$$= 1 + \cos(x - y) - \sin y - \sin x \cos y - \cos x$$

By Hadamard's inequality:

$$(\det A)^2 < (\sin^2 x + \cos^2 x + 1)(1 + \sin^2 y + \cos^2 y)(1^2 + 1^2 + 1^2)$$

$$(1 + \cos(x - y) - \sin y - \sin x \cos y - \cos x)^2 < 2 \cdot 2 \cdot 3 = 12$$

SOLUTION G.078.

$$\frac{\tan 20^\circ \tan 30^\circ}{\tan 10^\circ \tan 50^\circ} = \frac{\sqrt{3}}{3\tan 50^\circ} \cdot \frac{\sin 20^\circ \cos 10^\circ}{\sin 10^\circ \cos 20^\circ} =$$

$$= \frac{\sqrt{3}}{3\tan 50^\circ} \cdot \frac{\sin 20^\circ \sin 40^\circ \sin 80^\circ}{\cos 80^\circ \cos 40^\circ \cos 20^\circ} \cdot \frac{\cos 40^\circ}{\sin 40^\circ} =$$

$$= \frac{\sqrt{3}}{3\tan 50^\circ} \cdot \frac{\sin 20^\circ \sin 40^\circ \sin 80^\circ}{\cos 20^\circ \cos 40^\circ \cos 80^\circ} \cdot \cot 40^\circ =$$

$$= \frac{\sqrt{3}\tan 50^\circ}{3\tan 50^\circ} \cdot \frac{8\sin^2 20^\circ \sin 40^\circ \sin 80^\circ}{8\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ} =$$

$$= \frac{\sqrt{3}}{3} \cdot \frac{8\sin^2 20^\circ \cdot \frac{1}{2}(\cos 40^\circ - \cos 120^\circ)}{4\sin 40^\circ \cos 40^\circ \cos 80^\circ} =$$

$$= \frac{\sqrt{3}}{3} \cdot \frac{4\sin^2 20^\circ \left(\cos 40^\circ + \frac{1}{2}\right)}{\sin 20^\circ} = \frac{\sqrt{3}}{3} \cdot 4\sin 20^\circ \left(\cos 40^\circ + \frac{1}{2}\right) =$$

$$= \frac{\sqrt{3}}{3}(4\sin 20^\circ \cos 40^\circ + 2\sin 20^\circ) =$$

$$= \frac{\sqrt{3}}{3}\left(4 \cdot \frac{1}{2}(\sin 60^\circ - \sin 20^\circ) + 2\sin 20^\circ\right) =$$

$$= \frac{\sqrt{3}}{3}(2\sin 60^\circ - 2\sin 20^\circ + 2\sin 20^\circ) =$$

$$= \frac{\sqrt{3}}{3} \cdot 2 \cdot \frac{\sqrt{3}}{2} = 1 \Rightarrow \tan 20^\circ \tan 30^\circ = \tan 10^\circ \tan 50^\circ \quad (1)$$

$$x\tan 20^\circ + y\tan 30^\circ \overset{AM-GM}{>} 2\sqrt{xy\tan 20^\circ \tan 30^\circ} \overset{(1)}{=}$$

$$= 2\sqrt{xy\tan 10^\circ \tan 50^\circ} = 2\sqrt{(x\tan 10^\circ)\cdot(y\tan 50^\circ)}$$

GM-HM

$$2 \cdot \frac{2}{\frac{1}{x\tan 10^\circ} + \frac{1}{y\tan 50^\circ}} = \frac{4}{\frac{\cot 10^\circ}{x} + \frac{\cot 50^\circ}{y}} = \frac{4xy}{x\cot 50^\circ + y\cot 10^\circ} \quad (2)$$

$$\textit{Analogous: } y\tan 20^\circ + z\tan 30^\circ > \frac{4yz}{y\cot 50^\circ + z\cot 10^\circ} \quad (3)$$

$$z\tan 20^\circ + x\tan 30^\circ > \frac{4zx}{z\cot 50^\circ + x\cot 10^\circ} \quad (4)$$

By adding (2); (3); (4):

$$(x+y+z)\left(\frac{\sqrt{3}}{3} + \tan 20^\circ\right) > 4\sum_{cyc}\frac{xy}{x\cot 50^\circ + y\cot 10^\circ}$$

SOLUTION G.079.

$$\sin 2x \le 1 \Rightarrow \sin x\cos x \le \frac{1}{2} \Rightarrow \sin^2 x\cos^2 x \le \frac{1}{4} \quad (1)$$

$$\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x) =$$

$$= 1 - 3\sin^2 x\cos^2 x \overset{(1)}{\ge} 1 - \frac{3}{4} = \frac{1}{4} \quad (2)$$

$$(\sin^2 x + \csc^2 x)^3 + (\cos^2 x + \sec^2 x)^3 =$$

$$= \sin^6 x + \cos^6 x + 3\left(\sin^2 x + \frac{1}{\sin^2 x}\right) + 3\left(\cos^2 x + \frac{1}{\cos^2 x}\right) + \frac{1}{\sin^6 x} + \frac{1}{\cos^6 x} \overset{(2)}{\ge}$$

$$\ge \frac{1}{4} + 3 + 3\left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}\right) + \left(\frac{1}{\sin^6 x} + \frac{1}{\cos^6 x}\right) \overset{AM-GM}{\ge} \frac{13}{4} +$$

$$+3\cdot\frac{2}{\sin x\cos x} + \frac{\sin^6 x + \cos^6 x}{\sin^6 x\cos^6 x} \overset{(1);(2)}{\ge} \frac{13}{4} + \frac{6}{\frac{1}{2}} + \frac{\frac{1}{4}}{\left(\frac{1}{4}\right)^3} = \frac{13}{5} + 12 + 16 = \frac{125}{4}$$

$$\sum_{cyc(x,y,z,t)}(\sin^2 x + \csc^2 x)^3 + \sum_{cyc(x,y,z,t)}(\cos^2 x + \sec^2 x)^3 \ge 4\cdot\frac{125}{4} = 125$$

Equality holds for $x = y = z = t = \frac{\pi}{4}$.

SOLUTION G.080.

$$\sin\frac{3\pi}{11} + \sin\frac{4\pi}{11} = 2\sin\frac{7\pi}{22}\cos\frac{\pi}{22} = 2\cos\left(\frac{\pi}{2} - \frac{7\pi}{22}\right)\cos\frac{\pi}{22} =$$

$$= 2\cos\frac{2\pi}{11}\cos\frac{\pi}{22} = 2\cos\frac{2\pi}{11}\sin\left(\frac{\pi}{2} - \frac{\pi}{22}\right) = 2\cos\frac{2\pi}{11}\sin\frac{5\pi}{11} \quad (1)$$

$$\frac{x^2}{\sin\frac{3\pi}{11}} + \frac{y^2}{\sin\frac{4\pi}{11}} \overset{BERGSTROM}{\ge} \frac{(x+y)^2}{\sin\frac{3\pi}{11} + \sin\frac{4\pi}{11}} \overset{(1)}{=} \frac{(x+y)^2}{2\cos\frac{2\pi}{11}\sin\frac{5\pi}{11}} \overset{AM-GM}{\ge}$$

$$\ge \frac{4xy}{2\cos\frac{2\pi}{11}\sin\frac{5\pi}{11}} \overset{AM-GM}{>} \frac{4xy}{2\left(\frac{\cos\frac{2\pi}{11} + \sin\frac{5\pi}{11}}{2}\right)^2} = \frac{8xy}{\left(\cos\frac{2\pi}{11} + \sin\frac{5\pi}{11}\right)^2} \ge$$

$$\geq \frac{8\cdot\frac{1}{8}}{\left(\cos\frac{2\pi}{11}+\sin\frac{5\pi}{11}\right)^2}=\frac{1}{\left(\cos\frac{2\pi}{11}+\sin\frac{5\pi}{11}\right)^2}$$

SOLUTION G.081.

$$\left(\sin^2 x+\frac{1}{\sin^2 x}\right)^{m+1}+\left(\cos^2 x+\frac{1}{\cos^2 x}\right)^{m+1}=$$

$$=\frac{\left(\sin^2 x+\frac{1}{\sin^2 x}\right)^{m+1}}{1^m}+\frac{\left(\cos^2 x+\frac{1}{\cos^2 x}\right)^{m+1}}{1^m}\geq$$

$$\overset{RADON}{\geq}\frac{\left(\sin^2 x+\cos^2 x+\frac{1}{\sin^2 x}+\frac{1}{\cos^2 x}\right)^{m+1}}{(1+1)^m}=$$

$$=\frac{\left(1+\frac{\cos^2 x+\sin^2 x}{\sin^2 x\cos^2 x}\right)^{m+1}}{2^m}=\frac{\left(1+\frac{1}{\sin^2 x\cos^2 x}\right)^{m+1}}{2^m}=$$

$$=\frac{1}{2^m}\left(1+\frac{4}{\sin^2 2x}\right)^{m+1}\geq\frac{1}{2^m}(1+4)^{m+1}=\frac{5^{m+1}}{2^m}$$

SOLUTION G.082.

$$\sin^2 x+\sin^2 y=\frac{1}{2}\Rightarrow 1-\cos^2 x+\sin^2 y=\frac{1}{2}$$

$$\cos^2 x=\tfrac{1}{2}+\sin^2 y.\ \textit{Analogous}\ \cos^2 y=\tfrac{1}{2}+\sin^2 x$$

$$\tan x=\sqrt{\frac{\sin^2 x}{\cos^2 x}}=\sqrt{\frac{\sin^2 x}{\frac{1}{2}+\sin^2 y}}$$

$$\tan y=\sqrt{\frac{\sin^2 y}{\cos^2 y}}=\sqrt{\frac{\sin^2 y}{\frac{1}{2}+\sin^2 x}}$$

$$\tan x\cdot\tan y=\sqrt{\frac{\sin^2 x\sin^2 y}{\left(\frac{1}{2}+\sin^2 y\right)\left(\frac{1}{2}+\sin^2 x\right)}}\overset{AM-GM}{\leq}\frac{1}{2}\left(\frac{\sin^2 x}{\frac{1}{2}+\sin^2 x}+\frac{\sin^2 y}{\frac{1}{2}+\sin^2 y}\right)\quad \textit{(1)}$$

$$\tan x=\sqrt{\frac{\sin^2 x}{\frac{1}{2}+\sin^2 y}\cdot 1}\overset{AM-GM}{<}\frac{1}{2}\left(\sin^2 x+\frac{1}{\frac{1}{2}+\sin^2 y}\right)\quad \textit{(2)}$$

$$\tan y=\sqrt{\frac{\sin^2 y}{\frac{1}{2}+\sin^2 x}\cdot 1}\overset{AM-GM}{<}\frac{1}{2}\left(\sin^2 y+\frac{1}{\frac{1}{2}+\sin^2 x}\right)\quad (3)$$

By adding (1); (2); (3):

$$\tan x\cdot\tan y+\tan x+\tan y<\frac{1}{2}(\sin^2 x+\sin^2 y)+\frac{\frac{1}{2}+\sin^2 x}{\frac{1}{2}+\sin^2 x}+\frac{\frac{1}{2}+\sin^2 y}{\frac{1}{2}+\sin^2 y}=$$

$$=\frac{1}{2}\cdot\frac{1}{2}+1+1=\frac{1}{4}+2=\frac{9}{4}<3$$

SOLUTION G.083.

$$\frac{\sin^2 x}{1+\sin^2 x}+\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{\sin^2 z}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}+$$

$$+\frac{1}{8\sin x\sin y\sin z}=\frac{1+\sin^2 x-1}{1+\sin^2 x}+\frac{1+\sin^2 y-1}{(1+\sin^2 x)(1+\sin^2 y)}+$$

$$+\frac{1+\sin^2 z-1}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}+\frac{1}{8\sin x\sin y\sin z}=$$

$$=1-\frac{1}{1+\sin^2 x}+\frac{1}{1+\sin^2 x}-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{1}{(1+\sin^2 x)(1+\sin^2 y)}-$$

$$-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}+\frac{1}{8\sin x\sin y\sin z}=$$

$$=1-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}+\frac{1}{8\sin x\sin y\sin z}\geq$$

$$\overset{AM-GM}{\geq}1-\frac{1}{2\sin x\cdot 2\sin y\cdot 2\sin z}+\frac{1}{8\sin x\sin y\sin z}=1$$

By hypothesis: LHS $\leq 1\Rightarrow$ *LHS* $=1$

$$\sin x=\sin y=\sin z\,;\,x=y=z=\frac{\pi}{2}$$

SOLUTION G.084.

$$\textit{Let be } A=\begin{pmatrix}\sin x & \cos x & 1\\ 1 & \sin y & \cos y\end{pmatrix}\in M_{2,3}(\mathbb{R})$$

$$A^T=\begin{pmatrix}\sin x & 1\\ \cos x & \sin y\\ 1 & \cos y\end{pmatrix}\in M_{3,2}(\mathbb{R})$$

$$A\cdot A^T=\begin{pmatrix}\sin x & \cos x & 1\\ 1 & \sin y & \cos y\end{pmatrix}\begin{pmatrix}\sin x & 1\\ \cos x & \sin y\\ 1 & \cos y\end{pmatrix}$$

$$A \cdot A^T = \begin{pmatrix} \sin^2 x + \cos^2 x + 1 & \sin x + \cos x \sin y + \cos y \\ \sin x + \sin y \cos y + \cos y & 1 + \sin^2 y + \cos^2 y \end{pmatrix}$$

$$A \cdot A^T = \begin{pmatrix} 2 & \sin x + \sin y \cos x + \cos y \\ \sin x + \sin y \cos x + \cos y & 2 \end{pmatrix}$$

By Binet Cauchy's theorem: $\det(A \cdot A^T) > 0$

$$4 - (\sin x + \sin y \cos x + \cos y)^2 > 0$$

$$(\sin x + \sin y \cos x + \cos y)^2 < 4$$

$$|\sin x + \sin y \cos x + \cos y| < 2$$

SOLUTION G.085.

Let be $f: (0, \infty) \to \mathbb{R}; f(x) = \frac{\tan^{-1} x}{\tan^{-1}(1+x)}$

$$f'(x) = \frac{\frac{1}{1+x^2}\tan^{-1}(1+x) - \frac{1}{1+(1+x)^2}\tan^{-1} x}{\left(\tan^{-1}(1+x)\right)^2}$$

$$f'(x) = \frac{(1+(1+x)^2)\tan^{-1}(1+x) - (1+x^2)\tan^{-1} x}{(1+x^2)(1+(1+x)^2)(\tan^{-1}(1+x))^2} > 0$$

$$\textit{because: } \left.\begin{matrix} 1+(1+x)^2 > 1+x^2 \\ \tan^{-1}(1+x) > \tan^{-1} x \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow (1+(1+x)^2)\tan^{-1}(1+x) > (1+x^2)\tan^{-1} x$$

f *increasing. By Mac Laurin's inequality*

$$\sqrt[3]{abc} \le \sqrt{\frac{ab+bc+ca}{3}} = \sqrt{\frac{3}{3}} = 1$$

$$f\left(\sqrt[3]{abc}\right) \le f(1) = \frac{\tan^{-1} 1}{\tan^{-1} 2} = \frac{\pi}{4\tan^{-1} 2}$$

$$\frac{\tan^{-1}\left(\sqrt[3]{abc}\right)}{\tan^{-1}\left(1+\sqrt[3]{abc}\right)} \le \frac{\pi}{4\tan^{-1} 2}$$

$$4(\tan^{-1} 2)\tan^{-1}\left(\sqrt[3]{abc}\right) \le \pi \tan^{-1}\left(1+\sqrt[3]{abc}\right)$$

SOLUTION G.086.

We use the relationship: $\cos^6 x = \frac{10+15\cos 2x + 6\cos 4x + \cos 6x}{32}$

$10 + 15\cos 2x + 6\cos 4x + \cos 6x = 32\cos^6 x$. *By summing:*

$$30 + 15\sum_{cyc}\cos 2x + 6\sum_{cyc}\cos 4x + \sum_{cyc}\cos 6x =$$

$$= 32(\cos^6 x + \cos^6 y + \cos^6 z) \overset{AM-GM}{\geq} 32 \cdot 3\sqrt[3]{\cos^6 x \cos^6 y \cos^6 z} =$$

$$= 96 \cos^2 x \cos^2 y \cos^2 z = 96 \cdot \left(\frac{\sqrt{2}}{2}\right)^2 = 96 \cdot \frac{1}{2} = 48$$

$$15 \sum_{cyc} \cos 2x + 6 \sum_{cyc} \cos 4x + \sum_{cyc} 6x \geq 48 - 30 = 18$$

SOLUTION G.087.

$$3 \sum_{cyc} \cos 2x + 4 \sum_{cyc} \sin x \sin y = 1 \Rightarrow 3 \sum (1 - 2\sin^2 x) + 4 \sum_{cyc} \sin x \sin y = 1$$

$$9 - 6 \sum_{cyc} \sin^2 x + 4 \sum_{cyc} \sin x \sin y = 1$$

$$3 \sum_{cyc} (\sin^2 x) - 2 \sum_{cyc} \sin x \sin y = 4$$

$$\sum_{cyc} (\sin x + \sin y - \sin z)^2 = 4$$

$$(\sin x + \sin y - \sin z)^2 + (\sin x - \sin y + \sin z)^2 + (-\sin x + \sin y + \sin z)^2 = 4$$

$$(\sin x + \sin y - \sin z)^2 + 4 + (-\sin x + \sin y + \sin z)^2 = 4$$

$$(\sin x + \sin y - \sin z)^2 + (-\sin x + \sin y + \sin z)^2 = 0$$

$$\begin{cases} \sin x + \sin y - \sin z = 0 \\ -\sin x + \sin y + \sin z = 0 \end{cases}$$

$$\sin y = 0 \Rightarrow y = k\pi; k \in \mathbb{Z}$$

$$\sin x = \sin z = 1 \Rightarrow x = z = (-1)^m \frac{\pi}{2} + m\pi; m \in \mathbb{Z}$$

SOLUTION G.088.

$$(a + a')(b + b')(c + c') = abc + a'b'c' + a'bc + ab'c + abc' + a'b'c + a'bc' + ab'c' \geq$$

$$\overset{AM-GM}{\geq} abc + a'b'c' + 6\sqrt[6]{(abca'b'c')^3} = abc + a'b'c' + 6\sqrt{abca'b'c'} =$$

$$= 8\sqrt{abca'b'c'} + abc + a'b'c' - 2\sqrt{abca'b'c'} =$$

$$= 64\sqrt{\frac{RR'}{64RR'} abca'b'c'} + \left(\sqrt{abc} - \sqrt{a'b'c'}\right)^2 \geq$$

$$\overset{EULER}{\geq} 64\sqrt{rr' \cdot \frac{abc}{4R} \cdot \frac{a'b'c'}{4R'}} + \left(\sqrt{abc} - \sqrt{a'b'c'}\right)^2 =$$

$$= 64\sqrt{rr'SS'} + 4\left(\sqrt{\frac{abc}{4R}\cdot R} - \sqrt{\frac{a'b'c'}{4R'}\cdot R'}\right)^2 =$$

$$= 64rr'\sqrt{ss'} + 4\left(\sqrt{SR} - \sqrt{S'R'}\right)^2 = 64rr'\sqrt{ss'} + 4\left(\sqrt{Rrs} - \sqrt{R'r's'}\right)^2$$

SOLUTION G.089.

$$\tan y = 2\sin^2 x = \frac{2\sin^2 x\cos^2 x}{\cos^2 x} = \frac{2\tan^2 x}{\frac{1}{\cos^2 x}} = \frac{2\tan^2 x}{1+\tan^2 x} \overset{AM-GM}{\leq} \frac{2\tan^2 x}{2\tan x} = \tan x$$

$$\tan y \leq \tan x \Rightarrow y \leq x \quad (1)$$

$$\tan z = \frac{3\sin^3 y\cos y}{\sin^4 y + \cos^2 y} = \frac{3\tan^3 y\cos^4 y}{\sin^4 y + \cos^2 y} = \frac{3\tan^3 y}{\tan^4 y + \frac{1}{\cos^2 y}} =$$

$$= \frac{3\tan^3 y}{\tan^4 y + \tan^2 y + 1} \overset{AM-GM}{\leq} \frac{3\tan^3 y}{3\sqrt[3]{\tan^6 y}} = \frac{3\tan^3 y}{3\tan^2 y} = \tan y$$

$$\tan z \leq \tan y \Rightarrow z \leq y \quad (2)$$

$$\tan x = \frac{4\sin^4 z\cos^2 z}{\sin^4 z + \cos^4 z} = \frac{4\tan^4 z}{\frac{1}{\cos^2 z}\left(\frac{\sin^4 z}{\cos^4 z} + 1\right)} = \frac{4\tan^4 z}{(\tan^4 z + 1)(1+\tan^2 z)} \overset{AM-GM}{\leq}$$

$$\leq \frac{4\tan^4 z}{2\tan^2 z\cdot 2\tan z} = \tan z$$

$$\tan x \leq \tan z \Rightarrow x \leq z \quad (3)$$

By (1); (2); (3): $z \leq y \leq x \leq z \Rightarrow x = y = z$

$$2\sin^2 x = \tan x\,;\, x \in \left(0, \frac{\pi}{2}\right) \Rightarrow 2\sin x = \frac{1}{\cos x} \Rightarrow \sin 2x = 1 \Rightarrow x = \frac{\pi}{4}$$

Solution: $x = y = z = \frac{\pi}{4}$

SOLUTION G.090.

$$\cos^7 x = \frac{1}{64}(\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x) \geq$$

$$\overset{AM-GM}{\geq} \frac{1}{64}\cdot 64\sqrt[64]{\cos 7x\cdot(\cos 5x)^7\cdot(\cos 3x)^{21}\cdot(\cos x)^{35}}$$

$$(\cos^7 x)^{64} \geq \cos 7x\cdot(\cos 5x)^7\cdot(\cos 3x)^{21}(\cos x)^{35}$$

$$\cos^{448-35} x \geq (\cos 3x)^{21}\cdot(\cos 5x)^7\cdot\cos 7x$$

$$\cos^{413} x \geq (\cos 3x)^{21}\cdot(\cos 5x)^7\cdot\cos 7x$$

SOLUTION G.091.

$$w_a \geq h_a; w_b \geq h_b; w_c \geq h_c$$

$$\sum \frac{aw_a^2}{h_a} \geq \sum \frac{ah_a^2}{h_a} = \sum ah_a = \sum a \cdot \frac{2S}{a} = \sum 2S = 6S = 6rs$$

It remains to prove: $6rs \geq 2r^2\sqrt{\frac{486r}{R}} \Leftrightarrow 3s \geq r\sqrt{\frac{486r}{R}} \Leftrightarrow 9s^2 \geq r^2 \cdot \frac{486r}{R} \Leftrightarrow s^2R \geq 54r^3$

$$s^2R \overset{Euler}{\geq} s^2 2r \overset{MITRINOVIC}{\geq} \left(3\sqrt{3}r\right)^2 \cdot 2r = 27r^2 \cdot 2r = 54r^3$$

SOLUTION G.092.

$$m_a \geq h_a; m_b \geq h_b; m_c \geq h_c$$

$$\sum \frac{am_a}{h_a} \geq \sum \frac{ah_a}{h_a} = \sum a = 2s$$

It remains to prove: $2s \geq 2\sqrt{3\sqrt{3}S} \Leftrightarrow s^2 \geq 3\sqrt{3}S$

$$s^2 \geq 3\sqrt{3}rs \Leftrightarrow s \geq 3\sqrt{3}r \ \textit{(Mitrinovic)}$$

SOLUTION G.093.

$$\frac{ar_a}{h_a} + \frac{br_b}{h_b} + \frac{cr_c}{h_c} = \frac{a \cdot \frac{S}{s-a}}{\frac{2S}{a}} + \frac{b \cdot \frac{S}{s-b}}{\frac{2S}{b}} + \frac{c \cdot \frac{S}{s-c}}{\frac{2S}{c}}$$

$$= \frac{1}{2}\left(\frac{a^2}{s-a} + \frac{b^2}{s-b} + \frac{c^2}{s-c}\right) \overset{BERGSTROM}{\geq}$$

$$\geq \frac{1}{2} \cdot \frac{(a+b+c)^2}{s-a+s-b+s-c} = \frac{(a+b+c)^2}{2s} = \frac{(a+b+c)^2}{a+b+c} = a+b+c$$

SOLUTION G.094.

Let be $f: (1, \infty) \to \mathbb{R}; f(x) = x^2 \sin\frac{1}{x};\ f'(x) = 2x\sin\frac{1}{x} + x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

$$f'(x) = x\cos\frac{1}{x}\left(2\tan\frac{1}{x} - \frac{1}{x}\right) > 0 \ \textit{because: } \tan\frac{1}{x} > \frac{1}{x}; 2\tan\frac{1}{x} > \tan\frac{1}{x} > \frac{1}{x}$$

$$f \textit{ increasing; } \sqrt{ab} \leq \frac{a+b}{2} \Rightarrow f\left(\sqrt{ab}\right) \leq f\left(\frac{a+b}{2}\right)$$

$$ab\sin\left(\frac{1}{\sqrt{ab}}\right) \leq \frac{(a+b)^2}{4}\sin\left(\frac{2}{a+b}\right)$$

$\frac{\sin\left(\frac{2}{a+b}\right)}{\sin\left(\frac{1}{\sqrt{ab}}\right)} \geq \frac{4ab}{(a+b)^2}$ *(1). Analogous:* $\frac{\sin\left(\frac{2}{b+c}\right)}{\sin\left(\frac{1}{\sqrt{bc}}\right)} \geq \frac{4bc}{(b+c)^2}$ *(2);* $\frac{\sin\left(\frac{2}{c+a}\right)}{\sin\left(\frac{1}{\sqrt{ca}}\right)} \geq \frac{4ca}{(c+a)^2}$ *(3)*

By multiplying (1); (2); (3): $\frac{\sin\left(\frac{2}{a+b}\right)\sin\left(\frac{2}{b+c}\right)\sin\left(\frac{2}{c+a}\right)}{\sin\left(\frac{1}{\sqrt{ab}}\right)\sin\left(\frac{1}{\sqrt{bc}}\right)\sin\left(\frac{1}{\sqrt{ca}}\right)} \geq \left(\frac{8abc}{(a+b)(b+c)(c+a)}\right)^2$

SOLUTION G.095.

Let be $A = \begin{pmatrix} 1 & 1 & 1 \\ m_a & m_b & m_c \\ h_a & h_b & h_c \end{pmatrix}$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ m_a & m_b & m_c \\ h_a & h_b & h_c \end{vmatrix} = m_b h_c + m_a h_b + m_c h_a - m_b h_a - m_c h_b - m_a h_c =$$

$$\begin{matrix} 1 & 1 & 1 \\ m_a & m_b & m_c \end{matrix}$$

$$= m_a(h_b - h_c) + m_b(h_c - h_a) + m_c(h_a - h_b)$$

By Hadamard's inequality: $(\det A)^2 < (1^1 + 1^2 + 1^2)(m_a^2 + m_b^2 + m_c^2)(h_a^2 + h_b^2 + h_c^2)$

$$\left(m_a(h_b - h_c) + m_b(h_c - h_a) + m_c(h_a - h_b)\right)^2 < 3(m_a^2 + m_b^2 + m_c^2)(h_a^2 + h_b^2 + h_c^2) =$$

$$= 3 \cdot \frac{3}{4}(a^2 + b^2 + c^2)(h_a^2 + h_b^2 + h_c^2) = \frac{9}{4}(a^2 + b^2 + c^2)(h_a^2 + h_b^2 + h_c^2)$$

SOLUTION G.096.

If $x < y \Rightarrow x\sin\alpha + (1 - \sin\alpha)y = \sin\alpha \cdot (x - y) + y \leq y \leq \sqrt{x^2 + y^2}$

If $x > y \Rightarrow x\sin\alpha + (1 - \sin\alpha)\,y = (1 - \sin\alpha)(y - x) + x \leq x \leq \sqrt{x^2 + y^2}$

$$x\sin\alpha + (1 - \sin\alpha)\,y \leq \sqrt{x^2 + y^2} \quad (1)$$

Analogous: $z\sin\alpha + (1 - \sin\alpha)t \leq \sqrt{z^2 + t^2}$ *(2)*

By adding (1); (2): $(x + z)\sin\alpha + (1 - \sin\alpha)(y + t) \leq \sqrt{x^2 + y^2} + \sqrt{z^2 + t^2} \overset{AM-QM}{\leq}$

$$\leq 2\sqrt{\frac{\left(\sqrt{x^2 + y^2}\right)^2 + \left(\sqrt{z^2 + t^2}\right)^2}{2}} = \sqrt{2^2 \cdot \frac{x^2 + y^2 + z^2 + t^2}{2}} = \sqrt{2(x^2 + y^2 + z^2 + t^2)}$$

SOLUTION G.097.

First, we prove that if $x \leq y \leq z$ *then:* $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{y}{x} + \frac{z}{y} + \frac{x}{z}$ *(1)*

$$\frac{x}{y} - \frac{y}{x} + \frac{y}{z} - \frac{z}{y} + \frac{z}{x} - \frac{x}{z} \geq 0;\ \frac{x^2 - y^2}{xy} + \frac{y^2 - z^2}{2y} + \frac{z^2 - x^2}{xz} \geq 0$$

$$z(x^2 - y^2) + x(y^2 - z^2) + y(z^2 - x^2) \geq 0$$

$$x^2 z - zy^2 + xy^2 - xz^2 + yz^2 - yx^2 \geq 0$$

$$xz(x-z)+y^2(x-z)+y(z-x)(z+x)\geq 0$$

$$(x-z)(xz+y^2-yz-yx)\geq 0$$

$$(x-z)[y(y-x)-z(y-x)]\geq 0$$

$$(x-z)(y-x)(y-z)\geq 0$$

$$(z-x)(y-x)(z-y)\geq 0 \text{ which is true because, } z-x\geq 0; y-x\geq 0, z-y\geq 0$$

$$\text{By } a\leq b\leq c\Rightarrow m_a\geq m_b\geq m_c\Rightarrow \sqrt[5]{m_a}\leq \sqrt[5]{m_b}\leq \sqrt[5]{m_c}$$

$$\text{We take in (1): } x=\sqrt[5]{m_a}; y=\sqrt[5]{m_b}; z=\sqrt[5]{m_c}$$

$$\frac{\sqrt[5]{m_a}}{\sqrt[5]{m_b}}+\frac{\sqrt[5]{m_b}}{\sqrt[5]{m_c}}+\frac{\sqrt[5]{m_c}}{\sqrt[5]{m_a}}\geq \frac{\sqrt[5]{m_b}}{\sqrt[5]{m_a}}+\frac{\sqrt[5]{m_c}}{\sqrt[5]{m_b}}+\frac{\sqrt[5]{m_a}}{\sqrt[5]{m_c}}$$

$$\sqrt[5]{\frac{m_a}{m_b}}+\sqrt[5]{\frac{m_b}{m_c}}+\sqrt[5]{\frac{m_c}{m_a}}-\sqrt[5]{\frac{m_a}{m_c}}-\sqrt[5]{\frac{m_b}{m_a}}-\sqrt[5]{\frac{m_c}{m_b}}<1$$

SOLUTION G.098.

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=x^5; f'(x)=5x^4; f''(x)=20x^3>0\Rightarrow f$ *concexe*

By Jensen's inequality: $\sum am_a^5=2s\sum\left(\frac{a}{a+b+c}m_a^5\right)\overset{JENSEN}{\geq} 2s\sum\left(\frac{a}{2s}m_a\right)^5=\frac{1}{(2s)^4}\cdot\left(\sum am_a\right)^5$

$$\frac{\sum am_a^5}{(\sum am_a)^5}\geq\frac{1}{(2s)^4}\overset{MITRINOVIC}{\geq}\frac{1}{\left(2\cdot\frac{3\sqrt{3}}{2}R\right)^4}=\frac{1}{729R^4}$$

SOLUTION G.099.

$$a\leq b\leq c\Rightarrow h_a\geq h_b\geq h_c\Rightarrow (\exists)p,q\in\mathbb{R}$$

$$h_a=ph_c; h_a=qh_c\ \ (p\geq q\geq 1)$$

$$\text{Let be } f:(0,\infty)\to\mathbb{R}$$

$$f(p)=p^{20}-q^{20}+1-(p-q+1)^{20}\ \ (p\geq q\geq 1); q-\text{fixed}$$

$$f'(p)=20p^{19}-20(p-q+1)^{19}\geq 0$$

$$\text{because: } q\geq 1\Rightarrow 0\geq -q+1\Rightarrow$$

$$p\geq p-q+1\Rightarrow p^{19}\geq (p-q+1)^{19}\Rightarrow$$

$$20p^{19}-20(p-q+1)^{19}\geq 0\Rightarrow f'(p)\geq 0$$

$$f \text{ increasing; } p\geq q\Rightarrow f(p)\geq f(q)$$

$$p^{20}-q^{20}+1-(p-q+1)^{20}\geq$$

$$\geq p^{20}-p^{20}+1-(p-p+1)^{20}=0$$

$$p^{20} - q^{20} + 1 \geq (p - q + 1)^{20}$$

$$(ph_c)^{20} - (qh_c)^{20} + h_c^{20} \geq (ph_c - qh_c + h_c)^{20}$$

$$h_a^{20} - h_b^{20} + h_c^{20} \geq (h_a - h_b + h_c)^{20}$$

SOLUTION G.100.

$$4\left(\sum_{cyc} m_a(h_b - h_c)\right)^2 \overset{CBS}{\leq} 4\left(\sum_{cyc} m_a^2\right)\left(\sum_{cyc}(h_b - h_c)^2\right) =$$

$$= 4 \cdot \frac{3}{4}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc}(h_b - h_c)\right)^2 = 3\left(\sum_{cyc} a^2\right)\left(\sum_{cyc}(h_b - h_c)\right)^2$$

Remains to prove:

$$3\left(\sum_{cyc} a^2\right)\left(\sum_{cyc}(h_b - h_c)\right)^2 < 9\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} h_a^2\right)$$

$$\left(\sum_{cyc}(h_b - h_c)\right)^2 < 3\sum_{cyc} h_a^2$$

$$2\sum_{cyc} h_a^2 - 2\sum_{cyc} h_a h_b < 3\sum_{cyc} h_a^2$$

$$-2\sum_{cyc} h_a h_b \leq \sum_{cyc} h_a^2 \quad (1)$$

But: $-2\sum_{cyc} h_a h_b < 0; \sum_{cyc} h_a^2 > 0$

hence (1) is true.

SOLUTION G.101.

Let be $f: (0, \infty) \to \mathbb{R}; f(x) = \sqrt[5]{x} = x^{\frac{1}{5}};$

$$f'(x) = \frac{1}{5}x^{-\frac{4}{5}}; f''(x) = -\frac{4}{25}x^{-\frac{9}{5}} < 0$$

f concave. By Jensen's inequality:

$$\frac{1}{3}\sum_{cyc} f\left(\frac{2(s-a)}{c}\right) \leq f\left(\frac{1}{3}\sum_{cyc}\frac{2(s-a)}{c}\right)$$

$$LHS=\sum_{cyc} f\left(\frac{2(s-a)}{c}\right)\le 3f\left(\frac{1}{3}\sum_{cyc}\frac{2(s-a)}{c}\right)$$

Remains to prove:

$$3f\left(\frac{1}{3}\sum_{cyc}\frac{2(s-a)}{c}\right)\le 3$$

$$\sqrt[5]{\frac{\frac{2(s-a)}{c}+\frac{2(s-b)}{a}+\frac{2(s-c)}{b}}{3}}\le 1$$

$$\frac{2(s-a)}{c}+\frac{2(s-b)}{a}+\frac{2(s-c)}{b}\le 3$$

$$\frac{s-a}{c}+\frac{s-b}{a}+\frac{s-c}{b}\le\frac{3}{2}$$

$$\frac{b+c-a}{2c}+\frac{c+a-b}{2a}+\frac{a+b-c}{2b}\le\frac{3}{2}$$

WLOG: $a\ge b\ge c\Rightarrow a-c\ge 0; b-c\ge 0; b-a\le 0$ ***(1)***

$$\frac{b-a}{c}+\frac{1}{2}+\frac{c-b}{a}+\frac{1}{2}+\frac{a-c}{b}+\frac{1}{2}\le\frac{3}{2}$$

$$\frac{b-a}{c}+\frac{c-b}{a}+\frac{a-c}{b}\le 0,\ \frac{b}{c}-\frac{c}{b}+\frac{c}{a}-\frac{a}{c}+\frac{a}{b}-\frac{b}{a}\le 0$$

$$\frac{b^2-c^2}{bc}+\frac{c^2-a^2}{ca}+\frac{a^2-b^2}{ab}\le 0,\ a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\le 0$$

$$b^2a-ac^2+bc^2-ba^2+ca^2-cb^2\le 0,\ (a-c)(b^2-ba-cb+ac)\le 0$$

$$(a-c)(b-c)(b-a)\le 0 \text{ true by (1)}$$

SOLUTION G.102.

By Radon's generalized inequality:

$$\frac{a^8}{r_br_c}+\frac{b^8}{r_cr_a}+\frac{c^8}{r_ar_b}\ge 3^{2-8}\cdot\frac{(a+b+c)^8}{r_br_c+r_cr_a+r_ar_b}=$$

$$=\frac{1}{3^6}\cdot\frac{(2s)^8}{s^2}=\frac{2^8}{3^6}\cdot s^6\overset{MITRINOVIC}{\ge}$$

$$\ge\frac{256}{3^6}\cdot\left(3\sqrt{3}\right)^6r^6=\frac{256}{3^6}\cdot 3^6\cdot 3^3r^6=256\cdot 27r^6=6912r^6$$

SOLUTION G.103.

$$\frac{1}{64}\left(\frac{a^8r_b}{b^2}+\frac{b^8r_c}{c^2}+\frac{c^8r_a}{a^2}\right)=\frac{1}{64}\left(\frac{a^8}{\frac{b^2}{r_b}}+\frac{b^8}{\frac{c^2}{r_c}}+\frac{c^8}{\frac{a^2}{r_a}}\right)\overset{RADON}{\geq}$$

$$\geq\frac{1}{64}\cdot\frac{1}{3^6}\cdot\frac{(a+b+c)^8}{\frac{b^2}{r_b}+\frac{c^2}{r_c}+\frac{a^2}{r_a}}=\frac{1}{2^6\cdot 3^6}\cdot\frac{(2s)^8}{4(R+r)}=$$

$$=\frac{1}{2^8\cdot 3^6}\cdot\frac{2^8s^8}{R+r}\overset{MITRINOVIC}{\geq}\frac{\left(3\sqrt{3}\right)^8r^8}{3^6(R+r)}=\frac{3^{12}\cdot r^8}{3^6(R+r)}=\frac{3^6r^8}{R+r}=\frac{729r^8}{R+r}$$

SOLUTION G.104.

By Wolstenholme's inequality if $x_1,x_2,x_3\in\mathbb{R}$ then in ΔABC holds:

$$x_1^2+x_2^2+x_3^2\geq 2x_1x_2\cos A+2x_2x_3\cos B+2x_3x_1\cos C$$

For $x_1=\sqrt{r_cr_a};\ x_2=\sqrt{r_ar_b},\ x_3=\sqrt{r_br_c}$

$$r_cr_a+r_ar_b+r_br_c\geq 2r_a\sqrt{r_br_c}\cos A+2r_b\sqrt{r_cr_a}\cos B+2r_c\sqrt{r_ar_b}\cos C\quad (1)$$

It is known that $r_ar_b+r_br_c+r_cr_a=s^2$

By (1):

$$2\sqrt{r_ar_br_c}\left(\sqrt{r_a}\cos A+\sqrt{r_b}\cos B+\sqrt{r_c}\cos C\right)\leq s^2$$

$$\sqrt{r_a}\cos A+\sqrt{r_b}\cos B+\sqrt{r_c}\cos C\leq\frac{s^2}{2\sqrt{r_ar_br_c}}=$$

$$=\frac{s^2}{2\sqrt{rs^2}}=\frac{s}{2\sqrt{r}}\overset{MITRINOVIC}{\leq}\frac{\frac{3\sqrt{3}}{2}R}{2\sqrt{r}}=\frac{3\sqrt{3}R}{4\sqrt{r}}$$

SOLUTION G.105.

By Wolstenholme's inequality if $x_1,x_2,x_3\in\mathbb{R}$ then in ΔABC holds:

$$x_1^2+x_2^2+x_3^2\geq 2x_1x_2\cos A+2x_2x_3\cos B+2x_3x_1\cos C$$

For $x_1=\frac{b}{\sqrt{r_b}};\ x_2=\frac{c}{\sqrt{r_c}};\ x_3=\frac{a}{\sqrt{r_a}}$:

$$\frac{b^2}{r_b}+\frac{c^2}{r_c}+\frac{a^2}{r_a}\geq\frac{2bc\cos A}{\sqrt{r_br_c}}+\frac{2ca\cos B}{\sqrt{r_cr_a}}+\frac{2ab\cos C}{\sqrt{r_ar_b}}\quad (1)$$

It is known that:

$$\frac{a^2}{r_a}+\frac{b^2}{r_b}+\frac{c^2}{r_c}=4(R+r).\ \textit{By (1)}$$

$$\frac{2bc\cos A}{\sqrt{r_b r_c}}+\frac{2ca\cos B}{\sqrt{r_c r_a}}+\frac{2ab\cos C}{\sqrt{r_a r_b}}\le 4(R+r)$$

$$\frac{b^2+c^2-a^2}{\sqrt{r_b r_c}}+\frac{c^2+a^2-b^2}{\sqrt{r_c r_a}}+\frac{a^2+b^2-c^2}{\sqrt{r_a r_b}}\le 4(R+r)$$

SOLUTION G.106.

Let R_1, R_2, R_3 be the distances from I – incentre to vertex A, B, respectively C and r_1, r_2, r_3 the distances from I to sides BC, CA respectively AB.

By Gueron's inequality (AMM-2001):

$$\lambda_1 R_1+\lambda_2 R_2+\lambda_3 R_3\ge 2\left(\sqrt{\lambda_1\lambda_2}r_1+\sqrt{\lambda_2\lambda_3}r_2+\sqrt{\lambda_3\lambda_1}r_1\right)$$

$$\frac{\lambda_1 r}{\sin\frac{A}{2}}+\frac{\lambda_2 r}{\sin\frac{B}{2}}+\frac{\lambda_3 r}{\sin\frac{C}{2}}\ge 2\left(\sqrt{\lambda_1\lambda_2}r+\sqrt{\lambda_2\lambda_3}r+\sqrt{\lambda_3\lambda_1}r\right)$$

$$\frac{\lambda_1}{\sin\frac{A}{2}}+\frac{\lambda_2}{\sin\frac{B}{2}}+\frac{\lambda_3}{\sin\frac{C}{2}}\ge 2\left(\sqrt{\lambda_1\lambda_2}+\sqrt{\lambda_2\lambda_3}+\sqrt{\lambda_3\lambda_1}\right)$$

For $\lambda_1=\frac{1}{a^2}; \lambda_2=\frac{1}{b^2}; \lambda_3=\frac{1}{c^2}$

$$\frac{1}{a^2\sin\frac{A}{2}}+\frac{1}{b^2\sin\frac{B}{2}}+\frac{1}{c^2\sin\frac{C}{2}}\ge 2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=\frac{2(a+b+c)}{abc}=\frac{4s}{abc}$$

Equality holds for $a=b=c$.

SOLUTION G.107.

Let R_1, R_2, R_3 be the distances from G – centroid to vertex A, B, respectively C and r_1, r_2, r_3 the distances from G to sides BC, CA respectively AB. By Gueron's inequality (AMM-2001):

$$\lambda_1 R_1+\lambda_2 R_2+\lambda_3 R_3\ge 2\left(\sqrt{\lambda_1\lambda_2}r_1+\sqrt{\lambda_2\lambda_3}r_2+\sqrt{\lambda_3\lambda_1}r_3\right)$$

$$\lambda_1\cdot\frac{2}{3}m_a+\lambda_2\cdot\frac{2}{3}m_b+\lambda_3\cdot\frac{2}{2}m_c\ge 2\left(\sqrt{\lambda_1\lambda_2}\cdot\frac{h_a}{3}+\sqrt{\lambda_2\lambda_3}\cdot\frac{h_b}{3}+\sqrt{\lambda_3\lambda_1}\cdot\frac{h_c}{3}\right)$$

$$\lambda_1 m_a+\lambda_2 m_b+\lambda_3 m_c\ge\sqrt{\lambda_1\lambda_2}h_a+\sqrt{\lambda_2\lambda_3}h_b+\sqrt{\lambda_3\lambda_1}h_c$$

For $\lambda_1=\frac{1}{\sqrt{b}}; \lambda_2=\frac{1}{\sqrt{c}}; \lambda_3=\frac{1}{\sqrt{a}}$

$$\frac{m_a}{\sqrt{b}}+\frac{m_b}{\sqrt{c}}+\frac{m_c}{\sqrt{c}}\ge\frac{h_a}{\sqrt[4]{bc}}+\frac{h_b}{\sqrt[4]{ca}}+\frac{h_c}{\sqrt[4]{ab}}$$

Equality holds for $a=b=c$.

SOLUTION G.108.

Let R_1, R_2, R_3 be the distances from I – incentre to vertex A, B respectively C and r_1, r_2, r_3 the distances from I to BC, CA respectively AB. By Gueron's inequality (AMM-2001):

$$\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 \geq 2\left(\sqrt{\lambda_1\lambda_2}r_1 + \sqrt{\lambda_2\lambda_3}r_2 + \sqrt{\lambda_3\lambda_1}r_3\right)$$

$$\frac{\lambda_1 r}{\sin\frac{A}{2}} + \frac{\lambda_2 r}{\sin\frac{B}{2}} + \frac{\lambda_3 r}{\sin\frac{C}{2}} \geq 2\left(\sqrt{\lambda_1\lambda_2}r + \sqrt{\lambda_2\lambda_3}r + \sqrt{\lambda_3\lambda_1}r\right)$$

$$\frac{\lambda_1}{\sin\frac{A}{2}} + \frac{\lambda_2}{\sin\frac{B}{2}} + \frac{\lambda_3}{\sin\frac{C}{2}} \geq 2\left(\sqrt{\lambda_1\lambda_2} + \sqrt{\lambda_2\lambda_3} + \sqrt{\lambda_3\lambda_1}\right)$$

For $\lambda_1 = x; \lambda_2 = y; \lambda_3 = z$: $\frac{x}{\sin\frac{A}{2}} + \frac{y}{\sin\frac{B}{2}} + \frac{z}{\sin\frac{C}{2}} \geq 2\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right)$

Equality holds for $x = y = z; A = B = C$.

SOLUTION G.109.

It is known that: $\frac{a}{m_b^2 m_c^2} + \frac{b}{m_c^2 m_a^2} + \frac{c}{m_a^2 m_b^2} = \frac{2(4R+r)}{rs^3}$ ***(1)***

By Wolstenholme's inequality:

$$2x_1x_2\cos A + 2x_2x_3\cos B + 2x_3x_1\cos C \leq x_1^2 + x_2^2 + x_3^2 \quad (2)$$

$x_1, x_2, x_3 \in \mathbb{R}$ in any triangle ABC.

