x)^{\cos^2 x}\cdot(\cos^2 y)^{\cos^2 y}}\le 4$$

SOLUTION G.058.

$$\Delta ABC\sim\Delta A'B'C'\Rightarrow\frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}=k\Rightarrow a'=ka;b'=kb;c'=kb;k>0$$

Inequality can be written:

$$4(m_a m_{a'}+m_b m_{b'}+m_c m_{c'})+k(a^2+b^2+c^2)\ge 4k\left(\sqrt{a^2b^2}+\sqrt{b^2c^2}+\sqrt{c^2a^2}\right)$$

$$\frac{m_{a'}}{m_a}=\frac{m_{b'}}{m_b}=\frac{m_{c'}}{m_c}=k$$

$$4k\left(m_a^2+m_b^2+m_c^2\right)+k(a^2+b^2+c^2)\ge 4k(ab+bc+ca)$$

$$4\cdot\frac{3}{4}(a^2+b^2+c^2)+(a^2+b^2+c^2)\ge 4(ab+bc+ca)$$

$$a^2+b^2+c^2\ge ab+bc+ca$$

$$(a-b)^2+(b-c)^2+(c-a)^2\ge 0$$

SOLUTION G.059.

By Tereshin's inequality: $m_a\ge\frac{b^2+c^2}{4R}\Rightarrow b^2+c^2\le 4Rm_a$

$$b^2+c^2\le\frac{2\cdot 2R\sin A\cdot m_a}{\sin A}\Rightarrow b^2+c^2\le\frac{2am_a}{\sin A}$$

$$\frac{2S}{am_a}\le\frac{2\cdot\frac{2S}{\sin A}}{b^2+c^2}\Rightarrow\frac{h_a}{m_a}\le\frac{2b}{b^2+c^2}$$

$$\left.\begin{array}{l}\dfrac{h_a}{m_a}\le\dfrac{2bc}{b^2+c^2}\\[2mm]\dfrac{h_b}{m_b}\le\dfrac{2ac}{a^2+c^2}\end{array}\right\}\Rightarrow\frac{h_ah_b}{m_am_b}\le\frac{4abc^2}{(b^2+c^2)(a^2+c^2)}$$

$$\sum_{cyc}\frac{h_a h_b}{m_a m_b}\le 4abc\sum_{cyc}\frac{c}{(b^2+c^2)(a^2+c^2)}=$$

$$=4\cdot 4Rrs\sum_{cyc}\left(\frac{c}{(b^2+c^2)(a^2+c^2)}\right)\overset{MITRINOVIC}{\le}16Rr\cdot\frac{3\sqrt{3}R}{2}\sum_{cyc}\left(\frac{c}{(b^2+c^2)(a^2+c^2)}\right)=$$

$$=24\sqrt{3}R^2 r\sum_{cyc}\left(\frac{c}{(b^2+c^2)(a^2+c^2)}\right)$$

SOLUTION G.060.

$$\frac{\log_{\sin A}\sin B}{\tan\frac{A}{2}}+\frac{\log_{\sin B}\sin C}{\tan\frac{B}{2}}+\frac{\log_{\sin C}\sin A}{\tan\frac{C}{2}}\ge$$

$$\overset{AM-GM}{\ge}3\sqrt[3]{\frac{\log_{\sin A}\sin B\cdot\log_{\sin B}\sin C\cdot\log_{\sin C}\sin A}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}}=\frac{3}{\sqrt[3]{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}}\ge$$

$$\ge\frac{3}{\sqrt[3]{3\sqrt{3}}}=\frac{3}{\sqrt[3]{\left(\sqrt{3}\right)^3}}=\frac{3}{\sqrt{3}}=3\sqrt{3}$$

SOLUTION G.061.