We take: $x_1 = \frac{\sqrt{b}}{m_c m_a}; x_2 = \frac{\sqrt{c}}{m_a m_b}; x_3 = \frac{\sqrt{a}}{m_b m_c}$ ***in (2):***

$$\frac{2\sqrt{b}\cdot\sqrt{c}\cos A}{m_a^2 m_b m_c} + \frac{2\sqrt{c}\cdot\sqrt{a}\cos B}{m_a m_b^2 m_c} + \frac{2\sqrt{a}\cdot\sqrt{b}\cos C}{m_a m_b m_c^2} \leq$$

$$\leq \frac{a}{m_b^2 m_c^2} + \frac{b}{m_c^2 m_a^2} + \frac{c}{m_a^2 m_b^2} \overset{(1)}{=} \frac{2(4R+r)}{rs^3}$$

$$\left(\frac{2\sqrt{bc}\cos A}{m_a} + \frac{2\sqrt{ca}\cos B}{m_b} + \frac{2\sqrt{ab}\cos C}{m_c}\right)\cdot\frac{1}{m_a m_b m_c} \leq \frac{2(4R+r)}{rs^3}$$

$$\frac{\sqrt{bc}\cos A}{m_a} + \frac{\sqrt{ca}\cos B}{m_b} + \frac{\sqrt{ab}\cos C}{m_c} \leq \frac{(4R+r)m_a m_b m_c}{rs^3}$$

SOLUTION G.110.

$$s^3 = s\cdot s^2 \overset{MITRINOVIC}{\geq} 3\sqrt{3}rs^2 \overset{GERRETSEN}{\geq} 3\sqrt{3}r\cdot(16Rr - 5r^2) = 3\sqrt{3}r^2(16R - 5r)$$

It remains to prove: $\frac{3\sqrt{3}r^2(4R+r)^3}{(2R-r)(2R+5r)} \leq 3\sqrt{3}r^2(16R - 5r)$

$(4R+r)^3 \leq (16R-5r)(2R-r)(2R+5r)$, $(4t+1)^3 \leq (16t-5)(2t-1)(2t+5)$

$$t=\frac{R}{r}\geq 2\ \textit{(EULER)},\ (4t+1)^3\leq(32t^2-26t+5)(2t+5)$$

$$64t^3+48t^2+12t+1\leq 64t^3+160t^2-52t^2-130t+10t+25$$

$$60t^2-132t+24\geq 0,\ 10t^2-22t+4\geq 0,\ 5t^2-11t+2\geq 0$$

$$5t^2-10t-t+2\geq 0,\ (t-2)(5t-1)\geq 0 \textit{ true because } t\geq 2$$

SOLUTION G.111.

By Bernoulli's inequality:

$$(1+x)^\alpha<1+\alpha x; x>-1; 0<\alpha<1$$

For $x=p-1; \alpha=1-q$

$$(1+p-1)^{1-q}<1+(1-q)(p-1)=1+p-1-pq+q$$

$$p^{1-q}<p+q-pq<p+q\Rightarrow\frac{p}{pq}<p+q$$

$$p^q>\frac{p}{p+q}\quad (1)$$

By (1):

$$(\sin x)^{\sin y+\sin z}>\frac{\sin x}{\sin x+\sin y+\sin z}\quad (2)$$

$$(\sin y)^{\sin z+\sin x}>\frac{\sin y}{\sin x+\sin y+\sin z}\quad (3)$$

$$(\sin z)^{\sin x+\sin y}>\frac{\sin z}{\sin x+\sin y+\sin z}\quad (4)$$

By adding (2); (3); (4):

$$\sum_{cyc}(\sin x)^{\sin y+\sin z}>1\Rightarrow\sum_{cyc}(\sin x)^{2\sin\frac{y+z}{2}\cos\frac{y-z}{2}}>1$$

$$\Rightarrow\sum_{cyc}(\sin^2 x)^{\sin\frac{y+z}{2}\cos\frac{y-z}{2}}>1$$

SOLUTIONS

ANALYSIS

SOLUTION AN.001.

Let be: $p=\frac{\frac{1}{3}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{2}\right)^2}};q=\frac{\frac{1}{2}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{2}\right)^2}};p^2+q^2=1.$ *Exists* $t\in\left(0,\frac{\pi}{2}\right)$ *such that*

$$p=\sin t\,;q=\cos t$$

$$x_{n+1}=\frac{1}{3}\cos x_{n-1}+\frac{1}{2}\sin x_{n-1}$$

$$x_{n+1}=\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{2}\right)^2}\left(\frac{\frac{1}{3}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{2}\right)^2}}\cos x_{n-1}+\frac{\frac{1}{2}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{2}\right)^2}}\sin x_{n-1}\right)$$

$$x_{n+1}=\sqrt{\frac{1}{9}+\frac{1}{4}}(\sin t\cos x_{n-1}+\cos t\sin x_{n-1})$$

$$|x_{n+1}|=\sqrt{\frac{13}{36}}|\sin(t+x_{n-1})|<\sqrt{\frac{13}{36}}$$

$$|x_1x_2\cdot\ldots\cdot x_n|=|x_1|\cdot|x_2|\cdot\ldots\cdot|x_n|<\left(\sqrt{\frac{13}{36}}\right)^n$$

$$0<|x_1x_2\ldots x_n|<\left(\sqrt{\frac{13}{36}}\right)^n$$

$$\Omega=\lim_{n\to\infty}(x_1\cdot x_2\cdot\ldots\cdot x_n)=0$$

SOLUTION AN.002.

Let be $f:\mathbb{R}\to\mathbb{R};a,b\in\mathbb{R};a<b;f$ *continuous,* f *periodical with* $T>0$ *period. Its known that:* $\lim_{n\to\infty}\int_a^b f(nx)\,dx=\frac{b-a}{T}\int_a^T f(x)\,dx$ *(1)*

We take $f(x)=x-[x];f:\mathbb{R}\to\mathbb{R}$

$$f(x+1)=|[x+1]-(x+1)|=|[x]+1-x-1|=|[x]-x|=f(x)$$

Replace f and T in (1):

$$\Omega = \lim_{n\to\infty}\int_0^{2018}(x-[x])\,dx = \frac{2018-0}{1}\int_0^1 (x-[x])\,dx =$$

$$= 2018\left(\frac{x^2}{2}\Big|_0^1 - 0\right) = \frac{2018}{2} = 1009$$

SOLUTION AN.003.

Let be $f:\mathbb{R}\to\mathbb{R}; a,b\in\mathbb{R}; a<b;$ ***f continuous; f periodical with*** $T>0$ ***period.***

Its known that: $\lim_{n\to\infty}\int_a^b f(nx)\,dx = \frac{b-a}{T}\int_a^T f(x)\,dx$ ***(1)***

We take $f(x) = \left|\left[x+\frac{1}{2}\right]-x\right|; f:\mathbb{R}\to\mathbb{R}$

$$f(x+1) = \left|\left[x+1+\frac{1}{2}\right]-(x+1)\right| = \left|\left[x+\frac{1}{2}\right]+1-x-1\right| = \left|\left[x+\frac{1}{2}\right]-x\right| = f(x)$$

Replace f and T in (1):

$$\Omega = \lim_{n\to\infty}\int_0^{2018}\left|\left[nx+\frac{1}{2}\right]-nx\right|dx = \frac{2018-0}{1}\int_0^1\left|\left[x+\frac{1}{2}\right]-x\right|dx =$$

$$= 2018\left(\int_0^{\frac{1}{2}}|0-x|dx + \int_{\frac{1}{2}}^1|1-x|\,dx\right) = 2018\left(\int_0^{\frac{1}{2}}x\,dx + \int_{\frac{1}{2}}^1(1-x)\,dx\right) =$$

$$= 2018\left(\frac{x^2}{2}\Big|_0^{\frac{1}{2}} + x\Big|_{\frac{1}{2}}^1 - \frac{x^2}{2}\Big|_{\frac{1}{2}}^1\right) = 2018\left(\frac{1}{8}+1-\frac{1}{2}-\frac{1}{2}+\frac{1}{8}\right) = \frac{2018}{4} = \frac{1009}{2}$$

SOLUTION AN.004.

$$\sin 2x \le 1 \Rightarrow \sin x\cos x \le \tfrac{1}{2} \Rightarrow \sin^2 x\cos^2 x \le \tfrac{1}{4} \quad \textit{(1)}$$

$$\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x) =$$

$$= 1 - 3\sin^2 x\cos^2 x \overset{(1)}{\ge} 1 - \tfrac{3}{4} = \tfrac{1}{4} \quad \textit{(2)}$$

$$(\sin^2 x + \csc^2 x)^3 + (\cos^2 x + \sec^2 x)^3 =$$

$$= \sin^6 x + \cos^6 x + 3\left(\sin^2 x + \frac{1}{\sin^2 x}\right) + 3\left(\cos^2 x + \frac{1}{\cos^2 x}\right) + \frac{1}{\sin^6 x} + \frac{1}{\cos^6 x} \overset{(2)}{\ge}$$

$$\ge \frac{1}{4} + 3 + 3\left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}\right) + \left(\frac{1}{\sin^6 x} + \frac{1}{\cos^6 x}\right) \overset{AM-GM}{\ge}$$

$$\geq \frac{13}{4}+3\cdot\frac{2}{\sin x\cos x}+\frac{\sin^6 x+\cos^6 x}{\sin^6 x\cos^6 x}\overset{(1),(2)}{\geq}\frac{13}{4}+\frac{6}{\frac{1}{2}}+\frac{\frac{1}{4}}{\left(\frac{1}{4}\right)^3}=\frac{13}{4}+12+16=\frac{125}{4}$$

$$4((\sin^2 x+\csc^2 x)^3+(\cos^2 x+\sec^2 x)^3)\geq 125$$

$$4\int_a^b((\sin^2 x+\csc^2 x)^3+(\cos^2 x+\sec^2 x)^3)\,dx\geq\int_a^b 125\,dx=125(b-a)$$

Equality holds for $a=b$.

SOLUTION AN.005.

$$Let\ a_n=\sum_{k=0}^n\frac{3^{n-k-1}(4n-4k-1)}{n-k+1}\binom{n+1}{k+1}$$

$$=\sum_{k=0}^n\frac{3^{n-k-1}(4n-4k+4-5)}{n-k+1}\binom{n+1}{k+1}$$

$$=\frac{4}{3}\sum_{k=0}^n 3^{(n+1)-(k+1)}\binom{n+1}{k+1}-\frac{5}{9}\sum_{k=0}^n 3^{n-k+1}\frac{1}{n-k+1}\binom{n+1}{k+1}$$

$$=\frac{4}{3}[(1+3)^{n+1}-3^{n+1}]-\frac{5}{9}\sum_{k=0}^n\int_0^3 x^{n-k}\binom{n+1}{k+1}dx$$

$$=\frac{4}{3}(4^{n+1}-3^{n+1})-\frac{5}{9}\int_0^3\left[\sum_{k=0}^n\binom{n+1}{k+1}x^{n+1-(k+1)}\right]dx$$

$$=\frac{4}{3}(4^{n+1}-3^{n+1})-\frac{5}{9}\int_0^3[(1+x)^{n+1}-x^{n+1}]\,dx$$

$$=\frac{4}{3}(4^{n+1}-3^{n+1})-\frac{5}{9(n+2)}[(4^{n+2}-1)-3^{n+2}]$$

$$=\left[\frac{1}{3}-\frac{5}{9(n+2)}\right]4^{n+2}-\left(4-\frac{5}{n+2}\right)3^n+\frac{5}{9(n+2)}$$

$$a_n=\frac{(3n+1)4^{n+2}}{9(n+2)}-\frac{(4n+3)}{n+2}(3^n)+\frac{5}{9(n+2)}$$

$$a_n\to\infty\ as\ n\to\infty$$

SOLUTION AN.006.

$$\textit{If } a>0 \textit{ then: } \lim_{x\to a}\frac{x^x-a^2}{x-a}\overset{L'H}{=}\lim_{x\to a}\left(\frac{x\cdot x'\cdot x^{x-1}+x'\cdot x^x\cdot\ln x}{1}\right)=\lim_{x\to a}x^x(1+\ln x)=$$

$$a^a(1+\ln a)\quad (1)$$

$$\textit{Let be } a_n=\sum_{k=1}^{n}\frac{1}{k^2};\ \lim_{n\to\infty}a_n=\frac{\pi^2}{6}$$

$$\Omega=\lim_{n\to\infty}n\left((a_n)^{a_n}-\left(\frac{\pi^2}{6}\right)^{\frac{\pi^2}{6}}\right)=\lim_{n\to\infty}\frac{(a_n)^{a_n}-\left(\frac{\pi^2}{6}\right)^{\frac{\pi^2}{6}}}{a_n-\frac{\pi}{6}}\cdot\frac{a_n-\frac{\pi}{6}}{\frac{1}{n}}\overset{(1)}{=}$$

$$=\left(\frac{\pi^2}{6}\right)^{\frac{\pi^2}{6}}\cdot\left(1+\ln\frac{\pi^2}{6}\right)\cdot\lim_{n\to\infty}\frac{a_{n+1}-a_n}{\frac{1}{n+1}-\frac{1}{n}}=$$

$$=\left(\frac{\pi^2}{6}\right)^{\frac{\pi^2}{6}}\cdot\left(1+\ln\frac{\pi^2}{6}\right)\cdot\lim_{n\to\infty}\frac{1}{(n+1)^2}\cdot\frac{+n(n+1)}{-1}=-\left(\frac{\pi^2}{2}\right)^{\frac{\pi^2}{6}}\cdot\left(1+\ln\frac{\pi^2}{6}\right)$$

SOLUTION AN.007.

$$si\,(x)=-\int_x^\infty\left(\frac{\sin t}{t}\right)dt;\ si'(x)=-\left[\lim_{x\to\infty}\frac{\sin x}{x}-\frac{\sin x}{x}\right]=\frac{\sin x}{x}$$

$$\int_\gamma^e\left(\frac{1}{x}(si(e^2x)-\sin(\pi x))\right)dx=\int_\gamma^e\left(\int_\pi^{e^2}si'(tx)dt\right)dx=\int_\pi^{e^2}\left(\int_\gamma^e si'(tx)dx\right)dt=$$

$$=\int_\pi^{e^2}\left(\frac{si\,(tx)}{t}\Big|_\gamma^e\right)dt=\int_\pi^{e^2}\frac{si\,(et)-si\,(\gamma t)}{t}dt=\int_\pi^{e^2}\frac{si\,(ex)-si(\gamma x)}{x}dx=$$

$$=\int_\pi^{e^2}\left(\frac{1}{x}(si(ex)-si\,(\gamma x)\right)dx$$

SOLUTION AN.008.

$$\Omega=\frac{3abcdefxyz-1}{(a^2+b^2+x^2+3)(c^2+d^2+y^2+3)(e^2+f^2+z^2+3)}\overset{AM-GM}{\leq}$$

$$\leq\frac{3abcdefxyz-1}{6\sqrt[6]{a^2b^2x^2}\cdot6\sqrt[6]{c^2d^2y^2}\cdot6\sqrt[6]{e^2f^2z^2}}=\frac{3abcdefxyz-1}{216\sqrt[3]{abcdefxyz-1}}=\frac{3\alpha^3-1}{216\alpha};$$

$$\alpha=abcdefxyz\in[0,1]$$

$$108\Omega \le 1 \Leftrightarrow 108 \cdot \frac{3\alpha^3 - 1}{216\alpha} \le 1 \Leftrightarrow 3\alpha^3 - 1 \le 2\alpha$$

$$3\alpha^3 - 3\alpha + \alpha - 1 \le 0 \Leftrightarrow 3\alpha(\alpha^2 - 1) + (\alpha - 1) \le 0$$

$$\Leftrightarrow (\alpha - 1)(3\alpha^2 + 3\alpha + 1) \le 0 \Leftrightarrow \alpha - 1 \le 0 \Leftrightarrow \alpha \le 1 \text{ which is true.}$$

$$108\int_0^1\int_0^1\int_0^1 \Omega dxdydz \le \int_0^1\int_0^1\int_0^1 dxdydz = 1$$

SOLUTION AN.009.

$$\ln e^{2\sqrt{3}(a+b+c)} \ge \ln\left((a^2 + a + 1)(b^2 + b + 1)(c^2 + c + 1)\right)^3$$

$$2\sqrt{3}(a + b + c) \ge 3(\ln(a^2 + a + 1) + \ln(b^2 + b + 1) + \ln(c^2 + c + 1))$$

$$\text{Let be } f: [0, \infty) \to \mathbb{R}; f(x) = 3\ln(x^2 + x + 1) - 2\sqrt{3}x$$

$$f'(x) = 3 \cdot \frac{2x + 1}{x^2 + x + 1} - 2\sqrt{3} = \frac{6x + 3 - 2\sqrt{3}(x^2 + x + 1)}{x^2 + x + 1}$$

$$f'(x) = \frac{-2\sqrt{3}x^2 + x(6 - 2\sqrt{3}) + 3 - 2\sqrt{3}}{(x^2 + x + 1)}$$

$$\Delta = \left(6 - 2\sqrt{3}\right)^2 + 4 \cdot 2\sqrt{3}\left(3 - 2\sqrt{3}\right) = 36 + 12 - 24\sqrt{3} + 24\sqrt{3} - 48 = 0$$

$$x_1 = x_2 = \frac{2\sqrt{3} - 6}{2\left(-2\sqrt{3}\right)} = \frac{2\sqrt{3} - 6}{-12} = \frac{3 - \sqrt{3}}{6}$$

$$f'(x) = \frac{-2\sqrt{3}\left(x - \frac{3 - \sqrt{3}}{6}\right)^2}{x^2 + x + 1} \le 0$$

$$\Rightarrow \max f(x) = f(0) = 0 \Rightarrow f(x) \ge 0 \Rightarrow f(a) \ge 0 \Rightarrow \sum f(a) \ge 0$$

$$\sum 3\ln(a^2 + a + 1) \ge 2\sqrt{3}\sum a$$

SOLUTION AN.010.

$$f(x) = x - \sin x\,; g(x) = \sin x + \cos x + 4$$

$$f'(x) = 1 - \cos x\,; g'(x) = \cos x - \sin x$$

$$f'(x)g(x) - f(x)g'(x) = (1 - \cos x)(\sin x + \cos x + 4) - (x - \sin x)(\cos x - \sin x) =$$

$$= \sin x + \cos x + 4 - 4\cos x \sin x - \cos^2 x - 4\cos x - x\cos x + x\sin x +$$

$$+ \sin x \cos x - \sin^2 x = \sin x + \cos x + 3 - 4\cos x + x(\sin x - \cos x) =$$

$$= (\sin x - \cos x)(1 + x) + 3 - 2\cos x$$

$$\Omega = \int \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}dx = \int \left(\frac{f(x)}{g(x)}\right)' dx =$$

$$= \frac{f(x)}{g(x)} + c = \frac{x - \sin x}{\sin x + \cos x + 4} + c$$

SOLUTION AN.011.

$$f(x) = x - \sin x\,;\, g(x) = \sin x + \cos x + 4$$

$$f'(x) = 1 - \cos x\,;\, g'(x) = \cos x - \sin x$$

$$f'(x)g(x) - f(x)g'(x) = (\sin x - \cos x)(1 + x) + 3 - 2\cos x$$

$$\Omega = \int \frac{f'(x)g(x) - f(x)g'(x)}{f(x)g(x)}dx =$$

$$= \int \frac{f'(x)}{f(x)}dx - \int \frac{g'(x)}{g(x)}dx = \ln|f(x)| - \ln|g(x)| + c =$$

$$= \ln\left|\frac{f(x)}{g(x)}\right| + c = \ln\left|\frac{x - \sin x}{\sin x + \cos x + 4}\right| + c$$

SOLUTION AN.012.

$$\left|\frac{1}{e^{2x^2}} - \int_0^1 \frac{dy}{e^{2y^2}}\right| = \left|\int_0^1 \left(\frac{1}{e^{2x^2}} - \frac{1}{e^{2y^2}}\right)dy\right| \le \int_0^1 \left|\frac{1}{e^{2x^2}} - \frac{1}{e^{2y^2}}\right|dy \overset{Lagrange}{=}$$

$$= \int_0^1 \left|-4ce^{-2c^2}\right| \cdot |x - y|dy \le 4\int_0^1 \left|ce^{-2c^2}\right|dy\,; |x - y| \le 1; c \in (0, 1)$$

$$\max_{c \in (0,1)} \left|4ce^{-2c^2}\right| = \left|-\frac{2}{\sqrt{e}}\right| = \frac{2\sqrt{e}}{e}$$

$$\left|\frac{1}{2x^2} - \int_0^1 \frac{dy}{e^{2y^2}}\right| \le \frac{2\sqrt{e}}{e}$$

SOLUTION AN.013.

$$[e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + 72]' =$$

$$= -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + e^{-x}(4x^3 + 12x^2 + 24x + 24) = -e^{-x}x^4 \quad (1)$$

$$\Omega = \int \frac{x^4 e^x dx}{e^{2x}(e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + 72)^2} =$$

$$= -\int \frac{-x^4 e^{-x} dx}{(e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + 72)^2} =$$

$$\overset{(1)}{=}-\int\frac{[e^{-x}(x^4+4x^3+12x^2+24x+24)+72]'}{[e^{-x}(x^4+4x^3+12x^2+24x+24)+72]^2}dx=$$

$$=\frac{1}{e^{-x}(x^4+4x^3+12x^2+24x+24)+72}+C$$

SOLUTION AN.014.

$$Let\ be\ g:\mathbb{R}\to\mathbb{R};\ g(x)=f(x)-\frac{x^3}{3}$$

$$g(x)+g(y)=f(x)+f(y)-\frac{x^3}{3}-\frac{y^3}{3}=f(x+y)-xy(x+y)-\frac{x^3}{3}-\frac{y^3}{3}=$$

$$=f(x+y)-\frac{x^3+3x^2y+3xy^2+y^3}{3}=f(x+y)-\frac{(x+y)^3}{3}=g(x+y)$$

$$g(x+y)=g(x)+g(y)$$

$$g\ continuous\Rightarrow g(x)=ax;\ a\in\mathbb{R}^*$$

$$f(x)=g(x)+\frac{x^3}{3}=ax+\frac{x^3}{3};$$

SOLUTION AN.015.

$$\frac{(x+y)^2+1}{xy+(x+y)\sqrt{3}}\overset{AM-GM}{\geq}\frac{(x+y)^2+1}{\left(\frac{x+y}{2}\right)^2+(x+y)\sqrt{3}}\geq 1\Leftrightarrow$$

$$\Leftrightarrow(x+y)^2+1\geq\frac{(x+y)^2}{4}+(x+y)\sqrt{3}$$

$$\frac{3(x+y)^2}{4}-(x+y)\sqrt{3}+1\geq 0$$

$$\left(\frac{\sqrt{3}(x+y)}{2}-1\right)^2\geq 0$$

$$\Omega(a)=\int_a^{2a}\int_a^{2a}\frac{(x+y)^2+1}{xy+(x+y)\sqrt{3}}dxdy\geq\int_a^{2a}\int_a^{2a}dxdy=(2a-a)(2a-a)=a^2$$

$$\Omega(a)+\Omega(b)+\Omega(c)\geq a^2+b^2+c^2\geq ab+bc+ca$$

SOLUTION AN.016.

$$\Omega(a)=\lim_{n\to\infty}n\int_0^1\frac{x^n}{(x+a)^2}dx=\frac{1}{(1+a)^2}$$

$$\sum_{cyc(a,b,c)} b^3\Omega(a) = \sum_{cyc(a,b,c)} \frac{b^3}{(1+a)^2} \overset{RADON}{\geq} \frac{(a+b+c)^3}{(1+a+1+b+1+c)^2} = \frac{2^3}{(3+2)^2} = \frac{8}{25}$$

SOLUTION AN.017.

Let be $f:(0,\infty)\to\mathbb{R}; f(x) = \frac{\log(ax+1)}{\log(bx+1)}$

$$f'(x) = \frac{a(bx+1)\log(bx+1) - b(ax+1)\log(ax+1)}{(ax+1)(bx+1)\log^2(bx+1)}$$

$$\textit{Let be } g:(0,\infty)\to\mathbb{R};$$

$$g(x) = a(bx+1)\log(bx+1) - b(ax+1)\log(ax+1)$$

$$g'(x) = ab\log\left(\frac{bx+1}{ax+1}\right) > 0 \textit{ because } b \geq a$$

$$g'(x) \geq g(0) = 0 \Rightarrow f'(x) \geq 0 \Rightarrow f \textit{ increasing}$$

$$\Rightarrow f(a) \leq f(b) \quad (\textit{because } a \leq b)$$

$$\frac{\log(a^2+1)}{\log(ab+1)} \leq \frac{\log(ab+1)}{\log(b^2+1)} \Rightarrow \frac{\log(a^2+1)}{\log\left(\frac{1}{c}+1\right)} \leq \frac{\log\left(\frac{1}{c}+1\right)}{\log(b^2+1)}$$

$$\textit{Analogous: } \frac{\log(a^2+1)}{\log\left(\frac{1}{b}+1\right)} \leq \frac{\log\left(\frac{1}{b}+1\right)}{\log(c^2+1)}$$

$$\textit{By multiplying: } \frac{\log^2(a^2+1)}{\log\left(\frac{b+1}{b}\right)\log\left(\frac{c+1}{c}\right)} \leq \frac{\log\left(\frac{b+1}{b}\right)\log\left(\frac{c+1}{c}\right)}{\log(b^2+1)\log(c^2+1)}$$

SOLUTION AN.018.

$$\textit{We prove that: } \tan x + 2\tan 2x + 4\tan 4x = \cot x - 8\cot 8x$$

$$\cot x - \tan x - 2\tan 2x - 4\tan 4x = \frac{1}{\tan x} - \tan x - 2\tan 2x - 4\tan 4x =$$

$$= \frac{1-\tan^2 x}{\tan x} - 2\tan 2x - 4\tan 4x = 2\left(\frac{1-\tan^2 x}{2\tan x} - \tan 2x\right) - 4\tan 4x =$$

$$= 2(\cot 2x - \tan 2x) - 4\tan 4x = 2\left(\frac{1}{\tan 2x} - \tan 2x\right) - 4\tan 4x =$$

$$= 2\left(\frac{1-\tan^2 2x}{\tan 2x}\right) - 4\tan 4x = 4\left(\frac{1-\tan^2 2x}{2\tan 2x} - \tan 4x\right) =$$

$$= 4(\cot 4x - 4\tan 4x) = 4(1 - \tan 4x) = 4\left(\frac{1}{\tan 4x} - \tan 4x\right) = 4\cdot\frac{1-\tan^2 4x}{\tan 4x} =$$

$$= 8 \cdot \frac{1 - \tan^2 4x}{2 \tan 4x} = 8 \cot 8x$$

$$\Omega = \int \frac{x(\tan x + 2\tan 2x + 4\tan 4x)}{\cot x - 8\cot 8x} dx = \int x\, dx = \frac{x^2}{2} + C$$

SOLUTION AN.019.

$$\Omega = \lim_{n\to\infty}\left(1 + \sum_{k=1}^{n}\frac{1}{k^2} - \frac{\pi^2}{6}\right)^{\frac{1}{\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}}\cdot\frac{\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}}{\frac{1}{n}}} = e^{\lim\limits_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}}{\frac{1}{n}}} = e^{\lim\limits_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n}-\frac{1}{n+1}}} =$$

$$= e^{\lim\limits_{n\to\infty}\frac{-n(n+1)}{(n+1)^2}} = e^{-1} = \frac{1}{e}$$

SOLUTION AN.020.

$$\Omega = \lim_{n\to\infty}\left(1 + \sum_{k=1}^{n}\sin\frac{1}{n+k} - \log 2\right)^{\frac{1}{\sum_{k=1}^{n}\sin\frac{1}{n+k}-\log 2}\cdot\frac{\sum_{k=1}^{n}\sin\frac{1}{n+k}-\log 2}{\frac{1}{n}}} =$$

$$= e^{\lim\limits_{n\to\infty}\frac{\sum_{k=1}^{n}\sin\frac{1}{n+k}-\log 2}{\frac{1}{n}}} = e^{\lim\limits_{n\to\infty}\frac{\sum_{k=1}^{n+1}\sin\frac{1}{n+1+k}-\sum_{k=1}^{n}\sin\frac{1}{n+k}}{\frac{1}{n+1}-\frac{1}{n}}} =$$

$$= e^{\lim\limits_{n\to\infty}\frac{-\sin\frac{1}{n+1}+\sin\frac{1}{2n+1}+\sin\frac{1}{2n+2}}{\frac{-1}{n(n+1)}}} =$$

$$= e^{\lim\limits_{n\to\infty}\left(\frac{\sin\frac{1}{n+1}}{\frac{1}{n+1}} - \frac{\sin\frac{1}{2n+1}}{\frac{1}{2n+1}}\cdot\frac{\frac{1}{n+1}}{\frac{1}{n+1}} - \frac{\sin\frac{1}{2n+2}}{\frac{1}{2n+2}}\cdot\frac{1}{\frac{1}{n+1}}\right)n} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

SOLUTION AN.021.

$$\left(\sqrt{f^2(x)+f^2(y)} + \sqrt{2f(x)f(y)}\right)^2 \overset{CBS}{\leq}$$

$$\leq (1^2+1^2)\left(\left(\sqrt{f^2(x)+f^2(y)}\right)^2 + \left(\sqrt{2f(x)f(y)}\right)^2\right) =$$

$$= 2\left(f^2(x)+f^2(y)+2f(x)f(y)\right) = 2\big(f(x)+f(y)\big)^2$$

$$\sqrt{f^2(x)+f^2(y)} + \sqrt{2f(x)f(y)} \leq \sqrt{2}\big(f(x)+f(y)\big)$$

$$\int_a^a\int_a^a\left(\sqrt{f^2(x)+f^2(y)} + \sqrt{2f(x)f(y)}\right)dxdy =$$

$$=\int_a^a\int_a^a\sqrt{f^2(x)+f^2(y)}\,dx\,dy+\int_a^a\int_a^a\sqrt{2f(x)f(y)}\,dx\,dy\le$$

$$\le\sqrt{2}\int_a^a\int_a^a\big(f(x)+f(y)\big)\,dx\,dy=\sqrt{2}\int_a^a\int_a^a f(x)\,dx\,dy+\sqrt{2}\int_a^a\int_a^a f(y)\,dx\,dy=$$

$$=\sqrt{2}a\int_a^a f(x)\,dx+\sqrt{2}a\int_a^a f(y)\,dy=2\sqrt{2}a\int_a^a f(x)\,dx$$

SOLUTION AN.022.

$$x^4+x^2+1<x^4+2x^2+1=(x^2+1)^2;x\in(0,1)$$

$$\frac{1}{x^4+x^2+1}>\frac{1}{(x^2+1)^2}\Rightarrow\frac{x^2+1}{x^4+x^2+1}>\frac{1}{x^2+1}$$

$$\int_0^{\frac{1}{n^7}}\frac{x^2+1}{x^4+x^2+1}dx>\int_0^{\frac{1}{n^7}}\frac{1}{x^2+1}dx=\tan^{-1}\left(\frac{1}{n^7}\right)$$

$$\Omega=\lim_{n\to\infty}n^{10}\int_0^{\frac{1}{n^7}}\frac{x^2+1}{x^4+x^2+1}dx\ge\lim_{n\to\infty}n^{10}\tan^{-1}\left(\frac{1}{n^7}\right)=$$

$$=\lim_{n\to\infty}\frac{\tan^{-1}\left(\frac{1}{n^7}\right)}{\frac{1}{n^7}}\cdot n^3=1\cdot\infty^3=\infty$$

SOLUTION AN.023.

$$\textit{If } x\in[0,1]\Rightarrow x\le1\Rightarrow x^{11}\le x^{10};\ -x^9\ge-x^{10}\Rightarrow-x^{10}\le-x^9 \textit{ and analogous for}$$

$$y,z\in[0,1]$$

$$2(x^{11}+y^{11}+z^{11})-(x^{10}y^9+y^{10}z^9+z^{10}x^9)\le$$

$$\le2(x^{10}+y^{10}+z^{10})-(x^{10}y^{10}+y^{10}z^{10}+z^{10}x^{10})=$$

$$=(x^{10}-1)(1-y^{10})+(y^{10}-1)(1-z^{10})+(z^{10}-1)(1-x^{10})+3\le3$$

$$\textit{If } x,y,z\le1 \textit{ exists } a,b,c\le0 \textit{ such that } x=t^a;y=t^b;z=t^c$$

$$2\sum(t^a)^{11}-\sum(t^a)^{10}\cdot\left(t^b\right)^9\le3$$

$$2\sum t^{11}a-\sum t^{10a+9b}\le3$$

$$2\sum\int_0^1 t^{11a}\,dt-\sum\int_0^1 t^{10a+9b}\,dt\le\int_0^1 3dt$$

$$2\sum\frac{1}{11a+1}-\sum\frac{1}{10a+9b+1}\le 3$$

$$2\sum\frac{1}{11a+1}\le 3+\sum\frac{1}{10a+9b+1}$$

SOLUTION AN.024.

$$x\in[0,1]\Rightarrow x\le 1\Rightarrow x+\pi\le 1+\pi\Rightarrow (x+\pi)^a\le(1+\pi)^a\Rightarrow$$

$$\Rightarrow\frac{1}{(x+\pi)^a}\ge\frac{1}{(1+\pi)^a}\Rightarrow\int_0^1\frac{x^n}{(x+\pi)^a}dx\ge\int_0^1\frac{x^n}{(1+\pi)^a}dx$$

$$\Omega(a)=\lim_{n\to\infty}n\int_0^1\frac{x^n}{(x+\pi)^a}dx\ge\lim_{n\to\infty}n\int_0^1\frac{x^n}{(1+\pi)^a}dx=$$

$$=\lim_{n\to\infty}\frac{n}{(1+\pi)^a}\cdot\frac{x^{n+1}}{n+1}\Big|_0^1=\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1}{(1+\pi)^a}=\frac{1}{(1+\pi)^a}$$

$$\sum(1+\pi)^b\,\Omega(a)=\sum(1+\pi)^b\cdot\frac{1}{(1+\pi)^a}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{(1+\pi)^b}{(1+\pi)^a}}=3\sqrt[3]{1}=3$$

SOLUTION AN.025.

$$\Omega=\lim_{n\to\infty}\left(1+\frac{6}{\pi^2}\sum_{k=1}^{n}\frac{1}{n^2}-1\right)^n=$$

$$=\lim_{n\to\infty}\left(1+\frac{6}{\pi^2}\sum_{k=1}^{n}\frac{1}{n^2}-1\right)^{\frac{n}{\frac{6}{\pi^2}\sum_{k=1}^{n}\frac{1}{n^2}-1}\cdot\frac{\frac{6}{\pi^2}\sum_{k=1}^{n}\frac{1}{n^2}-1}{1}}=e^{\lim\limits_{n\to\infty}\frac{\frac{6}{\pi^2}\sum_{k=1}^{n}\frac{1}{n^2}-1}{\frac{1}{n}}}$$

$$=e^{\lim\limits_{n\to\infty}\left(\frac{6}{\pi^2}\cdot\frac{\sum_{k=1}^{n+1}\frac{1}{k^2}-\sum_{k=1}^{n}\frac{1}{k^2}}{\frac{1}{n+1}-\frac{1}{n}}\right)}=e^{\frac{6}{\pi^2}\lim\limits_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{n-n-1}{n(n+1)}}}=e^{\frac{6}{\pi^2}\lim\limits_{n\to\infty}\frac{-n}{n+1}}=e^{-\frac{6}{\pi^2}}$$

SOLUTION AN.026.

$$\Omega=\lim_{n\to\infty}\sqrt[7]{(\log n)^6}\left(\sqrt[7]{1+\frac{1}{2}+\cdots+\frac{1}{n}}-\sqrt[7]{\log n}\right)$$

$$\Omega = \lim_{n\to\infty} \sqrt[7]{(\log n)^6}\left(\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n}{\sqrt[7]{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)^6}+\cdots+\sqrt[7]{(\log n)^6}}\right)$$

$$\Omega = \gamma \lim_{n\to\infty} \frac{1}{\sqrt[7]{\left(\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\log n}\right)^6}+\sqrt[7]{\left(\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\log n}\right)^5}+\cdots+1}$$

$$\Omega = \frac{\gamma}{\underbrace{1+1+\cdots+1}_{for\ "7"\ times}} = \frac{\gamma}{7}$$

We've used:

$$\lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\log n} = \lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n+\log n}{\log n} =$$

$$= 1+\lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n}{\log n} = 1+\frac{\gamma}{\infty} = 1$$

SOLUTION AN.027.

$$f:\left[\frac{\sqrt{3}}{3},1\right]\to\mathbb{R}; f(x)=\frac{x}{\tan^{-1}x};\ f'(x)=\frac{\tan^{-1}x-\frac{x}{1+x^2}}{(\tan^{-1}x)^2}$$

$$\textit{Let be } g:\left[\frac{\sqrt{3}}{3},1\right]\to\mathbb{R}; g(x)=\tan^{-1}x-\frac{x}{1+x^2}$$

$$g'(x)=\frac{1}{1+x^2}-\frac{1+x^2-2x^2}{(1+x^2)^2}=\frac{1+x^2-1+x^2}{(1+x^2)^2}=\frac{2x^2}{(1+x^2)^2}>0\Rightarrow$$

$$\Rightarrow g'(x)>0;\ g\ \textit{strictly increasing} \Rightarrow g(x)\ge g\left(\frac{\sqrt{3}}{3}\right)=\frac{\pi}{6}-\frac{\frac{\sqrt{3}}{3}}{1+\frac{1}{3}}=$$

$$=\frac{\pi}{6}-\frac{\sqrt{3}}{4}=\frac{4\pi-6\sqrt{3}}{24}=\frac{2\left(2\pi-3\sqrt{3}\right)}{24}>0$$

$$f'(x)=\frac{g(x)}{(\tan^{-1}x)^2}>0\Rightarrow f\ \textit{strictly increasing}$$

By Mac-Laurin's inequality: $\sqrt[3]{abc}\le\sqrt{\frac{ab+bc+ca}{3}}\Rightarrow\frac{\sqrt[3]{abc}}{\tan^{-1}\left(\sqrt[3]{abc}\right)}\le\frac{\sqrt{\frac{ab+bc+ca}{3}}}{\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)}$

$$\sqrt[3]{abc}\cdot\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)\le\sqrt{\frac{ab+bc+ca}{3}}\tan^{-1}\left(\sqrt[3]{abc}\right)$$

SOLUTION AN.028.

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R};\ f(x)=\sin x\tan x-x^2;\ f'(x)=\sin x+\frac{\sin x}{\cos^2 x}-2x$$

$$f''(x)=\frac{\cos^4 x-2\cos^3 x-\cos^2 x+2}{\cos^3 x}$$

$$f''(x)=\frac{(\cos x-1)(\cos^3 x-1+\cos^2 x-1-\cos x)}{\cos^3 x}\ge 0$$

f convexe. By Jensen's inequality: $f\left(\frac{A+B+C}{3}\right)\le\frac{1}{3}\left(f(A)+f(B)+f(C)\right)$

$$f\left(\frac{\pi}{3}\right)\le\frac{1}{3}\left(\sum\sin A\tan A-\sum A^2\right)$$

$$\sin\frac{\pi}{3}\tan\frac{\pi}{3}-\left(\frac{\pi}{3}\right)^2+\frac{1}{3}\sum A^2\le\frac{1}{3}\sum\sin A\tan A$$

$$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{3}+\frac{1}{3}(A^2+B^2+C^2)\le\frac{1}{3}\sum\sin A\tan A+\frac{\pi^2}{9}$$

$$\frac{1}{2}+\frac{1}{3}(A^2+B^2+C^2)\le\frac{1}{3}\sum\sin A\tan A+\frac{\pi^2}{9}$$

$$3(A^2+B^2+C^2)+\frac{9}{2}\le 3\sum\sin A\tan A+\pi^2$$

SOLUTION AN.029.

Lemma: *If* $x_1,x_2,\dots,x_n\ge 1;n\in\mathbb{N};n\ge 1$ *then:*

$$\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}+x_1x_2\dots x_n\ge x_1+x_2+\dots+x_n+\frac{1}{x_1x_2\cdot\ldots\cdot x_n}\quad (1)$$

Proof:

For $n=1$: $\frac{1}{x_1}+x_1\ge x_1+\frac{1}{x_1}$ *(true)*

By induction:

$P(n)$: $\sum_{k=1}^{n}\frac{1}{x_k}+\prod_{k=1}^{n}x_k\ge\sum_{k=1}^{n}x_k+\frac{1}{\prod_k^n x_k}$ *(true)*

$P(n+1)$: $\sum_{k=1}^{n+1}\frac{1}{x_k}+\prod_{k=1}^{n+1}x_k\ge\sum_{k=1}^{n+1}x_k+\frac{1}{\prod_{k=1}^{n+1}x_k}$ *(to prove)*

Denote: $u=x_1x_2\dots x_n$

$$\sum_{k=1}^{n+1}\frac{1}{x_k}+\prod_{k=1}^{n+1}x_k=\sum_{k=1}^{n}\frac{1}{x_k}+\frac{1}{x_{n+1}}+ux_{n+1}=\sum_{k=1}^{n}\frac{1}{x_k}+u+\frac{1}{x_{n+1}}+ux_{n+1}-u\overset{P(n)}{\geq}$$

$$\geq\sum_{k=1}^{n}x_k+\frac{1}{u}+\frac{1}{x_{n+1}}+ux_{n+1}-u\geq\sum_{k=1}^{n+1}x_k+\frac{1}{ux_{n+1}}$$

Remains to prove:

$$\frac{1}{u}+\frac{1}{x_{n+1}}+ux_{n+1}-u\geq x_{n+1}+\frac{1}{ux_{n+1}}$$

$$x_{n+1}+u+u^2x_{n+1}^2-u^2x_{n+1}\geq ux_{n+1}^2+1$$

$$x_{n+1}+u+u^2x_{n+1}^2-u^2x_{n+1}-ux_{n+1}^2-1\geq 0$$

$$u^2x_{n+1}(x_{n+1}-1)+(x_{n+1}-1)-u(x_{n+1}-1)(x_{n+1}+1)\geq 0$$

$$(x_{n+1}-1)(u^2x_{n+1}+1-ux_{n+1}-u)\geq 0$$

$$(x_{n+1}-1)[ux_{n+1}(u-1)-(y-1)]\geq 0$$

$$(x_{n+1}-1)(u-1)(ux_{n+1}-1)\geq 0$$

Which is true because:

$$x_{n+1}\geq 1; u\geq 1; ux_{n+1}-1\geq 0$$

$$P(n)\to P(n+1)$$

Back to the problem:

We take in (1): $n=3; x_1=f(x); x_2=f(y); x_3=f(z)$

$$LHS=\frac{1}{f(x)}+\frac{1}{f(y)}+\frac{1}{f(z)}+f(x)f(y)f(z)$$

By integrating:

$$\int_1^a\int_1^a\int_1^a LHS\,dx\,dy\,dz=\int_1^a\int_1^a\int_1^a\left(\frac{1}{f(x)}+\frac{1}{f(y)}+\frac{1}{f(z)}+f(x)f(y)f(z)\right)dx\,dy\,dz=$$

$$=\int_1^a\int_1^a\int_1^a\frac{dxdydz}{f(x)}+\int_1^a\int_1^a\int_1^a\frac{dx\,dy\,dz}{f(y)}+\int_1^a\int_1^a\int_1^a\frac{dx\,dy\,dz}{f(z)}+$$

$$+\int_1^a\int_1^a\int_1^a f(x)f(y)f(z)\,dx\,dy\,dz=$$

$$=\left(\int_1^a\frac{1}{f(x)}dx\right)\cdot(a-1)^2+\left(\int_1^a\frac{1}{f(y)}dy\right)\cdot(a-1)^2+\left(\int_1^a\frac{1}{f(z)}dz\right)\cdot(a-1)^2+$$

$$+\left(\int_1^a f(x)\,dx\right)\left(\int_1^a f(y)\,dy\right)\left(\int_1^a f(z)\,dz\right) = 3(a-1)^2\left(\int_1^a \frac{dx}{f(x)}\right) + \left(\int_1^a f(x)\,dx\right)^3 \quad (2)$$

$$RHS = f(x) + f(y) + f(z) + \frac{1}{f(x)f(y)f(z)}$$

By integrating:

$$\int_1^a\int_1^a\int_1^a RHS\,dx\,dy\,dz = \int_1^a\int_1^a\int_1^a \left(f(x) + f(y) + f(z) + \frac{1}{f(x)f(y)f(z)}\right)dx\,dy\,dz =$$

$$= \int_1^a\int_1^a\int_1^a f(x)\,dx\,dy\,dz + \int_1^a\int_1^a\int_1^a f(y)\,dx\,dy\,dz + \int_1^a\int_1^a\int_1^a f(z)\,dx\,dy\,dz +$$

$$+ \int_1^a\int_1^a\int_1^a \frac{dx\,dy\,dz}{f(x)f(y)f(z)} = (a-1)^2\cdot\int_1^a f(x)\,dx + (a-1)^2\cdot\int_1^a f(y)\,dy +$$

$$+(a-1)^2\int_1^a f(z)\,dz + \left(\int_1^a \frac{dx}{f(x)}\right)\left(\int_1^a \frac{dy}{f(y)}\right)\left(\int_1^a \frac{dz}{f(z)}\right) =$$

$$= 3(a-1)^2\int_1^a f(x)\,dx + \left(\int_1^a \frac{dx}{f(x)}\right)^3 \quad (3)$$

$$LHS \ge RHS \Rightarrow \int_1^a\int_1^a\int_1^a LHS\,dx\,dy\,dz \ge \int_1^a\int_1^a\int_1^a RHS\,dx\,dy\,dz$$

By (2); (3):

$$3(a-1)^2\left(\int_1^a \frac{dx}{f(x)}\right) + \left(\int_1^a f(x)\,dx\right)^3 \ge 3(a-1)^2\int_1^a f(x)\,dx + \left(\int_1^a \frac{dx}{f(x)}\right)^3$$

$$3(a-1)^2\int_1^a \left(\frac{1}{f(x)} - f(x)\right)dx \ge \left(\int_1^a \frac{dx}{f(x)}\right)^3 - \left(\int_1^a f(x)\,dx\right)^3$$

SOLUTION AN.030.

Let be: $a_n = \frac{1}{n!};\, n \ge 0;\ \sum_{k=0}^{\infty} a_k = e$

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$Let\ be\ x_n = \sqrt{n+1};\, n \ge 0$$

$$\frac{1}{x_n} - \frac{1}{x_n+1} = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}} = \frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{(n+1)(n+2)}} =$$

$$=\frac{n+2-n-1}{\sqrt{(n+1)(n+2)}\left(\sqrt{n+2}+\sqrt{n+1}\right)}=\frac{1}{(n+2)\sqrt{n+1}+(n+1)\sqrt{n+2}}$$

Let be $\sigma_n = a_0x_0 + a_1x_1 + \cdots + a_nx_n;\ \Omega = \sum_{k=0}^{\infty}\sigma_k$

$$\sigma_n = a_0x_0\left(\frac{1}{x_0}-\frac{1}{x_1}\right)+(a_0x_0+a_1x_1)\left(\frac{1}{x_1}-\frac{1}{x_2}\right)+\cdots$$

$$\ldots+(a_0x_0+a_1x_1+\cdots+a_nx_n)\left(\frac{1}{x_n}-\frac{1}{x_{n+1}}\right)$$

$$\sigma_n = a_0x_0\left(\frac{1}{x_0}-\frac{1}{x_1}\right)+\left(\frac{1}{x_1}-\frac{1}{x_2}\right)+\cdots+\left(\frac{1}{x_n}-\frac{1}{x_{n+1}}\right)+$$

$$+a_1x_1\left(\frac{1}{x_1}-\frac{1}{x_2}\right)+\left(\frac{1}{x_2}-\frac{1}{x_3}\right)+\cdots+\left(\frac{1}{x_n}-\frac{1}{x_{n+1}}\right)+\cdots+a_nx_n\left(\frac{1}{x_n}-\frac{1}{x_{n+1}}\right)$$

$$\sigma_n = a_0x_0\left(\frac{1}{x_0}-\frac{1}{x_{n+1}}\right)+a_1x_1\left(\frac{1}{x_1}-\frac{1}{x_{n+1}}\right)+\cdots+a_nx_n\left(\frac{1}{x_n}-\frac{1}{x_{n+1}}\right)$$

$$\sigma_n = S_n-\frac{a_0x_0+a_1x_1+\cdots+a_nx_n}{x_{n+1}}$$

$$\sigma_n = S_n-\frac{S_0x_0+(S_1-S_0)x_1+(S_2-S_1)x_2+\cdots+(S_n-S_{n-1})x_n}{x_{n+1}}$$

$$\sigma_n = S_n+\frac{S_0(x_1-x_0)+S_1(x_3-x_1)+\cdots+S_{n-1}(x_n-x_{n-1})+S_n(x_{n+1}-x_n)-S_nx_{n+1}}{x_{n+1}}$$

$$\sigma_n = \frac{S_0(x_1-x_0)+S_1(x_2-x_1)+\cdots+S_n(x_{n+1}-x_n)}{x_{n+1}}$$

Let be $t_{nm} = \frac{x_{m+1}-x_m}{x_{n+1}} = \frac{\sqrt{m+2}-\sqrt{m+1}}{\sqrt{n+2}} \to 0;\ m \in \overline{0,n};\ m - \textit{fixed}$

$$T_n = t_{n0}+t_{n1}+\cdots+t_{nm} = -\frac{1}{\sqrt{n+2}}+1\to 1$$

By Toeplitz's theorem:

$$\lim_{n\to\infty}\sum_{k=0}^{n}(t_{k0}S_0+t_{k1}S_1+\cdots+t_{kn}S_n)=\lim_{n\to\infty}\sum_{k=0}^{n}\sigma_k=\lim_{n\to\infty}\sum_{k=0}^{n}a_k=e\Rightarrow\Omega=e$$

SOLUTION AN.031.

$$\tan^{-1}(k+1)-\tan^{-1}(k-2)=\tan^{-1}\left(\frac{k+1-(k-2)}{1+(k+1)(k-2)}\right)=$$

$$=\tan^{-1}\left(\frac{1}{1+k^2-2k+k-2}\right)=\tan^{-1}\left(\frac{1}{k^2-k-1}\right)$$

$$\lim_{n\to\infty} x_n = \frac{3\pi}{2} - \lim_{n\to\infty}\left(\sum_{k=1}^{n}(\tan^{-1}(k+1) - \tan^{-1}(k-2))\right) =$$

$$= \frac{3\pi}{2} - \lim_{n\to\infty}\left(\tan^{-1}(n-1) + \tan^{-2} n + \tan^{-1}(n+1) - \tan^{-1}(-1) - \tan^{-1} 0 - \tan^{-1} 1\right) =$$

$$= \frac{3\pi}{2} - \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{4} - 0 - \frac{\pi}{4}\right) = 0$$

$$\Omega = \lim_{n\to\infty}(1+x_n)^n = \lim_{n\to\infty}(1+x_n)^{\frac{1}{x_n}(nx_n)} =$$

$$= e^{\lim_{n\to\infty} nx_n} = e^{\lim_{n\to\infty}\frac{x_n}{\frac{1}{n}}} = e^{\lim_{n\to\infty}\frac{x_{n+1}-x_n}{\frac{1}{n+1}-\frac{1}{n}}} = e^{\lim_{n\to\infty}\frac{\tan^{-1}\frac{3}{(n+1)^2-(n+1)-1}}{\frac{n-n-1}{n(n+1)}}} =$$

$$= e^{\lim_{n\to\infty}\frac{\tan^{-1}(n+2)-\tan^{-1}(n-1)}{\frac{-1}{n(n+1)}}} e^{-\lim_{x\to\infty}\frac{\tan^{-1}(x+2)-\tan^{-1}(x-1)}{\frac{1}{x^2+x}}} = e^{-\lim_{x\to\infty}\frac{\frac{1}{1+(x+2)^2}-\frac{1}{1+(x-1)^2}}{\frac{-2x-1}{x^2(x+1)^2}}} =$$

$$= e^{\lim_{x\to\infty}\frac{1+(x-1)^2-1-(x+2)^2}{(1+(x+2)^2)((1+(x+1)^2)}\cdot\frac{x^2(x+1)^2}{2x+1}} =$$

$$= e^{\lim_{x\to\infty}\frac{x^2(x+1)^2}{(x^2+4x+5)(x^2+2x+2)}\cdot\frac{x^2-2x+1-x^2-4x-4}{2x+1}} = e^{1\cdot\lim_{x\to\infty}\frac{-6x-3}{2x+1}} = e^{-\frac{6}{2}} = e^{-3} = \frac{1}{e^3}$$

SOLUTION AN.032.

Let be $x_n = \sum_{k=1}^{n}\frac{1}{k(2k+1)}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{2k+1}\right) = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - 2\left(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}\right)\right) =$$

$$= \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n + \log n -$$

$$-2\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n+1} - \log(2n+1) - 1 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2n} + \log(2n+1)\right) =$$

$$= \gamma + \lim_{n\to\infty}\left(\log n - 2\gamma + 2\left(1 + \frac{1}{2} + \cdots + \frac{1}{2n} - \log 2n + \log 2n - \log(2n+1)\right)\right)$$

$$= \gamma - 2\gamma + \lim_{n\to\infty}\left(\log n - 2\left(-1 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2n} + \log(2n+1)\right)\right) =$$

$$= -\gamma + \lim_{n\to\infty}\left(\log n + 2 + 1 + \frac{1}{2} + \cdots + \frac{1}{n} - 2\log(2n+1)\right) =$$

$$= -\gamma + \lim_{n\to\infty}\left(2 + 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n + 2\log n - 2\log(2n+1)\right) =$$

$$= -\gamma + 2 + \gamma + 2 \lim_{n\to\infty} \log \frac{n}{2n+1} = 2 + 2\log\frac{1}{2} = 2 - 2\log 2$$

$$\Omega = \lim_{n\to\infty} (1 + 2\log 2 - 2 + x_n)^n =$$

$$= \lim_{n\to\infty} (1 + 2\log 2 - 2 + x_n)^{\frac{1}{2\log 2 - 2 + x_n} \cdot n(2\log 2 - 2 + x_n)}$$

$$= e^{\lim_{n\to\infty} \frac{2\log 2 - 2 + x_n}{\frac{1}{n}}} \overset{L'H}{=} e^{\lim_{n\to\infty} \frac{x_{n+1} - x_n}{\frac{1}{n+1} - \frac{1}{n}}} = e^{\lim_{n\to\infty} \frac{\frac{1}{(n+1)(2n+3)}}{\frac{-1}{n(n+1)}}} = e^{-\lim_{n\to\infty} \frac{n}{2n+3}} = e^{-\frac{1}{2}}$$

SOLUTION AN.033.

Let be: $a_n = \sum_{k=1}^{n} \frac{1}{(2k-1)^2}$; $b_n = \sum_{k=1}^{n} \frac{1}{k^2}$; $b_{2n} = \sum_{k=1}^{2n} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{(2k)^2} + \sum_{k=1}^{n} \frac{1}{(2k-1)^2}$

$$b_{2n} = \frac{1}{4} b_n + a_n \Rightarrow a_n = b_{2n} - \frac{1}{4} b_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_{2n} - \frac{1}{4} \lim_{n\to\infty} b_n = \frac{\pi^2}{6} - \frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}$$

$$\Omega = \lim_{n\to\infty} \left(\frac{\sqrt[n]{\prod_{k=1}^{n} (e+k)}}{n} \cdot n \left(\frac{\pi^2}{8} - \sum_{k=1}^{n} \frac{1}{(2k-1)^2} \right) \right) =$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{\prod_{k=1}^{n} (e+k)}{n^n}} \cdot \lim_{n\to\infty} n \left(\frac{\pi^2}{8} - \sum_{k=1}^{n} \frac{1}{(2k-1)^2} \right) =$$

$$\overset{CAUCHY-D'ALEMBERT}{=} \frac{\prod_{k=1}^{n} (e+k)}{(n+1)^{n+1}} \cdot \frac{n^n}{\prod_{k=1}^{n} (e+k)} \cdot \lim_{n\to\infty} \frac{\frac{\pi^2}{8} - \sum_{k=1}^{n} \frac{1}{(2k-1)^2}}{\frac{1}{n}} =$$

$$\overset{CESARO-STOLZ}{=} \lim_{n\to\infty} \frac{e+n+1}{\left(\frac{n+1}{n}\right)^n} \cdot \frac{1}{n+1} \lim_{n\to\infty} \frac{-\frac{1}{(2n+1)^2}}{\frac{1}{n+1} - \frac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{e+n+1}{n+1} \cdot \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} \cdot \lim_{n\to\infty} \frac{n(n+1)}{(2n+1)^2} = 1 \cdot \frac{1}{e} \cdot \frac{1}{4} = \frac{1}{4e}$$

SOLUTION AN.034.

$$\omega(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{2^n} \tanh\left(\frac{x}{2^n}\right) \right) =$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} \coth\left(\frac{x}{2^{n-1}}\right) - \frac{1}{2^n} \coth\left(\frac{x}{2^n}\right) \right) = \frac{1}{x} - \frac{1}{x} + \coth(x) = \coth(x)$$

We used the identity:

$$\frac{1}{2}\tanh\left(\frac{x}{2}\right)=\coth x-\frac{1}{2}\coth\left(\frac{x}{2}\right)\quad (1)$$

$$\frac{\sinh\left(\frac{x}{2}\right)}{2\cosh\left(\frac{x}{2}\right)}=\frac{\cosh x}{\sinh x}-\frac{\cosh\left(\frac{x}{2}\right)}{2\sinh\left(\frac{x}{2}\right)}$$

$$\sinh^2\left(\frac{x}{2}\right)=\cosh(x)-\cosh^2\left(\frac{x}{2}\right)$$

$$\cosh(x)=\cosh^2\left(\frac{x}{2}\right)-\sinh^2\left(\frac{x}{2}\right)$$

$$\cosh(x)=\cosh(x)$$

We multiply (1) with $\frac{1}{2}$ succesively and add the relationships: (replacing simultaneous x with $\frac{x}{2};\frac{x}{2^2};\ldots$)

$$\frac{1}{2}\tanh\left(\frac{x}{2}\right)=\coth x-\frac{1}{2}\coth\left(\frac{x}{2}\right)$$

$$\frac{1}{2^2}\tanh\left(\frac{x}{2^2}\right)=\frac{1}{2}\coth\left(\frac{x}{2}\right)-\frac{1}{2^2}\coth\left(\frac{x}{2^2}\right)$$

$$\frac{1}{2^3}\tanh\left(\frac{x}{2^3}\right)=\frac{1}{2^2}\coth\left(\frac{x}{2^2}\right)-\frac{1}{2^3}\coth\left(\frac{x}{2^3}\right)$$

$$-----------------------$$

$$\frac{1}{2^n}\tanh\left(\frac{x}{2^n}\right)=\frac{1}{2^{n-1}}\coth\left(\frac{x}{2^{n-1}}\right)-\frac{1}{2^n}\coth\left(\frac{x}{2^n}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\tanh\left(\frac{x}{2^n}\right)\right)=\coth x-\lim_{n\to\infty}\left(\frac{1}{2^n}\coth\left(\frac{x}{2^n}\right)\right)=\coth x-\lim_{n\to\infty}\frac{\coth\left(\frac{x}{2^n}\right)}{2^n}=$$

$$=\coth x-\lim_{n\to\infty}\left(\frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)}\cdot\frac{\cosh\left(\frac{x}{2^n}\right)}{x}\right)=\coth x-\frac{1}{x}$$

$$\omega(x)=\frac{1}{x}+\coth x-\frac{1}{x}=\coth x$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to0\\x>0}}\big(x^n\omega(x)\omega(2x)\cdot\ldots\cdot\omega(nx)\big)}=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to0\\x>0}}(x^n\coth x\coth 2x\cdot\ldots\cdot\coth nx)}$$

$$=\lim_{n\to\infty}\sqrt[n]{\lim_{\substack{x\to0\\x>0}}\left(\frac{x}{\sinh x}\cdot\frac{2x}{\sinh x}\cdot\ldots\cdot\frac{nx}{\sinh nx}\cdot\frac{\cosh x\cdot\ldots\cdot\cosh nx}{n!}\right)}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = \lim_{n\to\infty} \left(\frac{1}{(n+1)!} \cdot \frac{n!}{1}\right) = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

SOLUTION AN.035.

Let be $f: (0, \infty) \to \mathbb{R}; f(x) = \frac{1}{1+e^x}; f'(x) = -\frac{e^x}{(1+e^x)^2}; f''(x) = -\frac{-e^x(1-e^x)}{(1+e^x)^3} > 0$

f *convexe; suppose that* a, b, c *are different in pairs.*

$$f(b) = f\left(\frac{c-b}{c-a} \cdot a + \left(\frac{c-b}{c-a}\right) \cdot c\right) \le \frac{c-b}{c-a} f(a) + \left(1 - \frac{c-b}{c-a}\right) f(c)$$

$$f(b) \le \frac{c-b}{c-a} f(a) + \frac{b-a}{c-a} f(c) \quad (1)$$

We used the identity: $b = \frac{c-b}{c-a} \cdot a + \left(1 - \frac{c-b}{c-a}\right) \cdot c$

$$(c-a)b = (c-b)a + (c-a-c+b)c$$

$$bc - ab = ac - ab - ac + bc$$

$$0 = 0$$

$$f(a-b+c) = f\left(\frac{b-a}{c-a} a + \frac{c-b}{c-a} c\right) \le \frac{b-a}{c-a} f(a) + \frac{c-b}{c-a} f(c) \quad (2)$$

We used the identity:

$$a - b + c = a - \left(\frac{c-b}{c-a} a + \frac{b-a}{c-a} c\right) + c = \frac{b-a}{c-a} a + \frac{c-b}{c-a} c$$

By adding (1); (2):

$$f(a-b+c) + f(b) \le f(a)\left(\frac{b-a}{c-a} + \frac{c-b}{c-a}\right) + f(c)\left(\frac{c-b}{c-a} + \frac{b-a}{c-a}\right) f(c)$$

$$f(a-b+c) + f(b) \le f(a) + f(c)$$

$$\frac{1}{1+e^{a-b+c}} + \frac{1}{1+e^b} \le \frac{1}{1+e^a} + \frac{1}{1+e^c}$$

Equality holds if $a = b = c$.

SOLUTION AN.036.

$$f: [1, e] \times [1, e] \times [1, e] \to \mathbb{R}$$

$$f(x, y, z) = x \log x + y \log y + z \log z + e^{x+y+z}$$

$$f'_x = \log x + 1 + e^{x+y+z}$$

$$f''_{xx} = \frac{1}{x} + e^{x+y+z} > 0$$

$$f''_{yy} = \frac{1}{y} + e^{x+y+z} > 0$$

$$f''_{zz} = \frac{1}{z} + e^{x+y+z} > 0$$

***f* – convexe in each variable on compact domain $[1,e] \times [1,e] \times [1,e]$**

By Gireaux's theorem

$$\min f = \min\{f(1,1,1); f(e,1,1); f(e,e,1); f(e,e,e)\} = e^3$$

$$\max f = \max\{f(1,1,1); f(e,1,1); f(e,e,1); f(e,e,e)\} = 3e + e^{3e}$$

Maximum and minimum is attained in one of the points:

$$(1,1,1); (e,1,1); (1,e,1); (1,1,e); (e,e,1); (e,1,e); (1,e,e); (e,e,e)$$

SOLUTION AN.037.

$$\frac{(a+b+c)^5 - a^5 - b^5 - c^5}{(a+b+c)^3 - a^3 - b^3 - c^3} = \frac{5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca)}{3(a+b)(b+c)(c+a)} =$$

$$= \frac{5}{3}(a^2+b^2+c^2+ab+c+ca) \quad (1)$$

$$\Omega = \int \frac{242(x+2)^5 - (x+1)^5 - (x+3)^5}{26(x+2)^3 - (x+1)^3 - (x+3)^3} dx =$$

$$= \int \frac{(x+1+x+2+x+3)^5 - (x+1)^5 - (x+2)^5 - (x+3)^5}{(x+1+x+2+x+3)^3 - (x+1)^3 - (x+2)^3 - (x+3)^3} dx =$$

$$\overset{(1)}{=} \frac{5}{3}\int \left((x+1)^2 + (x+2)^2 + (x+3)^2 + (x+1)(x+2) + (x+2)(x+3) + (x+1)(x+3)\right) dx =$$

$$= \frac{5}{3}\int (x^2+2x+1+x^2+4x+4+x^2+6x+9+x^2+3x+2+x^2+5x+6+x^2+4x+3)\, dx =$$

$$= \frac{5}{3}\int (6x^2+24x+25)\, dx = \frac{5}{3}\cdot 6\cdot\frac{x^3}{3} + \frac{5}{3}\cdot 24\cdot\frac{x^2}{2} + \frac{5}{3}\cdot 25x + C =$$

$$= \frac{10}{3}x^3 + 20x^2 + \frac{125}{3}x + C$$

SOLUTION AN.038.

$$\cos(11\cos^{-1}x) + \cos(9\cos^{-1}x) = 2\cos(10\cos^{-1}x)\cos(\cos^{-1}x) = 2x\cos(10\cos^{-1}x)$$

$$\Omega(a) = \frac{4}{\pi}\int_0^a \frac{2x\cos(10^{-1}x)\,dx}{(x^2+a^2)\cdot 2x\cos(10^{-1}x)} =$$

$$= \frac{4}{\pi}\int_0^a \frac{1}{x^2+a^2}dx = \frac{4}{\pi}\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_0^a = \frac{4}{\pi a}\tan^{-1}(1) = \frac{4}{\pi a}\cdot\frac{\pi}{4} = \frac{1}{a}$$

$$\left(\Omega(a)+\Omega(b)+\Omega(c)\right)\left(6 + \frac{1}{\Omega^3(a)} + \frac{1}{\Omega^3(b)} + \frac{1}{\Omega^3(c)}\right) =$$

$$= \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(6+a^3+b^3+c^3) \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{1}{abc}} \cdot 9\sqrt[9]{1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot a^3b^3c^3} =$$

$$= 27 \cdot \frac{1}{\sqrt[3]{abc}} \cdot \sqrt[3]{abc} = 27$$

SOLUTION AN.039.

$$\frac{2xy}{x+y} \overset{HM-GM}{\leq} \sqrt{xy} \quad (1)$$

$$\frac{2(x+y)\cdot 1}{(x+y)+1} \overset{HM-GM}{\leq} \sqrt{(x+y)\cdot 1} \quad (2)$$

$$\frac{2xy}{x+y} \cdot \frac{2(x+y)}{x+y+1} \leq \sqrt{xy(x+y)}$$

$$\frac{4xy}{x+y+1} \leq \sqrt{x^2y+xy^2}$$

$$\frac{1}{\sqrt{x^2y+xy^2}} \leq \frac{x+y+1}{4xy} = \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{xy}\right)$$

$$\int_1^a \int_1^b \frac{dxdy}{\sqrt{x^2y+xy^2}} < \frac{1}{4}\int_1^a \left(\int_1^b \frac{1}{x}dx\right)dy + \frac{1}{4}\int_1^b \left(\int_1^a \frac{1}{y}dy\right)dx + \frac{1}{4}\left(\int_1^b \frac{1}{x}dx\right)\left(\int_1^1 \frac{1}{y}dy\right) =$$

$$= \frac{1}{4}\log b \cdot (a-1) + \frac{1}{4}\log a \cdot (b-1) + \frac{1}{4}\log b \cdot \log a =$$

$$= \log\sqrt[4]{ab(a-1)(b-1)} + \log\sqrt{a} \cdot \log\sqrt{b}$$

SOLUTION AN.040.

$$\sqrt{\log_{yz} x} = \sqrt{\frac{\ln x}{\ln y + \ln z}} \cdot 1 \overset{GM-HM}{\geq} \frac{2}{\frac{\ln y+\ln z}{\ln x}+1} = \frac{2\ln x}{\ln x + \ln y + \ln z} \quad (1)$$

Analogous: $\sqrt{\log_{zx} y} \geq \frac{2\ln y}{\ln z+\ln x}$ *(2);* $\sqrt{\log_{xy} z} \geq \frac{2\ln z}{\ln x+\ln y}$ *(3)*

By adding (1); (2); (3):

$$\sqrt{\log_{yz} x} + \sqrt{\log_{zx} y} + \sqrt{\log_{xy} z} \geq \frac{2(\ln x+\ln y+\ln z)}{\ln x+\ln y+\ln z} = 2$$

$$\int_a^b\int_a^b\int_a^b \left(\sqrt{\log_{yz} x} + \sqrt{\log_{zx} y} + \sqrt{\log_{xy} z}\right) \geq \int_a^b\int_a^b\int_a^b 2\,dx\,dy\,dz = 2(b-a)^3$$

SOLUTION AN.041.

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\Omega_n}=\lim_{n\to\infty}\frac{\Omega_{n+1}}{\Omega_n}=\lim_{n\to\infty}\frac{\binom{n+1}{7}+2\binom{n}{7}+\dots+(n-5)\binom{7}{7}}{\binom{n}{7}+2\binom{n-1}{7}+\dots+(n-6)\binom{7}{7}}=$$

$$\overset{STOLS-CESARO}{=}\lim_{n\to\infty}\frac{\Omega_{n+2}-\Omega_{n+1}}{\Omega_{n+1}-\Omega_n}=\lim_{n\to\infty}\frac{\binom{n+2}{7}+\dots+\binom{7}{7}}{\binom{n+1}{7}+\binom{n}{7}+\dots+\binom{7}{7}}=\lim_{n\to\infty}\frac{\binom{n+3}{7}}{\binom{n+2}{7}}=$$

$$=\lim_{n\to\infty}\frac{(n+3)!}{7!\,(n+3-7)!}\cdot\frac{7!\,(n+2-7)!}{(n+2)!}=\lim_{n\to\infty}\frac{(n+2)!\cdot(n+3)\cdot(n-5)!}{(n-5)!\cdot(n-4)\cdot(n+2)!}=$$

$$=\lim_{n\to\infty}\frac{n+3}{n-4}=1$$

SOLUTION AN.042.

Let be $f:\mathbb{R}\to\mathbb{R};\ f(x)=e^{x^2};\ f'(x)=2xe^{x^2};\ f''(x)=2e^{x^2}+4x^2e^{x^2}>0$

f *convexe. Suppose that* a,b,c *are different in pairs.*

$$bc-ab=ac-ab-ac+bc$$

$$(c-a)b=(c-b)a+(c-a-c+b)c$$

$$b=\frac{c-b}{c-a}a+\left(1-\frac{c-b}{c-a}\right)c$$

$$f(b)=f\left(\frac{c-b}{c-a}\cdot a+\left(1-\frac{c-b}{c-a}\right)c\right)\overset{f\ convexe}{\leq}\frac{c-b}{c-a}f(a)+\frac{b-a}{c-a}f(c)\quad (1)$$

$$a-b+c=a-\left(\frac{c-b}{c-a}a+\frac{b-a}{c-a}c\right)+c=\frac{b-a}{c-a}a+\frac{c-b}{c-b}c$$

$$f(a-b+c)=f\left(\frac{b-a}{c-a}a+\frac{c-b}{c-a}c\right)\overset{f\ convexe}{\leq}\frac{b-a}{c-a}f(a)+\frac{c-b}{c-a}f(c)\quad (2)$$

By adding (1); (2): $f(a-b+c)+f(b)\leq\left(\frac{b-a}{c-a}+\frac{c-b}{c-a}\right)f(a)+\left(\frac{c-b}{c-a}+\frac{b-a}{c-a}\right)f(c)$

$$f(a-b+c)+f(b)\leq f(a)+f(c)$$

$$e^{(a-b+c)^2}+e^{b^2}\leq e^{a^2}+e^{c^2}$$

Equality holds for $a=b=c$.

SOLUTION AN.043.

$$x^y=\big(1+(x-1)\big)^y\overset{Bernoulli}{\geq}1+y(x-1)\quad (1)$$

$$y^x=\big(1+(y-1)\big)^x\overset{Bernoulli}{\geq}1+x(y-1)\quad (2)$$

By adding (1); (2): $x^y+y^x\geq 1+y(x-1)+1+x(y-1)=$

$$= 1 + xy - y + 1 + xy - x = 1 + xy + (x-1)(y-1) \geq 1 + xy$$

$$\int_a^b \int_a^b (x^y + y^x)\, dx\, dy \geq \int_a^b \int_a^b (1 + xy) dx\, dy =$$

$$= \int_a^b \int_a^b dx\, dy + \int_a^b \int_a^b xy\, dx\, dy = (b-a)^2 + \frac{(b-a)^2}{2} \cdot \frac{(b-a)^2}{2} =$$

$$= (b-a)^2 \left(1 + \frac{(b-a)^2}{4}\right)$$

$$4 \int_a^b \int_a^b (x^y + y^x)\, dx\, dy \geq (b-a)^2 (4 + (b-a)^2)$$

SOLUTION AN.044.

$$\Omega(x,y) = \sum_{n=1}^{\infty} \frac{2n^2 + (2x + 2y + 5)n + 2xy + 6x - y}{3^n (n+y)(n+y+1)(n+y+2)} =$$

$$= \sum_{n=1}^{\infty} \left(\frac{n+x}{3^{n-1}(n+y)(n+y+1)} - \frac{n+x+1}{3^n (n+y+1)(n+y+2)} \right) = \frac{x+1}{(y+1)(y+2)}$$

$$\Omega(x,y) \cdot \Omega(y,x) = \frac{x+1}{(y+1)(y+2)} \cdot \frac{y+1}{(x+1)(x+2)} = \frac{1}{(x+2)(y+2)} =$$

$$= \frac{1}{(x+1+1)(y+1+1)} \overset{AM-GM}{\leq} \frac{1}{3\sqrt[3]{x \cdot 1 \cdot 1}} \cdot \frac{1}{3\sqrt[3]{y \cdot 1 \cdot 1}} = \frac{1}{9\sqrt[3]{xy}}$$

SOLUTION AN.045.

$$\Omega(a,b,c) = \sum_{n=1}^{\infty} \frac{an^2 + bn + c}{n!} = a + b + c + \sum_{n=2}^{\infty} \frac{an^2 + bn + c}{n!} =$$

$$= a + b + c + \sum_{n=1}^{\infty} \frac{an(n-1) + (b+a)n + c}{n!} =$$

$$= a + b + c + \sum_{n=1}^{\infty} \left(\frac{a}{(n-2)!} + \frac{b+a}{(n-1)!} + \frac{c}{n!} \right) =$$

$$= a + b + c + ae + (b+a)(e-1) + c(e-2) =$$

$$= a(1 + e + e - 1) + b(1 + e - 1) + c(e-1) = 2ae + be + c(e-1)$$

$$\Omega(a,b,c) + \Omega(b,c,a) + \Omega(c,a,b) =$$

$$= 2e(a+b+c) + e(a+b+c) + (a+b+c)(e-1) =$$

$$= (a+b+c)(2e + e + e - 1) =$$

$$= (4e-1)(a+b+c) \overset{AM-GM}{\geq} 3(4e-1)\sqrt[3]{abc}$$

SOLUTION AN.046.

$$\lim_{x\to\infty} x\left(\left(\frac{x+1}{x}\right)^{x+a} - e\right) = \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^{x+a} - e}{\frac{1}{x}} = \lim_{y\to 0} \frac{(1+y)^{\frac{1}{y}+a} - e}{y} =$$

$$= \lim_{y\to 0}\left[-\frac{1}{y^2}(1+y)^{\frac{1}{y}+a}\cdot \ln(1+y) + \left(\frac{1}{y}+a\right)(1+y)^{\frac{1}{y}+a-1}\right] =$$

$$= \lim_{y\to 0}(1+y)^{\frac{1}{y}+a}\left[\frac{-\ln(1+y)}{y^2} + \frac{\frac{1}{y}+a}{1+y}\right]$$

$$= e\lim_{y\to 0}\frac{y+ay^2-(1+y)\ln(1+y)}{y^2+y^3} = e\lim_{y\to 0}\frac{1+2ay-1-\ln(1+y)}{3y^2+2y} =$$

$$= e\lim_{y\to 0}\frac{2ay-\ln(1+y)}{3y^2+2y} = e\lim_{y\to 0}\frac{2a-\frac{1}{1+y}}{6y+2} = e\cdot\frac{2a-1}{2} = e\left(a-\frac{1}{2}\right)$$

$$\Omega(a)+\Omega(b)+\Omega(c) = e\left(a+b+c-\frac{3}{2}\right) = e(a+b+c) - \frac{3e}{2} =$$

$$= \frac{2e(a+b+c)-3e}{2} = \frac{3e+2-3e}{2} = 1$$

$$1 = \Omega(a)+\Omega(b)+\Omega(c) \geq 3\sqrt[3]{\Omega(a)\Omega(b)\Omega(c)}$$

$$1 \geq 27\Omega(a)\Omega(b)\Omega(c)$$

$$\Omega(a)\Omega(b)\Omega(c) \leq \frac{1}{27}$$

SOLUTION AN.047.

$$\Omega = \lim_{n\to\infty}\sqrt[n^4]{a_n} \Rightarrow \ln\Omega = \lim_{n\to\infty}\frac{\ln a_n}{n^4} = \lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{(n+1)^4-n^4} =$$

$$= \lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{n^4\left(\left(1+\frac{1}{n}\right)^4-1\right)} = \lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{n^3}\cdot\lim_{n\to\infty}\frac{n^3}{3n^3+3n^2+1} =$$

$$= \lim_{n\to\infty}\frac{\ln\frac{a_{n+1}}{a_n}}{n^3}\cdot\frac{1}{4} = \frac{1}{4}\lim_{n\to\infty}\frac{\ln\frac{a_{n+2}}{a_{n+1}} - \ln\frac{a_{n+1}}{a_n}}{(n+1)^3-n^3} = \frac{1}{4}\lim_{n\to\infty}\frac{\ln\frac{a_{n+2}\cdot a_n}{a_{n+1}^2}}{3n^2+3n+1} =$$

$$=\frac{1}{4}\lim_{n\to\infty}\frac{\ln\frac{a_{n+2}\cdot a_n}{a_{n+1}^2}}{n^2}\cdot\frac{n^2}{3n^2+3n+1}=$$

$$=\frac{1}{4}\cdot\frac{1}{3}\lim_{n\to\infty}\frac{\ln\frac{a_{n+2}\cdot a_n}{a_{n+1}^2}}{n^2}=\frac{1}{12}\lim_{n\to\infty}\frac{\ln\frac{a_{n+3}a_{n+1}}{a_{n+2}^2}-\ln\frac{a_{n+2}\cdot a_n}{a_{n+1}^2}}{(n+1)^2-n^2}=$$

$$=\frac{1}{12}\lim_{n\to\infty}\frac{\ln\left(\frac{a_{n+3}\cdot a_{n+1}}{a_{n+2}^2}\right)-\ln\left(\frac{a_{n+2}\cdot a_n}{a_{n+1}^2}\right)}{2n+1}=\frac{1}{12}\lim_{n\to\infty}\frac{\ln\left(\frac{a_{n+3}\cdot a_{n+1}}{a_{n+2}^2}\cdot\frac{a_{n+1}^2}{a_{n+2}\cdot a_n}\right)}{n}\cdot\frac{n}{2n+1}$$

$$=\frac{1}{24}\lim_{n\to\infty}\frac{\ln\left(\frac{a_{n+3}\cdot a_{n+1}^3}{a_n\cdot a_{n+2}^3}\right)}{n}=\frac{1}{24}\lim_{n\to\infty}\frac{\ln\left(\frac{a_{n+4}\cdot a_{n+2}^3}{a_{n+1}\cdot a_{n+3}^3}\right)-\ln\left(\frac{a_{n+3}\cdot a_{n+1}^3}{a_n\cdot a_{n+2}^3}\right)}{n+1-n}=$$

$$=\frac{1}{24}\lim_{n\to\infty}\ln\left(\frac{a_{n+4}\cdot a_{n+2}^3}{a_{n+1}\cdot a_{n+3}^3}\cdot\frac{a_n\cdot a_{n+2}^3}{a_{n+3}\cdot a_{n+1}^3}\right)=\frac{1}{24}\lim_{n\to\infty}\ln\left(\frac{a_n\cdot a_{n+2}^6\cdot a_{n+4}}{a_{n+1}^4\cdot a_{n+3}^4}\right)=\frac{1}{24}\cdot 10=\frac{5}{12}$$

$$\ln\Omega=\frac{5}{12}\Rightarrow\Omega=\ln e^{\frac{5}{12}}$$

$$\Omega=\sqrt[12]{e^5}$$

SOLUTION AN.048.

Let be $f:(0,1)\to\mathbb{R};\ f(x)=\log(1-x)+x;\ f'(x)=\frac{-1}{1-x}+1;\ f''(x)=\frac{-1}{(1-x)^2}<0;$

f *concave* $\Rightarrow f'$ *decreasing* $\Rightarrow\frac{f\left(\frac{x}{n}\right)-f(0)}{\frac{x}{n}-f(0)}>\frac{f(x)-f(0)}{x-0}$ *because* $\frac{x}{n}<x$ *for* $n\geq 2$.

$$\frac{f\left(\frac{x}{n}\right)}{\frac{x}{n}}>\frac{f(x)}{x}\Rightarrow nf\left(\frac{x}{n}\right)>f(x)\Rightarrow f\left(\frac{x}{n}\right)>\frac{1}{n}f(x)\Rightarrow f\left(\frac{\frac{1}{k}}{n}\right)>\frac{1}{n}f\left(\frac{1}{k}\right)\Rightarrow f\left(\frac{1}{nk}\right)-\frac{1}{n}f\left(\frac{1}{k}\right)>0$$

(1)

$$\sum_{k=2}^{a}\left(-f\left(\frac{1}{nk}\right)-\frac{1}{n}f\left(\frac{1}{k}\right)\right)=\sum_{k=2}^{a}f\left(\frac{1}{nk}\right)-\frac{1}{n}\sum_{k=2}^{a}f\left(\frac{1}{k}\right)=$$

$$=\sum_{k=2}^{a}\left(\log\left(1-\frac{1}{nk}\right)+\frac{1}{nk}\right)-\frac{1}{n}\sum_{k=2}^{a}\left(\log\left(1-\frac{1}{k}\right)+\frac{1}{k}\right)=\sum_{k=2}^{a}\log\left(\frac{nk-1}{nk}\right)+\frac{1}{n}\sum_{k=2}^{a}\frac{1}{k}-$$

$$-\frac{1}{n}\sum_{k=2}^{a}\log\left(\frac{k-1}{k}\right)-\frac{1}{n}\sum_{k=2}^{a}\frac{1}{k}=\log\prod_{k=2}^{a}\frac{(nk-1)}{nk}-\frac{1}{n}\log\prod_{k=2}^{a}\left(\frac{k-1}{k}\right)=\log\frac{\prod_{k}^{a}(kn-1)}{n^{a-1}\cdot a!}-$$

$$-\frac{1}{n}\log\left(\frac{1}{a}\right)=\frac{\log a}{n}+\log\left(\frac{\prod_{k=2}^{a}(kn-1)}{n^{a-1}\cdot a!}\right)=\log\frac{a^{\frac{1}{n}}\prod_{k=2}^{a}(kn-1)}{n^{a-1}\cdot a!}\overset{(1)}{>}0$$

$$For\ n = a \Rightarrow \log\frac{a^{\frac{1}{a}}\cdot\prod_{k=2}^{a}(ka-1)}{a^{a-1}\cdot a!} > \log 1$$

$$\prod_{k=2}^{a}(ka-1) > \frac{a^{a-1}\cdot a!}{a^{\frac{1}{a}}} = \frac{a^a\cdot a!}{a\cdot a^{\frac{1}{a}}}\ (2)$$

Analogous:

$$\prod_{k=2}^{b}(kb-1) > \frac{b^{b-1}\cdot b!}{b^{\frac{1}{b}}} = \frac{b^b\cdot b!}{b\cdot b^{\frac{1}{b}}}\quad (3)$$

$$\prod_{k=2}^{a}(ka-1)+\prod_{k=2}^{b}(kb-1) \overset{AM-GM}{\geq} 2\sqrt{\prod_{k=2}^{a}(ka-1)\cdot\prod_{k=2}^{b}(kb-1)} \overset{(2);(3)}{>} 2\sqrt{\frac{a!\cdot b!\cdot a^a\cdot b^b}{ab\cdot\sqrt[ab]{a^b\cdot b^a}}}$$

SOLUTION AN.049.

$$Let\ be\ f:[0,\infty)\to\mathbb{R}; f(x)=e^{x^2}; f'(x)=2xe^{x^2}; f''(x)=2e^{x^2}+4x^2e^{x^2}=$$

$$=2e^{x^2}(1+2x^2)>0\Rightarrow f\ convexe.$$

By Hermite – Hadamard inequality: $f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b e^{x^2}\,dx\leq\frac{f(a)+f(b)}{2}$

$$e^{\left(\frac{a+b}{2}\right)^2}\leq\frac{1}{b-a}\int_a^b e^{x^2}\,dx\leq\frac{e^{b^2}+e^{a^2}}{2}$$

$$e^{\left(\frac{a+b}{2}\right)^2}>e^{a^2}(1+ab-a^2)\Leftrightarrow e^{\left(\frac{a+b}{2}\right)^2-a^2}>1+ab-a^2$$

$$e^{\left(\frac{a+b}{2}\right)^2-a^2}>1+\left(\frac{a+b}{2}\right)^2-a^2>1+ab-a^2\Leftrightarrow$$

$$\Leftrightarrow(a+b)^2-4a^2>4ab-4a^2\Leftrightarrow(a-b)^2>0$$

It remains to prove: $\frac{e^{b^2}+e^{a^2}}{2}<(1-ab+b^2)e^{b^2}$

$$1+e^{a^2-b^2}<2(1-ab+b^2)$$

$$e^{a^2-b^2}<1+2(b^2-ab)$$

$$1<e^{b^2-a^2}\big(1+2b(b-a)\big)\ \textit{which is true because:}$$

$$e^{b^2-a^2}>1\ and\ 1+2b(b-a)>1$$

SOLUTION AN.050.

$$\sum f(x)\,(f^2(y)-4f(y)+4)=0$$

$$\sum f(x)\,(f(y)-2)^2=0$$

$$f(x), f(y), f(z) \in \{0, 2\}$$

$$f(x) = \begin{cases} 0; x \in D_1 \\ 2; x \in D_2 \end{cases}; D_1 \cap D_2 = \emptyset, D_1 \cap D_2 = \mathbb{R}$$

SOLUTION AN.051.

$$0 \le x \le \frac{\pi}{4}; \ -\frac{\pi^2}{4} \le -\pi x \le 0 \quad (1)$$

$$0 \le 4x^2 \le \frac{\pi}{4} \quad (2)$$

By adding (1); (2): $-\frac{\pi^2}{4} \le 4x^2 - \frac{\pi}{x} \le \frac{\pi^2}{4}$

$$4n^2 - \frac{\pi^2}{4} \le 4x^2 - \pi x + 4n^2 \le 4n^2 + \frac{\pi^2}{4} \quad (3)$$

$$0 \le x \le \frac{\pi}{4} \Rightarrow 0 \le \tan x \le 1 \Rightarrow 1 \le 1 + \tan x \le 2$$

$$0 \le \ln(1 + \tan x) \le \ln 2$$

Multiplying (3) with $\ln(1 + \tan x)$*:*

$$\left(4n^2 - \frac{\pi^2}{4}\right)\ln(1 + \tan x) \le (4x^2 - \pi x + 4n^2)\ln(1 + \tan x) \le$$

$$\le \left(4n^2 + \frac{\pi^2}{4}\right)\ln(1 + \tan x)$$

By integration: $\left(4n^2 - \frac{\pi^2}{4}\right) \cdot \frac{\pi \ln 2}{8} \le \Omega_n \le \left(4n^2 + \frac{\pi^2}{4}\right) \cdot \frac{\pi \ln 2}{8}$

$$\frac{\pi \ln 2}{8} \lim_{n\to\infty} \frac{4n^2 - \frac{\pi^2}{4}}{\frac{n(n+1)}{2}} \le \Omega \le \frac{\pi \ln 2}{8} \lim_{n\to\infty} \frac{4n^2 + \frac{\pi^2}{4}}{\frac{n(n+1)}{2}}$$

$\pi \ln 2 \le \pi \le \pi \ln 2$; $\Omega = \pi \ln 2$. *We used the fact:*

$$I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\, dx = \frac{\pi \ln 2}{8}$$

$$x = \frac{\pi}{4} - y; dx = -dy$$

$$I = \int_{\frac{\pi}{4}}^{0} \ln\left(1 + \tan\left(\frac{\pi}{4} - y\right)\right)(-dy) = \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{\tan\frac{\pi}{4} - \tan y}{1 + \tan\frac{\pi}{4}\tan y}\right)dy =$$

$$=\int_0^{\frac{\pi}{4}}\ln\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan x}\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\ln 2\,dx-\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx\Rightarrow I=\ln 2\cdot\frac{\pi}{4}-I\Rightarrow 2I=\frac{\pi\ln 2}{4}$$

$$I=\frac{\pi\ln 2}{8}$$

SOLUTION AN.052.

Let be $f:[0,\infty)\to\mathbb{R};\ f(x)=\frac{e^{ax}}{x^2+a^2}$

$$f'(x)=\frac{ae^{ax}(x^2+a^2)-2xe^{ax}}{(x^2+a^2)^2}=\frac{e^{ax}(ax^2-2x+a^3)}{(x^2+a^2)^2}$$

$$f'(x)=0\Rightarrow ax^2-2x+a^3=0$$

$$\Delta=4-4a^4=4(1-a)(1+a)(1+a^2)\le 0$$

because $a\ge 1$***. Hence*** $f'(x)\ge 0,(\forall)\ x\ge 0$.

$$\min f(x)=f(0)=\frac{1}{a^2}$$

$$f(x)\ge f(0)\Rightarrow\frac{e^{ax}}{x^2+a^2}\ge\frac{1}{a^2}\Rightarrow e^{ax}\ge\frac{x^2+a^2}{a^2}$$

$$e^{ax}>1+\left(\frac{x}{a}\right)^2;(\forall)x\ge 1\quad (1)$$

Take in (1): $x=b\Rightarrow e^{ab}>1+\left(\frac{b}{a}\right)^2$ ***(2)***

Analogous: $e^{bc}>1+\left(\frac{c}{b}\right)^2$ ***(3)***

$$e^{ca}>1+\left(\frac{a}{c}\right)^2\quad (4)$$

By adding (2); (3); (4): $e^{ab}+e^{bc}+e^{ca}>3+\left(\frac{b}{a}\right)^2+\left(\frac{c}{b}\right)^2+\left(\frac{a}{c}\right)^2\ge$

$$\ge 3+\frac{b}{a}\cdot\frac{c}{b}+\frac{c}{b}\cdot\frac{a}{c}+\frac{a}{c}\cdot\frac{b}{a}=3+\frac{c}{a}+\frac{b}{c}+\frac{a}{b}$$

SOLUTION AN.053.

$$\beta(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b+1)}$$

$$\beta(a+1,b+1)=\int_0^1 x^a(1-x)^b dx=\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}$$

$$\Gamma(a+1)=a!\Rightarrow \beta(a+1,b+1)=\frac{a!\,b!}{(a+b+1)!}$$

$$(\forall)x\ge 0;\ x\ge \sin x\Rightarrow \sin^{-1}x\ge x;\ (\forall)x\in[0,1]$$

$$\frac{1}{3}\sum\int_0^1 \sin^{-1}\left(x^a(1-x)^b\right)dx\ge \frac{1}{3}\sum\int_0^1 x^a(1-x)^b\,dx=$$

$$=\frac{1}{3}\sum\beta(a+1,b+1)=\frac{1}{3}\sum\frac{a!\,b!}{(a+b+1)!}\overset{AM-GM}{\ge}$$

$$\ge\frac{1}{3}\cdot 3\sqrt[3]{\prod\frac{a!\,b!}{(a+b+1)!}}=\sqrt[3]{\prod\frac{(a!)^2}{(b+a+1)!}}$$

SOLUTION AN.054.

Let be $f:(0,\infty)\to(0,\infty);\ f(x)=\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}$ ***and*** $F:(0,\infty)\to(0,\infty)$

F – primitive of f.

$$\lim_{x\to\infty}\left(x^7\int_0^{\frac{1}{x^7}}f(x)\,dx\right)=\lim_{x\to\infty}\frac{F\left(\frac{1}{x^7}\right)-F(0)}{\frac{1}{x^7}}=$$

$$=\lim_{x\to\infty}\frac{\frac{-7x^6}{x^{14}}F'\left(\frac{1}{x^7}\right)}{\frac{-7x^6}{x^{14}}}=\lim_{x\to\infty}f\left(\frac{1}{x^7}\right)=f(0)=\frac{1}{9}$$

SOLUTION AN.055.