If N – Nagel's point in ΔABC is the origin of the system of rectangular axis. Lets denote x,y,z the affixes of A,B respectively C

$A(x),B(y),C(z);x,y,z$ – different in pairs

$$C=AB=|y-x|;b=|AC|=|z-x|;a=BC=|z-y|$$

$$|x|=NA=\sqrt{(b-c)^2+4r^2}$$

$$|y|=NB=\sqrt{(c-a)^2+4r^2}$$

$$|z|=NC=\sqrt{(a-b)^2+4r^2}$$

$$1=|1|=\left|\frac{xy}{(z-x)(z-y)}+\frac{yz}{(x-y)(x-z)}+\frac{zx}{(y-x)(y-z)}\right|\le$$

$$\le\left|\frac{xy}{(z-x)(z-y)}\right|+\left|\frac{yz}{(x-y)(x-z)}\right|+\left|\frac{zx}{(y-x)(y-z)}\right|=$$

$$=\frac{|x|\cdot|y|}{|z-x|\cdot|z-y|}+\frac{|y|\cdot|z|}{|x-y|\cdot|x-z|}+\frac{|z|\cdot|x|}{|y-x|\cdot|y-z|}$$

$$|x|\cdot|y|\cdot|x-y|+|y|\cdot|z|\cdot|y-z|+|z|\cdot|x|\cdot|z-x|\ge|z-x|\cdot|z-y|\cdot|x-y|$$

$$\sum_{cyc(a,b,c)} a\sqrt{((b-c)^2+4r^2)((c-a)^2+4r^2)} \geq abc$$

SOLUTION G.062.

$$\sin A\,;\sin B\,;\sin C\,;\cos A\,;\cos B\,;\cos C \in [0,1]$$

$$2^{\sin A}=(1+1)^{\sin A} \overset{Bernoulli}{\geq} 1+\sin A>1+\sin^2 A \quad (1)$$

$$2^{\cos A}=(1+1)^{\cos A} \overset{Bernoulli}{\geq} 1+\cos A>1+\cos^2 A \quad (2)$$

By adding (1); (2): $2^{\sin A}+2^{\cos A}>1+\sin^2 A+1+\cos^2 A=3$

$$2^{\sin A}+2^{\cos A}>3 \quad (3)$$

Analogous: $2^{\sin B}+2^{\cos B}>3 \quad (4);\ 2^{\sin C}+2^{\cos C}>3 \quad (5)$

By adding (3); (4); (5): $2^{\sin A}+2^{\sin B}+2^{\sin C}+2^{\cos A}+2^{\cos B}+2^{\cos C}>9$

SOLUTION G.063.

Let $\Delta\, A_0B_0C_0$ *be the orthic triangle of* ΔABC *with sides:* $a_0=a\cos A\,;b_0=b\cos B\,;$ $c_0=c\cos C\,;R_0=\frac{R}{2}$ *– circumradii;* $r_0=2R\cos A\cos B\cos C$ *inradii.*

By Bandila's inequality (1985) in $\Delta A_0B_0C_0$*:*

$$\frac{1}{3}\left(\frac{a_0}{b_0}+\frac{b_0}{c_0}+\frac{c_0}{a_0}\right)\leq\frac{R_0}{2r_0}$$

$$\frac{1}{3}\left(\frac{a\cos A}{b\cos B}+\frac{b\cos B}{c\cos C}+\frac{c\cos C}{a\cos A}\right)\leq\frac{\frac{R}{2}}{2\cdot 2R\cos A\cos B\cos C}$$

$$\frac{a\cos A}{b\cos B}+\frac{b\cos B}{c\cos C}+\frac{c\cos C}{a\cos A}\leq\frac{3}{8R\cos A\cos B\cos C}$$

SOLUTION G.064.

Let $\Delta A_0B_0C_0$ *be the orthic triangle of* $\Delta\ ABC$ *with sides:* $a_0=a\cos A\,;b_0=b\cos B\,;$ $c_0=c\cos C$*; circumradii* $R_0=\frac{R}{2}$*; inradii* $r_0=2R\cos A\cos B\cos C$*.*

By Leunberger's inequality from 1960:

$$\frac{\sqrt{3}}{R_0}\leq\frac{1}{a_0}+\frac{1}{b_0}+\frac{1}{c_0}\leq\frac{\sqrt{3}}{2r_0}$$

$$\frac{\sqrt{3}}{\frac{R}{2}}\leq\frac{1}{a\cos A}+\frac{1}{b\cos B}+\frac{1}{c\cos C}\leq\frac{\sqrt{3}}{2\cdot 2R\cos A\cos B\cos C}$$

$$\frac{2\sqrt{3}}{R}\leq\frac{1}{a\cos A}+\frac{1}{b\cos B}+\frac{1}{c\cos C}\leq\frac{\sqrt{3}}{4R\cos A\cos B\cos C}$$

SOLUTION G.065.

$$(\cos x+i\sin x)^5=\cos^5 x-10\cos^3 x\sin^2 x+5\cos x\sin^4 x+$$
$$+i\left(5\cos^4 x\sin x-10\cos^2 x\sin^3 x+\sin^5 x\right)\quad (1)$$
$$(\cos x+i\sin x)^5=\cos 5x+i\sin 5x\quad (2)$$
By (1); (2): $\sin 5x=5\cos^4 x\sin x-10\cos^2 x\sin^3 x+\sin^5 x$
$$\sin 5x=5\sin x\,(1-\sin^2 x)^2-10(1-\sin^2 x)\sin^3 x+\sin^5 x$$
$$\sin 5x=5\sin x-10\sin^3 x+5\sin^5 x-10\sin^3 x+10\sin^5 x+\sin^5 x$$
$$\sin 5x=16\sin^5 x-20\sin^3 x+5\sin x\quad (3)$$
But $\sin 5x\le 5x;\ (\forall)x\ge 0$

By (3): $16\sin^5 x-20\sin^3 x+5\sin x\le 5x$
$$16\sin^5 x+5\sin x\le 5x+20\sin^3 x\le 5x+20x^3$$
$$16\sin^5 x+5\sin x\le 5x(4x^2+1)$$
$$\sin x\,(16\sin^4 x+5)\le 5x(4x^2+1)$$

SOLUTION G.066.

By Murray-Klamkin's duality principle if P is a point in ΔABC's plane and $PA=x$; $PB=y$; $PC=z$ then ax, by, cz are the sides of another triangle.

Let be $P=H$- orthocenter of ΔABC
$$PA=2R|\cos A|;\ PB=2R|\cos B|;\ PC=2R|\cos C|$$
In that case: $2Ra|\cos A|;\ 2Rb|\cos B|;\ 2Rc|\cos C|$ can be sides of a triangle.

By Padoa's inequality in any triangle:
$$\prod_{cyc}(2Ra|\cos A|+2Rb|\cos B|-2Rc|\cos C|)\le\prod_{cyc}(2Ra|\cos A|)$$
$$\prod_{cyc}(a|\cos A|+b|\cos B|-c|\cos C|)\le|abc\cdot\cos A\cos B\cos C|$$

SOLUTION G.067.

By $a,b,c\in(0,1)\Rightarrow a+b+c<3$. It remains to prove:
$$\tan^{-1}x-\tan^{-1}\left(\sqrt{x^2+y^2}\right)\le\tan^{-1}\left(\sqrt{x^2+y^2}\right)-\tan^{-1}y\quad (1)$$
$$f:(0,1)\to\mathbb{R};\ f(x)=\tan^{-1}x;\ f'(x)=\frac{1}{1+x^2}>0;\ f \text{ increasing;}$$
$$x\le\sqrt{x^2+y^2}\Rightarrow f(x)\le f\left(\sqrt{x^2+y^2}\right)\Rightarrow\tan^{-1}x\le\tan^{-1}\left(\sqrt{x^2+y^2}\right)$$

$$y \le \sqrt{x^2+y^2} \Rightarrow f(y) \le f\left(\sqrt{x^2+y^2}\right) \Rightarrow \tan^{-1} y \le \tan^{-1}\left(\sqrt{x^2+y^2}\right)$$

$$\tan^{-1} x \le \tan^{-1}\left(\sqrt{x^2+y^2}\right) \quad (2)$$

$$-\tan^{-1}\left(\sqrt{x^2+y^2}\right) \le -\tan^{-1} y \quad (3)$$

By adding (2); (3) ⇒ (1):

$$\tan^{-1} x - \tan^{-1}\left(\sqrt{x^2+y^2}\right) \le \tan^{-1}\left(\sqrt{x^2+y^2}\right) - \tan^{-1} y$$

SOLUTION G.068.