If $x\in(0,1)$ ***then*** $\ln x\le x-1\Rightarrow \ln\frac{1}{x}\le\frac{1}{x}-1\Rightarrow -\ln x\le\frac{1-x}{x}\Rightarrow \ln x\ge\frac{x-1}{x}$

$$\ln(1-x)\ge\frac{(1-x)-1}{1-x}\Rightarrow \ln(1-x)\ge\frac{x}{x-1}$$

$$\frac{1}{\ln(1-x)}\le\frac{x-1}{x}=1-\frac{1}{x}\Rightarrow\int_a^b\frac{dx}{\ln(1-x)}\le\int_a^b dx-\int_a^b\frac{1}{x}dx=b-a-(\ln b-\ln a)$$

$$\frac{1}{b-a}\int_a^b\frac{dx}{\ln(1-x)}\le 1-\frac{\ln b-\ln a}{b-a}\quad (1)$$

By the inequality logharitmic mean greater than geometric mean:

$$\frac{b-a}{\ln b-\ln a}>\sqrt{ab}\Rightarrow\frac{1}{\sqrt{ab}}>\frac{\ln b-\ln a}{b-a}$$

$$\frac{\ln b-\ln a}{b-a}<\frac{1}{\sqrt{ab}}\quad (2)$$

By adding (1); (2): $\frac{\ln b-\ln a}{b-a}+\frac{1}{b-a}\int_a^b\frac{dx}{\ln(1-x)}<1+\frac{1}{\sqrt{ab}}-\frac{\ln b-\ln a}{b-a}$

$$\frac{2(\ln b-\ln a)}{b-a}+\frac{1}{b-a}\int_a^b\frac{dx}{\ln(1-x)}<1+\frac{1}{\sqrt{ab}}$$

SOLUTION AN.056.

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=\frac{x^2+1}{x^4+x^2+1}$ *and* $F:(0,\infty)\to\mathbb{R}$ *his primitive.*

$$\lim_{x\to\infty}x^8\cdot\int_0^{\frac{1}{x^5}}\frac{x^2+1}{x^4+x^2+1}dx=\lim_{x\to\infty}\frac{F\left(\frac{1}{x^5}\right)-F(0)}{\frac{1}{x^8}}=\lim_{x\to\infty}\frac{\frac{-5x^4}{x^{10}}F'\left(\frac{1}{x^5}\right)}{\frac{-8x^7}{x^{16}}}=$$

$$=\frac{5}{8}\lim_{x\to\infty}\frac{1}{x^6}\cdot x^9 f\left(\frac{1}{x^5}\right)=\frac{5}{8}\lim_{x\to\infty}x^3\cdot\frac{\left(\frac{1}{x^5}\right)^2+1}{\left(\frac{1}{x^5}\right)^4+\left(\frac{1}{x^5}\right)^2+1}=$$

$$=\frac{5}{8}\lim_{x\to\infty}x^3\cdot\frac{1+x^{10}}{x^{10}}\cdot\frac{x^{20}}{1+x^{10}+x^{20}}=\frac{5}{3}\cdot\infty\cdot1\cdot1=\infty\quad (1)$$

Let be $(x_n)_{n\ge0}$ *a sequence such that* $\lim_{x\to\infty}x_n=\infty$

By (1): $\lim_{x\to\infty}x_n^8\int_1^{\frac{1}{x_n^5}}\frac{x^2+1}{x^4+x^2+1}dx=\infty$

For $x_n=n$ *we obtain* $\Omega=\infty$.

SOLUTION AN.057.

$$5<8\Rightarrow n+5<n+8\Rightarrow\frac{1}{n+8}<\frac{1}{n+5}$$

$$F:\left[\frac{1}{n+8};\frac{1}{n+5}\right]\to\mathbb{R}; F(x)=7^x$$

By Lagrange's theorem: $(\exists)C_n\in\left(\frac{1}{n+8};\frac{1}{n+5}\right)$

$$F\left(\frac{1}{n+5}\right)-F\left(\frac{1}{n+8}\right)=F'(C_n)\left(\frac{1}{n+5}-\frac{1}{n+8}\right)$$

$$7^{\frac{1}{n+5}}-7^{\frac{1}{n+8}}=7^{C_n}\cdot\log7\cdot\frac{n+8-n-5}{(n+5)(n+8)}$$

$$\Omega=\lim_{n\to\infty}n^2\left(\sqrt[n+5]{7}-\sqrt[n+8]{7}\right)=\lim_{n\to\infty}\frac{3n^2}{(n+5)(n+8)}\cdot 7^{C_n}\log 7=3\log 7$$

because:

$$\frac{1}{n+8}<C_n<\frac{1}{n+5}\Rightarrow\lim_{n\to\infty}C_n=0$$

SOLUTION AN.058.

If $p,q>0$ ***then:*** $\left(\frac{p+q}{p^2+q^2}\right)^2\le\frac{1}{pq}\Rightarrow pq(p+q)^2\le(p^2+q^2)^2$

$$pq(p^2+2pq+q^2)\le p^4+2p^2q^2+q^4$$

$$p^4-p^3q-pq^3+q^4\ge 0\Rightarrow p^3(p-q)-q^3(p-q)\ge 0$$

$$(p-q)^2(p^2+pq+q^2)\ge 0$$

For $p=\cos^2 x\,;q=\tan x$: $\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2\le\frac{1}{\cos^2 x\tan x}=\frac{\frac{1}{\cos^2 x}}{\tan x}$

$$\int_a^b\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2dx\le\int_a^b\frac{(\tan x)'}{\tan x}dx=\log\left|\frac{\tan b}{\tan a}\right|$$

SOLUTION AN.059.

Let be $F:\mathbb{R}\to\mathbb{R};F(x)=\frac{e^x}{\sqrt{e^x+1}};F'(x)=\frac{e^{2x}+2e^x}{2(e^x+1)^{\frac{3}{2}}};F''(x)=\frac{1}{2}\cdot\frac{\frac{1}{2}\cdot e^{3x}+e^{2x}+2e^x}{(e^x+1)^{\frac{5}{2}}}>0$

F convexe. By Jensen's inequality:

$$\frac{1}{\sqrt{2}}=F(0)=F\left(\frac{\ln 1}{3}\right)=F\left(\frac{\ln\frac{a}{b}+\ln\frac{b}{c}+\ln\frac{c}{a}}{3}\right)\le\frac{1}{3}\left(F\left(\ln\frac{a}{b}\right)+F\left(\ln\frac{b}{c}\right)+F\left(\ln\frac{c}{a}\right)\right)=$$

$$=\frac{1}{3}\left(\frac{\frac{a}{b}}{\sqrt{\frac{a}{b}+1}}+\frac{\frac{b}{c}}{\sqrt{\frac{b}{c}+1}}+\frac{\frac{c}{a}}{\sqrt{\frac{c}{a}+1}}\right)=\frac{1}{3}\cdot\left(\frac{a}{\sqrt{ab+b^2}}+\frac{b}{\sqrt{bc+c^2}}+\frac{c}{\sqrt{ca+a^2}}\right)$$

$$\frac{a}{\sqrt{b(a+b)}}+\frac{b}{\sqrt{c(b+c)}}+\frac{c}{\sqrt{a(c+a)}}\ge\frac{3\sqrt{2}}{2}$$

Let be $a=f(x);b=g(x);c=h(c)$.

$$\sum\frac{f(x)}{\sqrt{f(y)\big(f(x)+f(y)\big)}}\ge\frac{3\sqrt{2}}{2}$$

$$\int_0^a\int_0^b\int_0^c\left(\sum\frac{f(x)}{\sqrt{f(y)\big(f(x)+f(y)\big)}}\right)dz\,dy\,dx\geq\int_0^a\int_0^b\int_0^c\frac{3\sqrt{2}}{2}dz\,dy\,dx=\frac{3\sqrt{2}}{2}abc\geq$$

$$\overset{GM-HM}{\geq}\frac{3\sqrt{2}}{2}\cdot\left(\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\right)^3=\frac{81\sqrt{2}}{2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}$$

SOLUTION AN.060.

$$3<4\Rightarrow\frac{1}{n+4}<\frac{1}{n+3}$$

$$F:\left[\frac{1}{n+4};\frac{1}{n+3}\right]\to\mathbb{R};F(x)=(\log 5)^x$$

$$F\left(\frac{1}{n+3}\right)-F\left(\frac{1}{n+4}\right)=(\log 5)^{C_n}\cdot\log(\log 5)\left(\frac{1}{n+3}-\frac{1}{n+4}\right)$$

$$(\log 5)^{\frac{1}{n+3}}-(\log 5)^{\frac{1}{n+4}}=(\log 5)^{C_n}\cdot\log(\log 5)\frac{1}{(n+3)(n+4)}$$

$$\frac{1}{n+4}<C_n<\frac{1}{n+3}\Rightarrow\lim_{n\to\infty}C_n=0$$

$$\Omega=\lim_{n\to\infty}n^2\left(\sqrt[n+3]{\log 5}-\sqrt[n+4]{\log 5}\right)=\lim_{n\to\infty}\frac{n^2}{(n+3)(n+3)}\cdot\lim_{n\to\infty}(\log 5)^{C_n}\cdot\log(\log 5)=$$

$$=1\cdot(\log 5)^0\cdot\log(\log 5)=\log(\log 5)$$

SOLUTION AN.061.

$$s(x)+3\sqrt[3]{p(x)q(x)r(x)}\overset{AM-GM}{\geq}4\sqrt[4]{s(x)\left(\sqrt[3]{p(x)q(x)r(x)}\right)^3}=4\sqrt[4]{s(x)p(x)q(x)r(x)}$$

$$\int_0^a\left(s(x)+3\sqrt[3]{p(x)q(x)r(x)}\right)dx\geq 4\int_0^4\sqrt[4]{s(x)p(x)r(x)}\,dx$$

$$\int_0^a s(x)\,dx\geq 4\int_0^a\sqrt[4]{p(x)q(x)r(x)s(x)}\,dx-3\int_0^a\sqrt[3]{p(x)q(x)r(x)}\,dx$$

SOLUTION AN.062.

$$\frac{\sqrt[4]{1+\sin 2x}-\sqrt[4]{1-\sin 2x}}{\sqrt[4]{1+\sin 2x}+\sqrt[4]{1-\sin 2x}}=\frac{\sqrt[4]{(\sin x+\cos x)^2}-\sqrt[4]{(\sin x-\cos x)^2}}{\sqrt[4]{(\sin x+\cos x)^2}+\sqrt[4]{(\sin x-\cos x)^2}}$$

$$=\frac{\sqrt{\sin x+\cos x}-\sqrt{\sin x-\cos x}}{\sqrt{\sin x+\cos x}+\sqrt{\sin x-\cos x}}=\frac{\left(\sqrt{\sin x+\cos x}-\sqrt{\sin x-\cos x}\right)^2}{\sin x+\cos x-\sin x+\cos x}=$$

$$= \frac{\sin x + \cos x + \sin x - \cos x - 2\sqrt{\sin^2 x - \cos^2 x}}{2\cos x} =$$

$$= \tan x - \frac{2}{2}\sqrt{\frac{\cos 2x}{\cos x}} < \tan x$$

$$\int_a^b \frac{\sqrt[4]{1+\sin 2x} - \sqrt[4]{1-\sin 2x}}{\sqrt[4]{1+\sin 2x} + \sqrt[4]{1-\sin 2x}} dx \le \int_a^b \tan x\, dx =$$

$$= \int_a^b \frac{\sin x}{\cos x} dx = -\log|\cos x|\Big|_a^b = -\log|\cos b| + \ln|\cos a| = \log\left|\frac{\cos a}{\cos b}\right|$$

SOLUTION AN.063.

By Kober's inequality: $\cos x \le 1 - \frac{x^2}{\pi}; x \in \left[0, \frac{\pi}{2}\right]$

$$x \in \left[0, \frac{\pi}{2}\right] \Rightarrow \sin x \in [0,1] \subset \left[0, \frac{\pi}{2}\right] \Rightarrow \cos(\sin x) \le 1 - \frac{\sin^2 x}{\pi}$$

Replace x *with* $f(x)$: $\cos(\sin f(x))\, dx < 1 - \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$

SOLUTION AN.064.

If $p, q > 0$ *then:* $\left(\frac{p+q}{p^2+q^2}\right)^2 \le \frac{1}{pq}$ *(1)*

$$(p+q)^2 pq \le (p^2+q^2)^2, \quad (p^2+2pq+q^2)pq \le p^4 + 2p^2q^2 + q^4$$

$$p^3q + 2p^2q^2 + pq^3 \le p^4 + 2p^2q^2 + q^4, \; p^4 - p^3q - pq^3 + q^4 \ge 0$$

$$p^3(p-q) - q^3(p-q) \ge 0$$

$$(p-q)(p^3-q^3) \ge 0 \Rightarrow (p-q)^2(p^2+pq+q^2) \ge 0$$

For $p = x^2+1; q = \tan^{-1} x$ *in (1):*

$$\left(\frac{x^2+1+\tan^{-1} x}{(x^2+1)^2 + (\tan^{-1} x)^2}\right)^2 \le \frac{1}{(x^2+1)\tan^{-1} x}$$

$$\Omega(a) = \int_{\frac{\pi}{4}}^{a} \left(\frac{x^2+1+\tan^{-1} x}{x^4+2x^2+1+(\tan^{-1} x)^2}\right) dx \le \int_{\frac{\pi}{4}}^{a} \frac{\frac{1}{x^2+1}}{\tan^{-1} x} dx =$$

$$= \frac{1}{2}(\tan^{-1} a)^2 - \frac{1}{2}\left(\tan^{-1}\frac{\pi}{4}\right)^2 = \frac{1}{2}((\tan^{-1} a)^2 - 1) \le \frac{1}{2}(a^2+1)$$

$$2\Omega(a) \le a^2 - 1$$

$$\left(1+2\Omega(a)\right)b^2 \le a^2b^2$$

$$\sum_{cyc(a,b,c)} \left(1+2\Omega(a)\right)b^2 \le \sum_{cyc(a,b,c)} a^2b^2 \le a^4+b^4+c^4$$

SOLUTION AN.065.

Let be $f:[a,b]\to\mathbb{R}; f(x)=\frac{1}{x}; f'(x)=-\frac{1}{x^2}; f''(x)=\frac{2}{x^3}>0 \Rightarrow f$ *convexe.*

By Hermite-Hadamard inequality:

$$\frac{1}{b-a}\int_a^b f(x)\,dx > \frac{f(a)+f(b)}{2} \Rightarrow \frac{1}{b-a}\int_a^b \frac{1}{x}dx > \frac{\frac{1}{a}+\frac{1}{b}}{2} = \frac{a+b}{2ab} \quad (1)$$

Let be $g:(0,1)\to\mathbb{R}; g(x)=x-\ln(1-x);$

$$g'(x)=1+\frac{1}{1-x}=\frac{1-x+1}{1-x}=\frac{2-x}{1-x}>0; (\forall)x\in(0,1)$$

$$g \text{ increasing} \Rightarrow g(x)>g(0)=0 \Rightarrow x>\ln(1-x) \Rightarrow \frac{1}{x}<\frac{1}{\ln(1-x)} \Rightarrow$$

$$\Rightarrow \frac{1}{b-a}\int_a^b \frac{1}{x}dx < \frac{1}{b-a}\int_a^b \frac{1}{\ln(1-x)}dx$$

$$\frac{2(\ln b-\ln a)}{b-a}+\frac{1}{b-a}\int_a^b \frac{dx}{\ln(1-x)} >> \frac{2}{b-a}\int_a^b \frac{1}{x}dx+\frac{1}{b-a}\int_a^b \frac{1}{x}dx=$$

$$=3\cdot\frac{1}{b-a}\int_a^b \frac{1}{x}dx \overset{(1)}{>} 3\cdot\frac{a+b}{2ab} > \frac{6}{a+b} \text{ because } (a+b)^2>4ab. \text{ Remains to prove:}$$

$$\frac{6}{a+b}>\frac{2}{a+b}+\frac{1}{2} \Leftrightarrow \frac{4}{a+b}>\frac{1}{2} \Leftrightarrow a+b<8 \text{ which is true because } a<b<1.$$

SOLUTION AN.066.

$$\int_0^1 e^{\sqrt[k]{x^2}}\,dx=\int_0^1 kt^{k-1}e^{t^2}dt$$

$$\sqrt[k]{x}=t; x=t^k; dx=kt^{k-1}dt$$

$$\int_0^1 (1+2x+3x^2+4x^3)e^{x^2}\,dx=\int_0^1\left(e^{x^2}+e^{(\sqrt{x})^2}+e^{(\sqrt[3]{x})^3}+e^{(\sqrt[4]{x})^4}\right)dx \ge$$

$$\ge 4\int_0^1 e^{\left(\frac{x+\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{4}\right)^2}dx=4\int_0^1 \sqrt[16]{e^{\left(x+\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}\right)^2}}\,dx$$

SOLUTION AN.067.

$$\Omega_n(a)=\sum_{k=0}^{n}(k^2-a^2+1)(a+k)!=$$

$$=\sum_{k=0}^{n}\left([(k+a+1)(k+a+2)-(2a+3)(k+a+1)+2a+2](a+k)!\right)=$$

$$=\sum_{k=0}^{n}\left((2a+2)(a+k)!+(a+k+2)!-(2a+3)(a+k+1)!\right)=$$

$$=(2a+2)\sum_{k=0}^{n}[(a+k)!-(a+k+1)!]+\sum_{k=0}^{n}[(a+k+2)!-(a+k+1)!]=$$

$$=(2a+2)\sum_{k=0}^{n}[(a+k)!-(a+k+1)!]+\sum_{k=0}^{n}[(a+k+2)!-(a+k+1)!]=$$

$$=(2a+2)(a!-(a+n+1)!)+(a+n+2)!-(a+1)!=$$

$$=a!\,(2a+2-a+1)+(a+n+1)!\,(a+n+2-2a-2)=$$

$$=(n-a)(a+n+1)!+(a+1)!$$

$$\Omega_n=\lim_{n\to\infty}\sqrt[n]{(n-a)(a+n+1)!+(a+1)!-(a+1)!}=$$

$$=\lim_{n\to\infty}\frac{(n+1-a)}{(n-a)}\cdot\frac{(a+n+2)!}{(a+n+1)!}=1\cdot\lim_{n\to\infty}(a+n+2)=\infty$$

SOLUTION AN.068.

$$f:(0,\infty)\to\mathbb{R};\, f(x)=e^{x^2};\, f'(x)=2xe^{x^2};\, f''(x)=(4x^2+2)e^{x^2}>0\Rightarrow f\ \textit{convexe}$$

By Hermite-Hadamard's inequality:

$$\frac{1}{a-x}\int_x^a e^{x^2}dx\le\frac{e^{x^2}+e^{a^2}}{2}<(a^2-ax)e^{a^2}+e^{x^2}\quad \textit{(to prove)}$$

$$e^{x^2}+e^{a^2}<(2a^2-2ax)e^{a^2}+2e^{x^2}\quad (1)$$

By Lagrange's theorem: $(\exists)c\in(x,a)$

$$-e^{x^2}+e^{a^2}=2ce^{c^2}(a-x)$$

$$f\ \textit{increasing}\Rightarrow f'(c)<f'(a)\Rightarrow 2ce^{c^2}<2ae^{a^2}$$

$$\frac{e^{a^2}-e^{x^2}}{a-x}=2ce^{c^2}<2ae^{a^2}$$

$$e^{a^2}-e^{x^2}<(a-x)2ae^{a^2}$$

$$e^{a^2}+e^{x^2}<(2a^2-2ax)e^{a^2}+2e^{x^2} \text{ ((1) is proved)}$$

By Hermite-Hadamard's inequality:

$$\frac{1}{a-x}\int_x^a e^{x^2}dx \geq e^{\left(\frac{a+x}{2}\right)^2}>(1+ax-x^2)e^{x^2} \text{ (to prove)}$$

$$\log e^{\left(\frac{a+x}{2}\right)^2}>\log\left((1+ax-x^2)e^{x^2}\right)$$

$$\left(\frac{a+x}{2}\right)^2>\log(1+ax-x^2)+x^2$$

$$\log(1+ax-x^2)\leq ax-x^2<\left(\frac{a+x}{2}\right)^2-x^2 \Leftrightarrow ax<\frac{a^2+2ax+x^2}{4}\Leftrightarrow (x-a)^2>0$$

SOLUTION AN.069.

$$\sin^3 x=\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$$

$$\frac{1}{3}\sin^3(3x)=\frac{1}{4}\sin 3x-\frac{1}{4\cdot 3}\sin(3^2x)$$

$$\frac{1}{3^2}\sin^3(3^2x)=\frac{1}{4\cdot 3}\sin(3^2x)-\frac{1}{4\cdot 3^2}\sin(3^3x)$$

— —

$$\frac{1}{3^n}\sin^3(3^nx)=\frac{1}{4\cdot 3^{n-1}}\sin(3^nx)-\frac{1}{4\cdot 3^n}\sin(3^{n+1}x)$$

$$\sum_{n=0}^{\infty}\frac{1}{3^n}\sin^3(3^nx)=\lim_{n\to\infty}\left(\frac{3}{4}\sin x-\frac{\sin(3^{n+1}x)}{4\cdot 3^n}\right)=$$

$$=\frac{3}{4}\sin x-\frac{1}{4}\lim_{n\to\infty}\frac{\sin(3^{n+1}x)}{3^n}=\frac{3}{4}\sin x$$

$$4\big(b\Omega(a)+c\Omega(b)+a\Omega(c)\big)=$$

$$=4\left(b\cdot\frac{3}{4}\sin(\sin a)+c\cdot\frac{3}{4}\sin(\sin b)+a\cdot\frac{3}{4}\sin(\sin C)\right)\leq$$

$$\leq 3(ab+bc+ca)\leq 3(a^2+b^2+c^2)$$

SOLUTION AN.070.

First, we prove:

$$\tan\left(\frac{x}{2^{n-1}}\right)\tan^2\left(\frac{x}{2^n}\right)=\tan\left(\frac{x}{2^{n-1}}\right)-2\tan\left(\frac{x}{2^n}\right) \quad (1)$$

$$2\tan\left(\frac{x}{2^n}\right)=\tan\left(\frac{x}{2^{n-1}}\right)\left(1-\tan^2\left(\frac{x}{2^n}\right)\right)$$

$$2\tan\left(\frac{x}{2^n}\right)=\frac{2\tan\frac{x}{2^n}}{1-\tan^2\left(\frac{x}{2^n}\right)}\cdot\left(1-\tan^2\left(\frac{x}{2^n}\right)\right)$$

$$2\tan\left(\frac{x}{2^n}\right)=2\tan\left(\frac{x}{2^n}\right)$$

Multiplying (1) with 2^{n-1}: $2^{n-1}\tan\left(\frac{x}{2^{n-1}}\right)\tan^2\left(\frac{x}{2^n}\right)=2^{n-1}\tan\left(\frac{x}{2^{n-1}}\right)-2^n\tan\left(\frac{x}{2^n}\right)$

$$\Omega(x)=\lim_{n\to\infty}\sum_{k=1}^{n}\left(2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)-2^k\tan\left(\frac{x}{2^k}\right)\right)$$

$$\Omega(x)=\tan x-\lim_{n\to\infty}2^n\tan\left(\frac{x}{2^n}\right)=\tan x-\lim_{n\to\infty}\left(\frac{\tan\left(\frac{x}{2^n}\right)}{\frac{x}{2^n}}\cdot x\right)=\tan x-x$$

$$\Omega(A)+\Omega(B)+\Omega(C)=\tan A+\tan B+\tan C-(A+B+C)=$$

$$=\tan A\cdot\tan B\cdot\tan C-\pi>ABC-\pi$$

SOLUTION AN.071.

$$\Omega=\lim_{p\to\infty}\sqrt[p]{\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k^2+kp-1}{(p+k+1)!}\right)}=\lim_{p\to\infty}\sqrt[p]{\sum_{k=1}^{\infty}\frac{k}{(p+k)!}-\sum_{k=1}^{\infty}\frac{k+1}{(p+k+1)!}}=$$

$$=\lim_{p\to\infty}\sqrt[p]{\frac{1}{(p+1)!}}=\lim_{p\to\infty}\frac{(p+1)!}{(p+2)!}=\lim_{p\to\infty}\frac{1}{p+2}=0$$

SOLUTION AN.072.

Let be $f:[0,1]\to\mathbb{R};\ f(x)=t^x$

$$f'(x)=xt^{x-1};\ f''(x)=x(x-1)t^{x-2}<0;\ f \text{ concave}$$

By Jensen's inequality: $\frac{af\left(\frac{b}{a}\right)+bf\left(\frac{c}{b}\right)+cf\left(\frac{a}{c}\right)+bf\left(\frac{a}{b}\right)+cf\left(\frac{b}{c}\right)+af\left(\frac{c}{a}\right)}{a+b+c+a+b+c}\le$

$$\le f\left(\frac{a\cdot\frac{b}{a}+b\cdot\frac{c}{b}+c\cdot\frac{a}{c}+b\cdot\frac{a}{b}+c\cdot\frac{b}{c}+a\cdot\frac{c}{a}}{a+b+c+a+b+c}\right)$$

$$a\cdot\left(\frac{b}{a}\right)^x+b\left(\frac{c}{b}\right)^x+c\left(\frac{a}{c}\right)^x+b\left(\frac{a}{b}\right)^x+c\left(\frac{b}{c}\right)^x+a\left(\frac{c}{a}\right)^x\le$$

$$\le(a+b+c+a+b+c)\cdot\left(\frac{b+c+a+b+c+a}{a+b+c+a+b+c}\right)^x=(3+3)\cdot 1^x=6$$

SOLUTION AN.073.

$$\sum_{\substack{n=1\\n\neq k}}^{\infty}\left(\frac{1}{n^2-k^2}\right)=\sum_{n=1}^{k-1}\left(\frac{1}{n^2-k^2}\right)+\sum_{n=k+1}^{\infty}\left(\frac{1}{n^2-k^2}\right)=$$

$$=\frac{1}{2k}\sum_{n=1}^{k-1}\left(\frac{1}{n-k}-\frac{1}{n+k}\right)+\frac{1}{2k}\sum_{n=k+1}^{\infty}\left(\frac{1}{n-k}-\frac{1}{n+k}\right)=$$

$$=-\frac{1}{2k}\sum_{n=1}^{k-1}\left(\frac{1}{k-n}+\frac{1}{k+n}\right)+\frac{1}{2k}\sum_{n=k+1}^{\infty}\left(\frac{1}{n-k}-\frac{1}{n+k}\right)=$$

$$=-\frac{1}{2k}\left(\frac{1}{k-1}+\frac{1}{k+1}+\frac{1}{k-2}+\frac{1}{k+2}+\cdots+\frac{1}{1}+\frac{1}{2k-1}\right)+$$

$$+\frac{1}{2k}\left[\left(1-\frac{1}{2k+1}\right)+\left(\frac{1}{2}-\frac{1}{2k+2}\right)+\left(\frac{1}{3}-\frac{1}{2k+3}\right)+\cdots\right]=$$

$$=-\frac{1}{2k}\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}+\frac{1}{k+1}+\cdots+\frac{1}{2k-1}\right)+$$

$$+\frac{1}{2k}\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}\right)=-\frac{1}{2k}\left(-\frac{1}{k}\right)+\frac{1}{4k^2}=\frac{3}{4k^2}$$

$$\Omega=\sum_{k=1}^{\infty}\left(\sum_{\substack{n=1\\n\neq k}}^{\infty}\left(\frac{1}{n^2-k^2}\right)\right)=\sum_{k=1}^{\infty}\left(\frac{3}{4k^2}\right)=\frac{3}{4}\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)=\frac{3}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{8}$$

SOLUTION AN.074.

$$\Omega=\int\left(\sum_{n=1}^{\infty}\left(3^n\sinh^3\left(\frac{x}{3^n}\right)\right)\right)dx=\frac{1}{4}\int\left(\sum_{n=1}^{\infty}\left(3^n\sinh\left(\frac{x}{3^{n-1}}\right)-3^{n+1}\sinh\left(\frac{x}{3^n}\right)\right)\right)dx=$$

$$=\frac{1}{4}\int(3\sinh(x)-3x)\,dx=\frac{3}{4}\cosh(x)-\frac{3}{4}\cdot\frac{x^2}{2}+c=\frac{3\cosh(x)}{4}-\frac{3x^2}{8}+c$$

SOLUTION AN.075.

Let be $m,n,p>0$ such that: $x=e^m;y=e^n;z=e^p$

$$\textit{Let be } f:(0,\infty)\to(0,\infty);f(x)=\frac{1}{1+e^x};\ f'(x)=-\frac{e^x}{(1+e^x)^2};f''(x)=\frac{e^x(e^x-1)}{(1+e^x)^3}>0$$

$$f\ \textit{convexe}\Rightarrow f\left(\frac{m+n+p}{3}\right)\le\frac{1}{3}\big(f(m)+f(n)+f(p)\big)$$

$$\frac{3}{1+e^{\frac{m+n+p}{3}}}\le\frac{1}{1+e^m}+\frac{1}{1+e^n}+\frac{1}{1+e^p}$$

$$\frac{3}{1+\sqrt[3]{xyz}} \le \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}$$

$$\int_a^b\int_a^b\int_a^b \left(\frac{3}{1+\sqrt[3]{xyz}}\right) dx\,dy\,dz \le \int_a^b\int_a^b\int_a^b \left(\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}\right) dx\,dy\,dz$$

$$\int_a^b\int_a^b\int_a^b \left(\frac{1}{1+\sqrt[3]{xyz}}\right) dx\,dy\,dz \le \frac{1}{3}(b-a)^2 \log(x+1)\Big|_a^b =$$

$$= \frac{1}{3}(b-a)^2 \log\left(\frac{b+1}{a+1}\right) = \log\left(\sqrt[3]{\frac{b+1}{a+1}}\right)^{(b-a)^2}$$

SOLUTION AN.076.

Let be $F: [a,b] \to \mathbb{R}$;

$$F(t) = \int_t^b e^{4t^2} dt - \frac{e^{4b^2} - e^{4t^2}}{8be^{b^2}}$$

$$F'(t) = -e^{4t^2} + \frac{8te^{4t^2}}{8be^{b^2}}$$

$$F'(t) = e^{4t^2}\left(\frac{t}{b} - 1\right) = \frac{e^{4t^2}}{b}(t-b)$$

$$t \le b \Rightarrow t-b \le 0 \Rightarrow F'(t) \le 0; F'(b) = 0$$

$$F \textit{ decreasing} \Rightarrow a \le b \Rightarrow F(a) \ge F(b)$$

$$\textit{But } F(b) = 0 \Rightarrow F(a) \ge 0$$

$$\int_a^b e^{4x^2}\,dx - \frac{e^{4b^2} - e^{4a^2}}{8be^{b^2}} \ge 0 \quad (1)$$

$$\textit{Let be } G: [a,b] \to \mathbb{R};\ G(t) = \frac{e^{4t^2} - e^{4a^2}}{8ae^{a^2}} - \int_a^t e^{4t^2}\,dt$$

$$G'(t) = \frac{1}{8ae^{a^2}} \cdot 8te^{4t^2} - e^{4t^2}$$

$$G'(t) = e^{4t^2}\left(\frac{t}{a} - 1\right) = e^{4t^2} \cdot \frac{t-a}{a} \ge 0$$

$$\textit{Because } t \ge a$$

$$G'(a) = 0; G \textit{ – increasing}$$

$$a \le b \Rightarrow G(a) \le G(b); G(a) = 0 \Rightarrow G(b) \ge 0 \Rightarrow \frac{e^{4b^2} - e^{4a^2}}{8ae^{a^2}} - \int_a^b e^{4x^2}\,dx \ge 0 \quad (2)$$

By (1); (2):

$$\frac{e^{4b^2}-e^{4a^2}}{8be^{b^2}} \le \int_a^b e^{4x^2}\,dx \le \frac{e^{4b^2}-e^{4a^2}}{8ae^{a^2}}$$

Equality holds for $a=b$

SOLUTION AN.077.

Let be $F:[a,b]\to\mathbb{R}$;

$$F(t)=\frac{\cos(\cos t)-\cos(\cos a)}{\sin a}-\int_a^t \sin(\cos t)\,dt$$

$$F'(t)=\frac{1}{\sin a}\cdot\sin t\cdot\sin(\cos t)-\sin(\cos t)$$

$$F'(t)=\sin(\cos t)\left(\frac{\sin t}{\sin a}-1\right)$$

$$F'(t)=\sin(\cos t)\cdot\frac{\sin t-\sin a}{\sin a}\ge 0$$

$F(a)=0$; ***F increasing because*** $t\ge a$

$$a<b\Rightarrow F(a)\le F(b)\Rightarrow F(b)\ge 0$$

$$\frac{\cos(\cos b)-\cos(\cos a)}{\sin a}-\int_a^b \sin(\cos t)\,dt\ge 0$$

$$\int_a^b \sin(\cos t)\,dt\le\frac{\cos(\cos b)-\cos(\cos a)}{\sin a}\quad (1)$$

Let be $G:[a,b]\to\mathbb{R}$;

$$G(t)=\int_t^b \sin(-\cos x)\,dx-\frac{-\cos(\cos b)+\cos(\cos t)}{\sin b}$$

$$G'(t)=-\sin(-\cos t)+\frac{\sin t\cdot\sin(-\cos t)}{\sin b},\ G'(t)=\sin(-\cos t)\left(-1+\frac{\sin t}{\sin b}\right)$$

$$G'(t)=\sin(\cos t)\left(1-\frac{\sin t}{\sin b}\right),\ G'(t)=\sin(\cos t)\left(\frac{\sin b-\sin t}{\sin b}\right)\ge 0$$

$$t\le b\Rightarrow \sin t\le\sin b\Rightarrow \sin b-\sin t\ge 0$$

$G'(t)\ge 0$; ***G increasing;*** $G(b)=0$

$$a\le b\Rightarrow G(a)\le G(b)=0$$

$$G(a)\le 0$$

$$\int_a^b \sin(-\cos x)\,dx - \frac{-\cos(\cos b) + \cos(\cos a)}{\sin b} \le 0$$

$$\int_a^b \sin(\cos x)\,dx \ge \frac{\cos(\cos b) - \cos(\cos a)}{\sin b} \quad (2)$$

By (1); (2):

$$\frac{\cos(\cos b) - \cos(\cos a)}{\sin a} \le \int_a^b \sin(\cos x)\,dx \le \frac{\cos(\cos b) - \cos(\cos a)}{\sin b}$$

Equality holds for $a = b$.

SOLUTION AN.078.

Let be $F: [a,b] \to \mathbb{R}; F(t) = \int_t^b t^{24}\,dt - \frac{b^{30} - t^{30}}{30b^5}$

$$F'(t) = -t^{24} + \frac{30t^{29}}{30b^5} = t^{24}\left(-1 + \frac{t^5}{b^5}\right) = t^{24}\left(\frac{t^5 - b^5}{b^5}\right) \le 0 \text{ because } t \le b$$

F decreasing; $F(b) = 0$

$$a \le b \Rightarrow F(a) \ge F(b) \Rightarrow F(a) \ge 0$$

$$\int_a^b t^{24}\,dt - \frac{b^{30} - a^{30}}{30b^5} \ge 0 \Rightarrow \frac{b^{25} - a^{25}}{25} \ge \frac{b^{30} - a^{30}}{30b^5}$$

$$\Rightarrow \frac{b^{25} - a^{25}}{5} \ge \frac{b^{30} - a^{30}}{6b^5} \quad (1)$$

Let be $G: [a,b] \to \mathbb{R}; G(t) = \frac{t^{30} - a^{30}}{30a^5} - \int_a^t t^{24}\,dt$

$$G'(t) = \frac{30t^{29}}{30a^5} - t^{24} = t^{24}\left(\frac{t^5}{a^5} - 1\right) = \frac{t^{24}(t^5 - a^5)}{a^5} \le 0 \quad \text{because } t \ge a$$

G increasing $\Rightarrow G(a) \le G(b); (a \le b)$

$$G(a) = 0 \Rightarrow G(b) \ge 0$$

$$\frac{b^{30} - a^{30}}{30a^5} - \int_a^b t^{24}\,dt \ge 0 \Rightarrow \frac{b^{30} - a^{30}}{30a^5} \ge \frac{b^{25} - a^{25}}{25} \Rightarrow \frac{b^{25} - a^{25}}{5} \le \frac{b^{30} - a^{30}}{6a^5} \quad (2)$$

By (1); (2):

$$\frac{b^{30} - a^{30}}{6b^5} \le \frac{b^{25} - a^{25}}{5} \le \frac{b^{30} - a^{30}}{6a^5} \quad (3)$$

Analogous by $c \le a$:

$$\frac{b^{30} - c^{30}}{6b^5} \le \frac{b^{25} - c^{25}}{5} \le \frac{b^{30} - c^{30}}{6c^5} \quad (4)$$

By multiplying (3); (4):

$$\frac{(b^{30}-a^{30})(b^{30}-c^{30})}{36b^{10}} \le \frac{(b^{25}-a^{25})(b^{25}-c^{25})}{25} \le \frac{(b^{30}-a^{30})(b^{30}-c^{30})}{36(ac)^5}$$

Equality holds for $a=b=c$.

SOLUTION AN.079.

Let be: $\Omega(n) = 6 - 2\sum_{i=2}^{n}\frac{1}{i+1}\binom{2i}{i} + 3\sum_{i=2}^{n}\binom{2i}{i}$

$$\Omega(n) = 6 + \sum_{i=2}^{n}\left(3 - \frac{2}{i+1}\right)\binom{2i}{i}$$

$$\Omega(n) = 6 + \sum_{i=2}^{n}\frac{3i+1}{i+1}\binom{2i}{i}$$

$$\Omega(n) = 2 + \frac{4}{2}\cdot 2 + \sum_{i=2}^{n}\frac{3i+1}{i+1}\binom{2i}{i}$$

$$\Omega(n) = 2 + \frac{3\cdot 1+1}{1+1}\cdot\binom{2}{1} + \sum_{i=2}^{n}\frac{3i+1}{i+1}\binom{2i}{i}$$

$$\Omega(n) = 2 + \sum_{i=1}^{n}\frac{3i+1}{i+1}\binom{2i}{i} = 2 + \sum_{i=1}^{n}\binom{2i}{i}\cdot\frac{4i+2-i-1}{i+1} =$$

$$= 2 + \sum_{i=1}^{n}\binom{2i}{i}\cdot\left(\frac{2(2i+1)}{i+1} - 1\right) = 2 + \sum_{i=1}^{n}\binom{2i}{i}\cdot\left(\frac{2(2i+1)(i+1)}{(i+1)^2} - 1\right)$$

$$\Omega(n) = 2 + \sum_{i=1}^{n}\binom{2i}{i}\left(\frac{(2i+1)(2i+2)}{(i+1)^2} - 1\right)$$

$$\Omega(n) = 2 + \sum_{i=1}^{n}\left(\frac{(2i+2)!}{((i+1)!)^2} - \frac{(2i)!}{(i!)^2}\right)$$

$$\Omega(n) = 2 + \sum_{i=1}^{n}\left(\binom{2i+2}{i+1} - \binom{2i}{i}\right)$$

$$\Omega(n) = 2 + \sum_{i=1}^{n}\binom{2i+2}{i+1} - \sum_{i=1}^{n}\binom{2i}{i}$$

$$\Omega(n) = 2 + \binom{2n+2}{n+1} - 2 = \binom{2n+2}{n+1}$$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\Omega(n)} = \lim_{n\to\infty} \frac{\Omega(n+1)}{\Omega(n)} = \lim_{n\to\infty} \left(\binom{2n+4}{n+2} \cdot \frac{1}{\binom{2n+2}{n+1}} \right) =$$

$$= \lim_{n\to\infty} \left(\frac{(2n+4)!}{\left((n+2)!\right)^2} \cdot \frac{\left((n+1)!\right)^2}{(2n+2)!} \right) = \lim_{n\to\infty} \left(\frac{(2n+2)!\,(2n+3)(2n+4)}{(n+2)^2 \cdot (2n+2)!} \right) =$$

$$= \lim_{n\to\infty} \left(\frac{(2n+3)(2n+4)}{(n+2)^2} \right) = 4$$

SOLUTION AN.080.

If $x, y \geq 1$ ***we prove that:*** $\frac{1}{1+x} + \frac{1}{1+y} \geq \frac{2}{1+\sqrt{xy}}$ ***(1)***

$$\frac{1}{1+x} + \frac{1}{1+y} - \frac{2}{1+\sqrt{xy}} \geq 0$$

$$(1+y)\left(1+\sqrt{xy}\right) + (1+x)\left(1+\sqrt{xy}\right) - 2(1+x)(1+y) \geq 0$$

$$1 + \sqrt{xy} + y + y\sqrt{xy} + 1 + \sqrt{xy} + x\sqrt{xy} - 2 - 2x - 2y - 2xy \geq 0$$

$$2\sqrt{xy} + y\sqrt{xy} + x\sqrt{xy} - 2xy - x - y \geq 0$$

$$-\left(x - 2\sqrt{xy} + y\right) + \sqrt{xy}\left(x + y - 2\sqrt{xy}\right) \geq 0$$

$$\left(x - 2\sqrt{xy} + y\right)\left(\sqrt{xy} - 1\right) \geq 0$$

$$\left(\sqrt{x} - \sqrt{y}\right)^2\left(\sqrt{xy} - 1\right) \geq 0$$

Which is true; $x, y \geq 1 \Rightarrow 1 \Rightarrow \sqrt{xy} \geq 1 \Rightarrow \sqrt{xy} - 1 \geq 0$

Analogous: $\frac{1}{1+y} + \frac{1}{1+z} \geq \frac{2}{1+\sqrt{yz}}$ ***(2)***

$\frac{1}{1+z} + \frac{1}{1+x} \geq \frac{2}{1+\sqrt{zx}}$ ***(3)***

By adding (1); (2); (3): $\frac{2}{1+x} + \frac{2}{1+y} + \frac{2}{1+z} \geq \frac{2}{1+\sqrt{xy}} + \frac{2}{1+\sqrt{y}}$

SOLUTION AN.081.

$$f(x); f(y); f(z) \in [1, \infty) \Rightarrow f(x) \geq 1; f(y) \geq 1 \Rightarrow$$

$$\Rightarrow f(x)(f(y) - 1) + f(y)(f(x) - 1) \geq 0$$

$$2f(x)f(y) - f(x) - f(y) \geq 0 \quad \textit{(1)}$$

$$f(x) \geq 1; f(y) \geq 1 \Rightarrow f(x)f(y) - 1 \geq 0 \quad \textit{(2)}$$

By (1); (2):

$$\sum_{cyc}(f(x)(fy)-1)\left(2f(x)f(y)-f(x)-f(y)\right)\geq 0$$

$$\sum_{cyc}\left(\frac{f(x)+f(y)+2f^2(x)f^2(y)-2f(x)f(y)-f^2(x)f(y)-f(x)f^2(y)}{2f(x)f(y)}\right)\geq 0$$

$$\sum_{cyc}\left(\frac{f(x)+f(y)}{2f(x)f(y)}+f(x)f(y)-1-\frac{f(x)}{2}-\frac{f(y)}{2}\right)\geq 0$$

$$\sum_{cyc}\frac{f(x)+f(y)}{2f(x)f(y)}+\sum_{cyc}f(x)f(y)\geq 3+\sum_{cyc}f(x)$$

$$\sum_{cyc}\frac{1}{f(x)}+\sum_{cyc}f(x)f(y)\geq 3+\sum_{cyc}f(x)$$

$$\left(\sum_{cyc}f(x)f(y)\right)\cdot\frac{1}{f(x)f(y)f(z)}+\sum_{cyc}f(x)f(y)\geq 3+\sum_{cyc}f(x)$$

$$1+\frac{1}{f(x)f(y)f(z)}\geq\frac{3+f(x)+f(y)+f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(x)}$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{3+f(x)+f(y)+f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(x)}\right)dx\,dy\,dz\leq$$

$$\leq\int_a^b\int_a^b\int_a^b dx\,dy\,dz+\int_a^b\int_a^b\int_a^b\left(\frac{dx\,dy\,dz}{f(x)f(y)f(z)}\right)=(b-a)^3+\left(\int_a^b\frac{dx}{f(x)}\right)^3$$

SOLUTION AN.082.

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix};A^n=\begin{pmatrix}1&n\\0&1\end{pmatrix}\quad\textit{(by induction)}$$

$$B=\begin{pmatrix}1&0\\1&1\end{pmatrix};B^n=\begin{pmatrix}1&0\\n&1\end{pmatrix}\quad\textit{(by induction)}$$

$$e^A=\sum_{n=0}^{\infty}\left(\frac{1}{n!}\cdot A^n\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}1&n\\0&1\end{pmatrix}=\begin{pmatrix}\sum_{n=0}^{\infty}\frac{1}{n!}&\sum_{n=0}^{\infty}\frac{n}{n!}\\0&\sum_{n=0}^{\infty}\frac{1}{n!}\end{pmatrix}=\begin{pmatrix}e&e\\0&e\end{pmatrix}$$

$$e^B=\sum_{n=0}^{\infty}\left(\frac{1}{n!}\cdot B^n\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}1&0\\n&1\end{pmatrix}=\begin{pmatrix}e&0\\e&e\end{pmatrix}$$

$$\det(e^B) = e^2; (e^B)^T = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix};$$

$$(e^B)^* = \begin{pmatrix} e & 0 \\ -e & e \end{pmatrix}; (e^B)^{-1} = \begin{pmatrix} \frac{1}{e} & 0 \\ -\frac{1}{e} & \frac{1}{e} \end{pmatrix}$$

$$\Omega = e^A \cdot (e^B)^{-1} = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix}\begin{pmatrix} \frac{1}{e} & 0 \\ -\frac{1}{e} & \frac{1}{e} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

SOLUTION AN.083.

With substitution $y = -x$

$$\Omega(a,b) = \int_{-\sin a}^{\sin a} \frac{dy}{-y^5 + \sin b + \sqrt{y^{10} + \sin^2 b}} = \int_{-\sin a}^{\sin a} \frac{dx}{-x^5 + \sin b + \sqrt{x^{10} + \sin^2 b}}$$

$$2\Omega(a,b) = \int_{-\sin a}^{\sin a} \left(\frac{1}{x^5 + \sin b + \sqrt{x^{10} + \sin^2 b}} + \frac{1}{-x^5 + \sin b + \sqrt{x^{10} + \sin^2 b}}\right) dx$$

$$2\Omega(a,b) = \int_{-\sin a}^{\sin a} \frac{2\sin b + 2\sqrt{x^{10} + \sin^2 b}}{\left(\sin b + \sqrt{x^{10} + \sin^2 b}\right)^2 - x^{10}} dx$$

$$2\Omega(a,b) = \int_{-\sin a}^{\sin a} \frac{2\left(\sin b + \sqrt{x^{10} + \sin^2 b}\right)dx}{\sin^2 b + \sin^2 b + 2\sin b\sqrt{x^{10} + \sin^2 b}}$$

$$2\Omega(a,b) = \int_{-\sin a}^{\sin a} \frac{2\left(\sin b + \sqrt{x^{10} + \sin^2 b}\right)dx}{2\sin b\left(\sin b + \sqrt{x^{10} + \sin^2 b}\right)}$$

$$2\Omega(a,b) = \frac{1}{\sin b}\int_{-\sin a}^{\sin a} dx = \frac{2\sin a}{\sin b}$$

$$\Omega(a,b) = \frac{\sin a}{\sin b}$$

$$\Omega(a,b) + \Omega(b,c) + \Omega(c,a) = \frac{\sin a}{\sin b} + \frac{\sin b}{\sin c} + \frac{\sin c}{\sin a} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{\sin a}{\sin b} \cdot \frac{\sin b}{\sin c} \cdot \frac{\sin c}{\sin a}} = 3$$

SOLUTION AN.084.