First, we prove that if $x \le y \le z$ then: $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \ge \frac{y}{x}+\frac{z}{y}+\frac{x}{z}$ *(1)*

$$\frac{x}{y}-\frac{y}{x}+\frac{y}{z}-\frac{z}{y}+\frac{z}{x}-\frac{x}{z} \ge 0$$

$$\frac{x^2-y^2}{xy}+\frac{y^2-z^2}{zy}+\frac{z^2-x^2}{xz} \ge 0$$

$$z(x^2-y^2)+x(y^2-z^2)+y(z^2-x^2) \ge 0$$

$$x^2z-zy^2+xy^2-xz^2+yz^2-yx^2 \ge 0$$

$$xz(x-z)+y^2(x-z)+y(z-x)(z+x) \ge 0$$

$$(x-z)(xz+y^2-yz-yx) \ge 0$$

$$(x-z)[y(y-x)-z(y-x)] \ge 0$$

$$(x-z)(y-x)(y-z) \ge 0$$

$(z-x)(y-x)(z-y) \ge 0$ *which is true because:* $z-x \ge 0; y-x \ge 0; z-y \ge 0$

By $a \le b \le c \Rightarrow \begin{cases} m_a \ge m_b \ge m_c \\ h_a \ge h_b \ge h_c \end{cases} \Rightarrow m_a h_a \ge m_b h_b \ge m_c h_c$

We take in (1): $x = m_c h_c; y = m_b h_b; z = m_a h_a$

$$\frac{m_c h_c}{m_b h_b}+\frac{m_b h_b}{m_a h_a}+\frac{m_a h_a}{m_c h_c} \ge \frac{m_b h_b}{m_c h_c}+\frac{m_a h_a}{m_b h_b}+\frac{m_c h_c}{m_a h_a}$$

$$\frac{\frac{2S}{c}\cdot m_c}{\frac{2S}{b}\cdot m_b}+\frac{\frac{2S}{b}\cdot m_b}{\frac{2S}{b}\cdot m_a}+\frac{\frac{2S}{a}\cdot m_a}{\frac{2S}{c}\cdot m_c} \ge \frac{\frac{2S}{b}\cdot m_b}{\frac{2S}{c}\cdot m_c}+\frac{\frac{2S}{a}\cdot m_a}{\frac{2S}{a}\cdot m_b}+\frac{\frac{2S}{c}\cdot m_c}{\frac{2S}{a}\cdot m_a}$$

$$\frac{bm_c}{cm_b}+\frac{am_b}{bm_a}+\frac{cm_a}{am_c} \ge \frac{cm_b}{bm_c}+\frac{bm_a}{am_b}+\frac{am_c}{cm_a}$$

SOLUTION G.069.

Inequality can be written:

$$\sqrt[3]{\frac{2R\sin A}{2R\sin B}}+\sqrt[3]{\frac{2R\sin B}{2R\sin C}}+\sqrt[3]{\frac{2R\sin C}{2R\sin A}}-\sqrt[3]{\frac{2R\sin A}{2R\sin C}}-\sqrt[3]{\frac{2R\sin B}{2R\sin A}}-\sqrt[3]{\frac{2R\sin C}{2R\sin B}}<1$$

$$\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{c}}+\sqrt[3]{\frac{c}{a}}-\sqrt[3]{\frac{a}{c}}-\sqrt[3]{\frac{b}{a}}-\sqrt[3]{\frac{c}{b}}<1$$