By Radon's inequality:

$$\sum_{k=1}^{\infty}\frac{x_k^3}{a_k^2}\ge\frac{(\sum_{k=1}^{\infty}x_k)^3}{(\sum_{k=1}^{\infty}a_k)^2};x_k,a_k>0;k\in\mathbb{N}^*$$

For $x_k=\frac{1}{k^{\alpha}};a_k=\frac{1}{k^{\beta}}$ *we obtain:*

$$\sum_{k=1}^{\infty}\frac{\frac{1}{k^{3\alpha}}}{\frac{1}{k^{2\beta}}}\ge\frac{\left(\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}}\right)^3}{\left(\sum_{k=1}^{\infty}\frac{1}{k^{\beta}}\right)^2}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^{3\alpha-2\beta}}\ge\frac{\left(\zeta(\alpha)\right)^3}{\left(\zeta(\beta)\right)^2}$$

$$\zeta(3\alpha-2\beta)\ge\frac{\left(\zeta(\alpha)\right)^3}{\left(\zeta(\beta)\right)^2}$$

$$\zeta(3\alpha-2\beta)\left(\zeta(\beta)\right)^2\ge\left(\zeta(\alpha)\right)^3$$

SOLUTION AN.085.

Let be: $I_k=\int_{-\frac{1}{k}}^{\frac{1}{k}}(2x^8+3x^6+1)\cos^{-1}(kx)\,dx$

With $y=-x$

$$I_k=\int_{\frac{1}{k}}^{-\frac{1}{k}}(2(-y)^8+3(-y)^6+1)\cos^{-1}(-ky)\,(-dy)$$

$$I_k=\int_{-\frac{1}{k}}^{\frac{1}{k}}(2y^8+3y^6+1)\,(\pi-\cos^{-1}(ky))dy$$

$$I_k=\int_{-\frac{1}{k}}^{\frac{1}{k}}(2y^8+3y^6+1)\cdot\pi dy-I_k$$

$$2I_k=\pi\left(\frac{2y^9}{9}\Bigg|_{-\frac{1}{k}}^{\frac{1}{k}}+\frac{3y^7}{7}\Bigg|_{-\frac{1}{k}}^{\frac{1}{k}}+\frac{2}{k}\right)$$

$$I_k=\frac{2\pi}{9k^9}+\frac{3\pi}{7k^7}+\frac{2\pi}{k}$$

$$\Omega_n = \sum_{k=1}^{n} I_k = \frac{2\pi}{9}\sum_{k=1}^{n}\frac{1}{k^9} + \frac{3\pi}{7}\sum_{k=1}^{n}\frac{1}{k^7} + \pi\cdot H_n$$

$$\Omega = \lim_{n\to\infty}(\Omega - \pi\cdot H_n) = \frac{2\pi}{9}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^9} + \frac{3\pi}{7}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^7} = \frac{2\pi}{9}\zeta(9) + \frac{3\pi}{7}\zeta(7)$$

SOLUTION AN.086.

$$\lim_{x\to\infty}\left(x^2\left(\frac{\pi}{2} - \tan^{-1}x\right) - x\right) = \lim_{x\to\infty} x\left(\frac{\pi x}{2} - x\tan^{-1}x - 1\right) =$$

$$= \lim_{x\to\infty}\frac{\frac{\pi x}{2} - x\tan^{-1}x - 1}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{\pi}{2} - \tan^{-1}x - \frac{x}{1+x^2}}{-\frac{1}{x^2}} =$$

$$\overset{L'H}{=} \lim_{x\to\infty}\frac{-\frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2}}{-\frac{2}{x^3}} = \lim_{x\to\infty}\frac{(1+x^2+1-x^2)x^3}{2(1+x^2)^2} = \lim_{x\to\infty}\frac{2x^3}{2(1+x^2)^2} = 0$$

$$\lim_{n\to\infty} H_n = \infty \Rightarrow \lim_{n\to\infty}\left(H_n^2\left(\frac{\pi}{2} - \tan^{-1}H_n\right) - H_n\right) = 0$$

SOLUTION AN.087.

$$\Omega(k) = \int_0^1 \frac{x^k - kx^{k-1} - x^{k-1} + 1}{x^{2k+1} + 2x^{k+1} + x^k + x + 1}dx =$$

$$= \int_0^1 \frac{x^k - (k+1)x^{k-1} + 1}{x^{k+1}(x^k+1) + x(x^k+1) + (x^k+1)}dx =$$

$$= \int_0^1 \frac{\big((k+1)x^k + 1\big)\big(x^k+1\big) - (k+1)x^{k-1}\big(x^{k+1}+x+1\big)}{(x^k+1)(x^{k+1}+x+1)}dx =$$

$$= \int_0^1\left(\frac{(k+1)x^k+1}{x^{k+1}+x+1} - \frac{(k+1)x^{k-1}}{x^k+1}\right)dx = \log\left|x^{k+1}+x+1\right|\Big|_0^1 - \frac{k+1}{k}\log\left|x^k+1\right|\Big|_0^1 =$$

$$= \log 3 - \frac{k+1}{k}\log 2$$

$$\Omega = \lim_{n\to\infty}\left(\log 2\cdot\log n - n\log\left(\frac{3}{2}\right) + \sum_{k=1}^{n}\Omega(k)\right) =$$

$$= \lim_{n\to\infty}\left(\log 2\cdot\log n - n\log\left(\frac{3}{2}\right) + \sum_{k=1}^{n}\left(\log 3 - \frac{k+1}{k}\log 2\right)\right) =$$

$$= \lim_{n\to\infty}\left(\log 2\cdot\log n - n\log 3 + n\log 2 + n\log 3 - \log 2\sum_{k=1}^{n}\frac{k+1}{k}\right) =$$

$$= \lim_{n\to\infty}\log 2\cdot\left(\log 2 - \sum_{k=1}^{n}\left(1+\frac{1}{k}\right) + n\right) =$$

$$= \log 2\cdot\lim_{n\to\infty}\left(\log n - n - \sum_{n=1}^{n}\frac{1}{k} + n\right) = -\log 2\cdot\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) =$$

$$= -\log 2\cdot\gamma = -\gamma\log 2$$

SOLUTION AN.088.

First, we prove that: $\frac{x^4+y^4}{(x^2+y^2)(x^2-xy+y^2)} \geq$ ***(1)***

$$x^4 + y^4 \geq (x^2+y^2)(x^2-xy+y^2)$$

$$x^4 + y^4 \geq x^4 - x^3y + x^2y^2 + y^2x^2 - xy^3 + y^4$$

$$x^3y + xy^3 - 2x^2y^2 \geq 0$$

$$xy(x^2 - 2xy + y^2) \geq 0$$

$$xy(x-y)^2 \geq 0 \text{ which is true.}$$

$$\text{Let be } f: [0,\infty) \to \mathbb{R}; f(x) = \tan^{-1} x$$

$$f'(x) = \frac{1}{1+x^2} > 0; f \text{ increasing}$$

$$\text{By (1)} \Rightarrow f\left(\frac{x^4+y^4}{(x^2+y^2)(x^2-xy+y^2)}\right) \geq f(1) = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{x^4+y^4}{(x^2+y^2)(x^2-xy+y^2)}\right) \geq \frac{\pi}{4} \quad \text{(2). Analogous: } \tan^{-1}\left(\frac{y^4+z^4}{(y^2+z^2)(y^2-yz+z^2)}\right) \geq \frac{\pi}{4} \quad (3)$$

$$\tan^{-1}\left(\frac{z^4+x^4}{(z^2+x^2)(z^2-zx+z^2)}\right) \geq \frac{\pi}{4} \quad (4)$$

By adding (2); (3); (4):

$$\sum_{cyc}\tan^{-1}\left(\frac{x^4+y^4}{(x^2+y^2)(x^2-xy+y^2)}\right) \geq \frac{3\pi}{4}$$

SOLUTION AN.089.

Let be $f: [0,\infty) \to \mathbb{R}; f(x) = (x+1)\log(x+1) - x - \frac{x^2}{2}$

$$f'(x) = \log(x+1) - x; f''(x) = \frac{-x}{1+x} \leq 0$$

$$f''(x) \leq 0 \Rightarrow f' \text{ decreasing} \Rightarrow f'(x) \leq f'(0) = 0$$

$$f'(x) < 0 \Rightarrow f \text{ decreasing} \Rightarrow f(x) \le f(0) = 0$$

$$(x+1)\log(x+1) - x - \frac{x^2}{2} \le 0$$

$$(x+1)\log(x+1) \le x + \frac{x^2}{2}$$

$$\log(x+1)^{x+1} \le \log e^{x+\frac{x^2}{2}}$$

$$(x+1)^{x+1} \le e^{x+\frac{x^2}{2}}$$

Let be $x \to \tan^2 x$

$$(\tan^2 x + 1)^{\tan^2 x + 1} \le e^{\tan^2 x + \frac{\tan^4 x}{2}}$$

$$\left(\frac{1}{\cos^2 x}\right)^{\frac{1}{\cos^2 x}} \le e^{\tan^2 x + \frac{\tan^4 x}{2}}$$

$$e^{\cos^2 x\left(\tan^2 x + \frac{\tan^4 x}{2}\right)} \ge \frac{1}{\cos^2 x}$$

$$\int_a^b e^{\cos^2 x}\left(\frac{\sin^2 x}{\cos^2 x} + \cos^2 x \cdot \frac{\sin^2 x}{2\cos^2 x} \cdot \tan^2 x\right)dx \ge \int_a^b \frac{1}{\cos^2 x}dx$$

$$\int_a^b \left(e^{\sin^2 x + \frac{\sin^4 x}{2\cos^2 x}}\right)dx \ge \tan b - \tan a$$

SOLUTION AN.090.

If $u, v > 0$ *then:* $\left(\frac{u}{v} + \frac{v}{u}\right) \cdot \frac{1}{u+v} \overset{AM-GM}{\ge} 2\sqrt{\frac{u}{v} \cdot \frac{v}{u}} \cdot \frac{1}{u+v} =$

$$= \frac{2}{u+v} = \frac{1}{\frac{u+v}{2}} \overset{AM-GM}{\ge} \frac{1}{\frac{2}{\frac{1}{u}+\frac{1}{v}}} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{v}\right) \quad (1)$$

We take in (1): $u = f(x); v = f(y)$

$$\left(\frac{f(x)}{f(y)} + \frac{f(y)}{f(x)}\right) \cdot \frac{1}{f(x) + f(y)} \ge \frac{1}{2}\left(\frac{1}{f(x)} + \frac{1}{f(y)}\right)$$

$$\int_a^b \int_a^b \left(\frac{f(x)}{f(y)} + \frac{f(y)}{f(x)}\right) \cdot \frac{dxdy}{f(x) + f(y)} \ge \frac{1}{2}\int_a^b \int_a^b \left(\frac{1}{f(x)} + \frac{1}{f(y)}\right)dxdy =$$

$$= \frac{(b-a)}{2}\int_a^b \frac{dx}{f(x)} = (b-a)\int \frac{dx}{f(x)}$$

SOLUTION AN.091.

$$(a-b)^2 \geq 0 \Rightarrow a^2 - 2ab + b^2 \geq 0$$

$$2a^2 - 4ab + 2b^2 \geq 0 \Rightarrow 2a^2 + 5ab + 2b^2 \geq 9ab$$

$$2a^2 + 4ab + ab + 2a^2 \geq 9ab$$

$$(2a+b)(a+2b) \geq 9ab$$

$$\frac{(2a+b)(a+2b)}{9ab} \geq 1 \quad (1)$$

$$\textit{Let be } f:(0,\infty) \to \mathbb{R}; f(x) = \tan^{-1} x$$

$$f'(x) = \frac{1}{1+x^2} > 0; f \textit{ increasing}$$

By (1):

$$\tan^{-1}\left(\frac{(2a+b)(a+2b)}{9ab}\right) \geq \tan^{-1} 1 = \frac{\pi}{4} \quad (2)$$

$$\textit{Analogous: } \tan^{-1}\left(\frac{(2b+c)(b+2c)}{9bc}\right) \geq \frac{\pi}{4} \quad (3); \tan^{-1}\left(\frac{(2c+a)(c+2a)}{9ca}\right) \geq \frac{\pi}{4} \quad (4)$$

By adding (2); (3); (4):

$$\sum_{cyc} \tan^{-1}\left(\frac{(2a+b)(a+2b)}{9ab}\right) \geq \frac{3\pi}{4}$$

SOLUTION AN.092.

$$(x-y)^4 + (x^2-y^2)^2 + xy(x-y)^2 \geq 0$$

$$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 + x^4 + y^4 + x^3y + xy^3 - 4x^2y^2 \geq 0$$

$$2x^4 - 3x^3y + 2x^2y^2 - 3xy^3 + 2y^4 \geq 0$$

$$3x^3y + 3xy^3 \leq 2x^4 + 2x^2y^2 + 2y^4$$

$$\frac{x^2+y^2}{x^4+x^2y^2+y^4} \leq \frac{2}{3xy}$$

$$\frac{3}{2} \cdot \frac{x^2+y^2}{x^4+x^2y^2+y^4} \leq \frac{1}{xy}$$

$$\frac{3}{2}\int_a^b \int_a^b \left(\frac{x^2+y^2}{x^4+x^2y^2+y^4}\right) dxdy \leq \int_a^b \int_a^b \frac{dxdy}{xy} = \left(\int_a^b \frac{1}{x} dx\right)^2 = \left(\log\left(\frac{b}{a}\right)\right)^2$$

SOLUTION AN.093.

$$(a^3-b^3)^2 + 3ab(a-b)^2(a^2+ab+b^2) \geq 0$$

$$(a^3-b^3)^2 + 3a^4b(b-a) + 3ab^4(a-b) \geq 0$$

$$a^6+b^6+3a^4b^2+3a^2b^4-2a^3b^3-3a^5b-3ab^5\geq 0$$

$$2a^6+6a^4b^2+6a^2b^4+2b^6\geq a^6+3a^5b+3a^4b^2+a^3b^3+a^3b^3+3a^2b^4+3ab^5+b^6$$

$$2(a^6+3a^4b^2+3a^2b^4+b^6)\geq (a^3+b^3)(a^3+3a^2b+3ab^2+b^3)$$

$$2(a^2+b^2)^3\geq (a^3+b^3)(a+b)^3$$

$$\left(\frac{a^2+b^2}{a+b}\right)^3\geq \frac{1}{2}(a^3+b^3)$$

$$\sum_{cyc}\left(\frac{a^2+b^2}{a+b}\right)^3\geq \sum_{cyc}\frac{1}{2}(a^3+b^3)=\sum_{cyc}a^3 \quad (1)$$

For $a=f(x);b=f(y);c=f(z)$ ***in (1):***

$$\sum_{cyc}\left(\frac{f^2(x)+f^2(y)}{f(x)+f(y)}\right)^3\geq \sum_{cyc}f^3(x)$$

$$\Omega(x,y,z)\geq f^3(x)+f^3(y)+f^3(z)$$

$$\int_a^b\int_a^b\int_a^b \Omega(x,y,z)\,dx\,dy\,dz\geq \int_a^b\int_a^b\int_a^b \left(f^3(x)+f^2(y)+f^3(z)\right)dxdydz=$$

$$=3(b-a)^2\int_a^b f^3(x)\,dx$$

SOLUTION AN.094.

If $u,v>0$ ***then:***

$$(u-v)^2(u+v)\geq 0$$

$$u^2(u-v)-v^2(u-v)\geq 0$$

$$u^2v^2+u^3+v^3+uv\geq u^2v^2+u^2v+uv^2+uv$$

$$u^2v^2+u^3+v^3+uv\geq uv(uv+u+v+1)$$

$$(u^2+v)(v^2+u)\geq uv(u+1)(v+1)$$

$$\frac{(u^2+v)(v^2+u)}{(u+1)(v+1)}\geq uv \quad (1)$$

In (1) we take $u=f(x);v=f(y)$:

$$\frac{\left(f^2(x)+f(y)\right)\left(f^2(y)+f(x)\right)}{\left(1+f(x)\right)\left(1+f(y)\right)}dxdy\geq f(x)f(y)$$

$$\int_a^b\int_a^b \frac{\left(f^2(x)+f(y)\right)\left(f^2(y)+f(x)\right)}{\left(1+f(x)\right)\left(1+f(y)\right)}dxdy \ge \int_a^b\int_a^b f(x)f(y)\,dxdy = \left(\int_a^b f(x)\,dx\right)^2$$

SOLUTION AN.095.

$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$. *By Young's inequality:*

$$\frac{x^2}{2}+\frac{y^3}{3}+\frac{z^6}{6}\ge xyz \quad (Equality\ for\ x=y=z)\ (1)$$

By integrating in (1):

$$\int_0^x \frac{x^2}{2}dx+\int_0^x \frac{y^3}{3}dx+\int_0^x \frac{z^6}{6}dx > yz\int_0^x x\,dx$$

$$\frac{x^3}{6}+\frac{y^3x}{3}+\frac{z^6x}{6} > \frac{x^2}{2}yz \quad (2)$$

$$\int_0^y \frac{x^2}{2}dx+\int_0^y \frac{y^3}{3}dy+\int_0^y \frac{z^6}{6}dy > xz\int_0^y y\,dy$$

$$\frac{x^2y}{2}+\frac{y^4}{12}+\frac{z^6y}{6} > x\cdot\frac{y^2}{2}\cdot z \quad (3)$$

$$\int_0^z \frac{x^2}{2}dz+\int_0^z \frac{y^3}{3}dy+\int_0^z \frac{z^6}{6}dz > xy\int_0^z z\,dz$$

$$\frac{x^2z}{2}+\frac{y^3z}{3}+\frac{z^7}{42} > xy\cdot\frac{z^2}{2} \quad (4)$$

By adding (2); (3); (4):

$$\frac{x^2}{2}\left(\frac{x}{3}+y+z\right)+\frac{y^3}{3}\left(x+\frac{y}{4}+z\right)+\frac{z^6}{6}\left(x+y+\frac{z}{7}\right) > \frac{xyz(x+y+z)}{2}$$

SOLUTION AN.096.

$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$. *By Young's inequality:* $abc\le\frac{a^2}{2}+\frac{b^3}{3}+\frac{c^6}{6}$ *(1)*

In (1) we take $a=\frac{1}{n^2}$; $b=\frac{1}{n^3}$; $c=\frac{1}{n^5}$

$$\frac{1}{2}\cdot\frac{1}{n^4}+\frac{1}{3}\cdot\frac{1}{n^9}+\frac{1}{6}\cdot\frac{1}{n^{30}} > \frac{1}{n^2}\cdot\frac{1}{n^3}\cdot\frac{1}{n^5}; n\ge 2$$

Pass to the sum:

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}+\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^9}+\frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{n^{30}} > \sum_{n=1}^{\infty}\frac{1}{n^{10}}$$

$$\frac{1}{2}\zeta(4)+\frac{1}{3}\zeta(9)+\frac{1}{6}\zeta(30)>\zeta(10)$$

SOLUTION AN.097.

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=1. \textit{ Let be } (x_n)_{n\geq 1}; x_n>0; \lim_{n\to\infty} x_n=0.$$

$$(\forall)\varepsilon>0; (\exists)N(\varepsilon)\in\mathbb{N}; (\forall)n\geq N(\varepsilon);$$

$$\left|\frac{\log(1+x_n)}{x_n}-1\right|<\varepsilon$$

$$-\varepsilon<\frac{\log(1+x_n)}{x_n}-1<\varepsilon$$

$$1-\varepsilon<\frac{\log(1+x_n)}{x_n}<1+\varepsilon$$

$$\textit{For } k\in\mathbb{N}^*; k-\textit{fixed}; x_n=\frac{n}{n^2+k^2}\to 0$$

$$(1-\varepsilon)\cdot\frac{n}{n^2+k^2}<\log\left(1+\frac{n}{n^2+k^2}\right)<(1+\varepsilon)\frac{n}{n^2+k^2}$$

$$(1-\varepsilon)\sum_{k=1}^{n}\frac{n}{n^2+k^2}<\sum_{k=1}^{n}\log\left(\frac{n^2+k^2+n}{n^2+k^2}\right)<(1+\varepsilon)\sum_{k=1}^{n}\frac{n}{n^2+k^2}; (\forall)\varepsilon>0$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}=\int_0^1\frac{1}{1+x^2}dx=\tan^{-1}x\Big|_0^1=\frac{\pi}{4}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\log\left(\frac{n^2+k^2+n}{n^2+k^2}\right)=\frac{\pi}{4}$$

$$\log\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{n^2+k^2+n}{n^2+k^2}\right)\right)=\frac{\pi}{4}$$

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{n^2+k^2+n}{n^2+k^2}\right)=e^{\frac{\pi}{4}}$$

SOLUTION AN.098.

By integral form of AM-GM:

$$e^{\frac{1}{1-0}\int_0^1\log(f(x))dx}\leq\frac{1}{1-0}\int_0^1 f(x)\,dx \quad (1)$$

$$e^{\frac{1}{1-0}\int_0^1\log(g(x))dx}\leq\frac{1}{1-0}\int_0^1 g(x)\,dx \quad (2)$$

$$e^{\frac{1}{1-0}\int_0^1\log(h(x))dx}\leq\frac{1}{1-0}\int_0^1 h(x)\,dx \quad (3)$$

By multiplying (1); (2); (3):

$$e^{\int_0^1 \log(f(x))dx+\int_0^1 \log(g(x))dx+\int_0^1 \log(h(x))dx} \le$$

$$\le \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right)\left(\int_0^1 h(x)\,dx\right) \overset{GM-HM}{\le}$$

$$\le \left(\frac{\int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx + \int_0^1 h(x)\,dx}{3}\right)^3 = \frac{1}{27}\left(\int_0^1 f(x)+g(x)+h(x)\,dx\right)^3$$

$$27e^{\int_0^1 \log(f(x)g(x)h(x))dx} \le \left(\int_0^1 \big(f(x)+g(x)+h(x)\big)\,dx\right)^3$$

SOLUTION AN.099.

Let be $f:[a,b]\to\mathbb{R}; f(x)=\frac{1}{x-a}\int_a^x(\tan^{-1}t)\,dt$

$$f'(x)=\frac{(\tan^{-1}x)(x-a)-\int_a^x(\tan^{-1}t)\,dt}{(x-a)^2}$$

$$\int_a^x(\tan^{-1}t)\,dt \overset{MVT}{=} (\tan^{-1}c)(x-a); 0<c<x$$

$$f'(x)=\frac{(\tan^{-1}x)(x-a)-(\tan^{-1}c)(x-a)}{(x-a)^2}=\frac{\tan^{-1}x-\tan^{-1}c}{x-a}>0$$

Because $g(x)=\tan^{-1}x\,; g:(a,b)\to\mathbb{R}$ ***is increasing***

$$c<x\Rightarrow \tan^{-1}x-\tan^{-1}c>0$$

$$x>a\Rightarrow x-a>0$$

$$f'(x)>0\Rightarrow f \textit{ increasing}\Rightarrow f(x)<f(b)$$

$$\Rightarrow f\big(\sqrt{ab}\big)<f(b)$$

$$\frac{1}{\sqrt{ab}-a}\int_a^{\sqrt{ab}}(\tan^{-1}t)\,dt<\frac{1}{b-a}\int_a^b(\tan^{-1}t)\,dt$$

$$\frac{\int_a^b(\tan^{-1}t)\,dt}{\int_a^{\sqrt{ab}}(\tan^{-1}t)dt}>\frac{\frac{1}{\sqrt{ab}-a}}{\frac{1}{b-a}}=\frac{b-a}{\sqrt{ab}-a}=$$

$$=\frac{(\sqrt{b}-\sqrt{a})(\sqrt{b}+\sqrt{a})}{\sqrt{a}(\sqrt{b}-\sqrt{a})}=\frac{\sqrt{b}+\sqrt{a}}{\sqrt{a}}=1+\sqrt{\frac{b}{a}}$$

SOLUTION AN.100.

Let be $f:[a,b]\to\mathbb{R}; f(x)=\frac{1}{x-a}\int_a^b(\tan^{-1}t)\,dt$

$$f'(x)=\frac{(\tan^{-1}x)(x-a)-\int_a^x(\tan^{-1}t)dt}{(x-a)^2}$$

$$\int_a^x(\tan^{-1}t)\,dt\overset{MVT}{=}(\tan^{-1}c)(x-a);0<c<x$$

$$f'(x)=\frac{(\tan^{-1}x)(x-a)-(\tan^{-1}c)(x-a)}{(x-a)^2}=\frac{\tan^{-1}x-\tan^{-1}c}{x-a}>0 \text{ beause the function}$$

$$g:(a,b)\to\mathbb{R}; g(x)=\tan^{-1}x$$

$$g'(x)=\frac{1}{1+x^2}>0 \text{ is increasing}$$

$$x>c\Rightarrow\tan^{-1}x-\tan^{-1}c>0$$

$$x>0\Rightarrow x-a>0$$

$$f'(x)>0\Rightarrow f \text{ increasing}\Rightarrow f(x)<f(b)\Rightarrow f\left(\frac{a+b}{2}\right)<f(b)$$

$$\frac{1}{\frac{a+b}{2}-a}\int_a^b(\tan^{-1}t)\,dt<\frac{1}{b-a}\int_a^b(\tan^{-1}t)\,dt$$

$$\frac{\int_a^{\frac{a+b}{2}}(\tan^{-1}t)\,dt}{\int_a^b(\tan^{-1}t)\,dt}<\frac{\frac{1}{b-a}}{\frac{1}{\frac{a+b-2a}{2}}}=\frac{\frac{1}{b-a}}{\frac{2}{b-a}}=\frac{1}{2}$$

SOLUTION AN.101.

Let be $f:[a,b)\to\mathbb{R}; f(x)=\frac{1}{b-x}\int_x^b(\log t)\,dt$

$$f'(x)=\frac{-(\log x)(b-x)+\int_x^b(\log t)dt}{(b-x)^2}\overset{MVT}{=}$$

$$=\frac{-(\log x)(b-x)+(\log c)(b-x)}{(b-x)^2}=\frac{\log c-\log x}{b-x}>0; x<c<b$$

$$\text{Because the function } g:[a,b]\to\mathbb{R}; g(x)=\log x$$

$$g'(x)=\frac{1}{x}>0 \text{ is increasing.}$$

$$c > x \Rightarrow \log c - \log x > 0$$

$$b > x \Rightarrow b - x > 0$$

$$f'(x) > 0 \Rightarrow f \text{ increasing} \Rightarrow f(\sqrt{ab}) > f(a)$$

$$\frac{1}{b-a}\int_x^b (\log t)\,dt < \frac{1}{b-\sqrt{ab}}\int_{\sqrt{ab}}^b (\log t)\,dt$$

$$\frac{\int_a^b (\log t)\,dt}{\int_{\sqrt{ab}}^b (\log t)\,dt} < \frac{\frac{1}{b-\sqrt{ab}}}{\frac{1}{b-a}} = \frac{b-a}{\sqrt{b}(\sqrt{b}-\sqrt{a})} = \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}} = 1 + \sqrt{\frac{a}{b}}$$

SOLUTION AN.102.

$$\left|\int_0^1 \log(1+x^2)\,dx - \frac{1}{n}\sum_{k=1}^n \log\left(1+\frac{k^2}{n^2}\right)\right| = \left|\sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\log(1+x^2) - \log\left(1+\frac{k^2}{n^2}\right)\right) dx\right| =$$

$$\overset{MVT}{=} \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left|f'(c_k)\left(x-\frac{k}{n}\right)\right| dx;\ c_k \in \left(\frac{k-1}{n}, \frac{k}{n}\right)$$

$$\text{where } f(x) = \log(1+x^2)\,;\ f: [0,1] \to \mathbb{R}$$

$$f'(x) = \frac{2x}{1+x^2} \le 1 \Rightarrow |f'(c_k)| \le 1$$

$$\left|\int_0^1 \log(1+x^2)\,dx - \frac{1}{n}\sum_{k=1}^n \log\left(1+\frac{k^2}{n^2}\right)\right| \le \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left|x-\frac{k}{n}\right| dx = \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\frac{k}{n}-x\right) dx =$$

$$= \sum_{k=1}^n \left(\frac{k}{n}\left(\frac{k}{n}-\frac{k-1}{n}\right) - \frac{1}{2}\left(\frac{k^2}{n^2}-\frac{(k-1)^2}{n^2}\right)\right) = \sum_{k=1}^n \left(\frac{k}{n^2}-\frac{2k-1}{2n^2}\right) = \sum_{k=1}^n \frac{1}{2n^2} = \frac{1}{2n}$$

SOLUTION AN.103.

$$\left|\sum_{k=1}^n \tan^{-1}\left(\frac{k+1}{\sqrt{3}}\right) - \int_0^n \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) dx\right| =$$

$$= \left|\sum_{k=1}^n \left(\tan^{-1}\left(\frac{k+1}{\sqrt{3}}\right) - \int_{k-1}^k \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) dx\right)\right| \le$$

$$\leq \sum_{k=1}^{n} \left| \tan^{-1}\left(\frac{k+1}{\sqrt{3}}\right) - \int_{k-1}^{k} \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) dx \right| =$$

$$= \sum_{k=1}^{n} \left| \int_{k-1}^{k} \left(\tan^{-1}\left(\frac{k+1}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) \right) dx \right| \overset{MVT}{=}$$

$$= \sum_{k=1}^{n} \left| \frac{1}{\sqrt{3}} \int_{k-1}^{k} \left(\frac{1}{(c_k+1)^2+3} \cdot (k-x) dx \right) \right| \leq$$

$$\overset{(c_k \in (x,k))}{\leq} \frac{1}{\sqrt{3}} \sum_{k=1}^{n} \int_{k-1}^{k} \left| \frac{1}{(c_k+1)^2+3} \right| \cdot (k-x) dx \leq$$

$$\leq \frac{1}{\sqrt{3}} \cdot \frac{1}{3} \sum_{k=1}^{n} \int_{k-1}^{k} (k-x)\, dx = \frac{1}{3\sqrt{3}} \sum_{k=1}^{n} \left(k - \frac{x^2}{2} \Big|_{k-1}^{k} \right) = \frac{1}{3\sqrt{3}} \sum_{k=1}^{n} \left(k - \frac{k^2-(k-1)^2}{2} \right) =$$

$$= \frac{1}{3\sqrt{3}} \sum_{k=1}^{n} \left(k - \frac{k^2-k^2+2k-1}{2} \right) = \frac{1}{3\sqrt{3}} \sum_{k=1}^{n} \left(k - \frac{2k-1}{2} \right) = \frac{1}{3\sqrt{3}} \sum_{k=1}^{n} \frac{2k-2k+1}{2} =$$

$$= \frac{1}{3\sqrt{3}} \sum_{k=1}^{n} \frac{1}{2} = \frac{n}{6\sqrt{3}} = \frac{n\sqrt{3}}{18}$$

SOLUTION AN.104.

Let be $\varphi: (0,\infty) \to \mathbb{R}; \varphi(x) = \log x$;

$\varphi'(x) = \frac{1}{x}$; $\varphi''(x) = -\frac{1}{x^2} < 0$; φ *- concave. By Jensen's inequality:*

$$\varphi\left(\frac{1}{b-a} \int_a^b \frac{x^2+1}{x^4+1} dx \right) \geq \frac{1}{b-a} \int_a^b \varphi\left(\frac{x^2+1}{x^4+1}\right) dx$$

$$\log\left(\frac{1}{b-a} \int_a^b \frac{x^2+1}{x^4+1} dx \right) \geq \frac{1}{b-a} \int_a^b \log\left(\frac{x^2+1}{x^4+1}\right) dx$$

$$\frac{1}{b-a} \int_a^b \frac{x^2+1}{x^4+1} dx \geq e^{\frac{1}{b-a} \int_a^b \log\left(\frac{x^2+1}{x^4+1}\right) dx} \quad (1)$$

Analogous: $\frac{1}{b-a} \int_a^b \frac{x^4+1}{x^6+1} dx \geq e^{\frac{1}{b-a} \int_a^b \log\left(\frac{x^4+1}{x^6+1}\right)} dx$ *(2);* $\frac{1}{b-a} \int_a^b \frac{x^6+1}{x^2+1} dx \geq e^{\frac{1}{b-a} \int_a^b \log\left(\frac{x^6+1}{x^2+1}\right) dx}$ *(3)*

By multiplying (1); (2); (3):

$$\frac{1}{(b-a)^3}\left(\int_a^b \frac{x^2+1}{x^4+1}dx\right)\left(\int_a^b \frac{x^4+1}{x^6+1}dx\right)\left(\int_a^b \frac{x^6+1}{x^2+1}dx\right) \geq$$

$$\geq e^{\frac{1}{b-a}\int_a^b \log\left(\left(\frac{x^2+1}{x^4+1}\right)\cdot\left(\frac{x^4+1}{x^6+1}\right)\cdot\left(\frac{x^6+1}{x^4+1}\right)\right)dx} = e^{\frac{1}{b-a}\cdot 0} = 1$$

$$\left(\int_a^b \frac{x^2+1}{x^4+1}dx\right)\left(\int_a^b \frac{x^4+1}{x^6+1}dx\right)\left(\int_a^b \frac{x^6+1}{x^2+1}\right) \geq (b-a)^3$$

SOLUTION AN.105.

Let be $\varphi: (0,\infty) \to \mathbb{R}; \varphi(x) = \log x$;

$\varphi'(x) = \frac{1}{x}; \varphi''(x) = -\frac{1}{x^2} < 0 \Rightarrow \varphi$ *concave. By Jensen's inequality:*

$$\varphi\left(\frac{1}{b-a}\int_a^b \frac{f_1(x)}{f_2(x)}dx\right) \geq \frac{1}{b-a}\int_a^b \varphi\left(\frac{f_1(x)}{f_2(x)}\right)dx$$

$$\log\left(\frac{1}{b-a}\int_a^b \frac{f_1(x)}{f_2(x)}dx\right) \geq \frac{1}{b-a}\int_a^b \log\left(\frac{f_1(x)}{f_2(x)}\right)dx$$

$$\frac{1}{b-a}\int_a^b \frac{f_1(x)}{f_2(x)}dx \geq e^{\frac{1}{b-a}\int_a^b \log\left(\frac{f_1(x)}{f_2(x)}\right)dx} \quad (1)$$

Analogous: $\frac{1}{b-a}\int_a^b \frac{f_2(x)}{f_3(x)}dx \geq e^{\frac{1}{b-a}\int_a^b \log\left(\frac{f_2(x)}{f_3(x)}\right)dx}$ *(2)*

$$\frac{1}{b-a}\int_a^b \frac{f_3(x)}{f_1(x)}dx \geq e^{\frac{1}{b-a}\int_a^b \log\left(\frac{f_3(x)}{f_1(x)}\right)dx} \quad (3)$$

By multiplying (1); (2); (3):

$$\frac{1}{(b-a)^3}\left(\int_a^b \frac{f_1(x)}{f_2(x)}dx\right)\left(\int_a^b \frac{f_2(x)}{f_3(x)}dx\right)\left(\int_a^b \frac{f_3(x)}{f_1(x)}dx\right) \geq$$

$$\geq e^{\frac{1}{b-a}\int_a^b \log\left(\frac{f_1(x)}{f_2(x)}\cdot\frac{f_2(x)}{f_3(x)}\cdot\frac{f_3(x)}{f_1(x)}\right)dx} = e^{\frac{1}{b-a}\cdot 0} = e^0 = 1$$

$$\left(\int_a^b \frac{f_1(x)}{f_2(x)}dx\right)\left(\int_a^b \frac{f_2(x)}{f_3(x)}dx\right)\left(\int_a^b \frac{f_3(x)}{f_1(x)}dx\right) \geq (b-a)^3$$

SOLUTION AN.106.

We prove by induction:

$$P(n): \frac{1}{\sqrt{3}} \cdot \frac{3}{5} \cdot \frac{7}{9} \cdot \ldots \cdot \frac{4n-1}{4n+1} < \frac{1}{\sqrt{4n+3}}$$

$$P(1): \frac{1}{\sqrt{3}} \cdot \frac{3}{5} < \frac{1}{\sqrt{6}} \Leftrightarrow 3\sqrt{6} < 5\sqrt{3} \Leftrightarrow 54 < 75$$

$$P(n+1): \frac{1}{\sqrt{3}} \cdot \frac{3}{5} \cdot \frac{7}{9} \cdot \ldots \cdot \frac{4n-1}{4n+1} \cdot \frac{4n+3}{4n+5} < \frac{1}{\sqrt{4n+6}} \quad \textit{(to prove)}$$

$$\frac{1}{\sqrt{3}} \prod_{k=1}^{n+1} \left(\frac{4k-1}{4k+1}\right) < \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{4n+3}} \cdot \frac{4n+3}{4n+5} < \frac{1}{\sqrt{4n+6}}$$

$$\frac{\sqrt{4n+3}}{\sqrt{3}(4n+5)} < \frac{1}{\sqrt{4n+6}} \Leftrightarrow (4n+3)(4n+6) < 3(4n+5)^2$$

$$16n^2 + 24n + 12n + 18 < 48n^2 + 120n + 75$$

$$\Leftrightarrow 0 < 32n^2 + 84n + 57 \quad \textit{(True)}$$

$$0 < \frac{1}{\sqrt{3}} \prod_{k=1}^{n} \left(\frac{4k-1}{4k+1}\right) < \frac{1}{\sqrt{4n+3}}$$

$$0 \le \frac{1}{\sqrt{3}} \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{4k-1}{4k+1}\right) \le 0 \Rightarrow \Omega = 0$$

SOLUTION AN.107.

We prove by induction:

$$P(n): \frac{1}{2^n} \cdot \frac{1}{2} \cdot \frac{7}{5} \cdot \frac{13}{8} \cdot \ldots \cdot \frac{6n-6}{3n-1} < \frac{1}{\sqrt{6n+1}}$$

$$P(1): \frac{1}{2} \cdot \frac{1}{2} < \frac{1}{\sqrt{7}} \Leftrightarrow \sqrt{7} < 4 \textit{ (True)}$$

$$P(n+1): \frac{1}{2^{n+1}} \cdot \frac{1}{2} \cdot \frac{7}{5} \cdot \frac{13}{8} \cdot \ldots \cdot \frac{6n-5}{3n-1} \cdot \frac{6n+1}{3n+2} < \frac{1}{\sqrt{6n+7}}$$

$$\frac{1}{2^{n+1}} \cdot \frac{1}{2} \cdot \frac{7}{5} \cdot \frac{13}{8} \cdot \ldots \cdot \frac{6n-5}{3n-1} \cdot \frac{6n+1}{3n+2} < \frac{1}{2} \cdot \frac{1}{\sqrt{6n+1}} \cdot \frac{6n+1}{3n+2} < \frac{1}{\sqrt{6n+7}} \Leftrightarrow$$

$$\Leftrightarrow \frac{6n+1}{4(3n+2)^2} < \frac{1}{6n+7}$$

$$\Leftrightarrow (6n+1)(6n+7) < 4(9n^2 + 12n + 4)$$

$$36n^2 + 42n + 6n + 7 < 36n^2 + 48n + 16$$

$$7 < 16$$

$$P(n) \to P(n+1)$$

$$0<\frac{1}{2^n}\prod_{k=1}^{n}\left(2-\frac{3}{3k-1}\right)<\frac{1}{\sqrt{6n+1}}$$

$$0\le\lim_{n\to\infty}\frac{1}{2^n}\prod_{k=1}^{n}\left(2-\frac{3}{3k-1}\right)\le 0\Rightarrow\Omega=0$$

SOLUTION AN.108.

$$\Omega=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{(25k^2+5k-6)(n-k+1)^2}\right)=\left(\sum_{n=1}^{\infty}\frac{1}{25k^2+5k-6}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)=$$

$$=\frac{1}{15}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{90}$$

SOLUTION AN.109.

$$\Omega=\lim_{m\to\infty}\sqrt[m]{\sum_{n=1}^{\infty}\left(\frac{2n^2+2nm+n-1}{(2n+2m+2)!!}\right)}=$$

$$=\lim_{m\to\infty}\sqrt[m]{\frac{1}{(2m+2)!!}}\overset{CDA}{=}\lim_{m\to\infty}\frac{1}{(2m+4)!!}\cdot\frac{(2m+2)!!}{1}=\lim_{n\to\infty}\frac{1}{2m+4}=0$$

SOLUTION AN.110.

$$\int_1^2 f^2(x)\,dx+\int_1^2\frac{dx}{f(x)}-2=\int_1^2 f^2(x)\,dx+\int_1^2\frac{dx}{f(x)}-\int_1^2 f(x)\,dx-\int_1^2 dx=$$

$$=\int_1^2\left(f^2(x)+\frac{1}{f(x)}-f(x)-1\right)dx=\int_1^2\frac{f^3(x)+1-f(x)(f(x)+1)}{f(x)}dx=$$

$$=\int_1^2\frac{(f(x)+1)\left(f^2(x)-f(x)+1-f(x)\right)}{f(x)}dx=\int_1^2\frac{(f(x)+1)(f^2(x)-2f(x)+1)}{f(x)}dx=$$

$$=\int_1^2\frac{(f(x)+1)(f(x)-1)^2}{f(x)}dx\ge 0$$

Equality holds for $f(x)=1;x\in[1,2]$

SOLUTION AN.111.

$$\int_1^2\left(f'(x)\right)^2dx+\int_1^2\frac{dx}{f'(x)}-2=\int_1^2\left(\left(f'(x)\right)^2+\frac{1}{f'(x)}\right)dx-(2-1)-1=$$

$$=\int_1^2 \frac{(f'(x))^3+1}{f'(x)}dx-(f(2)-f(1))-1=\int_1^2 \frac{(f'(x))^3+1}{f'(x)}dx-\int_1^2 f'(x)\,dx-\int_1^2 dx=$$

$$=\int_1^2\left(\frac{(f'(x))^3+1}{f'(x)}-(f'(x)+1)\right)dx=$$

$$=\int_1^2 \frac{(f'(x)+1)\left((f'(x))^2-f'(x)+1-f'(x)\right)}{f'(x)}dx=$$

$$=\int_1^2 \frac{(f'(x)+1)(f'(x)-1)^2}{f'(x)}dx\geq 0$$

Equality holds for $f(x)=x; x\in[1,2]$.

SOLUTION AN.112.

If $a,b>0$ then: $(a-b)^2(a^2+ab+b^2)\geq 0$

$$(a-b)(a^3-b^3)\geq 0\Rightarrow a^3(a-b)-b^3(a-b)\geq 0$$

$$a^4+2a^2b^2+b^4\geq a^3b+ab^3+2a^2b^2$$

$$(a^2+b^2)^2\geq(a+b)^2ab$$

$$a^2+b^2\geq(a+b)\sqrt{ab}$$

$$\frac{a^2+b^2}{a+b}\geq\sqrt{ab}$$

For $a=f'(x); b=f'(y)$

$$\frac{(f'(x))^2+(f'(y))^2}{f'(x)+f'(y)}\geq\sqrt{f'(x)f'(y)}\quad (1)$$

Analogous: $\frac{(f'(y))^2+(f'(z))^2}{f'(y)+f'(z)}\geq\sqrt{f'(y)f'(z)}$ *(2);* $\frac{(f'(z))^2+(f'(x))^2}{f'(z)+f'(x)}\geq\sqrt{f'(z)f'(x)}$ *(3)*

By multiplying (1); (2); (3):

$$\prod_{cyc}\left(\frac{(f'(x))^2+(f'(y))^2}{f'(x)+f'(y)}\right)\geq f'(x)f'(y)f'(z)$$

By integrating:

$$\int_1^2\int_1^2\int_1^2\left(\prod_{cyc}\left(\frac{(f'(x))^2+(f'(y))^2}{f'(x)+f'(y)}\right)\right)dx\,dy\,dz\geq$$

$$\geq \int_1^2\int_1^2\int_1^2 f'(x)f'(y)f'(z)dx\,dy\,dz = \left(\int_1^2 f'(x)dx\right)^2 = \big(f(2)-f(1)\big)^3 =$$

$$=(2-1)^3=1$$

SOLUTION AN.113.

$$\lim_{n\to\infty}\ln\sqrt[n^4]{x_n}=\lim_{n\to\infty}\frac{\ln x_n}{n^4}=\lim_{n\to\infty}\frac{\ln\left(\frac{x_{n+1}}{x_n}\right)}{(n+1)^4-n^4}=\lim_{n\to\infty}\frac{\ln\left(\frac{x_{n+1}}{x_n}\right)}{4n^3+6n^2+4n+1}=$$

$$=\lim_{n\to\infty}\frac{\ln\left(\frac{x_{n+2}}{x_{n+1}}\right)-\ln\left(\frac{x_{n+1}}{x_n}\right)}{4(n+1)^3-4n^3+6(n+1)^2-6n^2+4(n+1)-4n}=$$

$$=\lim_{n\to\infty}\frac{\ln\left(\frac{x_{n+2}}{x_{n+1}}\cdot\frac{x_n}{x_{n+1}}\right)}{12n^2+24n+14}=\lim_{n\to\infty}\frac{\ln\left(\frac{x_{n+3}x_{n+1}}{x_{n+2}^2}\right)-\ln\left(\frac{x_{n+2}x_n}{x_{n+1}^2}\right)}{12(n+1)^2-12n^2+24(n+1)-24n}=$$

$$=\lim_{n\to\infty}\frac{\ln\left(\frac{x_{n+3}x_{n+1}}{x_{n+2}^2}\cdot\frac{x_{n+1}^2}{x_{n+2}x_n}\right)}{24n+12n+24}=\lim_{n\to\infty}\frac{\ln\left(\frac{x_{n+3}x_{n+1}^3}{x_{n+2}^3x_n}\right)}{24n+36}=$$

$$=\frac{1}{24}\lim_{n\to\infty}\left[\ln\left(\frac{x_{n+4}x_{n+2}^3}{x_{n+3}^3x_{n+1}}\cdot\frac{x_{n+2}^3x_n}{x_{n+3}x_{n+1}^3}\right)\right]=\frac{1}{24}\lim_{n\to\infty}\ln\left(\frac{x_{n+4}x_{n+2}^6x_n}{x_{n+3}^4x_{n+1}^4}\right)=$$

$$=\frac{1}{24}\ln e^{24}=\frac{1}{24}\cdot 24=1$$

$$\ln\Omega=1\Rightarrow\Omega=e$$

SOLUTION AN.114.

$$\lim_{x\to 0}\left(\frac{1}{x}-\log(x+1)^{\frac{1}{x^2}}\right)=$$

$$=\lim_{x\to 0}\left(\frac{1}{x}-\frac{\log(1+x)}{x^2}\right)=\lim_{x\to 0}\frac{x-\log(1+x)}{x^2}=$$

$$=\lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x}=\lim_{x\to 0}\frac{1+x-1}{2x(1+x)}=\lim_{x\to 0}\frac{1}{2(x+1)}=\frac{1}{2}\quad \mathit{(1)}$$

$$x_n=\sum_{k=0}^{n}\frac{2^k}{5^n}\binom{n}{2n-k}=\frac{1}{5^n}\sum_{k=0}^{n}2^k\binom{n}{2n-k}=\frac{1}{5^n}\cdot 4^n=\left(\frac{4}{5}\right)^n$$

$$\lim_{n\to\infty}x_n=0\overset{(1)}{\Rightarrow}\lim_{n\to\infty}\left(\frac{1}{x_n}-\log(x_n+1)^{\frac{1}{x_n^2}}\right)=\frac{1}{2}$$

SOLUTION AN.115.

Let be $u, v \in (0,1]$***. By Bernoulli's inequality:***

$$u^{1-v} = (1+u-1)^{1-v} \le 1 + (u-1)(1-v) = 1 + u - uv - 1 + v = u + v - uv$$

$$\frac{u}{u^v} \le u + v - uv \Rightarrow u^v \ge \frac{u}{u+v-uv} \quad (1)$$

$$\textit{Analogous: } v^u \ge \frac{v}{u+v-uv} \quad (2)$$

$$\textit{By adding (1); (2): } u^v + v^u \ge \frac{u+v}{u+v-uv}$$

$$\frac{u+v}{u^v+v^u} \le u + v - uv \quad (3)$$

$$\textit{We take in (3): } u = f(x); v = f(y)$$

$$\frac{f(x)+f(y)}{\left(f(x)\right)^{f(y)} + \left(f(y)\right)^{f(x)}} + f(x)f(y) \le f(x) + f(y)$$

$$\int_a^b \int_a^b \frac{f(x)+f(y)}{\left(f(x)\right)^{f(y)} + \left(f(y)\right)^{f(x)}} dx\,dy + \int_a^b \int_a^b f(x)f(y)\,dx\,dy \le \int_a^b \int_a^b \left(f(x)+f(y)\right) dx\,dy$$

$$\int_a^b \int_a^b \frac{f(x)+f(y)}{\left(f(x)\right)^{f(y)} + \left(f(y)\right)^{f(x)}} dx\,dy + \left(\int_a^b f(x)\,dx\right)^2 \le 2(b-a)\int_a^b f(x)\,dx$$

SOLUTION AN.116.

$$\sqrt{\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{y^2}} \ge \frac{\sqrt{3}}{2}\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{y^2} \ge \frac{3}{4}\left(\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2}\right)$$

$$\frac{4}{x^2} + \frac{4}{xy} + \frac{4}{y^2} \ge \frac{3}{x^2} + \frac{6}{xy} + \frac{3}{y^2}$$

$$\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} \ge 0 \Leftrightarrow \left(\frac{1}{x} - \frac{1}{y}\right)^2 \ge 0$$

$$\int_a^b \left(\sqrt{\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{y^2}}\right) dx \ge \frac{\sqrt{3}}{2}\int_a^b \frac{1}{x} dx + \frac{\sqrt{3}}{2}\int_a^b \frac{1}{y} dx = \frac{\sqrt{3}}{2}\log\left(\frac{b}{a}\right) + \frac{\sqrt{3}}{2}(b-a)\cdot\frac{1}{y}$$

$$\int_a^b \left(\int_a^b \left(\sqrt{\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{y^2}}\right) dx\right) dy \ge \frac{\sqrt{3}}{2}\log\left(\frac{b}{a}\right)\cdot(b-a) + \frac{\sqrt{3}}{2}(b-a)\cdot\log\left(\frac{b}{a}\right) =$$

$$= \sqrt{3} \log \left(\frac{b}{a}\right)^{b-a}$$

SOLUTION AN.117.

$$\text{For } x > 0: (x-1)^2(x^4 + x^3 + x^2 + x + 1) \geq 0$$

$$(x-1)(x^5 - 1) \geq 0 \Rightarrow x^5(x-1) - (x-1) \geq 0$$

$$x^6 - x^5 + 1 - x \geq 0$$

$$(x^4 + 2x^2 + 1)x \leq x^6 + 2x^3 + 1$$

$$(x^2+1)^2 x \leq (x^3+1)^2$$

$$(x^2+1)\sqrt{x} \leq x^3 + 1 \Rightarrow \frac{\sqrt{x}}{x^3+1} \leq \frac{1}{x^2+1} \quad (1)$$

Replace x in (1) with $f(x)$ and multiplying with $f'(x)$.

$$\frac{f'(x)\sqrt{f(x)}}{f^3(x)+1} \leq \frac{f'(x)}{f^2(x)+1}$$

$$\int_a^b \frac{f'(x)\sqrt{f(x)}}{f^3(x)+1} dx \leq \int_a^b \frac{f'(x)}{f^2(x)+1} dx =$$

$$= \tan^{-1}(f(b)) - \tan^{-1}(f(a)) = \tan^{-1}\left(\frac{f(b)-f(a)}{1+f(a)f(b)}\right)$$

Equality holds for $a = b$.

SOLUTION AN.118.

$$\text{For } x > 0: (x-1)^2(x^4 + x^3 + x^2 + x + 1) \geq 0$$

$$(x-1)(x^5 - 1) \geq 0 \Rightarrow x^5(x-1) - (x-1) \geq 0$$

$$x^6 - x^5 + 1 - x \geq 0 \Rightarrow (x^4 + 2x^2 + 1)x \leq x^6 + 2x^3 + 1$$

$$(x^2+1)^2 x \leq (x^3+1)^2$$

$$(x^2+1)\sqrt{x} \leq x^3 + 1 \Rightarrow \frac{\sqrt{x}}{x^3+1} \leq \frac{1}{x^2+1} \quad (1)$$

Replacing x in (1) with e^{x^2} and multiplying with $2xe^{x^2}$.

$$\frac{2xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1} \leq \frac{2xe^{x^2}}{e^{2x^2}+1}$$

$$\int_a^b \frac{2xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2+1}} dx \leq \int_a^b \frac{2xe^{x^2}}{e^{2x^2}+1} dx = \tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})$$

$$\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}dx \le \frac{1}{2}\left(\tan^{-1}\left(e^{b^2}\right)-\tan^{-1}\left(e^{a^2}\right)\right)=\frac{1}{2}\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right)$$

SOLUTION AN.119.

$$\Omega=\int \frac{x\cos x-\cos x-x\sin x-\sin x}{x^2+\sin 2x+1}dx=$$

$$=\int \frac{(\cos x-\sin x)x-\sin x-\cos x}{x^2+1+\sin 2x}dx=$$

$$=\int\left(\frac{(\sin x+\cos x)'x-x'(\sin x+\cos x)}{x^2}\cdot\frac{x^2}{x^2+(\sin x+\cos x)^2}\right)dx=$$

$$=\int\frac{\left(\frac{\sin x+\cos x}{x}\right)'}{1+\left(\frac{\sin x+\cos x}{x}\right)^2}dx=\int\left(\tan^{-1}\left(\frac{\sin x+\cos x}{x}\right)\right)dx=\tan^{-1}\left(\frac{\sin x+\cos x}{x}\right)+C$$

SOLUTION AN.120.

$$\lim_{n\to\infty}\frac{H_n}{n}=\lim_{n\to\infty}\frac{H_{n+1}-H_n}{n+1-n}=\lim_{n\to\infty}\frac{1}{n+1}=0 \quad (1)$$

$$\lim_{\substack{x\to 0\\ x>0}} x^{\frac{1}{x^2}}=e^{\lim_{\substack{x\to 0\\ x>0}}\frac{\ln x}{x^2}}=e^{\frac{1}{0+}\cdot\ln(0_+)}=e^{\infty\cdot(-\infty)}=e^{-\infty}=\frac{1}{e^{\infty}}=\frac{1}{\infty}=0 \quad (2)$$

By (1); (2):

$$\lim_{n\to\infty}\left(\frac{H_n}{n}\right)^{\frac{n^2}{H_n^2}}=0\Rightarrow\lim_{n\to\infty}\frac{\tan\left(\left(\frac{H_n}{n}\right)^{\frac{n^2}{H_n}}\right)}{\left(\frac{H_n}{n}\right)^{\frac{n^2}{H_n}}}=1$$

$$\Omega=\lim_{n\to\infty}\frac{\tan\left(\left(\frac{H_n}{n}\right)^{\frac{n^2}{H_n^2}}\right)}{\left(\frac{H_n}{n}\right)^{\frac{n^2}{H_n}}}\cdot\frac{\left(\frac{H_n}{n}\right)^{\frac{n^2}{H_n^2}}}{\left(\tan\left(\frac{H_n}{n}\right)\right)^{\frac{n^2}{H_n^2}}}=1\cdot\lim_{n\to\infty}\frac{\left(\frac{H_n}{n}\right)^{\frac{n^2}{H_n^2}}}{\left(\tan\left(\frac{H_n}{n}\right)\right)^{\frac{n^2}{H_n^2}}}=\lim_{n\to\infty}\left(\frac{\frac{H_n}{n}}{\tan\left(\frac{H_n}{n}\right)}\right)^{\frac{n^2}{H_n^2}}=$$

$$=\lim_{n\to\infty}\left(1+\frac{\frac{H_n}{n}}{\tan\left(\frac{H_n}{n}\right)}-1\right)^{\frac{n^2}{H_n^2}}=\lim_{n\to\infty}\left(1+\frac{\frac{H_n}{n}-\tan\left(\frac{H_n}{n}\right)}{\tan\left(\frac{H_n}{n}\right)}\right)^{\frac{n^2}{H_n^2}}=$$

$$=\lim_{x\to 0}\left(1+\frac{x-\tan x}{\tan x}\right)^{\frac{\tan x}{x-\tan x}\cdot\frac{x-\tan x}{\tan x}\cdot\frac{1}{x^2}}=e^{\lim_{x\to 0}\frac{x-\tan x}{x^2\tan x}}=e^{\lim_{x\to 0}\frac{1-\frac{1}{\cos^2 x}}{2x\tan x+x^2\cdot\frac{1}{\cos^2 x}}}=$$

$$= e^{\lim\limits_{x\to 0}\frac{-\sin^2 x}{2x\sin x\cos x+x^2}} = e^{-\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\lim\limits_{x\to 0}\frac{\sin x}{\sin 2x+x}} = e^{-1\cdot\lim\limits_{x\to 0}\frac{\cos x}{2\cos x+1}} = e^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{e}}$$

SOLUTION AN.121.

Let be $F: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$;

$$F(x) = \frac{\int_0^x \sqrt[3]{t}\sin t\,dt}{\int_0^x \sqrt[3]{t}\cos t\,dt}$$

$$F'(x) = \frac{\sqrt[3]{x}\sin x\int_0^x \sqrt[3]{t}\cos t\,dt - \sqrt[3]{3}\cos x\int_0^x \sqrt[3]{t}\sin t\,dt}{\left(\int_0^x \sqrt[3]{t}\cos t\,dt\right)^2}$$

$$F'(x) = \frac{1}{\left(\int_0^x \sqrt[3]{t}\cos t\,dt\right)^2}\cdot\int_0^x \sqrt[3]{xt}\cos x\sin t\,(\tan x - \tan t)dt$$

$g: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$; $g(x) = \tan x$ ***is increasing on*** $\left(0, \frac{\pi}{2}\right)$ ***because*** $g'(x) = \frac{1}{\cos^2 x} > 0$

If $0 < t \le x < \frac{\pi}{2} \Rightarrow g(t) \le g(x) \Rightarrow \tan x \ge \tan t \Rightarrow F'(x) \ge 0 \Rightarrow F$ ***increasing***

$$0 < a \le \sqrt{ab} \le \frac{a+b}{2} \le b < \frac{\pi}{2} \Rightarrow F(\sqrt{ab}) \le F\left(\frac{a+b}{2}\right)$$

$$\frac{\int_0^{\sqrt{ab}} \sqrt[3]{x}\sin x\,dx}{\int_0^{\sqrt{ab}} \sqrt[3]{x}\cos x\,dx} \le \frac{\int_0^{\frac{a+b}{2}} \sqrt[3]{x}\sin x\,dx}{\int_0^{\frac{a+b}{2}} \sqrt[3]{x}\cos x\,dx}$$

$$\left(\int_0^{\sqrt{ab}} \sqrt[3]{x}\sin x\,dx\right)\left(\int_0^{\frac{a+b}{2}} \sqrt[3]{x}\cos x\,dx\right) \le \left(\int_0^{\sqrt{ab}} \sqrt[3]{x}\cos x\,dx\right)\left(\int_0^{\frac{a+b}{2}} \sqrt[3]{x}\sin x\,dx\right)$$

Equality holds for $a = b$.

SOLUTION AN.122.

$$\Omega = \lim_{n\to\infty}\frac{\left(\sqrt[n]{1+\sin x} + \sqrt[n]{1-\sin x} - 2\right)(n+1)}{\left(\sqrt[n+1]{1+\sin x} + \sqrt[n+1]{1-\sin x} - 2\right)n} = \lim_{n\to\infty}\frac{\Omega_1(n)}{\Omega_2(n)}$$

$$\Omega_1(n) = (n+1)\left(\sqrt[n]{1+\sin x} + \sqrt[n]{1-\sin x} - 2\right) =$$

$$= \frac{\sqrt[n]{1+\sin x} - 1}{\frac{1}{n+1}} + \frac{\sqrt[n]{1-\sin x} - 1}{\frac{1}{n+1}} =$$

$$= \frac{e^{\frac{\ln(1+\sin x)}{n+1}} - 1}{\frac{\ln(1+\sin x)}{n+1}} \cdot \frac{\ln(1+\sin x)}{n+1} \cdot n + \frac{e^{\frac{\ln(1-\sin x)}{n+1}} - 1}{\frac{\ln(1-\sin x)}{n+1}} \cdot \frac{\ln(1-\sin x)}{n+1} \cdot n$$

$$\lim_{n\to\infty} \Omega_1(n) = 1 \cdot \ln(1+\sin x) \cdot \lim_{n\to\infty} \frac{n}{n+1} + 1 \cdot \ln(1-\sin x) \cdot \lim_{n\to\infty} \frac{n}{n+1}$$

$$\lim_{n\to\infty} \Omega_1(n) = \ln(1+\sin x) + \ln(1-\sin x)$$

Analogous:

$$\lim_{n\to\infty} \Omega_2(n) = \ln(1+\sin x) + \ln(1-\sin x)$$

$$\Omega = \lim_{n\to\infty} \frac{\Omega_1(n)}{\Omega_2(n)} = \lim_{n\to\infty} \frac{\ln(1+\sin x) + \ln(1-\sin x)}{\ln(1+\sin x) + \ln(1-\sin x)} = 1$$

SOLUTION AN.123.

First, we prove that if $a, b, c > 0$ then:

$$\frac{4a}{a^2+bc} \le \frac{b+c}{bc} \quad (1)$$

$$4abc \le (b+c)(a^2+bc) \Leftrightarrow a^2b + a^2c + b^2c + bc^2 - 4abc \ge 0$$

$$b(a^2 - 2ac + c^2) + c(a^2 - 2ba + b^2) \ge 0$$

$$b(a-c)^2 + c(a-b)^2 \ge 0 \text{ which is true.}$$

In (1) we take: $a = f(x); b = f(y); c = f(z)$ and multiplying with $f'(y)f'(z)$.

$$\frac{f(x)f'(y)f'(z)}{f^2(x) + f(y)f(z)} \le \frac{1}{4}\left(f'(z) \cdot \frac{f'(y)}{f(y)} + f'(y) \cdot \frac{f'(z)}{f(z)}\right)$$

By integrating:

$$\int_a^b \int_a^b \int_a^b \frac{f(x)f'(y)f'(z)}{f^2(x) + f(y)f(z)} dx\, dy\, dz \le \frac{1}{4}\left(\int_a^b dx\right)\left(\int_a^b f'(z)\, dz\right)\left(\int_a^b \frac{f'(y)}{f(y)} dy\right) +$$

$$+ \frac{1}{4}\left(\int_a^b dx\right)\left(\int_a^b f'(y)\, dy\right)\left(\int_a^b \frac{f'(z)}{f(z)} dx\right) =$$

$$= \frac{1}{4}(b-a) \cdot f\big((b) - f(a)\big)\big(\log(f(b)) - \log(f(a))\big) +$$

$$+ \frac{1}{4}(b-a)\big(f(b) - f(a)\big)\big(\log(f(b)) - \log(f(a))\big) =$$

$$= \frac{1}{2}(b-a)\big(f(b) - f(a)\big)\log\left(\frac{f(b)}{f(a)}\right) = \log\left(\sqrt{\frac{f(b)}{f(a)}}\right)^{(b-a)(f(b)-f(a))}$$

SOLUTION AN.124.

$$\Omega=\lim_{n\to\infty}\left(1+\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}\right)^n=\lim_{n\to\infty}\left(1+\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}\right)^{\frac{1}{\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}}\cdot\frac{\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}}{\frac{1}{n}}}=$$

$$=e^{\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}}{\frac{1}{n}}}\overset{CESARO-STOLZ}{=}=e^{\lim_{n\to\infty}\frac{\frac{1}{(n+1)^4}}{\frac{1}{n+1}-\frac{1}{n}}}=e^{\lim_{n\to\infty}\frac{\frac{1}{(n+1)^4}}{\frac{n-n-1}{n(n+1)}}}=$$

$$=e^{\lim_{n\to\infty}-\frac{n}{(n+1)^3}}=e^0=1$$

SOLUTION AN.125.

$$\lim_{n\to\infty}H_n=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n+\log n\right)=$$

$$=\gamma+\log\infty=\infty\quad (1)$$

$$\Omega=\lim_{n\to\infty}H_n^2\left[\left(\frac{1+H_n}{H_n}\right)^{H_n}-\log\left(\frac{1+H_n}{H_n}\right)^{eH_n}\right]=\lim_{x\to\infty}x^2\left[\left(\frac{1+x}{x}\right)^x-\log\left(\frac{1+x}{x}\right)^{ex}\right]=$$

$$=\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^x-ex\log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}}=$$

$$=\lim_{x\to\infty}\frac{x\left(-\frac{1}{x^2}\right)\left(1+\frac{1}{x}\right)^{x-1}+\left(1+\frac{1}{x}\right)^x\log\left(1+\frac{1}{x}\right)-e\log\left(1+\frac{1}{x}\right)+\frac{e}{x+1}}{-\frac{2}{x^3}}=$$

$$=\lim_{x\to\infty}\frac{\left[\left(1+\frac{1}{x}\right)^x-e\right]\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right]}{-\frac{2}{x^3}}=$$

$$=-\frac{1}{2}\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^x-e}{\frac{1}{x}}\cdot\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}}{\frac{1}{x^2}}=$$

$$=\frac{1}{2}\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\left(\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}}{\frac{1}{x^2}}\right)^2=\frac{1}{2}\cdot e\cdot\left(\lim_{x\to\infty}\frac{-\frac{1}{x^2}\cdot\frac{x}{x+1}+\frac{1}{(x+1)^2}}{-\frac{2}{3}}\right)^2=$$

$$=\frac{e}{2}\left(\lim_{x\to\infty}\frac{-\frac{1}{x(x+1)}+\frac{1}{(x+1)^2}}{-\frac{2}{x^3}}\right)^2=\frac{e}{2}\left(\lim_{x\to\infty}\frac{x^2}{2(x+1)^2}\right)^2=\frac{e}{2}\cdot\frac{1}{4}=\frac{e}{8}$$

SOLUTION AN.126.

$$\tan x > x;\ (\forall)x\in\left(0,\frac{\pi}{2}\right)\Rightarrow \tan^2 x > x^2$$

$$\sin^2 x > x^2\cos^2 x \Rightarrow \sin^2 x > x^2(1-\sin^2 x)$$

$$\sin^2 x\,(1+x^2) > x^2 \Rightarrow \sin^2 x > \frac{x^2}{1+x^2}$$

$$\sin x > \frac{x}{\sqrt{1+x^2}} \Rightarrow \sin(e^x) > \frac{e^x}{\sqrt{1+e^{2x}}}$$

$$\int_a^b \sin(e^x)\,dx \ge \int_a^b \frac{e^x}{\sqrt{1+e^{2x}}}dx =$$

$$=\log\left(e^b+\sqrt{1+e^{2b}}\right)-\log\left(e^a+\sqrt{1+e^{2a}}\right)=\log\left(\frac{e^b+\sqrt{1+e^{2b}}}{e^a+\sqrt{1+e^{2a}}}\right)$$

SOLUTION AN.127.

$$\sum_{k=1}^{\infty}\frac{1}{(k^2+k)^2}=\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)}\right)^2=\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)^2=$$

$$=\sum_{k=1}^{\infty}\frac{1}{k^2}+\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}-2\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=$$

$$=\frac{\pi^2}{6}+\frac{\pi^2}{6}-2\sum_{k=1}^{\infty}\left(\frac{1}{k_1}-\frac{1}{k+1}\right)-1=\frac{\pi^2}{3}-3$$

$$\Omega=\lim_{n\to\infty}\left(1+3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)^{\frac{1}{3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}}\cdot\frac{3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}}{\frac{1}{n}}}=$$

$$=e^{\lim\limits_{n\to\infty}\frac{3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}}{\frac{1}{n}}}\overset{CESARO-STOLZ}{=}=e^{\lim\limits_{n\to\infty}\frac{\frac{1}{\left((n+1)^2+(n+1)\right)^2}}{\frac{1}{n+1}-\frac{1}{n}}}=e^{\lim\limits_{n\to\infty}\frac{\frac{1}{(n+1)^2(n+2)^2}}{-\frac{1}{n(n+1)}}}=$$

$$=e^{\lim\limits_{n\to\infty}-\frac{n}{(n+1)(n+2)^2}}=e^0=1$$

SOLUTION AN.128.

Denote:

$$L(a)=\lim_{x\to\infty}\left(x^2\left(a^{\frac{1}{x}}-1\right)-x\ln a\right)$$

$$L(a)=\lim_{x\to\infty}\frac{xa^{\frac{1}{x}}-x-\ln a}{\frac{1}{x}}=\lim_{x\to\infty}\frac{a^{\frac{1}{x}}+x\cdot\frac{-1}{x^2}\cdot\ln a\cdot a^{\frac{1}{x}}-1}{-\frac{1}{x^2}}=$$

$$=\lim_{x\to\infty}\frac{a^{\frac{1}{x}}-\frac{1}{x}\cdot\ln a\cdot a^{\frac{1}{x}}-1}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{-\frac{1}{x^2}a^{\frac{1}{x}}\ln a+\frac{1}{x^2}a^{\frac{1}{x}}\ln a+\frac{1}{x^3}\ln a\cdot a^{\frac{1}{x}}}{\frac{2}{x^3}}=$$

$$=\lim_{x\to\infty}\frac{a^{\frac{1}{x}}\ln^2 a}{2}=\frac{\ln^2 a}{2}$$

$$\Omega(a,b)=\lim_{x\to\infty}\frac{x\left(a^{\frac{1}{x}}-1\right)-\ln a}{x\left(b^{\frac{1}{x}}-1\right)-\ln b}=$$

$$=\lim_{x\to\infty}\frac{x^2\left(a^{\frac{1}{x}}-1\right)-x\ln a}{x^2\left(b^{\frac{1}{x}}-1\right)-x\ln b}=\frac{L(a)}{L(b)}=\frac{\ln^2 a}{2}\cdot\frac{2}{\ln^2 b}=\frac{\ln^2 a}{\ln^2 b}$$

$$\ln a\cdot\Omega(a,b)+\ln b\cdot\Omega(b,c)+\ln c\cdot\Omega(c,a)=$$

$$=\frac{\ln^3 a}{\ln^2 b}+\frac{\ln^3 b}{\ln^2 c}+\frac{\ln^3 c}{\ln^2 a}\overset{RADON}{\geq}\frac{(\ln a+\ln b+\ln c)^3}{(\ln a+\ln b+\ln c)^2}=\ln a+\ln b+\ln c=\ln(abc)$$

SOLUTION AN.129.

$$\left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right]\geq\left[\frac{\sqrt{k}+\sqrt{k}+\sqrt{k}}{3}\right]=\left[\sqrt{k}\right]\quad (1)$$

$$0<1\Rightarrow k^2+2k<k^2+2k+1\Rightarrow$$

$$\sqrt{k^2+2k}<k+1\Rightarrow 2\sqrt{k(k+2)}<2k+2$$

$$2k+2+2\sqrt{k(k+2)}<4k+4$$

$$k+(k+2)+2\sqrt{k(k+2)}<4(k+1)$$

$$\left(\sqrt{k}+\sqrt{k+2}\right)^2<4(k+1)$$

$$\sqrt{k}+\sqrt{k+2}<2\sqrt{k+1}$$

$$\left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right]<\left[\frac{2\sqrt{k+1}+\sqrt{k+1}}{3}\right]=\left[\sqrt{k+1}\right]\Rightarrow$$

$$\Rightarrow\left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right]\le\left[\sqrt{k}\right]\quad (2)$$

$$\text{By (1); (2)}\Rightarrow\left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right]=\left[\sqrt{k}\right]\quad (3)$$

$$\sum_{k=1}^{n}\left(\frac{1}{k[\sqrt{k}]}\cdot\left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right]\right)\overset{(3)}{=}\sum_{k=1}^{n}\frac{1}{k}$$

$$\Omega=\lim_{n\to\infty}\left(\log(2n+1)-\sum_{k=1}^{n}\frac{1}{k}\right)=$$

$$=\lim_{n\to\infty}\left(\log(2n+1)-\log n+\log n-\sum_{k=1}^{n}\frac{1}{k}\right)=\gamma+\lim_{n\to\infty}\log\left(\frac{2n+1}{n}\right)=$$

$$=\gamma+\log 2$$

SOLUTION AN.130.

$$\textit{If } i=1\Rightarrow\left[\frac{i^2+i+1}{i^2-i+1}\right]=\left[\frac{1^2+1+1}{1^2-1+1}\right]=3$$

$$\textit{If } i=2\Rightarrow\left[\frac{i^2+i+1}{i^2-i+1}\right]=\left[\frac{2^2+2+1}{2^2-2+1}\right]=\left[\frac{7}{3}\right]=2$$

$$\textit{If } i\ge 3 \textit{ then:}$$

$$1<\frac{i^2+i+1}{i^2+i+1}<2\Leftrightarrow\begin{cases}i^2-i+1<i^2+i+1\\ i^2+i+1<2i^2-2i+2\end{cases}\Leftrightarrow\begin{cases}2i>0\\ i^2-3i+1>0\end{cases}$$

$$\Delta=9-4=5$$

$$i_{1,2}=\frac{3\pm\sqrt{5}}{2}$$

$$i\in\left(\frac{3+\sqrt{5}}{2},\infty\right); i\ge 3\,.\ \textit{True.}$$

$$\Rightarrow\sum_{i=1}^{n}\left[\frac{i^2+i+1}{i^2-i+1}\right]=3+2+\sum_{i=3}^{n}\left[\frac{i^2+i+1}{i^2-i+1}\right]=3+2+\sum_{i=3}^{n}1=5+(n-2)=n+3$$

$$\omega(k)=k+3$$

$$\Omega=\lim_{n\to\infty}\left(\log(3n+1)-\sum_{k=1}^{n}\frac{1}{\omega(k)}\right)=\lim_{n\to\infty}\left(\log(3n+1)-\sum_{k=1}^{n}\frac{1}{k+3}\right)=$$

$$=\lim_{n\to\infty}\left(\log(3n+1)-\frac{1}{4}-\frac{1}{5}-\cdots-\frac{1}{n+3}\right)=$$

$$= \lim_{n\to\infty}\left(\log(3n+1) - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n+3} + \log(n+3) + 1 + \frac{1}{2} + \frac{1}{3} - \log(n+3)\right) =$$

$$= 1 + \frac{1}{2} + \frac{1}{3} - \gamma + \lim_{n\to\infty}\log\left(\frac{3n+1}{n+3}\right) = \frac{6+3+2}{6} - \gamma + \log 3 = \frac{11}{6} + \log 3 - \gamma$$

SOLUTION AN.131.

First, we prove that if $i \geq 3$ ***then:*** $1 < \frac{i-\sqrt{i}}{i-\sqrt{i+\sqrt{i}}} < 2$ ***(1)***

$$\begin{cases} i - \sqrt{i+\sqrt{i}} < i - \sqrt{i} \\ i - \sqrt{i} < 2i - 2\sqrt{i+\sqrt{i}} \end{cases} \Leftrightarrow \begin{cases} \sqrt{i} < \sqrt{i+\sqrt{i}} \\ 2\sqrt{i+\sqrt{i}} < i + \sqrt{i} \end{cases}$$

$$\Leftrightarrow \begin{cases} i < i + \sqrt{i} \\ 4i + 4\sqrt{i} < i^2 + 2i\sqrt{i} + i \end{cases} \Leftrightarrow\Leftrightarrow \begin{cases} \sqrt{i} > 0 \\ i\sqrt{i} + 2i + \sqrt{i} - 4\sqrt{i} - 4 > 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} i > 0 \\ \sqrt{i}(i-3) + 2(i-2) > 0 \end{cases}$$

Which are true because $i \geq 3$***.***

$$x_n = \left[\frac{1-\sqrt{1}}{1-\sqrt{1+\sqrt{1}}}\right] + \left[\frac{2-\sqrt{2}}{2-\sqrt{2+\sqrt{2}}}\right] + \sum_{i=3}^{n}\left[\frac{i-\sqrt{i}}{i-\sqrt{i+\sqrt{i}}}\right]$$

$$\left[\frac{2-\sqrt{2}}{2-\sqrt{2+\sqrt{2}}}\right] = \left[\frac{(2-\sqrt{2})(2+\sqrt{2+\sqrt{2}})}{4-2-\sqrt{2}}\right] = \left[2+\sqrt{2+\sqrt{2}}\right] = 3$$

$$x_n = 0 + 3 + \sum_{i=3}^{n} 1 \quad \textit{(by (1))}$$

$$x_n = 3 + (n-2) = n+1; \lim_{n\to\infty} x_n = \infty$$

$$\Omega = \lim_{n\to\infty}\left(\frac{1 + x_n^2\log\left(\frac{1+x_n}{x_n}\right)}{x_n}\right)^{x_n} = \lim_{x\to\infty}\left(\frac{1}{x} + x\log\left(1+\frac{1}{x}\right)\right)^x =$$

$$= \lim_{x\to\infty} e^{x\log\left(\frac{1}{x}+x\log\left(1+\frac{1}{x}\right)\right)} = e^{\lim\limits_{x\to\infty}\frac{\log\left(\frac{1}{x}+x\log\left(1+\frac{1}{x}\right)\right)}{\frac{1}{x}}} =$$

$$= e^{\lim\limits_{x\to\infty}\frac{-\frac{1}{x^2}+\log\left(1+\frac{1}{x}\right)+x\left(-\frac{1}{x^2}\right)\cdot\frac{x}{x+1}}{-\frac{1}{x^2}}} = e^1 \cdot e^{\lim\limits_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}}{-\frac{1}{x^2}}} =$$

$$= e \cdot e^{\lim\limits_{x\to\infty}\frac{-\frac{1}{x^2}\cdot\frac{x}{x+1}+\frac{1}{(x+1)^2}}{\frac{2}{x^3}}} = e \cdot e^{\lim\limits_{x\to\infty}\frac{-\frac{1}{x(x+1)}+\frac{1}{(x+1)^2}}{\frac{2}{x^3}}} =$$

$$= e \cdot e^{\lim\limits_{x\to\infty}\frac{-x-1+x}{x(x+1)^2}\cdot\frac{x^3}{2}} = e \cdot e^{-\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$$

SOLUTION AN.132.

Let be $f:\mathbb{R}\to\mathbb{R}; f(x) = \tan^{-1} x + \frac{x^2}{2} - \frac{x}{1+x^2}$

$$f'(x) = \frac{1}{1+x^2} + x - \frac{1+x^2-2x^2}{(1+x^2)^2} =$$

$$= x + \frac{1+x^2-1+x^2}{(1+x^2)^2} = x + \frac{2x^2}{(1+x^2)^2} = x\left(1 + \frac{2x}{(1+x^2)^2}\right)$$

$$\left|\frac{2x}{(1+x^2)^2}\right| \le \frac{2|x|}{1+x^2} \overset{AM-GM}{\le} \frac{2|x|}{2|x|} = 1$$

$$\left|\frac{2x}{(1+x^2)^2}\right| \le 1 \Rightarrow \frac{2x}{1+x^2} \ge -1 \Rightarrow 1 + \frac{2x}{1+x^2} \ge 0$$

$$\operatorname{sgn} f'(x) = \operatorname{sgn} x$$

$$\min f(x) = f(0) = 0 \Rightarrow f(x) \ge 0; (\forall)x \in \mathbb{R}$$

$$\tan^{-1} x \ge \frac{x}{1+x^2} - \frac{x^2}{2}$$

$$\int_a^b (\tan^{-1} x)\, dx \ge \int_a^b \frac{x}{1+x^2}\, dx - \frac{1}{6}(b^3 - a^3)$$

$$6\int_a^b (\tan^{-1} x)\, dx \ge 3\log\left(\frac{1+b^2}{1+a^2}\right) + a^3 - b^3$$

SOLUTION AN.133.

If $n > 1$ *then:*

$$n^3 < n^3 + 2n + 1 < n^3 + 3n^2 + 3n + 1$$

$$\sqrt[3]{n^3} < \sqrt[3]{n^3+2n+1} < \sqrt[3]{(n+1)^3}$$

$$n < \sqrt[3]{n^3+2n+1} < n+1$$

$$\left[\sqrt[3]{n^3+2n+1}\right] = n$$

$$\omega = \sum_{n=1}^{\infty} \frac{1}{\left[\sqrt[3]{(n^3+2n+1)^2}\right]^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Omega = \lim_{n\to\infty} n\left(\omega - \sum_{k=1}^{\infty} \frac{1}{k^2}\right) = \lim_{n\to\infty} \frac{\omega - \sum_{k=1}^{\infty} \frac{1}{k^2}}{\frac{1}{n}} \overset{CESARO-STOLZ}{=}$$

$$=\lim_{n\to\infty}\frac{-\frac{1}{(n+1)^2}}{\frac{1}{n+1}-\frac{1}{n}}=\lim_{n\to\infty}\frac{-\frac{1}{(n+1)^2}}{-\frac{1}{n(n+1)}}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)=1$$

SOLUTION AN.134.

Let be $I_n=\int_0^1 2^{nx}dx\,;n\ge 1$

$$I_n^2=\left(\int_0^1 2^{nx}\,dx\right)^2=\left(\int_0^1\left(\sqrt{2^{(n-k)x}}\cdot\sqrt{2^{(n+k)x}}\right)dx\right)^2\le$$

$$\le\left(\int_0^1 2^{(n-k)x}dx\right)\left(\int_0^1 2^{(n+k)x}\,dx\right)=I_{n-k}\cdot I_{n+k}$$

$$I_n^2\le I_{n-k}\cdot I_{n+k};0\le k\le n$$

$$I_n^2\le I_{n-1}\cdot I_{n+1};I_n^2\le I_{n-2}\cdot I_{n+2};\dots;I_n^2\le I_0\cdot I_{2n}$$

$$I_n^{2n}\le I_0\cdot I_1\cdot I_2\cdot\dots\cdot I_{n-1}\cdot I_{n+1}\cdot\dots\cdot I_{2n}$$

$$I_n^{2n+1}\le I_0\cdot I_1\cdot I_2\cdot\dots\cdot I_{2n}$$

$$\left(\frac{2^{nx}}{n\log 2}\Big|_0^1\right)^{2n+1}\le\left(\frac{2^x}{\log 2}\Big|_0^1\right)\cdot\left(\frac{2^{2x}}{2\log 2}\Big|_0^1\right)\cdot\left(\frac{2^{3x}}{3\log 2}\Big|_0^1\right)\cdot\dots\cdot\left(\frac{2^{2nx}}{2n\log 2}\Big|_0^1\right)$$

$$\left(\frac{2^n-1}{n\log 2}\right)^{2n+1}\le\frac{(2-1)(2^2-1)(2^3-1)\cdot\dots\cdot(2^{2n}-1)}{(2n)!\cdot(\log 2)^{2n}}$$

$$\frac{1}{\log 2}\left(\frac{2^n-1}{n}\right)^{2n+1}\le\frac{1\cdot 3\cdot 7\cdot\dots\cdot(2^{2n}-1)}{(2n)!}$$

SOLUTION AN.135.

$$x_{n+3}-5x_{n+2}+7x_{n+1}-3x_n=0\Rightarrow$$

$$x_{n+3}-4x_{n+2}+3x_{n+1}=x_{n+2}-4x_{n+1}+3x_n;$$

$$x_{n+2}-4x_{n+1}+3x_n=x_{n+1}-4x_n+3x_{n-1}$$

$$x_{n+1}-4x_n+3x_{n-1}=x_n-4x_{n-1}+3x_{n-2}$$

– –

$$x_{n+3}-4x_{n+2}+3x_{n+1}=x_3-4x_2+3x_1=$$

$$=10-4\cdot 4+3\cdot 2=0$$

$$x_{n+3}-4x_{n+2}+3x_{n+1}=0$$

$$x_{n+3}-x_{n+2}=3(x_{n+2}-x_{n+1})=3^2(x_{n+1}-x_n)=$$

$$= 3^3(x_n - x_{n-1}) = \cdots = 3^{n+1}(x_2 - x_1) =$$

$$= 3^{n+1}(4-2) = 2 \cdot 3^{n+1}$$

$$x_3 - x_2 = 3(x_2 - x_1)$$

$$x_4 - x_3 = 3^2(x_2 - x_1)$$

$$x_5 - x_4 = 3^3(x_2 - x_1)$$

$$- - - - - - - - - - - -$$

$$x_n - x_{n-1} = 3^{n-2}(x_2 - x_1)$$

By adding:

$$x_n - x_2 = (3 + 3^2 + \cdots + 3^{n-2})(x_2 - x_1)$$

$$x_n = x_2 + \frac{3(3^{n-2}-1)}{3-1}(x_2 - x_1)$$

$$x_n = 4 + 3(3^{n-2}-1) \cdot \frac{4-2}{2}$$

$$x_n = 3^{n-1} + 1; n \geq 1$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}(3^{n-1}+1) = \infty$$

$$\Omega = \lim_{n\to\infty}\left(x_n^2\left(3^{\frac{1}{x_n}} - 1\right) - x_n \log 3\right) =$$

$$= \lim_{x\to\infty}\left(x^2\left(3^{\frac{1}{x}} - 1\right) - x\log 3\right) = \lim_{x\to\infty}\frac{x\left(3^{\frac{1}{x}} - 1\right) - \log 3}{\frac{1}{x}} =$$

$$= \lim_{x\to\infty}\frac{3^{\frac{1}{x}} - 1 + x\left(-\frac{1}{x^2}\right)\cdot 3^{\frac{1}{x}}\log 3}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{3^{\frac{1}{x}} - 1 - \frac{1}{x}\cdot 3^{\frac{1}{x}}\cdot\log 3}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to\infty}\frac{-\frac{1}{x^2}\cdot 3^{\frac{1}{x}}\log 3 + \frac{1}{x^2}\cdot 3^{\frac{1}{x}}\log 3 + \frac{1}{x^3}\cdot 3^{\frac{1}{x}}\log^2 3}{\frac{2}{x^3}} =$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x^3}\cdot 3^{\frac{1}{x}}\cdot\log^2 3}{\frac{2}{x^3}} = \lim_{x\to\infty}\frac{3^{\frac{1}{x}}\log^2 3}{2} = \frac{\log^2 3}{2}$$

SOLUTION AN.136.

$$\sum_{k=1}^{p}\left\{\frac{kp}{p+1}\right\}=\sum_{k=1}^{p}\left(\frac{kp}{p+1}-\left[\frac{kp}{p+1}\right]\right)=\frac{p}{p+1}\sum_{k=1}^{p}k-\sum_{k=1}^{p}\left[\frac{kp+p-p}{p+1}\right]=$$

$$=\frac{p}{p+1}\cdot\frac{p(p+1)}{2}-\sum_{k=1}^{p}\left[\frac{k(p+1)}{p+1}-\frac{p}{p+1}\right]=\frac{p^2}{2}-\sum_{k=1}^{p}\left[k-\frac{p+1-1}{p+1}\right]=$$

$$=\frac{p^2}{2}-\sum_{k=1}^{p}\left[k-1+\frac{1}{p+1}\right]=\frac{p^2}{2}-\sum_{k=1}^{p}\left((k-1)+\left[\frac{1}{p+1}\right]\right)=$$

$$=\frac{p^2}{2}-\frac{p(p+1)}{2}+p=\frac{p^2-p^2-p+2p}{2}=\frac{p}{2}$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{p}\frac{1}{\frac{p}{2}}-2\log(2n+1)\right)=\lim_{n\to\infty}\left(2\sum_{p=1}\frac{1}{p}-2\log(2n+1)\right)=$$

$$=2\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log(2n+1)\right)=$$

$$=2\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n+\log n-\log(2n+1)\right)=$$

$$=2\left(\gamma+\lim_{n\to\infty}\log\left(\frac{n}{2n+1}\right)\right)=2\left(\gamma+\log\frac{1}{2}\right)=2(\gamma-\log 2)$$

SOLUTION AN.137.

$$\lim_{n\to\infty}\omega(n)=\lim_{n\to\infty}\left(n\prod_{i=2}^{n}\left(\frac{i^3+1}{i^3-1}\right)\right)=\lim_{n\to\infty}\left(n\cdot\prod_{i=2}^{n}\frac{(i+1)(i^2-i+1)}{(i-1)(i^2+i+1)}\right)=$$

$$=\lim_{n\to\infty}\left(n\cdot\prod_{i=2}^{n}\frac{i+1}{i-1}\cdot\prod_{i=2}^{n}\frac{i^2-i+1}{i^2+i+1}\right)=$$

$$=\lim_{n\to\infty}\left(n\cdot\frac{3\cdot4\cdot5\cdot\ldots\cdot(n-1)n(n+1)}{1\cdot2\cdot3\cdot\ldots\cdot(n-1)}\cdot\frac{3\cdot7\cdot13\cdot\ldots\cdot(n^2-n+1)}{7\cdot13\cdot21\cdot\ldots\cdot(n^2+n+1)}\right)=$$

$$=\lim_{n\to\infty}\left(n\cdot\frac{n(n+1)}{1\cdot2}\cdot\frac{3}{n^2+n+1}\right)=\frac{3}{2}\lim_{n\to\infty}\frac{n^2(n+1)}{n^2+n+1}=\infty$$

$$\Omega=\lim_{n\to\infty}\left(\omega^2(n)\left(1+\frac{1}{\omega(n)}\right)^{\frac{1}{\omega(n)}}-\omega^2(n)\cos\left(\frac{1}{\omega^2(n)}\right)\right)=$$

$$= \lim_{x\to\infty}\left(x^2\left(1+\frac{1}{x}\right)^{\frac{1}{x}} - x^2\cos\frac{1}{x^2}\right) = \lim_{y\to 0}\left(\frac{(1+y)^y - \cos y^2}{y^2}\right) =$$

$$= \lim_{y\to 0}\left(\frac{y(1+y)^{y-1} + (1+y)^y\ln(1+y) + 2y\sin y^2}{2y}\right) =$$

$$= \lim_{y\to 0}\frac{(y+1)^{y-1}}{2} + \lim_{y\to 0}\frac{(1+y)^y}{2}\cdot\lim_{y\to 0}\frac{\ln(1+y)}{y} + \lim_{y\to 0}\sin y^2 =$$

$$= \frac{1}{2} + \frac{1}{2}\cdot 1 + \sin 0 = 1$$

SOLUTION AN.138.

$$\tan^{-1}\left(\frac{k+2}{k+1}\right) - \tan^{-1}\left(\frac{k+1}{k}\right) = \tan^{-1}\left(\frac{\frac{k+2}{k+1} - \frac{k+1}{k}}{1 + \frac{k+2}{k+1}\cdot\frac{k+1}{k}}\right) =$$

$$= \tan^{-1}\left(\frac{k^2+2k-k^2-2k-1}{k(k+1)}\cdot\frac{k(k+1)}{k^2+k+k^2+3k+2}\right) =$$

$$= \tan^{-1}\left(-\frac{1}{2k^2+4k+1}\right) = -\tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$$

$$\tan^{-1}\left(\frac{k+2}{k+1}\right) + \tan^{-1}\left(\frac{k+1}{k}\right) =$$

$$= \tan^{-1}\left(\frac{\frac{k+2}{k+1} + \frac{k+1}{k}}{1 - \frac{k+2}{k+1}\cdot\frac{k+1}{k}}\right) = \tan^{-1}\left(\frac{k^2+2k+k^2+2k+1}{k(k+1)}\cdot\frac{k(k+1)}{k^2+k-k^2-3k-2}\right)$$

$$= \tan^{-1}\left(\frac{2k^2+4k+1}{-2k-2}\right) = -\tan^{-1}\left(\frac{2k^2+4k+1}{2(k+1)}\right)$$

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{k+2}{k+1}\right) - \tan^{-1}\left(\frac{k+1}{k}\right)\right)\cdot\left(\tan^{-1}\left(\frac{k+2}{k+2}\right) + \tan^{-1}\left(\frac{k+1}{k}\right)\right)\right) =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\left(\tan^{-1}\left(\frac{k+2}{k+1}\right)\right)^2 - \left(\tan^{-1}\left(\frac{k+1}{k}\right)\right)^2\right)\right) =$$

$$= \lim_{n\to\infty}\left(\left(\tan^{-1}\left(\frac{n+2}{n+1}\right)\right)^2 - \left(\tan^{-1}\left(\frac{1+1}{1}\right)\right)^2\right) = (\tan^{-1}1)^2 - (\tan^{-1}2)^2$$

SOLUTION AN.139.

$$a^3f(x) + b^3f(y) + c^3f(z) = f(x)f(y)f(z)$$

$$1=\frac{a^3}{f(y)f(z)}+\frac{b^3}{f(z)f(x)}+\frac{c^3}{f(x)f(y)}\overset{AM-GM}{\geq}$$

$$\geq\frac{a^3}{\left(\frac{f(y)+f(z)}{2}\right)^2}+\frac{b^3}{\left(\frac{f(z)+f(x)}{2}\right)^2}+\frac{c^3}{\left(\frac{f(x)+f(y)}{2}\right)^2}\overset{RADON}{\geq}\frac{(a+b+c)^3}{\left(f(x)+f(y)+f(z)\right)^2}$$

$$1\geq\frac{(a+b+c)^3}{\left(f(x)+f(y)+f(z)\right)^2}$$

$$\left(f(x)+f(y)+f(z)\right)^2\geq(a+b+c)^3$$

$$f(x)+f(y)+f(z)\geq(a+b+c)\sqrt{a+b+c}$$

$$\int_\alpha^\beta\int_\alpha^\beta\int_\alpha^\beta\left(f(x)+f(y)+f(z)\right)dx\,dy\,dz\geq\int_\alpha^\beta\int_\alpha^\beta\int_\alpha^\beta(a+b+c)\sqrt{a+b+c}\,dx\,dy\,dz$$

$$3(\beta-\alpha)^2\int_\alpha^\beta f(x)\,dx\geq(a+b+c)\sqrt{a+b+c}\cdot(\beta-\alpha)^3$$

$$\int_\alpha^\beta f(x)\,dx\geq\frac{(\beta-\alpha)(a+b+c)\sqrt{a+b+c}}{3}$$

SOLUTION AN.140.