If a,b,c are sides in a triangle then $\sqrt[3]{a},\sqrt[3]{b},\sqrt[3]{c}$ are also sides in a triangle because:

$$\sqrt[3]{b}+\sqrt[3]{c}>\sqrt[3]{a}\quad \textit{(and analogs)}$$

$$\sqrt[3]{b}+\sqrt[3]{c}\overset{MAHLER}{>}\sqrt[3]{b+c}>\sqrt[3]{a}$$

$$\textit{Denote } a'=\sqrt[3]{a},b'=\sqrt[3]{b},c'=\sqrt[3]{c}$$

Inequality to prove becomes: $\frac{a'}{b'}+\frac{b'}{c'}+\frac{c'}{a'}-\frac{a'}{c'}-\frac{b'}{a'}-\frac{c'}{b'}=\frac{a'-c'}{b'}+\frac{b'-a'}{c'}+\frac{c'-b'}{a'}=$

$$=\frac{1}{a'b'c'}\left(a'^2c'-c'^2a'+b'^2a'-a'^2b'+c'^2b'-b'^2c'\right)=$$

$$=\frac{1}{a'b'c'}(a'b'c'-a'c'^2-a'^2b'+a'^2c'-b'^2c'+b'c'^2+a'b'^2-a'b'c')=$$

$$=\frac{1}{a'b'c'}\Big(a'c'(b'-c')-a'^2(b'-c')-b'c'(b'-c')+a'b'(b'-c')\Big)=$$

$$=\frac{1}{a'b'c'}(b'-c')(a'c'-a'^2-b'c'+a'b')=\frac{1}{a'b'c'}(b'-c')\big(a'(c'-a')-b'(a'-b')\big)=$$

$=\frac{1}{a'b'c'}(b'-c')(c'-a')(a'-b')<1$ ***because:*** $b'-c'<a';c'-a'<b';a'-b'<c'$

SOLUTION G.070.

In $\Delta ABC: m_a<\frac{b+c}{2}\Leftrightarrow m_a^2<\frac{(b+c)^2}{4}$ ***(1)***

$$4m_a^2<(b+c)^2\Leftrightarrow 2b^2+2c^2-a^2<b^2+c^2+2bc\Leftrightarrow b^2-2bc+c^2<a^2\Leftrightarrow$$

$$\Leftrightarrow (b-c)^2<a^2\Leftrightarrow |b-c|<a \textit{ which is true.}$$

$$\sum_{cyc}\frac{b^2+c^2+2bc}{b^2+c^2-a^2}=\sum_{cyc}\frac{(b+c)^2}{b^2+c^2-a^2}=4\sum_{cyc}\frac{(b+c)^2}{4(b^2+c^2-a^2)}\overset{(1)}{>}4\sum_{cyc}\frac{m_a^2}{b^2+c^2-a^2}=$$

$$=\sum_{cyc}\frac{2b^2+2c^2-a^2}{b^2+c^2-a^2}=\frac{1}{2}\sum_{cyc}\frac{4(b^2+c^2-a^2)+(c^2+a^2-b^2)+(a^2+b^2-c^2)}{b^2+c^2-a^2}=$$

$$=\frac{1}{2}\left(\sum_{cyc}\frac{4(b^2+c^2-a^2)}{b^2+c^2-a^2}+\sum_{cyc}\left(\frac{(c^2+a^2-b^2)+(a^2+b^2-c^2)}{b^2+c^2-a^2}\right)\right)=$$

$$=\frac{1}{2}\left(4\cdot 3+\sum_{cyc}\frac{c^2+a^2-b^2}{b^2+c^2-a^2}+\sum_{cyc}\frac{a^2+b^2-c^2}{b^2+c^2-a^2}\right)=$$

$$=6+\frac{1}{2}\left[\left(\frac{c^2+a^2-b^2}{b^2+c^2-a^2}+\frac{b^2+c^2-a^2}{c^2+a^2-b^2}\right)+\left(\frac{a^2+b^2-c^2}{c^2+a^2-b^2}+\frac{c^2+a^2-b^2}{a^2+b^2-c^2}\right)+ +\left(\frac{b^2+c^2-a^2}{a^2+b^2-c^2}+\frac{a^2+b^2-c^2}{b^2+c^2-a^2}\right)\right]\geq$$

$$\geq 6+\frac{1}{2}(2+2+2)=6+3=9$$

SOLUTION G.071.

If a,b,c are sides in an acute triangle then a^2,b^2,c^2 can be sides in a triangle with area: $F=abc\sqrt{\frac{(a^2+b^2+c^2)\cos A\cos B\cos C}{2}}$. By Pedoe's inequality:

$$(a^2)^2(b^2+c^2-a^2)+(b^2)^2(a^2+c^2-b^2)+(c^2)^2(a^2+b^2-c^2)\geq 16FS$$

$$\sum_{cyc}a^4(b^2+c^2-a^2)\geq 16Sabc\sqrt{\frac{(a^2+b^2+c^2)\cos A\cos B\cos C}{2}}=$$

$$=64RS^2\sqrt{\frac{(a^2+b^2+c^2)\cos A\cos B\cos C}{2}}=$$

$$=32RS^2\sqrt{2(a^2+b^2+c^2)\cos A\cos B\cos C}$$

SOLUTION G.072.

$$\sum_{cyc}\frac{\left(\frac{2}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)^2}{\frac{1}{ab}+\frac{2}{bc}+\frac{1}{ca}}\overset{BERGSTROM}{\geq}\frac{\left(\frac{4}{ab}+\frac{4}{bc}+\frac{4}{ca}\right)^2}{\frac{4}{ab}+\frac{4}{bc}+\frac{4}{ca}}=\frac{4}{ab}+\frac{4}{bc}+\frac{4}{ca}=$$

$$=\frac{4}{abc}(a+b+c)=\frac{4\cdot 2s}{4Rrs}=\frac{2}{Rr}\overset{EULER}{\geq}\frac{2}{R\cdot\frac{R}{2}}=\frac{4}{R^2}$$

SOLUTION G.073. ***If $u,v,w>0$ then $(u^7;v^7;w^7)$***

$\left(\frac{1}{v^6+w^6};\frac{1}{w^6+u^6};\frac{1}{u^6+v^6}\right)$ are same oriented

$$(u^7-v^7)\left(\frac{1}{v^6+w^6}-\frac{1}{w^6+u^6}\right)=\frac{(u^7-v^7)(w^6+u^6-v^6-w^6)}{(v^6+w^6)(w^6+u^6)}=$$

$$=\frac{(u^7-v^7)(u^6-v^6)}{(v^6+w^6)(w^6+u^6)}\geq 0.$$ ***By Cebyshev's inequality:***

$$\sum_{cyc}\frac{u^7}{v^6+w^6}\ge\frac{1}{3}\left(\sum_{cyc}u^7\right)\left(\sum_{cyc}\frac{1}{v^6+w^6}\right)\overset{AM-HM}{\ge}\frac{1}{3}\left(\sum_{cyc}u^7\right)\cdot\frac{9}{2\sum_{cyc}u^6}=$$

$$=\frac{3}{2}\cdot\frac{u^7+v^7+w^7}{u^6+v^6+w^6}\ge\frac{u+v+w}{2}\ \textit{because}\ 3(u^7+v^7+w^7)\ge(u^6+v^6+w^6)(u+v+w)$$

Last inequality it's Cebyshev's:

$$u^7+v^7+w^7=u^6\cdot u+v^6\cdot v+w^6\cdot w\ge\frac{1}{3}(u^6+v^6+w^6)(u+v+w)$$