Let be $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R};$

$$f(x)=e^x(4\cot^3x+\cot^2x+\cot x-2)=-e^x(-4\cot^3x-\cot^2x-\cot x+2)=$$

$$=-e^x(-4\cot x-4\cot^3x-3-3\cot^2x+2\cot^2x+3\cot x+5)=$$

$$=-e^x[-4\cot x(1+\cot^2x)-3(1+\cot^2x)+2\cot^2x+3\cot x+5]=$$

$$=-e^x\left(-\frac{4}{\sin^2x}\cot x+3\frac{-1}{\sin^2x}+2\cot^2x+3\cot x+5\right)=$$

$$=(e^x)'(2\cot^2x+3\cot x+5)+e^x(2\cot^2x+3\cot x+5)'=$$

$$=[e^x(2\cot^2x+3\cot x+5)]'$$

$$\Omega=\int[e^x(2\cot^2x+3\cot x+5)]'\,dx=e^x(2\cot^2x+3\cot x+5)+C$$

SOLUTION AN.141.

$$\Omega=\lim_{n\to\infty}\left(n\left(x_n^b-b^{x_n}\right)\right)=\lim_{n\to\infty}\frac{x_n^b-b^{x_n}}{\frac{1}{n}}=$$

$$\overset{CESARO}{=} \lim_{n\to\infty}\left(\frac{b^{x_{n+1}}-b^{x_n}+x_n^b-x_{n+1}^b}{x_{n+1}-x_n}\cdot\frac{1}{\frac{1}{n+1}-\frac{1}{n}}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{b^{x_{n+1}}+x_n^b-b^{x_n}-x_{n+1}^b}{x_{n+1}-x_n}\cdot n(n+1)(x_{n+1}-x_n)\right)=$$

$$=a\cdot\lim_{n\to\infty}\frac{b^{x_{n+1}}+x_n^b-b^{x_n}-x_{n+1}^b}{x_{n+1}-x_n}=\left(\lim_{n\to\infty}\frac{b^{x_{n+1}}-b^{x_n}}{x_{n+1}-x_n}+\lim_{n\to\infty}\frac{x_n^b-x_{n+1}^b}{x_{n+1}-x_n}\right)=$$

$$=a\left(\lim_{n\to\infty}\frac{b^{x_n}(b^{x_{n+1}-x_n}-1)}{x_{n+1}-x_n}+\lim_{n\to\infty}\frac{x_{n+1}^b\left(\left(\frac{x_n}{x_{n+1}}\right)^b-1\right)}{x_{n+1}-x_n}\right)=$$

$$=a\left(b^b\log b+b^b\lim_{n\to\infty}\left(\frac{\left(\frac{x_n}{x_{n+1}}\right)^b-1}{\frac{x_n}{x_{n+1}}-1}\cdot\left(\frac{x_n}{x_{n+1}}-1\right)\cdot\frac{1}{x_{n+1}-x_n}\right)\right)=$$

$$=a\left(b^b\log b-b^b\cdot b\cdot\frac{1}{b}\right)=a\cdot b^b(\log b-1)=a\cdot b^b\log\left(\frac{b}{e}\right)$$

SOLUTION AN.142.

$x,y\in[a,b]$. By Schweitzer inequality:

$$(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)\le\frac{(a+b)^2}{ab}$$

$$\frac{(x+y)^2}{xy}\le\frac{(a+b)^2}{ab}$$

$$ab(x+y)^2\le xy(a+b)^2$$

$$\sqrt{ab}(x+y)\le\sqrt{xy}(a+b)\quad (1)$$

Analogous:

$$\sqrt{ab}(y+z)\le\sqrt{yz}(a+b)\quad (2)$$

$$\sqrt{ab}(z+t)\le\sqrt{zt}(a+b)\quad (3)$$

$$\sqrt{ab}(t+x)\le\sqrt{tx}(a+b)\quad (4)$$

By adding (1); (2); (3); (4):

$$2\sqrt{ab}(x+y+z+t)\le(a+b)\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}\right)$$

$$\frac{x+y+z+t}{\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}}\le\frac{a+b}{2\sqrt{ab}}=$$

$$= \frac{a+b}{2} \cdot \frac{1}{\sqrt{ab}} \overset{GM-HM}{\leq} \frac{a+b}{2} \cdot \frac{1}{\frac{2}{\frac{1}{a}+\frac{1}{b}}} = \frac{a+b}{2} \cdot \frac{1}{\frac{2ab}{a+b}} = \frac{(a+b)^2}{4ab}$$

$$\frac{x+y+z+t}{\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}} \leq \frac{(a+b)^2}{4ab}$$

$$\int_a^b \int_a^b \int_a^b \int_a^b \frac{(x+y+z+t)dx\,dy\,dz\,dt}{\sqrt{xy}+\sqrt{yz}+\sqrt{zt}+\sqrt{tx}} \leq$$

$$\leq \int_a^b \int_a^b \int_a^b \int_a^b \frac{(a+b)^2}{4ab} dx\,dy\,dz\,dt = \frac{(b+a)^2(b-a)^4}{4ab}$$

SOLUTION AN.143.

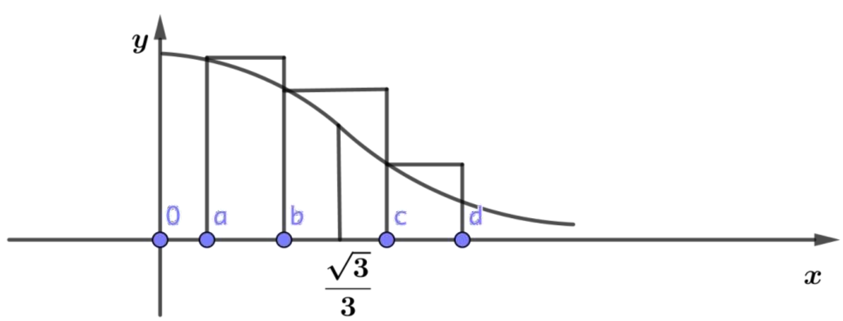
Let be $f: [0,\infty) \to \mathbb{R}; f(x) = \frac{1}{1+x^2}$; $f'(x) = -\frac{2x}{(1+x^2)^2} \leq 0$

$$\lim_{x\to\infty} f(x) = 0; f(0) = 1$$

$$f''(x) = \frac{-2(1+x^2)^2 + 2x \cdot 2 \cdot 2x(1+x^2)}{(1+x^2)^4}$$

$$f''(x) = \frac{-2(1+x^2)+8x^2}{(1+x^2)^4} = \frac{2(4x^2-x^2-1)}{(1+x^2)^3}$$

$$f''(x) = \frac{2(3x^2-1)}{(1+x^2)^3}$$



$$(b-a)f(a) + (c-b)f(b) + (d-c)f(c) \geq \int_a^d f(x)\,dx$$

$$\frac{b-a}{1+a^2} + \frac{c-b}{1+b^2} + \frac{d-c}{1+c^2} \geq \int_a^d \frac{1}{1+x^2} dx = \tan^{-1} d - \tan^{-1} a$$

SOLUTION AN.144.

$$\lim_{n\to\infty} H_n = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)=\infty$$

$$\lim_{n\to\infty}\frac{1}{H_n}=\frac{1}{\infty}=0$$

$$\Omega=\lim_{n\to\infty}\frac{\left(\pi+\frac{1}{H_n}\right)^{\pi^{\left(\pi+\frac{1}{H_n}\right)}}-\pi^{\left(\pi+\frac{1}{H_n}\right)^{\pi}}}{\frac{1}{H_n}}$$

$$\Omega=\lim_{y\to 0}\frac{(\pi+y)^{\pi(\pi+y)}-\pi^{(y+\pi)^{\pi}}}{y}$$

$$\textit{Denote } x=y+\pi$$

$$\Omega=\lim_{x\to\pi}\frac{x^{\pi^x}-\pi^{y^{\pi}}}{x-\pi}$$

$$\Omega=\lim_{x\to\pi}\frac{x^{\pi x}-\pi^{\pi x}}{x-\pi}-\lim_{x\to\pi}\frac{e^{x^{\pi}\log\pi}-e^{\pi^x\log\pi}}{x-\pi}$$

$$\Omega=\lim_{x\to\pi}\frac{e^{\pi^x\log\pi}-e^{\pi^x\log\pi}}{x-\pi}-\lim_{x\to\pi}\frac{e^{x^{\pi}\log\pi}-e^{\pi^x\log\pi}}{x-\pi}$$

$$\Omega=\lim_{x\to\pi}\frac{e^{\pi^x\log\pi}\left(e^{\pi^x(\log x-\log\pi)}-1\right)}{\pi^x\cdot(\log x-\log\pi)}\cdot\frac{\pi^x(\log x-\log\pi)}{x-\pi}-$$

$$-\lim_{x\to\pi}\frac{e^{\pi^x\log\pi}\left(e^{x^{\pi}\log\pi-\pi^x\log\pi}-1\right)}{\log\pi(x^{\pi}-\pi^x)}\cdot\frac{\log\pi\,(x^{\pi}-\pi^x)}{x-\pi}$$

$$\Omega=e^{\pi^{\pi}\log\pi}\cdot\pi^{\pi}\cdot\lim_{x\to\pi}\frac{\log x-\log\pi}{x-\pi}-e^{\pi^x\log\pi}\cdot\log\pi\cdot\lim_{x\to\pi}\frac{x^{\pi}-\pi^x}{x-\pi}$$

$$\Omega=e^{\pi^{\pi}\log\pi}\cdot\pi^{\pi}\cdot\frac{1}{\pi}-e^{\pi^x\log\pi}\cdot\log\pi\cdot\lim_{x\to\pi}\frac{\pi x^{\pi-1}-\pi^x\log\pi}{1}$$

$$\Omega=e^{\pi^{\pi}\log\pi}\cdot\pi^{\pi-1}-e^{\pi^{\pi}\log\pi}\cdot\log\pi\,(\pi^{\pi}-\pi^{\pi}\log\pi)$$

$$\Omega=\pi^{\pi^{\pi}}\cdot\pi^{\pi-1}-\pi^{\pi^{\pi}}\cdot\pi^{\pi}\log\pi+\pi^{\pi^{\pi}}\cdot\pi^{\pi}\log^2\pi$$

$$\Omega=\pi^{\pi^{\pi}}(\pi^{\pi-1}-\pi^{\pi}\cdot\log\pi+\pi^{\pi}\log^2\pi)$$

$$\Omega=\pi^{\pi^{\pi}}\cdot\pi^{\pi}\left(\frac{1}{\pi}-\log\pi+\log^2\pi\right)$$

$$\Omega=\pi^{\pi+\pi^{\pi}}\left(\frac{1}{\pi}-\log\pi+\log^2\pi\right)$$

SOLUTION AN.145.

In Hlawka's inequality:

$$|u+v|+|v+w|+|w+u|\le|u|+|v|+|w|+|u+v+w|$$

we take: $u=\log f(x)\,;v=\log f(y)\,;w=\log f(z)$

$$f(x),f(y),f(z)>1\Rightarrow u,v,w>0$$

$$e^x>x+1;x>0\Rightarrow x>\log(x+1)\Rightarrow\log(x)<x-1\quad (1)$$

$$|\log f(x)+\log f(y)|+|\log f(y)+\log f(z)|+|\log f(z)+\log f(x)|\le$$

$$\le|\log f(x)|+|\log f(y)|+|\log f(z)|+|\log f(x)\,f(y)f(z)|\le$$

$$\overset{(1)}{\le} f(x)-1+f(y)-1+f(z)-1+f(x)f(y)f(z)-1$$

$$\log\big(f(x)\cdot f(y)\cdot f(y)\cdot f(z)\cdot f(z)\cdot f(x)\big)\le$$

$$\le-4+f(x)+f(y)+f(z)+f(x)f(y)f(z)$$

$$\log\big(f(x)f(y)f(z)\big)^2+4\le f(x)+f(y)+f(z)+f(x)f(y)f(z)$$

$$4+2\log\big(f(x)f(y)f(z)\big)\le f(x)+f(y)+f(z)+f(x)f(y)f(z)$$

$$\int_a^b\int_a^b\int_a^b\Big(4+2\log\big(f(x)f(y)f(z)\big)\Big)\,dx\,dy\,dz\le$$

$$\le\int_a^b\int_a^b\int_a^b\big(f(x)+f(y)+f(z)+f(x)f(y)f(z)\big)\,dxdydz$$

$$4(b-a)^3+6(b-a)^2\int_a^b\log f(x)\,dx\le3(b-a)^2\int_a^b f(x)\,dx+\left(\int_a^b f(x)\,dx\right)^3$$

SOLUTION AN.146.

By induction we prove that:

$$\frac{1\cdot\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[k]{k!}}{(k+1)!}\le\frac{1}{2^k}\quad (1)$$

Suppose (1) true. We must prove that:

$$\frac{1\cdot\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[k]{k!}\cdot\sqrt[k+1]{(k+1)!}}{(k+2)!}\le\frac{1}{2^{k+1}}\quad (2)$$

$$\frac{1\cdot\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[k]{k!}\cdot\sqrt[k+1]{(k+1)!}}{(k+2)!}=$$

$$=\frac{1\cdot\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[k]{k!}}{(k+1)!}\cdot\frac{\sqrt[k+1]{(k+1)!}}{k+2}\le$$

$$\le\frac{1}{2^k}\cdot\frac{\sqrt[k+1]{(k+1)!}}{k+2}\le\frac{1}{2^{k+1}}\quad \textit{(to prove)}$$

$$\frac{\sqrt[k+1]{(k+1)!}}{k+2}=\frac{\sqrt[k+1]{1\cdot 2\cdot 3\cdot\ldots\cdot k(k+1)}}{k+2}\overset{AM-GM}{<}$$

$$<\frac{1}{k+2}\cdot\frac{1+2+3+\cdots+k+(k+1)}{k+1}==\frac{(k+1)(k+2)}{2(k+1)(k+2)}=\frac{1}{2}.\ \textit{Hence (1)}\Rightarrow\textit{(2)}$$

$$P(k)\to P(k+1)$$

$$\frac{1\cdot\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[k]{k!}}{(k+1)!}<\frac{1}{2^k}$$

$$\sum_{k=1}^{n}\frac{1\cdot\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[k]{k!}}{(k+1)!}<\sum_{k=1}^{n}\frac{1}{2^k}=\frac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^n-1\right)}{\frac{1}{2}-1}=1-\left(\frac{1}{2}\right)^n$$

$$\frac{1}{H_n}\sum_{k=1}^{n}\frac{1\cdot\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[k]{k!}}{(k+1)!}<\frac{1}{H_n}\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$0\le\Omega\le\lim_{n\to\infty}\frac{1}{H_n}\left(1-\left(\frac{1}{2}\right)^n\right)=\frac{1}{\infty}(1-0)=0$$

$$\Omega=0$$

SOLUTION AN.147.

$$\textit{Let be } A=\begin{pmatrix}1 & u & v & w\\ u & v & w & 1\end{pmatrix}\in M_{2,4}(\mathbb{R})$$

$$A^T=\begin{pmatrix}1 & u\\ u & v\\ v & w\\ w & 1\end{pmatrix}\in M_{4,2}(\mathbb{R}).\ \textit{By Binet – Cauchy's theorem:}$$

$$\det(A\cdot A^T)\ge 0$$

$$A\cdot A^T=\begin{pmatrix}1 & u & v & w\\ u & v & w & 1\end{pmatrix}\begin{pmatrix}1 & u\\ u & v\\ v & w\\ w & 1\end{pmatrix}=$$

$$=\begin{pmatrix}1+u^2+v^2+w^2 & u+uv+vw+w\\ u+uv+vw+w & u^2+v^2+w^2+1\end{pmatrix}$$

$$\det(A\cdot A^T)\ge 0\Rightarrow (1+u^2+v^2+w^2)^2\ge(u+uv+vw+w)^2$$

$$1+u^2+v^2+w^2\ge u+uv+vw+w$$

Replacing $u = f(x), v = f(y), w = f(z)$*. By integrating:*

$$\int_a^b\int_a^b\int_a^b \left(1 + f^2(x) + f^2(y) + f^2(z)\right) dx\,dy\,dz \ge$$

$$\ge \int_a^b\int_a^b\int_a^b \left(f(x) + f(x)f(y) + f(y)f(x) + f(z)\right) dx\,dy\,dz$$

$$\int_a^b\int_a^b\int_a^b dx\,dy\,dz + 3\int_a^b\int_a^b\int_a^b f^2(x)\,dx\,dy\,dz \ge$$

$$\ge 2\int_a^b\int_a^b\int_a^b f(x)\,dx\,dy\,dz + 2\int_a^b\int_a^b\int_a^b f(x)\,f(y)dx\,dy$$

$$(b-a)^3 + 3(b-a)^2\int_a^b f^2(x)dx \ge 2(b-a)^2\int_a^b f(x)dx + 2(b-a)\left(\int_a^b f(x)dx\right)^2$$

By dividing with $b - a > 0$*:*

$$(b-a)^2 + 3(b-a)\int_a^b f^2(x)\,dx \ge 2(b-a)\int_a^b f(x)\,dx + 2\left(\int_a^b f(x)\,dx\right)^2$$

SOLUTION AN.148.

The characteristic equation is:

$$\lambda^3 - 14\lambda^2 + 65\lambda - 100 = 0 \Leftrightarrow (\lambda - 4)(\lambda - 5)^2 = 0$$

Eigen values are $\lambda_1 = 4; \lambda_2 = \lambda_3 = 5$

$$x_n = A\lambda_1^n + B\lambda_2^n + C_n\lambda_2^n = A \cdot 4^n + (B + C_n) \cdot 5^n$$

By $x_1 = 4A + (B + C) \cdot 5 = 14;$

$$x_2 = 16A + (B + 2C) \cdot 25 = 81;$$

$$x_3 = 64A + (B + 3C) \cdot 125 = 564 \textit{ results } A = B = C = 1$$

$$x_n = 4^n + (1 + n) \cdot 5^n$$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{x_{n+1}}{x_n}} \overset{CDA}{=} \lim_{n\to\infty} \frac{\frac{x_{n+2}}{x_{n+1}}}{\frac{x_{n+1}}{x_n}} = \lim_{n\to\infty} \frac{x_{n+2} \cdot x_n}{(x_{n+1})^2} =$$

$$= \lim_{n\to\infty} \frac{(4^{n+2} + (3+n)5^{n+2}) \cdot (4^n + (1+n) \cdot 5^n)}{(4^{n+1} + (2+n)5^{n+1})^2} =$$

$$= \lim_{n\to\infty} \frac{5^{2n+2}\left(\left(\frac{4}{5}\right)^{n+2} + n + 3\right)\left(\left(\frac{4}{5}\right)^{n} + n + 1\right)}{5^{2n+2}\left(\left(\frac{4}{5}\right)^{n+1} + n + 2\right)^2} =$$

$$= \lim_{n\to\infty} \frac{n^2\left(\left(\frac{4}{5}\right)^{n+2}\cdot\frac{1}{n} + 1 + \frac{3}{n}\right)\left(\left(\frac{4}{5}\right)^{n}\cdot\frac{1}{n} + 1 + \frac{1}{n}\right)}{n^2\left(\left(\frac{4}{5}\right)^{n+1}\cdot\frac{1}{n} + 1 + \frac{2}{n}\right)^2} = 1$$

SOLUTION AN.149.

$$\textit{Let be } f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R};\ f(x) = \log\left(\frac{\sin x}{x}\right)$$

$$f'(x) = \frac{\left(\frac{\sin x}{x}\right)'}{\frac{\sin x}{x}} = \frac{x}{\sin x}\cdot\frac{x\cos x - \sin x}{x^2} = \frac{x\cos x - \sin x}{x\sin x}$$

$$f'(x) = \cot x - \frac{1}{x};\ f''(x) = -\frac{1}{\sin^2 x} + \frac{1}{x^2} = \frac{\sin^2 x - x^2}{x^2\sin^2 x}$$

$$\sin x < x;\ (\forall)x\in\left(0,\frac{\pi}{2}\right) \Rightarrow \sin^2 x < x^2 \Rightarrow \sin^2 x - x^2 < 0$$

$$f''(x) < 0 \Rightarrow f \textit{ concave on } \left(0,\frac{\pi}{2}\right)$$

By Jensen's inequality:

$$f\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}\right) \ge \frac{1}{a+b+c}\left(af(x_1) + bf(x_2) + cf(x_3)\right)$$

$$(\forall)a,b,c > 0;\ (\forall)x_1,x_2,x_3\in\left(0,\frac{\pi}{2}\right)$$

We take: $x_1 = b;\ x_2 = c;\ x_3 = a$:

$$f\left(\frac{ab+bc+ca}{a+b+c}\right) \ge \frac{1}{a+b+c}\left(af(b) + bf(a) + cf(a)\right)$$

$$(a+b+c)\log\left(\frac{\sin\left(\frac{ab+bc+ca}{a+b+c}\right)}{\frac{ab+bc+ca}{a+b+c}}\right) \ge a\log\left(\frac{\sin b}{b}\right) + b\log\left(\frac{\sin c}{c}\right) + c\log\left(\frac{\sin a}{a}\right)$$

$$\log\left(\frac{\sin\left(\frac{ab+bc+ca}{a+b+c}\right)}{\frac{ab+bc+ca}{a+b+c}}\right)^{a+b+c} \ge \log\left(\left(\frac{\sin b}{b}\right)^a\cdot\left(\frac{\sin c}{c}\right)^b\cdot\left(\frac{\sin a}{a}\right)^c\right)$$

$$\left(\frac{\sin\left(\frac{ab+bc+ca}{a+b+c}\right)}{\frac{ab+bc+ca}{a+b+c}}\right)^{a+b+c} \geq \left(\frac{\sin b}{b}\right)^a \cdot \left(\frac{\sin c}{c}\right)^b \cdot \left(\frac{\sin a}{a}\right)^c$$

SOLUTION AN.150.

$$\frac{\sin x}{\sin y} + \frac{\sin x + \sin y}{\sin z} = \frac{\sin x}{\sin y} + \frac{\sin x}{\sin z} + \frac{\sin y}{\sin z} \geq$$

$$\overset{AM-GM}{\geq} 3\sqrt[3]{\frac{\sin x}{\sin y} \cdot \frac{\sin x}{\sin z} \cdot \frac{\sin y}{\sin z}} = 3\sqrt[3]{\left(\frac{\sin x}{\sin z}\right)^2}$$

It remains to prove that:

$$3\sqrt[3]{\left(\frac{\sin x}{\sin z}\right)^2} > \frac{6}{\pi}\sqrt[3]{\left(\frac{x}{z}\right)^2} \Leftrightarrow 27\left(\frac{\sin x}{\sin z}\right)^2 > \frac{27 \cdot 8}{\pi^3} \cdot \left(\frac{x}{z}\right)^2$$

$$\pi^3\left(\frac{\sin x}{\sin z}\right)^2 > 8\left(\frac{x}{z}\right)^2$$

$$\pi\sqrt{\pi} \cdot \frac{\sin x}{\sin z} > 2\sqrt{2} \cdot \frac{x}{z}$$

$$\pi\sqrt{\pi}\frac{\sin x}{x} > 2\sqrt{2}\frac{\sin z}{z} \quad (1)$$

$$\sin x \overset{JORDAN}{>} \frac{2x}{\pi} \Rightarrow \pi\frac{\sin x}{x} > 2$$

$$\pi\sqrt{\pi}\frac{\sin x}{x} > 2\sqrt{\pi} \quad (2)$$

$$\sin z < z \Rightarrow \frac{\sin z}{z} < 1 \Rightarrow 2\sqrt{2}\frac{\sin z}{z} < 2\sqrt{2} < 2\sqrt{\pi} \quad (3)$$

By (2); (3) ⇒ (1)

SOLUTION AN.151.

The characteristic equation for this sequence is:

$$\lambda^4 - 10\lambda^3 + 36\lambda^2 - 54\lambda + 27 = 0$$

$$\lambda^4 - \lambda^3 - 9\lambda^3 + 9\lambda^2 + 27\lambda^2 - 27\lambda + 27 = 0$$

$$\lambda^3(\lambda-1) - 9\lambda^2(\lambda-1) + 27\lambda(\lambda-1) - 27(\lambda-1) = 0$$

$$(\lambda-1)(\lambda^3 - 9\lambda^2 + 27\lambda - 27) = 0,\ (\lambda-1)(\lambda-3)^3 = 0$$

$$\lambda_1 = 1; \lambda_2 = \lambda_3 = \lambda_4 = 3$$

$$x_n = A \cdot \lambda_1^n + B\lambda_2^n + Cn\lambda_3^n + Dn^2\lambda_3^n$$

$$x_n = A \cdot 1^n + B \cdot 3^n + Cn \cdot 3^n + Dn^2 \cdot 3^n$$

$$x_n = A + B\cdot 3^n + Cn\cdot 3^n + Dn^2\cdot 3^n$$

$$x_1 = A + 3B + 3C + 3D = 10$$

$$x_2 = A + 9B + 18C + 36D = 64$$

$$x_3 = A + 27B + 81C + 243D = 352$$

$$x_4 = A + 81B + 324C + 1296D = 1702$$

with solution $A = B = C = D = 1$

Hence: $x_n = 1 + 3^n + n\cdot 3^n + n^2\cdot 3^n$

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\frac{x_n\cdot x_{n+3}}{x_{n+1}\cdot x_{n+2}}} \overset{CAUCHY-D'ALEMBERT}{=}$$

$$= \lim_{n\to\infty}\left(\frac{x_{n+1}\cdot x_{n+4}}{x_{n+2}\cdot x_{n+3}}\cdot\frac{x_{n+1}\cdot x_{n+2}}{x_n\cdot x_{n+3}}\right) = \lim_{n\to\infty}\left(\frac{x_{n+4}\cdot(x_{n+1})^2}{x_n\cdot(x_{n+3})^2}\right) =$$

$$= \lim_{n\to\infty}\frac{(1+3^{n+4}+(n+4)3^{n+4}+(n+4)^2 3^{n+4})(1+3^{n+1}+(n+1)3^{n+1}+(n+1)^2 3^{n+1})^2}{(1+3^n+n\cdot 3^n+n^2\cdot 3^n)(1+3^{n+3}+(n+3)3^{n+3}+(n+3)^2 3^{n+2})^2}$$

$$= \lim_{n\to\infty}\frac{(n+4)^2 3^{n+4}((n+1)^2\cdot 3^{n+1})^2}{n^2\cdot 3^n\cdot((n+3)^2\cdot 3^{n+3})^2} = \lim_{n\to\infty}\frac{3^{n+4}\cdot 3^{2n+2}}{3^n\cdot 3^{2n+6}} = 1$$

SOLUTION AN.152.

By Bernoulli's inequality:

$$(1+x)^\alpha < 1+\alpha x;\ x > -1;\ 0 < \alpha < 1$$

For $x = p-1;\ \alpha = 1-q$

$$(1+p-1)^{1-q} < 1+(1-q)(p-1) = 1+p-1-pq+q < p+q$$

$$p^{1-q} < p+q \Rightarrow \frac{p}{p^q} < p+q \Rightarrow p^q > \frac{p}{p+q}\quad (1)$$

By (1):

$$(\sin x)^{2\cos^2 x} = (\sin^2 x)^{\cos^2 x} > \frac{\sin^2 x}{\sin^2 x+\cos^2 x}\quad (1)$$

$$(\cos x)^{2\sin^2 x} = (\cos^2 x)^{\sin^2 x} > \frac{\cos^2 x}{\sin^2 x+\cos^2 x}\quad (2)$$

By adding (1); (2):

$$(\sin x)^{2\cos^2 x} + (\cos x)^{2\sin^2 x} > 1 > \frac{1}{1+\cos^2 x}\quad (3)$$

Because $1 > \frac{1}{1+\cos^2 x} \Leftrightarrow \cos^2 x > 0$

Integrating (3):

$$\int_a^b \left((\sin x)^{2\cos^2 x} + (\cos x)^{2\sin^2 x}\right) dx \geq \int_a^b \frac{dx}{1+\cos^2 x} =$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)\Big|_a^b = \frac{1}{\sqrt{2}}\left(\tan^{-1}\left(\frac{\tan b}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{\tan a}{\sqrt{2}}\right)\right)$$

SOLUTION AN.153.

By Bernoulli's inequality: $(1+x)^{\alpha} < 1 + \alpha x; x > -1; 0 < \alpha < 1$

For $x = p - 1; \alpha = 1 - q$

$$(1 + p - 1)^{1-q} < 1 + (1-q)(p-1)$$

$$p^{1-q} < 1 + p - 1 - pq + q$$

$$\frac{p}{p^q} < p + q - pq < p + q$$

$$\frac{p^q}{p} > \frac{1}{p+q} \Rightarrow p^q > \frac{p}{p+q} \quad (1)$$

By (1):

$$(\tan x)^{\cot x} > \frac{\tan x}{\tan x + \cot x} \quad (2)$$

$$(\cot x)^{\tan x} > \frac{\cot x}{\cot x + \tan x} \quad (3)$$

By adding (2); (3):

$$(\tan x)^{\cot x} + (\cot x)^{\tan x} > \frac{\tan x + \cot x}{\tan x + \cot x} = 1$$

By integrating:

$$\int_a^b \left((\tan x)^{\cot x} + (\cot x)^{\tan x}\right) dx \geq \int_a^b dx = b - a$$

FAMOUS THEOREMS

CAUCHY – SCHWARZ'S INEQUALITY

$$(ax+by)^2 \le (a^2+b^2)(x^2+y^2);\ a,b,x,y \in \mathbb{R}$$

$$(ax+by+cz)^2 \le (a^2+b^2+c^2)(x^2+y^2+z^2);\ a,b,c,x,y,z \in \mathbb{R}$$

$$\left(\sum_{i=1}^{n} a_i x_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} x_i^2\right); a_i, x_i \in \mathbb{R}, i \in \overline{1,n}$$

$$\frac{a^2}{x}+\frac{b^2}{y} \ge \frac{(a+b)^2}{x+y}; a,b \in \mathbb{R}; x,y \in (0,\infty)$$

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z} \ge \frac{(a+b+c)^2}{x+y+z}; a,b,c \in \mathbb{R}; x,y,z \in (0,\infty)$$

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z} \ge \frac{(a+b+c)^2}{ax+by+cz}; a,b,c,x,y,z \in (0,\infty)$$

$$\frac{a_1^2}{x_1}+\frac{a_2^2}{x_2}+\cdots+\frac{a_n^2}{x_n} \ge \frac{(a_1+a_2+\cdots+a_n)^2}{x_1+x_2+\cdots+x_n}; a_i \in \mathbb{R}; x_i > 0; i \in \overline{1,n}$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}; (\forall) a,b,c \in (0,\infty)$$

$$\frac{a}{b+nc}+\frac{b}{c+na}+\frac{c}{a+nb} \ge \frac{3}{n+1}; a,b,c \in (0,\infty);\ n \in \mathbb{N}^*$$

MINKOWSKI'S INEQUALITY

$$\sqrt{(x+a)^2+(y+b)^2} \le \sqrt{x^2+y^2}+\sqrt{a^2+b^2}$$

$$\sqrt{(x+y+z)^2+(a+b+c)^2} \le \sqrt{x^2+a^2}+\sqrt{y^2+b^2}+\sqrt{z^2+c^2}$$

$$\sqrt{(x+a)^2+(y+b)^2+(z+c)^2} \le \sqrt{x^2+y^2+z^2}+\sqrt{a^2+b^2+c^2}$$

$$\sqrt{(x_1+a_1)^2+(x_2+a_2)^2+\cdots+(x_n+a_n)^2} \le \sqrt{x_1^2+x_2^2+\cdots+x_n^2}+\sqrt{a_1^2+a_2^2+\cdots+a_n^2}$$

$$x_i; a_i \in \mathbb{R}; i \in \overline{1,n}\,; n \in \mathbb{N}^*$$

$$\left(\sum_{i=1}^{n} |x_i+y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

$$p > 1; x_i, y_i \in \mathbb{R}; i \in \overline{1,n}; n \in \mathbb{N}^*$$

HÖLDER'S INEQUALITY

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\ge\frac{(a+b+c)^3}{3(x+y+z)};a,b,c,x,y,z\in(0,\infty)$$

$$\frac{a^4}{x}+\frac{b^4}{y}+\frac{c^4}{z}\ge\frac{(a+b+c)^4}{9(x+y+z)};a,b,c,x,y,z\in(0,\infty)$$

$$\frac{a^n}{x}+\frac{b^n}{y}+\frac{c^n}{z}\ge\frac{(a+b+c)^n}{3^{n-2}(x+y+z)};a,b,c,x,y,z\in(0,\infty)$$

$$\frac{a^n}{x}+\frac{b^n}{y}\ge\frac{(a+b)^n}{2^{n-2}(x+y)};a,b,x,y\in(0,\infty);\ n\ge 2;n\in\mathbb{N}$$

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}\ge\frac{(x+y+z)^3}{(a+b+c)^2};x,y,z,a,b,c\in(0,\infty)$$

$$\frac{x^4}{a^3}+\frac{y^4}{b^3}+\frac{z^4}{c^3}\ge\frac{(x+y+z)^4}{(a+b+c)^3};x,y,z,a,b,c\in(0,\infty)$$

$$\frac{x^{n+1}}{a^n}+\frac{y^{n+1}}{b^n}+\frac{z^{n+1}}{c^n}\ge\frac{(x+y+z)^{n+1}}{(a+b+c)^n};x,y,z,a,b,c\in(0,\infty);\ n\in\mathbb{N}$$

$$\left(\sum_{i=1}^n a_i^3\right)\left(\sum_{i=1}^n b_i^3\right)\left(\sum_{i=1}^n c_i^3\right)\ge\left(\sum_{i=1}^n a_ib_ic_i\right)^3;a_i,b_i,c_i\in[0,\infty);n\in\mathbb{N}^*$$

$$\left(\sum_{i=1}^n a_i^4\right)\left(\sum_{i=1}^n b_i^4\right)\left(\sum_{i=1}^n c_i^4\right)\left(\sum_{i=1}^n d_i^4\right)\ge\left(\sum_{i=1}^n a_ib_ic_id_i\right)^4;a_i,b_i,c_i,d_i\in\mathbb{R};n\in\mathbb{N}^*$$

$$\sum_{i=1}^n|x_iy_i|\le\left(\sum_{i=1}^n|x_i|^p\right)^{\frac{1}{p}}\cdot\left(\sum_{i=1}^n|y_i|^q\right)^{\frac{1}{q}};p,q\in(1,\infty)$$

$$\frac{1}{p}+\frac{1}{q}=1;x_i,y_i\in\mathbb{R};i\in\overline{1,n};n\in\mathbb{N}^*$$

HUYGENS' INEQUALITY

$$(1+a_1)(1+a_2)\ge\left(1+\sqrt{a_1a_2}\right)^2;a_1,a_2\in[0,\infty)$$

$$(1+a_1)(1+a_2)(1+a_3)\ge\left(1+\sqrt[3]{a_1a_2a_3}\right)^3;a_1,a_2,a_3\in[0,\infty)$$

$$(1+a_1)(1+a_2)(1+a_3)(1+a_4)\ge\left(1+\sqrt[4]{a_1a_2a_3a_4}\right)^4;a_1,a_2,a_3,a_4\in[0,\infty)$$

$$\prod_{i=1}^n(1+x_i)\ge\left(1+\sqrt[n]{x_1x_2\cdots x_n}\right)^n;a_i\in[0,\infty);\ n\in\mathbb{N};n\ge 2$$

$$(a_1+b_1)(a_2+b_2)\geq\left(\sqrt{a_1a_2}+\sqrt{b_1b_2}\right)^2$$

$$(a_1+b_1)(a_2+b_2)(a_3+b_3)\geq\left(\sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}\right)^3$$

$$(a_1+b_1)(a_2+b_2)\cdot\ldots\cdot(a_n+b_n)\geq\left(\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}+\sqrt[n]{b_1b_2\cdot\ldots\cdot b_n}\right)^n$$

$$(a_1+b_1+c_1)(a_2+b_2+c_2)\geq\left(\sqrt{a_1a_2}+\sqrt{b_1b_2}+\sqrt{c_1c_2}\right)^2$$

$$(a_1+b_1+c_1)(a_2+b_2+c_2)\cdot\ldots\cdot(a_n+b_n+c_n)$$
$$\geq\left(\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}+\sqrt[n]{b_1b_2\cdot\ldots\cdot b_n}+\sqrt[n]{c_1c_2\cdot\ldots\cdot c_n}\right)^n$$

HÖLDER'S INEQUALITY GENERALIZED

$$\prod_{i=1}^{n}\left(\sum_{j=1}^{n}x_{ij}\right)^{w_j}\geq\sum_{i=1}^{n}\left(\prod_{j=1}^{n}x_{ij}^{w_i}\right)$$

$$w_1+w_2+\cdots+w_n=1$$

CEBYSHEV'S INEQUALITY

$$\begin{cases}(x_1\leq x_2)\wedge(y_1\leq y_2)\ or\ (x_1\geq x_2)\wedge(y_1\geq y_2)\\ x_1y_1+x_2y_2\geq\dfrac{1}{2}(x_1+x_2)(y_1+y_2)\end{cases}$$

$$\begin{cases}(x_1\leq x_2\leq x_3)\wedge(y_1\leq y_2\leq y_3)\ or\ (x_1\geq x_2\geq x_3)\wedge(y_1\geq y_2\geq y_3)\\ x_1y_1+x_2y_2+x_3y_3\geq\dfrac{1}{3}(x_1+x_2+x_3)(y_1+y_2+y_3)\end{cases}$$

$$\begin{cases}(x_1\leq x_2\leq\cdots\leq x_n)\wedge(y_1\leq y_2\leq\cdots\leq y_n)\ or\ (x_1\geq x_2\geq\cdots\geq x_n)\wedge(y_1\geq y_2\geq\cdots\geq y_n)\\ x_1y_1+x_2y_2+\cdots+x_ny_n\geq\dfrac{1}{n}(x_1+x_2+\cdots+x_n)(y_1+y_2+\cdots+y_n)\end{cases}$$

$$\begin{cases}(x_1\leq x_2)\wedge(y_1\geq y_2)\ or\ (x_1\geq x_2)\wedge(y_2\leq y_2)\\ x_1y_1+x_2y_2\leq\dfrac{1}{2}(x_1+x_2)(y_1+y_2)\end{cases}$$

$$\begin{cases}(x_1\leq x_2\leq x_3)\wedge(y_1\geq y_2\geq y_3)\ or\ (x_1\geq x_2\geq x_3)\wedge(y_1+y_2+y_3)\\ x_1y_1+x_2y_2+x_3y_3\leq\dfrac{1}{3}(x_1+x_2+x_3)(y_1+y_2+y_3)\end{cases}$$

$$\begin{cases}(x_1\leq x_2\leq\cdots\leq x_n)\wedge(y_1\geq y_2\geq\cdots\geq y_n)\ or\ (x_1\geq x_2\geq\cdots\geq x_n)\wedge(y_1\leq y_2\leq\cdots\leq y_n)\\ x_1y_1+x_2y_2+\cdots+x_ny_n\leq\dfrac{1}{n}(x_1+x_2+\cdots+x_n)(y_1+y_2+\cdots+y_n)\\ x_i,y_i\in\mathbb{R};n\in\mathbb{N}^*;i\in\overline{1,n}\end{cases}$$

$$\begin{cases}\sum_{i=1}^{n} f(a_i)g(b_i)p_i \geq \left(\sum_{i=1}^{n} f(a_i)p_i\right)\left(\sum_{i=1}^{n} g(b_i)p_i\right) \geq \sum_{i=1}^{n} f(a_i)g(b_{n-i+1})\,p_i \\ for\ a_1 \leq a_2 \leq \cdots \leq a_n; b_1 \leq b_2 \leq \cdots \leq b_n \\ p_i \geq 0; i \in \overline{1,n}; n \in \mathbb{N}^*; p_1 + p_2 + \cdots + p_n = 1\end{cases}$$

f, g **nonincreasing**

SCHUR'S INEQUALITIES

$$\begin{cases}a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0 \\ a,b,c \in [0,\infty); r > 0\end{cases}$$

$$\begin{cases}a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a) \\ abc \geq (-a+b+c)(a-b+c)(a+b-c) \\ (a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca) \\ (a-b)^2(a+b-c) + (b-c)^2(b+c-a) + (c-a)^2(c+a-b) \geq 0 \\ a^2 + b^2 + c^2 + \dfrac{9abc}{a+b+c} \geq 2(ab+bc+ca) \\ \dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b} + \dfrac{4abc}{(a+b)(b+c)(c+a)} \geq 2\end{cases}$$

$$a,b,c \in (0,\infty)$$

$$\begin{cases}a^4 + b^4 + c^4 + abc(a+b+c) \geq ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2) \\ a,b,c \in [0,\infty)\end{cases}$$

HADAMARD'S INEQUALITIES

$$\begin{cases}\sum_{i=1}^{n} \lambda_i^2 \leq \sum_{i=1}^{n}\left(\sum_{j=1}^{n} A_{ij}^2\right) \qquad ; A \in M_n(\mathbb{R}) \\ \lambda_1, \lambda_2, \cdots, \lambda_n \text{ eigenvalues of } A; n \in \mathbb{N}^*\end{cases}$$

$$\begin{cases}\sum_{i=1}^{k} a_{ii} \leq \sum_{i=1}^{k} \lambda_i\,; 1 \leq k \leq n; n \in \mathbb{N}^* \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\ eigenvalues\ of\ A = (a_{ij}) \\ a_{11} \geq a_{22} \geq \cdots \geq a_{nn}\end{cases}$$

MILNES' INEQUALITY

$$(a_1 + b_1 + a_2 + b_2)\left(\frac{a_1 b_1}{a_1 + b_1} + \frac{a_2 b_2}{a_2 + b_2}\right) \leq (a_1 + a_2)(b_2 + b_2)$$

$$(a_1 + b_1 + a_2 + b_2 + a_3 + b_3)\left(\frac{a_1 b_1}{a_1 + b_1} + \frac{a_2 b_2}{a_2 + b_2} + \frac{a_3 b_3}{a_3 + b_3}\right)$$

$$\leq (a_1 + a_2 + a_3) \cdot (b_1 + b_2 + b_3)$$

$$\left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right)\le\left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}b_i\right)$$

$$i\in\overline{1,n}; n\ge 2; a_i>0; b_i>0$$

REARRANGEMENTS INEQUALITY

$$\begin{cases}\displaystyle\sum_{i=1}^{n}a_ib_i\ge\sum_{i=1}^{n}a_i\ge\sum_{i=1}^{n}a_ib_{n-i+1}\\ a_1\le a_2\le\cdots\le a_n; b_1\le b_2\le\cdots\le b_n\\ \pi=\begin{pmatrix}1 & 2 & \cdots & i & \cdots & n\\ \cdots & \cdots & \pi(i) & \cdots & \cdots & \cdots\end{pmatrix}\in S_n\end{cases}; n\in\mathbb{N}^*$$

$$\begin{cases}\displaystyle\sum_{i=1}^{n}f_i(b_i)\ge\sum_{i=1}^{n}f_i\left(b_{\pi(i)}\right)\ge\sum_{i=1}^{n}f_i(b_{n-i+1})\\ if\ \left(f_{i+1}(x)-f_i(x)\right) f-nondecreasing; 1\le i\le n\end{cases}$$

STIRLING'S INEQUALITY

$$e\left(\frac{n}{e}\right)^n\le\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}}\le n!\le\sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}\le en\left(\frac{n}{e}\right)^n$$

MAHLER'S INEQUALITY

$$\sqrt{(x_1+y_1)(x_2+y_2)}\ge\sqrt{x_1x_2}+\sqrt{y_1y_2}$$

$$\sqrt[3]{(x_1+y_1)(x_2+y_2)(x_3+y_3)}\ge\sqrt[3]{x_1x_2x_3}+\sqrt[3]{y_1y_2y_3}$$

$$\sqrt[4]{(x_1+y_1)(x_2+y_2)(x_3+y_3)(x_4+y_4)}\ge\sqrt[4]{x_1x_2x_3x_4}+\sqrt[4]{y_1y_2y_3y_4}$$

$$\prod_{i=1}^{n}(x_i+y_i)^{\frac{1}{n}}\ge\prod_{i=1}^{n}x_i^{\frac{1}{n}}+\prod_{i=1}^{n}y_i^{\frac{1}{n}}$$

$$x_i,y_i>0; i\in\overline{2,n}; n\in\mathbb{N}; n\ge 2$$

WEIERSTRASS' INEQUALITY

$$\prod_{i=1}^{n}(1-x_i)^{w_i}\ge 1-\sum_{i=1}^{n}w_ix_i\,; x_i\le 1; w_i\ge 1\ \text{ or } w_i\le 0; i\in\overline{1,n}$$

$$\prod_{i=1}^{n}(1-x_i)^{w_i}\le 1-\sum_{i=1}^{n}w_ix_i\,; w_i\in[0,1];\ \sum_{i=1}^{n}w_i\le 1\,;$$

$$x_i\in(-\infty,1];\ i\in\overline{1,n}; n\in\mathbb{N}^*$$

YOUNG'S INEQUALITY

$$\left(\frac{1}{px^p}+\frac{1}{qx^q}\right)^{-1}\le xy\le \frac{x^p}{p}+\frac{y^q}{q}$$

$$x,y\in(0,\infty);\ p,q\in(0,\infty);\frac{1}{p}+\frac{1}{q}=1$$

JENSEN'S INEQUALITY

$$\begin{cases} f: I\to\mathbb{R}; f \text{ convexe on } I;\\ a,b,c,x_1x_2,\cdots,x_n\in I; n\in\mathbb{N}^*\\ f\left(\frac{a+b}{2}\right)\le\frac{f(a)+f(b)}{2}\\ f\left(\frac{a+b+c}{3}\right)\le\frac{f(a)+f(b)+f(c)}{3}\\ f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)\le\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}\end{cases}$$

$$\begin{cases} f: I\to\mathbb{R}; f \text{ concave on } I;\\ f\left(\frac{a+b}{2}\right)\ge\frac{f(a)+f(b)}{2}\\ f\left(\frac{a+b+c}{3}\right)\ge\frac{f(a)+f(b)+f(c)}{3}\\ f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)\ge\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}\end{cases}$$

$$\begin{cases} f: I\to\mathbb{R}; f \text{ convexe on } I;\\ p_i\ge 0; p_1+p_2+\cdots+p_n=1\\ f(p_1x_1+p_2x_2+\cdots+p_nx_n)\le p_1f(x_1)+p_2f(x_2)+\cdots+p_nf(x_n)\end{cases}$$

$$\begin{cases} f: I\to\mathbb{R}; f \text{ concave on } I\\ p_i\ge 0; p_1+p_2+\cdots+p_n=1\\ f(p_1x_1+p_2x_2+\cdots+p_nx_n)\ge p_1f(x_1)+p_2f(x_2)+\cdots+p_nf(x_n)\end{cases}$$