In inequality: $\sum_{cyc}\frac{u^7}{v^6+w^6}\ge\frac{u+v+w}{2}$ ***we replace*** $u=xy; v=yz; w=zx$

$$\sum_{cyc}\frac{(xy)^7}{(yz)^6+(zx)^6}\ge\frac{xy+yz+zx}{2}\quad (1)$$

For $x=\frac{AI}{a}; y=\frac{BI}{b}; z=\frac{CI}{c}$ ***in (1):*** $\sum_{cyc}\frac{\left(\frac{AI}{a}\cdot\frac{BI}{b}\right)^7}{\left(\frac{BI}{b}\cdot\frac{CI}{c}\right)^6+\left(\frac{CI}{c}\cdot\frac{AI}{a}\right)^6}\ge\frac{1}{2}\left(\frac{AI}{a}\cdot\frac{BI}{b}+\frac{BI}{b}\cdot\frac{CI}{c}+\frac{CI}{c}\cdot\frac{AI}{a}\right)\overset{HAYASHY}{\ge}\frac{1}{2}$

Using: $AI=\frac{r}{\sin\frac{A}{2}}; BI=\frac{r}{\sin\frac{B}{2}}; CI=\frac{r}{\sin\frac{C}{2}}$

$$\sum_{cyc}\frac{\left(\frac{r}{a\sin\frac{A}{2}}\cdot\frac{r}{b\sin\frac{B}{2}}\right)^7}{\left(\frac{r}{b\sin\frac{B}{2}}\cdot\frac{r}{c\sin\frac{C}{2}}\right)^6+\left(\frac{r}{c\sin\frac{C}{2}}\cdot\frac{r}{a\sin\frac{A}{2}}\right)^6}\ge\frac{1}{2}$$

$$\sum_{cyc}\frac{\left(\frac{1}{ab\sin\frac{A}{2}\sin\frac{B}{2}}\right)^7}{\left(\frac{1}{bc\sin\frac{B}{2}\sin\frac{C}{2}}\right)^6+\left(\frac{1}{ca\sin\frac{C}{2}\sin\frac{A}{2}}\right)^6}\ge\frac{1}{2r^2}$$

SOLUTION G.074.

Let be $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}; f(x)=\frac{\sin x-\cos x}{x}$

$$f'(x)=\frac{(\cos x+\sin x)x-(\sin x-\cos x)}{x^2}$$

Let be $g:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}$

$$g(x)=(\cos x+\sin x)x-(\sin x-\cos x)$$

$$g(x)=x\cos x+x\sin x-\sin x+\cos x$$

$$g'(x) = \cos x - x\sin x + \sin x + x\cos x - \cos x - \sin x$$

$$g'(x) = x(\cos x - \sin x);\ g'(x) = 0 \Rightarrow \cos x = \sin x \Rightarrow x = \frac{\pi}{4}$$

$$\max g(x) = f\left(\frac{\pi}{4}\right)$$

$$g(0) = 1; g\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 > 0$$

$$\inf g(x) = \frac{\pi}{2} - 1 \Rightarrow g(x) > 0 \Rightarrow f'(x) > 0 \Rightarrow f \textit{ increasing (1)}$$

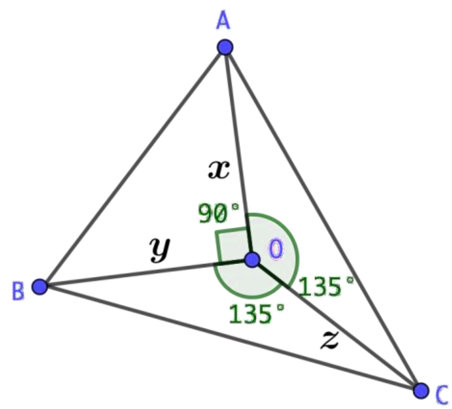
$$\sqrt{ab} \le \frac{a+b}{2} \overset{(1)}{\Rightarrow} f(\sqrt{ab}) \le f\left(\frac{a+b}{2}\right)$$

$$\frac{\sin(\sqrt{ab}) - \cos(\sqrt{ab})}{\sqrt{ab}} \le \frac{\sin\left(\frac{a+b}{2}\right) - \cos\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}$$

$$(a+b)\left(\sin(\sqrt{ab}) - \cos(\sqrt{ab})\right) \le 2\sqrt{ab}\left(\sin\left(\frac{a+b}{2}\right) - \cos\left(\frac{a+b}{2}\right)\right)$$

Equality holds if $a = b$.

SOLUTION G.075.



$$OA = x; OB = y; OC = z$$

$$\mu(\widehat{AOB}) = 90°; \mu(\widehat{BOC}) = 135°;\ \mu(\widehat{AOC}) = 135°$$

$$AB^2 = x^2 + y^2$$

$$BC^2 = y^2 + z^2 - 2yz\cos 135° = y^2 + z^2 + yz\sqrt{2}$$

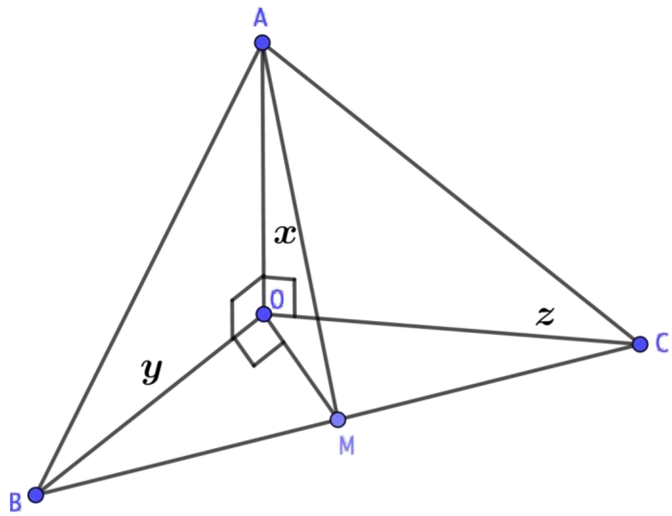
$$AC^2 = x^2 + z^2 - 2xz\cos 135° = x^2 + z^2 + xz\sqrt{2}$$

$$\textit{In } \Delta ABC: BC + AC > AB$$

$$\sqrt{x^2 + z^2 + xz\sqrt{2}} + \sqrt{y^2 + z^2 + yz\sqrt{2}} \ge \sqrt{x^2 + y^2}$$

$$\textit{Equality holds if } x = y = z = 0$$

SOLUTION G.076.



$$OA = x; OB = y; OC = z; OA \perp (OBC)$$

$$OM = \frac{yz}{\sqrt{y^2+z^2}}; AM = \sqrt{x^2 + \frac{y^2z^2}{y^2+z^2}}$$

$$AM = \frac{\sqrt{x^2y^2+y^2z^2+z^2x^2}}{\sqrt{y^2+z^2}}$$

$$S[ABC] = \frac{AM \cdot BC}{2} = \frac{1}{2} \cdot \sqrt{y^2+z^2} \cdot \frac{\sqrt{x^2y^2+y^2z^2+z^2x^2}}{\sqrt{y^2+z^2}}$$

$$S[ABC] = \frac{\sqrt{x^2y^2+y^2z^2+z^2x^2}}{2}$$

$$\textit{By Mitrinovic's inequality in } \Delta ABC\text{: } s \ge r\sqrt{3} \Leftrightarrow s \ge \frac{S}{s}\sqrt{3} \Leftrightarrow s^2 \ge S\sqrt{3}$$

$$\frac{\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2}{4} \ge \frac{\sqrt{3}}{2}\sqrt{x^2y^2+y^2z^2+z^2x^2}$$

$$\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2 \ge 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$$

SOLUTION G.077.

$$\textit{Let be } A = \begin{pmatrix} \sin x & \cos x & 1 \\ 1 & \sin y & \cos y \\ 1 & 1 & 1 \end{pmatrix}$$

$$\det A = \sin x \sin y + \cos x \cos y + 1 - \sin y - \sin x \cos y - \cos x =$$