WEIGHTED MEANS INEQUALITY

$$\begin{cases} w=w_1+w_2\\ \frac{w_1x_1+w_2x_2}{w_1+w_2}\ge\sqrt[w_1+w_2]{x_1^{w_1}\cdot x_2^{w_2}}\ge\frac{w}{\frac{w_1}{x_1}+\frac{w_2}{x_2}}\end{cases}$$

$$\begin{cases} w=w_1+w_2+w_3\\ \frac{w_1x_1+w_2x_2+w_3x_3}{w_1+w_2+w_3}\ge\sqrt[w_1+w_2+w_3]{x_1^{w_1}\cdot x_2^{w_2}\cdot x_3^{w_3}}\ge\frac{w}{\frac{w_1}{x_1}+\frac{w_2}{x_2}+\frac{w_3}{x_3}}\end{cases}$$

$$\begin{cases} \dfrac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i} \geq \sqrt[\sum_{i=1}^n w_i]{\prod_{i=1}^n x_i^{w_i}} \geq \dfrac{\sum_{i=1}^n w_i}{\sum_{i=1}^n \frac{w_i}{x_i}} \\ w_i > 0; i \in \overline{1,n}; n \in \mathbb{N}^*. \text{Convention}: 0^0 = 1 \end{cases}$$

MEANS INEQUALITY

$$\begin{cases} \min\{x_i\} \leq \dfrac{n}{\sum_{i=1}^n \frac{1}{x_i}} \leq \sqrt[n]{\prod_{i=1}^n x_i} \leq \dfrac{\sum_{i=1}^n x_i}{n} \leq \sqrt{\dfrac{1}{n}\sum_{i=1}^n x_i^2} \leq \max\{x_i\} \\ x_i \in (0,\infty); i \in \overline{1,n}; i \in \overline{1,n}; n \in \mathbb{N}^* \end{cases}$$

POWER MEANS INEQUALITY

$$\begin{cases} \sqrt{a|x_1|^2 + b|x_2|^2} \leq \sqrt[3]{a|x_1|^3 + b|x_2|^3} \\ a, b, \in [0,\infty); a + b = 1; x_1, x_2 \in \mathbb{R} \end{cases}$$

$$\begin{cases} \sqrt[p]{a|x_1|^p + b|x_2|^p} \leq \sqrt[q]{a|x_1|^9 + b|x_2|^q} \\ p \geq q > 0; a, b \in [0,\infty); a + b = 1; x_1, x_2 \in \mathbb{R} \end{cases}$$

$$\begin{cases} \sqrt{a|x_1|^2 + b|x_2|^2 + c|x_3|^2} \leq \sqrt[3]{a|x_1|^3 + b|x_2|^3 + c|x_3|^3} \\ a, b, c \in [0,\infty); a + b + c = 1; x_1, x_2, x_3 \in \mathbb{R} \end{cases}$$

$$\begin{cases} \sqrt[p]{a|x_1|^2 + b|x_2|^2 + c|x_3|^2} \leq \sqrt[q]{a|x_1|^3 + b|x_2|^3 + c|x_3|^3} \\ a, b, c \in [0,\infty); a + b + c = 1; x_1, x_2, x_3 \in \mathbb{R} \end{cases}$$

$$\begin{cases} \sqrt[p]{\sum_{i=1}^n w_i |x_i|^p} \leq \sqrt[q]{\sum_{i=1}^n w_i |x_i|^q} \\ p, q \in [0,\infty); p \leq q; w_i \in [0,\infty) \\ i \in \overline{1,n}; n \in \mathbb{N}^*; w_1 + w_2 + \cdots + w_n = 1 \\ M(p) = \sqrt[p]{\sum_{i=1}^n w_i |x_i|^p} \\ \lim_{p\to 0} M(p) = \prod_{i=1}^n |x_i|^{w_i} \\ \lim_{x\to -\infty} M(p) = \min\{x_1, x_2, \cdots, x_n\} \\ \lim_{x\to \infty} M(p) = \max\{x_1, x_2, \dots, x_n\} \end{cases}$$

LEHMER'S INEQUALITY

$$\frac{w_1|x_1|^p+w_2|x_2|^p}{w_1|x_1|^{p-1}+w_2|x_2|^{p-1}}\le\frac{w_1|x_1|^q+w_2|x_2|^q}{w_1|x_1|^{q-1}+w_2|x_2|^{q-1}}$$

$$\frac{w_1|x_1|^p+w_2|x_2|^p+w_3|x_3|^p}{w_1|x_1|^{p-1}+w_2|x_2|^{p-1}+w_3|x_3|^{p-1}}\le\frac{w_1|x_1|^q+w_2|x_2|^q+w_3|x_3|^q}{w_1|x_1|^{q-1}+w_2|x_2|^{q-1}+w_3|x_3|^{q-1}}$$

$$\frac{\sum_{i=1}^n w_i|x_i|^p}{\sum_{i=1}^n w_i|x_i|^{p-1}}\le\frac{\sum_{i=1}^n w_i|x_i|^q}{\sum_{i=1}^n w_i|x_i|^{q-1}}$$

$$p\le q; w_i\ge 0; i\in\overline{1,n}; n\in\mathbb{N}$$

CARLEMAN'S INEQUALITY

$$\sum_{k=1}^n\left(\prod_{i=1}^k|a_i|\right)^{\frac{1}{k}}\le e\sum_{k=1}^n|a_k|, n\in\mathbb{N}^*;\ a_1,a_2,\dots,a_n\in\mathbb{R}$$

SUM & PRODUCT INEQUALITY

$$\begin{cases}\displaystyle\sum_{j=1}^m\prod_{i=1}^n a_{ij}\ge\sum_{j=1}^m\prod_{i=1}^n a_{i\pi(j)}\\ \displaystyle\prod_{j=1}^m\sum_{i=1}^n a_{ij}\le\prod_{j=1}^m\sum_{i=1}^n a_{i\pi(j)}\\ 0\le a_{i1}\le a_{i2}\le\cdots\le a_{im}; i\in\overline{1,n}\\ \pi=\begin{pmatrix}1&2&3&\cdots&i&\cdots&n\\ \dots&\dots&\dots&\pi(i)&\dots&\dots&\dots\end{pmatrix}\end{cases}$$

$$\begin{cases}\left|\displaystyle\prod_{i=1}^n a_i-\prod_{i=1}^n b_i\right|\le\sum_{i=1}^n|a_i-b_i|\,; |a_i|\le 1; |b_i|\le 1\\ i\in\overline{1,n}; n\in\mathbb{N}^*; a_i,b_i\in\mathbb{R};\end{cases}$$

$$\prod_{i=1}^n(\alpha+a_i)\ge(1+\alpha)^n;\prod_{i=1}^n a_i\ge 1; a_i>0;\alpha>0$$

SQUARE ROOT INEQUALITY

$$\left\{2\sqrt{x+1}-2\sqrt{x}<\frac{1}{\sqrt{x}}<\sqrt{x+1}-\sqrt{x-1}<2\sqrt{x}-2\sqrt{x-1}; x\ge 1\right.$$

LOGARITHM MEAN INEQUALITY

$$\begin{cases}\sqrt{xy}\le\left(\dfrac{\sqrt{x}+\sqrt{y}}{2}\right)\sqrt[4]{xy}\le\dfrac{x-y}{\ln x-\ln y}\le\dfrac{\left(\sqrt{x}+\sqrt{y}\right)^2}{2}\le\dfrac{x+y}{2};\\ x,y\in(0,\infty)\end{cases}$$

HEINZ' INEQUALITY

$$\left\{\sqrt{xy} \le \frac{x^{1-\alpha}y^{\alpha} + x^{\alpha}y^{1-\alpha}}{2} \le \frac{x+y}{2}; x, y > 0, \alpha \in [0,1]\right.$$

MACLAURIN'S INEQUALITY

$$\begin{cases} \dfrac{a+b+c}{3} \ge \sqrt{\dfrac{ab+ac+ca}{3}} \ge \sqrt[3]{abc} \\ \dfrac{a+b+c+d}{4} \ge \sqrt{\dfrac{ab+ac+ad+bc+bd+cd}{6}} \ge \sqrt[3]{\dfrac{abc+abd+bcd+acd}{4}} \ge \sqrt[4]{abcd} \\ \dfrac{\sum x_i}{n} \ge \sqrt{\dfrac{\sum x_i x_j}{\binom{n}{2}}} \ge \sqrt[3]{\dfrac{\sum x_i x_j x_k}{\binom{n}{3}}} \ge \cdots \\ \cdots \ge \sqrt[n-1]{\dfrac{\sum x_{i_1} x_{i_2} \cdot \ldots \cdot x_{i_{n-1}}}{\binom{n}{n-1}}} \ge \sqrt[n]{x_{i_1} x_{i_2} \cdot \ldots \cdot x_{i_n}} \end{cases}$$

$$\begin{cases} S_k^2 \ge S_{k-1}S_{k+1};\ \sqrt[k]{S_k} \ge \sqrt[k+1]{S_{k+1}}; 1 \le k < n \\ S_k = \dfrac{1}{\binom{n}{k}} \cdot \sum\limits_{1 \le i_1 < \cdots < i_k \le n} a_{i_1} a_{i_2} \cdot \ldots \cdot a_{i_k}; a_i \ge 0 \end{cases}$$

BERNOULLI'S INEQUALITIES

$$(1+x)^r \ge 1 + rx; x \ge -1; r \in (-\infty, 0] \cup [1, \infty)$$

$$(1+x)^r \le 1 + rx; x \ge -1; r \in [0,1]$$

$$(1+x)^r \le 1 + (2^r - 1)x; x \in [0,1]; r \in (-\infty, 0] \cup [1, \infty)$$

$$(1+x)^n \le \frac{1}{1-nx}; x \in [-1, 0];\ n \in \mathbb{N}$$

$$(1+x)^r \le 1 + \frac{rx}{1-(r-1)x}; x \in \left[-1, \frac{1}{r-1}\right];\ r > 1$$

$$(1+nx)^{n+1} \ge (1+(n+1)x)^n; x \in \mathbb{R}; n \in \mathbb{N}$$

$$(a+b)^n \le a^n + nb(a+b)^{n-1}; a, b \ge 0; n \in \mathbb{N}$$

$$\left(1+\tfrac{x}{p}\right)^p \ge \left(1+\tfrac{x}{q}\right)^q \text{ for } \begin{cases} (i) x > 0; p > q > 0 \\ (ii) -p < -q < x < 0 \\ (iii) -q > -p > x > 0 \end{cases}$$

$$\left(1+\frac{x}{p}\right)^p \le \left(1+\frac{x}{q}\right)^q \text{ for } \begin{cases} (iv) q < 0 < p, -q > x > 0 \\ (v) q < 0 < p, -p < x < 0 \end{cases}$$

TRIGONOMETRIC INEQUALITIES

$$x - \frac{x^3}{2} \le x\cos x \le \frac{x\cos x}{1 - x^{\frac{2}{3}}} \le x\sqrt[3]{\cos x} \le x - \frac{x^3}{6} \le x\cos\frac{x}{\sqrt{3}} \le \sin x$$

HYBERBOLIC INEQUALITIES

$$x\cos x \le \frac{x^3}{\sin h^2 x} \le x\cos^2\left(\frac{x}{2}\right) \le \sin x \le (x\cos x + 2x)/3 \le \frac{x^2}{\sin hx}$$

$$\frac{2x}{\pi} \le \sin x \le x\cos\frac{x}{2} \le x \le x + \frac{x^3}{3} \le \tan x\,; x \in \left[0, \frac{\pi}{2}\right]$$

$$\cos hx + \alpha \sin hx \le e^{x\left(\alpha + \frac{x}{2}\right)}; x \in \mathbb{R}; \alpha \in [-1, 1]$$

ACZEL'S INEQUALITY

$$\begin{cases} \left(a_1 b_1 - \sum_{i=1}^n a_i b_i\right)^2 \ge \left(a_1^2 - \sum_{i=1}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \\ \text{dacă } a_1^2 > \sum_{i=1}^n a_i^2 \text{ sau } b_1^2 > \sum_{i=2}^n b_i^2 \end{cases}$$

ABEL'S INEQUALITY

$$\begin{cases} b_1 \min_k \sum_{i=1}^k a_i \le \sum_{i=1}^n a_i b_i \le b_1 \max_k \sum_{i=1}^k a_i \\ b_1 \ge b_2 \ge \cdots \ge b_n \ge 0; n \in \mathbb{N}^* \end{cases}$$

KY FAN'S INEQUALITY

$$\begin{cases} \frac{\prod_{i=1}^n x_i^{a_i}}{\prod_{i=1}^n (1 - x_i)^{a_i}} \le \frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i (1 - x_i)}; x_i \in \left[0, \frac{1}{2}\right] \\ a_i \in [0, 1]; a_1 + a_2 + \cdots + a_n = 1 \end{cases}$$

SHAPIRO'S INEQUALITY

$$\begin{cases} \sum_{i=1}^n \frac{x_i}{x_{i+1} + x_{i+2}} \ge \frac{n}{2}; x_i > 0; x_{n+1} = x_1; x_{n+2} = x_2 \\ n \le 12, n \text{ odd or } n \le 23, n\ even; n \in \mathbb{N}^*; n \ge 3 \end{cases}$$

CHONG'S INEQUALITY

$$\begin{cases} \sum_{i=1}^{n} \frac{a_i}{a_{\pi(i)}} \geq n;\ \prod_{i=1}^{n} a_i^{a_i} \geq \prod_{i=1}^{n} a_i^{a_{\pi(a_i)}};\ a_i > 0 \\ \pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & i & \cdots & n \\ \cdots & \cdots & \cdots & \pi(i) & \cdots & \cdots & \cdots \end{pmatrix} \in S_n;\ n \in \mathbb{N}^* \end{cases}$$

POLYA – SZEGO'S INEQUALITY

$$\frac{(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)}{(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2} \leq \frac{1}{4}\left(\sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}}\right)^2$$

$$0 < a < A;\ 0 < b < B;\ a_1, a_2, \ldots, a_n \in [a, A],$$

$$b_1, b_2, \ldots, b_n \in [b, B];\ n \geq 2;\ p_1, p_2, \ldots, p_n > 0$$

$$\frac{(p_1a_1^2 + p_2a_2^2 + \cdots + p_na_n^2)(p_1b_1^2 + p_2b_2^2 + \cdots + p_nb_n^2)}{(p_1a_1b_1 + p_2a_2b_2 + \cdots + p_na_nb_n)^2} \leq \frac{1}{4}\left(\sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}}\right)^2$$

SCHWEITZER'S INEQUALITY

$$(x_1 + x_2)\left(\frac{1}{x_1} + \frac{1}{x_2}\right) \leq \frac{(a+b)^2}{ab}$$

$$(x_1 + x_2 + x_3)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) \leq \frac{9(a+b)^2}{4ab}$$

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \leq \frac{n^2(a+b)^2}{4ab}$$

$$x_1, x_2, \ldots, x_n \in [a, b];\ b > a > 0;\ n \geq 2$$

KANTOROVICI'S INEQUALITY

$$(p_1x_1 + p_2x_2)\left(\frac{p_1}{x_1} + \frac{p_2}{x_2}\right) \leq \frac{(a+b)^2}{4ab}(p_1 + p_2)^2$$

$$(p_1x_1 + p_2x_2 + p_3x_3)\left(\frac{p_1}{x_1} + \frac{p_2}{x_2} + \frac{p_3}{x_3}\right) \leq \frac{(a+b)^2}{4ab}(p_1 + p_2 + p_3)^2$$

$$(p_1x_1 + p_2x_2 + \cdots + p_nx_n)\left(\frac{p_1}{x_1} + \frac{p_2}{x_2} + \cdots + \frac{p_n}{x_n}\right) \leq \frac{(a+b)^2}{4ab}(p_1 + p_2 + \cdots + p_n)^2$$

$$x_1, x_2, \ldots, x_n \in [a, b], p_1, p_2, \ldots, p_n > 0, b > a > 0, n \geq 2$$

CALLEBAUT'S INEQUALITY

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^{1+x} b_i^{1-x}\right)\left(\sum_{i=1}^{n} a_i^{1-x} b_i^{1+x}\right) \le$$

$$\le \left(\sum_{i=1}^{n} a_i^{1+y} b_i^{1-y}\right)\left(\sum_{i=1}^{n} a_i^{1-y} b_i^{1+y}\right) \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

$$0 < x < y < 1; a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n > 0; n \ge 2$$

DIAZ – METCALF'S INEQUALITY

$$\sum_{i=1}^{n} a_i^2 + mM \sum_{i=1}^{n} b_i^2 \le (m+M) \sum_{i=1}^{n} a_i b_i$$

$$m \le \frac{a_i}{b_i} \le M; i \in \overline{1,n}, m, M, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}^*; n \ge 2$$

TURKEVICIU'S INEQUALITY

$$a^4 + b^4 + c^4 + d^4 + 2abcd \ge a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2c^2 + b^2d^2 + c^2d^2;$$

$$a, b, c, d > 0$$

LYNESS – CARLITZ'S INEQUALITY

$$a, b, c \in \mathbb{R}; a^3 + b^3 + c^3 = 0$$

$$(a^2 + b^2 + c^2)^3 \le [(a-b)^2 + (b-c)^2 + (c-a)^2](a^4 + b^4 + c^4)$$

CAUCHY – BINET'S IDENTITY

$$\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} b_i y_i\right) - \left(\sum_{i=1}^{n} a_i y_i\right)\left(\sum_{i=1}^{n} b_i x_i\right) = \frac{1}{2} \sum_{i,j=1}^{n} (a_i b_j - a_j b_j)(x_i y_j - x_j y_i)$$

$$a_i, b_i, x_i, y_i \in \mathbb{R}; i \in \overline{1,n}; n \ge 2$$

LAGRANGE'S IDENTITY

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \frac{1}{2} \sum_{i,j=1}^{n} (a_i b_j - a_j b_i)^2$$

$$a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}; n \geq 2$$

WEIGHTED CEBYSHEV'S INEQUALITY

$$\left(\sum_{i=1}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i x_i y_i\right) \geq \left(\sum_{i=1}^{n} p_i x_i\right)\left(\sum_{i=1}^{n} p_i y_i\right)$$

$(x_1, x_2, \dots, x_n); (y_1, y_2, \dots, y_n)$ – same orientation

$$p_1, p_2, \dots, p_n > 0; n \geq 2$$

GRUSS' INEQUALITY

$$\left|\sum_{i=1}^{n} p_i a_i b_i - \left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right)\right| \leq \frac{1}{4}(H-h)(K-k)$$

$$p_1 + p_2 + \dots + p_n = 1; p_1, p_2, \dots, p_n > 0;$$

$$a_1, a_2, \dots, a_n \in [h, H]; b_1, b_2, \dots, b_n \in [k, K]; n \geq 2$$

CÂRTOAJE'S INEQUALITY

$$\frac{n}{1 + \frac{a_1 + a_2 + \dots + a_n}{n}} \leq \frac{1}{1+a_1} + \frac{1}{1+a_2} + \dots + \frac{1}{1+a_n} \leq \frac{n}{1 + \sqrt[n]{a_1 a_2 \dots a_n}}$$

$$a_1, a_2, \dots, a_n \in (0, 1); n \geq 2$$

DICU'S INEQUALITY

$$\sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1\left(\frac{a_2}{a_1}\right)^x + a_2\left(\frac{a_3}{a_2}\right)^x + \dots + a_n\left(\frac{a_1}{a_n}\right)^x}{n} \leq \frac{a_1 + a_2 + \dots + a_n}{n}$$

$$x \in [0, 1]; a_1, a_2, \dots, a_n > 0; n \geq 3$$

DRÎMBE'S REFINEMENT FOR CBS INEQUALITY

$$\left(\sum_{i=1}^{n} \sqrt{a_i b_i}\right)^2 \leq \left(\sum_{i=1}^{n} (a_i^x + b_i^x)^{\frac{1}{x}}\right)\left(\sum_{i=1}^{n} (a_i^{-x} + b_i^{-x})^{-\frac{1}{x}}\right) \leq$$

$$\leq \left(\sum_{i=1}^{n} (a_i^y + b_i^y)^{\frac{1}{y}}\right)\left(\sum_{i=1}^{n} (a_i^{-y} + b_i^{-y})^{-\frac{1}{y}}\right) \leq$$

$$\leq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right); 0 < x < y$$

$$a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n > 0; n \geq 2$$

MITRINOVIC'S INEQUALITY

$$3\min(a,b,c) \leq a+b+c \pm \sqrt{a^2+b^2+c^2-ab-bc-ca} \leq 3\max(a,b,c)$$

WEIGHTED JENSEN'S INEQUALITY

$$f\left(\frac{p_1x_1+p_2x_2+\dots+p_nx_n}{p_1+p_2+\dots+p_n}\right) \leq \frac{p_1f(x_1)+p_2f(x_2)+\dots+p_nf(x_n)}{p_1+p_2+\dots+p_n}$$

f convexe on I; $p_1, p_2, \dots, p_n > 0$; $x_1, x_2, \dots, x_n \in I$; $n \geq 2$

BORDEN'S INEQUALITY

$$\frac{\left(\sum_{i=1}^n x_i\right)^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i^{x_i}} \cdot \frac{\left(\sum_{i=1}^n y_i\right)^{\sum_{i=1}^n y_i}}{\prod_{i=1}^n y_i^{y_i}} \leq \frac{\left(\sum_{i=1}^n (x_i+y_i)\right)^{\sum_{i=1}^n (x_i+y_i)}}{\prod_{i=1}^n (x_i+y_i)^{(x_i+y_i)}}$$

KARAMATA'S INEQUALITY

$$f(x_1)+f(x_2)+\dots+f(x_n) \geq f(y_1)+f(y_2)+\dots+f(y_n)$$

f convexe on I; $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in I$; $n \geq 2$

$$x_1 \geq x_2 \geq \dots \geq x_n; y_1 \geq y_2 \geq y_3 \geq \dots y_n;$$

$$x_1 \geq y_1; x_1+x_2 \geq y_1+y_2; \dots; x_1+x_2+\dots+x_{n-1} \geq y_1+y_2+\dots+y_{n-1}$$

$$x_1+x_2+\dots+x_n = y_1+y_2+\dots+y_n$$

DRÎMBE'S INEQUALITY

$$f(x^2)+f(y^2)+f(z^2) \geq f(xy)+f(yz)+f(zx)$$

f increasing and convexe on $[0, a^2)$; $x, y, z \in [0, a^2)$

EXPONENTIAL INEQUALITIES

$$e^x \geq \left(1+\frac{x}{n}\right)^n \geq 1+x; \left(1+\frac{x}{n}\right)^n \geq e^x\left(1-\frac{x^2}{n}\right); n > 1; |x| \leq n$$

$$e^x \geq x^e; (\forall) x \in \mathbb{R}$$

$$\frac{x^n}{n!}+1 \leq e^x \leq \left(1+\frac{x}{n}\right)^{n+\frac{x}{2}}; x, n \in (0, \infty)$$

$$e^x \geq 1+x+\frac{x^2}{2}; x \geq 0; e^x \leq 1+x+\frac{x^2}{2}; x \leq 0$$

$$e^{-x} \leq 1-\frac{x}{2}; x \in [0, 1, 59]$$

$$2^{-x} \leq 1-\frac{x}{2}; x \in [0, 1]$$

$$\frac{1}{2-x} < x^x < x^2 - x + 1; x \in (0,1)$$

$$x^{\frac{1}{r}}(x-1) \le rx\left(x^{\frac{1}{r}} - 1\right);\ x, r \in [1,\infty)$$

$$x^y + y^x > 1; e^x > \left(1 + \frac{x}{y}\right)^y > e^{\frac{xy}{x+y}}; x, y \in (0,\infty)$$

$$2 - y - e^{-x-y} \le 1 + x \le y + e^{x-y}; x, y \in \mathbb{R}$$

$$e^x \le x + e^{x^2}; x, y \in \mathbb{R}$$

LOGARITHMIC INEQUALITIES

$$\frac{x-1}{x} \le \ln x \le \frac{x^2-1}{2x} \le x - 1; \ln x \le n\left(x^{\frac{1}{n}} - 1\right); x, n \in (0,\infty)$$

$$\frac{2x}{1+x} \le \ln(1+x) \le \frac{x}{\sqrt{x+1}}; x \ge 0$$

$$\frac{2x}{1+x} \ge \ln(1+x) \ge \frac{x}{\sqrt{x+1}}; x \in (-1,0]$$

$$\ln(n+1) < \frac{1}{n} + \ln n \le \sum_{i=1}^{n} \frac{1}{i} \le 1 + \ln n$$

$$\ln(1+x) \ge \frac{x}{2}; x \in [0,2,51]$$

$$\ln(1+x) \le \frac{x}{2}; x \in (-1,0] \cup (2,51,\infty)$$

$$\ln(1+x) \ge x - \frac{x^2}{2} + \frac{x^3}{4}; x \in [0;0,45]$$

$$\ln(1+x) \le x - \frac{x^2}{2} + \frac{x^3}{4}; x \in (-\infty,0) \cup (0,45;\infty)$$

$$\ln(1-x) \ge -x - \frac{x^2}{2} - \frac{x^3}{2}; x \in [0,0,43]$$

$$\ln(1-x) \le -x - \frac{x^2}{2} - \frac{x^3}{2}; x \in (-\infty,0) \cup (0,43;1)$$

BINOMIAL INEQUALITIES

$$\max\left\{\frac{n^k}{k^k}; \frac{(n-k+1)^k}{k!}\right\} \le \binom{n}{k} \le \frac{n^k}{k!} \le \frac{(en)^k}{k^k}$$

$$\binom{n}{k} \le \frac{n^n}{k^k(n-k)^{n-k}} \le 2^n$$

$$\frac{n^k}{4k!} \le \binom{n}{k} \text{ for } \sqrt{n} \ge k \ge 0$$

$$\frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{8n}\right) \le \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{9n}\right)$$

$$\binom{n_1}{k_1}\binom{n_2}{k_2} \le \binom{n_1+n_2}{k_1+k_2} \text{ for } n_1 \ge k_1 \ge 0; n_2 \ge k_2 \ge 0$$

$$\frac{\sqrt{\pi}}{2}G \le \binom{n}{\alpha n} \le G; G = \frac{2^n H(\alpha)}{\sqrt{2\pi n\alpha(1-\alpha)}};$$

$$H(x) = -\log_2(x^x(1-x)^{1-x})$$

$$\sum_{i=1}^{d}\binom{n}{i} \le n^d + 1; \sum_{i=0}^{d}\binom{n}{i} \le 2^n; n \ge d \ge 0$$

$$\sum_{i=0}^{d}\binom{n}{i} \le \left(\frac{en}{d}\right)^d; n \ge d \ge 1$$

$$\sum_{i=0}^{d}\binom{n}{i} \le \binom{n}{d}\left(1+\frac{d}{n-2d+1}\right); \frac{n}{2} \ge d \ge 0$$

$$\binom{n}{\alpha n} \le \sum_{i=0}^{\alpha n}\binom{n}{i} \le \frac{1-\alpha}{1-2\alpha}\binom{n}{\alpha n}; \alpha \in \left(0,\frac{1}{2}\right)$$

USEFUL INEQUALITIES

$$x^3+y^3+z^3 \ge 3xyz; x,y,z \in [0,\infty)$$

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2} \ge \frac{1}{1+xy}; x,y \in (0,\infty);\ xy \ge 1$$

$$\frac{x^3+y^3}{x^2+xy+y^2} \ge \frac{x+y}{3}; x,y \in (0,\infty)$$

$$\frac{x^3}{x^2+xy+y^2} \ge \frac{2x-y}{3}; x,y \in (0,\infty)$$

$$\sqrt{x^2+xy+y^2} \ge \frac{\sqrt{3}}{2}(x+y);\ x,y \in [0,\infty)$$

$$\sqrt{x^2+xy+y^2} \le \sqrt{3(x^2-xy+y^2)}; x,y \in \mathbb{R}$$

BASIC INEQUALITIES

I. Inequalities can be proved by equivalence:

1) $a^2+b^2 \ge \frac{(a+b)^2}{2} \ge 2ab$ ***(Basic BCS inequality for 2 numbers)***

$\rightarrow \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}}$ ***(Basic RMS-AM-GM-HM inequality for $a, b > 0$)***

$\rightarrow \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \leq a + b - \frac{2ab}{a+b}$ *with* $a, b \geq 0$ *such that* $a + b > 0$

2) $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \geq ab + bc + ca$

$$\rightarrow \begin{cases} * \, 3(a^2b^2 + b^2c^2 + c^2a^2) \geq (ab + bc + ca)^2 \geq 3abc(a + b + c) \\ * \, \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq \sqrt{3(a^2 + b^2 + c^2)} \geq a + b + c \geq \sqrt{3(ab + bc + ca)} \; for \; a, b, c > 0 \end{cases}$$

3) $a^2 + b^2 + c^2 \geq 2ab + 2ac - 2bc$

4) $a^3 + b^3 + c^3 \geq 3abc \; \forall a, b, c$ *such that* $a + b + c \geq 0$ ***(special case: $a, b, c > 0$ – Cauchy's inequality for 3 non-negative numbers)***

5) $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$ ***(Basic Bunyakovsky's inequality)***

6) $a^3 + b^3 \geq ab(a + b) \; \forall a, b$ such that $a + b \geq 0$

$\rightarrow 4(a^3 + b^3) \geq (a + b)^3 \geq 4ab(a + b)$ *for* $a + b \geq 0$

7) $2(a^2 + b^2 - ab)^2 \geq a^4 + b^4 \geq 2ab(2a^2 - 3ab + 2b^2) \forall a, b$

$\rightarrow \sqrt{a^2 - ab + b^2} \geq \frac{a^2+b^2}{a+b} = a + b - \frac{2ab}{a+b}$ *with* $a + b \neq 0$

8) $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{(a+b)^2}$ *with real numbers* a, b *such that* $ab > 0$ *(special case:* $a, b > 0$*)*

9) $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$ *with* $a, b, c > 0$ ***(Nesbitt's inequality)*** → *This is even true for real numbers* $a; b; c$ *such that* $ab + bc + ca > 0$.

10) $(a + b)(b + c)(c + a) \geq \frac{8}{9}(a + b + c)(ab + bc + ca) \geq 8abc$ *for* $a, b, c \geq 0$.

$$\rightarrow \begin{cases} * \, abc \geq (b + c - a)(c + a - b)(a + b - c) \; for \; a, b, c \; are \; sides \; of \; a \; triangle \\ * \, (a + b + c)^3 \geq 27abc \; for \; a, b, c \geq 0 \; (\boldsymbol{Cauchy's \; inequality \; for \; 3 \; non-negative \; variables}) \end{cases}$$

11) $\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$ *with* $x, y > 0$ ***(Basic Bunyakovsky Cauchy-Schwarz inequality)***

* *Also, from this inequality we have the chain:* $\frac{a^2}{b} + \frac{b^2}{a} \geq \sqrt{2(a^2 + b^2)} \geq a + b \geq 2\sqrt{ab}$ *for* $a, b > 0$.

12) $\begin{cases} * \, 2(ax + by) \geq (a + b)(x + y) \geq 2(ay + bx) \; with \; a \geq b, x \geq y \\ * \, 3(ax + by + cz) \geq (a + b + c)(x + y + z) \geq 3(az + by + cx) \; with \; a \geq b \geq c, x \geq y \geq z \end{cases}$ ***(Basic Chebyshev's inequality)***

13) $a^2b + b^2c + c^2a \geq ab^2 + bc^2 + ca^2$ *with* $a \geq b \geq c$

** More general:* $a^nb + b^nc + c^na \geq ab^n + bc^n + ca^n$ *for* $a \geq b \geq c \geq 0, n \in Z^+$

14) $2(a^n + b^n) \geq (a^x + b^x)(a^y + b^y)$ *with* $a, b > 0; x, y, n \in Z^+: x + y = n$

$\rightarrow a^{m+n} + b^{m+n} \geq a^mb^n + a^nb^m$ *for* $a, b \geq 0, m, n \in Z^+$

15) $a^2 + ab + b^2 \geq \frac{3}{4}(a+b)^2; a^2 - ab + b^2 \geq \frac{1}{4}(a+b)^2\ \forall a, b \rightarrow 3 \geq \frac{a^2-ab+b^2}{a^2+ab+b^2} \geq \frac{1}{3}$ *with real numbers* $a, b: a^2 + b^2 > 0.$

16) $\sqrt{a^2+b^2} + \sqrt{c^2+d^2} \geq \sqrt{(a+c)^2 + (b+d)^2}$ ***(Basic Minkovsky's inequality)***

17*) $\frac{1}{(a+1)\text{^}2} + \frac{1}{(b+1)^2} \geq \frac{1}{ab+1}$ *with* $a, b \geq 0$ *(The equality happens iff* $a = b = 1$*)*

18) *Consider* $f(a;b) = \frac{1}{1+a^2} + \frac{1}{1+b^2} - \frac{2}{1+ab}$ *with* $a, b > 0$. *If* $ab \geq 1$ *then* $f(a;b) \geq 0$*; if* $ab \leq 1$ *then* $f(a;b) \leq 0.$

19) $\frac{1}{a^2-1} + \frac{1}{b^2-1} \geq \frac{2}{ab-1}$ *with* $a, b > 1$

20) $(a^2+b^2+c^2)(a+b+c) \geq 3.\max\{a^2b + b^2c + c^2a; ab^2 + bc^2 + ca^2\}$ *with* $a, b, c \geq 0$

21) $(a^2+b^2+c^2)^2 \geq (a+b+c)(a^2b+b^2c+c^2a)\ \forall a, b, c > 0$

22) $\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \geq \frac{3}{2} \geq \frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a}$ *with* $a \geq b \geq c > 0$

23*)$(a^2+b^2+c^2)^2 \geq 3(a^3b + b^3c + c^3a)$ ***(Vasile's inequality)***

24) $\frac{1}{1+x^2} \geq 1 - \frac{x}{2}$ *with* $\geq 0 \rightarrow \frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{1}{1+c^2} \geq \frac{3}{2}\ \forall a, b, c \geq 0: a + b + c = 3.$

25) $(a+b+c)^3 \geq a^3 + b^3 + c^3 + 24abc\ \forall a, b, c > 0$

II. Some familiar Inequalities, lemmas and techniques: (ascending by higher level)

a) For junior – early – senior:

1) $\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geq \frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1a_2\ldots a_n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$ *with* $a_i > 0\ \ \forall i = 1; 2; \ldots; n$ *and* $n \in Z^+, n \geq 2$

(RMS-AM-GM-HM inequality for n positive numbers)

$\rightarrow$ *If* $x_1; x_2; \ldots; x_n$ *are positive real numbers that* $x_1 + x_2 \ldots + x_n = 1$ *then with same condition for* a_i *we have:*

$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq a_1^{x_1} \cdot a_2^{x_2} \ldots a_n^{x_n}$ ***(Weighted AM-GM inequality)***

2) *For real numbers* $a_i; b_i$ $(i = 1; 2; \ldots; n)$ *we have:*

$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$

(Cauchy-Schwarz inequality) $\rightarrow \frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n}\geq\frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}\quad\forall b_i>0$

$$\rightarrow \frac{x_1^2}{x_2}+\frac{x_2^2}{x_3}+\cdots+\frac{x_{n-1}^2}{x_n}+\frac{x_n^2}{x_1}\geq\sqrt{n(x_1^2+x_2^2+\cdots+x_n^2)}\geq x_1+x_2+\cdots+x_n;\forall x_i>0$$

3) *Let* $a_1\geq a_2\geq\cdots\geq a_n$

$$\begin{cases} *\ If\ b_1\geq b_2\geq\cdots\geq b_n\ then: n(a_1b_1+a_2b_2+\cdots+a_nb_n)\geq(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n)\\ *\ If\ b_1\leq b_2\leq\cdots\leq b_n\ then: n(a_1b_1+a_2b_2+\cdots+a_nb_n)\leq(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n)\end{cases}$$

(Chebyshev's inequality)

4) *If* $x_{i.j}\geq 0\ (i=1;2;..;n, j=1;2;\dots;m)$ *then we have:*

$$(x_{1.1}+x_{1.2}+\cdots+x_{1.n})(x_{2.1}+x_{2.2}+\cdots+x_{2.n})\dots(x_{m.1}+x_{m.2}+\cdots+x_{m.n})\geq$$
$$\geq\left(\sqrt[m]{x_{1.1}x_{2.1}\dots x_{m.1}}+\sqrt[m]{x_{1.2}x_{2.2}\dots x_{m.2}}+\cdots+\sqrt[m]{x_{1.n}x_{2.n}\dots x_{m.n}}\right)^m$$

(Hölder's inequality)

Ex: $*\ (a+b)(c+d)\geq\left(\sqrt{ac}+\sqrt{bd}\right)^2\ \forall a,b,c,d\geq 0$

$*\ (a+b+c)(m+n+p)(x+y+z)\geq\left(\sqrt[3]{amx}+\sqrt[3]{bny}+\sqrt[3]{cpz}\right)^3\ \forall a,b,c,m,n,p,x,y,z\geq 0$

$$\rightarrow\begin{cases} *\ \frac{a_1^m}{a_2^m}+\frac{a_2^m}{a_3^m}+\cdots+\frac{a_n^m}{a_1^m}\geq\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_n}{a_1}\ for\ a_i>0\ (i=1;2;\dots;n); m\in Z^+, n\geq 2\\ *\ \sqrt[n]{\frac{a_1^n+a_2^n+\cdots+a_n^n}{n}}\geq\frac{a_1+a_2+\cdots+a_n}{n}\ with\ same\ condition\ (\textbf{Power Mean Inequality})\\ *\ \frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2}\geq\frac{n^3}{(a_1+a_2+\cdots a_n)^2}\ for\ a_i>0\ (i=1;2;\dots;n)\rightarrow\frac{1}{a_1^n}+\frac{1}{a_2^n}+\cdots+\frac{1}{a_n^n}\geq\frac{n^{n+1}}{(a_1+a_2+\cdots+a_n)^n}\end{cases}$$

$\rightarrow$ *Let* a_i *and* $b_i>0(i=1;2;\dots;n)$ *and real* $p,q>0$ *such that* $p+q=pq$*. Then we have:*

$\left(a_1^p+a_2^p+\cdots+a_n^p\right)^{\frac{1}{p}}\cdot\left(b_1^q+b_2^q+..+b_n^q\right)^{\frac{1}{q}}\geq a_1b_1+a_2b_2+\cdots+a_nb_n$ ***(General Hölder's inequality)***

5) *Let* $a_i,b_i>0\ (i=1;2;\dots;n)$ *and any* $k>1$*. Then we have:*

$$\left(a_1^k+b_1^k\right)\left(a_2^k+b_2^k\right)\dots\left(a_n^k+b_n^k\right)\geq\sqrt[k]{(a_1+a_2+\cdots+a_n)^k+(b_1+b_2+\cdots+b_n)^k}$$

(Minkovsky's inequality) $\rightarrow$ *similarly for 3 variables* $a_i,b_i,c_i>0$.

6) *For any* $x\geq -1$ *we have:* $\begin{cases} *\ (1+x)^r\geq 1+xr\ \ for\ \ r\geq 1\ and\ r\leq 0\\ *\ (1+x)^r\leq 1+xr\ for\ 0\leq r\leq 1\end{cases}$ ***(Bernoulli's inequality)***

7) *For any positive integer* m *and* $a,b,c\geq 0$ *we have:*

$$a^m(a-b)(a-c)+b^m(b-c)(b-a)+c^m(c-a)(c-b)\geq 0$$

(Schur's inequality) → *This is also true for real $m \geq 1$ and equality happens iff $a = b = c$ or*

$(a; b; c) \sim (0; k; k)$ *with $k > 0$.*

Δ Case $m = 1$ – Schur deg 3: All forms: (p, q, r will be discussed in later part)

$$* \ a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a) \rightarrow (a+b+c)^3 + 9abc \geq$$

$$\geq 4(a+b+c)(ab+bc+ca) \rightarrow a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca) \rightarrow$$

$$\rightarrow (a+b+c)(a^2+b^2+c^2+ab+bc+ca) \geq 3[ab(a+b) + bc(b+c) + ca(c+a)]$$

$* \ abc \geq (b+c-a)(c+a-b)(a+b-c)$ *(Well – known result)*

$* \ (b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \geq 0$

$* \ 3(a^3+b^3+c^3) \geq (a+b+c)[2(a^2+b^2+c^2) - ab - bc - ca]$

$* \ 4(a^3+b^3+c^3) + 15abc \geq (a+b+c)^3$

$* \ \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 2$

$* \ \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3 \geq a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ *for $a, b, c > 0$: $abc = 1$.*

$* \ (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq (a^2b^2 + b^2c^2 + c^2a^2)(ab+bc+ca)$

Δ Case $m = 2$ – Schur deg 4: All forms:

$* \ a^4 + b^4 + c^4 + abc(a+b+c) \geq ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2) \rightarrow$

$\rightarrow a^4 + b^4 + c^4 + 2abc(a+b+c) \geq (a^2+b^2+c^2)(ab+bc+ca)$

$$* \ 2(ab+bc+ca) - (a^2+b^2+c^2) \leq \frac{6abc(a+b+c)}{a^2+b^2+c^2+ab+bc+ca} \leq \frac{9abc}{a+b+c}$$

$* \ [(b-c)(b+c-a)]^2 + [(c-a)(c+a-b)]^2 + [(a-b)(a+b-c)]^2 \geq 0$

→ *Let $a, b, c, x, y, z \geq 0$. Then we have:*

$x(a-b)(a-c) + y(b-c)(b-a) + z(c-a)(c-b) \geq 0$ *iff $a \geq b \geq c$ and:*

$* \ x \geq y \curlyvee z \geq y$ $\qquad$ $* \ ax + cz \geq by$

$* \ x + z \geq y$ $\qquad$ $* \ \sqrt{x} + \sqrt{z} \geq \sqrt{y}$

$* \ ax \geq by \curlyvee cz \geq by$

(General Vornicu – Schur inequality)

8) $\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \geq \frac{9}{4(ab+bc+ca)}$ *for $a, b, c \geq 0$, no 2 of which are 0. (Iran 96 inequality)*

9) $(a^2+b^2+c^2)^2 \geq 3(a^3b + b^3c + c^3a)$ ***(Vasile's inequality)*** → *The equality happens iff*

$a = b = c$ and also for $(a; b; c) = \left(k \cdot \sin^2 \frac{4\pi}{7}; k \cdot \sin^2 \frac{2\pi}{7}; k \cdot \sin^2 \frac{\pi}{7}\right)$ or any cyclic permutation.

10) *Let* a_i *and* b_i $(i = 1; 2; \dots; n)$ *such that:*

$* \ a_1 \geq a_2 \geq \dots \geq a_n \geq 0, b_1 \geq b_2 \geq \dots \geq b_n \geq 0$

$* \ a_1 \geq b_1; a_1 + a_2 \geq b_1 + b_2; \dots; a_1 + a_2 + \dots + a_{n-1} \geq b_1 + b_2 + \dots + b_{n-1}$

$* \ a_1 + a_2 + \dots + a_{n-1} + a_n = b_1 + b_2 + \dots + b_{n-1} + b_n$

For $x_1 \geq 0$ *we have:* $\sum_{sym} x_{t_1}^{a_1} x_{t_2}^{a_2} \dots x_{t_n}^{a_n} \geq \sum_{sym} x_{t_1}^{b_1} x_{t_2}^{b_2} \dots x_{t_n}^{b_n}$, *where* $(t_1; t_2; \dots; t_n)$ *are all the permutations of* $(1; 2; \dots; n)$

(Muirhead's inequality)

E.g: $a^3 + b^3 \geq ab(a + b) \to a^3b^0 + b^3a^0 \geq a^2b^1 + a^1b^2$

$a^2 + b^2 + c^2 \geq ab + bc + ca \to a^2b^0 + b^2c^0 + c^2a^0 \geq a^1b^2 + b^1c^1 + c^1a^1$

$a^4 + b^4 + c^4 \geq abc(a + b + c) \to a^4b^0c^0 + b^4c^0a^0 + c^4a^0b^0 \geq$

$\geq a^2b^1c^1 + b^2c^1a^1 + c^2a^1b^1$

b) For senior and higher classes:

1) $*$ *If* $\left[\begin{array}{l} a_1 \geq a_2 \geq \dots \geq a_n; b_1 \geq b_2 \geq \dots \geq b_n \\ a_1 \leq a_2 \leq \dots \leq a_n; b_1 \leq b_2 \leq \dots \leq b_n \end{array}\right.$ *and* $(k_1; k_2; \dots; k_n)$ *is an arbitrary permutation of* $(1; 2; \dots; n)$ *then:* $a_1b_1 + a_2b_2 + \dots + a_nb_n \geq a_1b_{k_1} + a_2b_{k_2} + \dots + a_nb_{k_n}$.

$*$ *If* $a_1 \geq a_2 \geq \dots \geq a_n$ *and* $b_1 \leq b_2 \leq \dots \leq b_n$ *then:*

$a_1b_1 + a_2b_2 + \dots + a_nb_n \leq a_1b_{k_1} + a_2b_{k_2} + \dots + a_nb_{k_n}$;

$n(a_1b_1 + a_2b_2 + \dots + a_nb_n) \leq (a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n)$

(Rearrangement inequality)

2) * *Convex function: If* $a, b \geq 0$ *such that* $a + b = 1$ *then* $f(x)$ *is called a convex function on* $I(a; b) \subset R$ *iff* $\forall x_1; x_2 \in I$ *we have:* $f(ax_1 + bx_2) \leq af(x_1) + bf(x_2)$

* *Concave function: If* $a, b \geq 0$ *such that* $a + b = 1$ *then* $f(x)$ *is called a concave function on* $I(a; b) \subset R$ *iff* $\forall x_1; x_2 \in I$ *we have:* $f(ax_1 + bx_2) \geq af(x_1) + bf(x_2)$

* *If* $f(x)$ *is a convex function on interval* $I \subset R$ *then for any* $x_i \in I$ $(i = 1; 2; \dots; n)$ *we have:*

$f\left(\frac{x_1+x_2+\dots+x_n}{n}\right) \leq \frac{f(x_1)+f(x_2)+\dots+f(x_n)}{n}$ ***(Classic Jensen's inequality)***

* *If* $f(x)$ *is a convex function on interval* $I \subset R$ *then for any* $x_1 \in I$ $(i = 1; 2; \dots; n)$ *and* $p_i > 0$ *we have:*

$$\frac{p_1f(x_1) + p_2f(x_2) + \dots + p_nf(x_n)}{p_1 + p_2 + \dots + p_n} \geq f\left(\frac{p_1x_1 + p_2x_2 + \dots + p_nx_n}{p_1 + p_2 + \dots + p_n}\right)$$

And if $f(x)$ *is a concave function then the inequality is reversed.* ***(General Jensen's inequality)***

$\to *$ *If* $f(x)$ *is a convex and continuous function on interval* $I \subset R$ *then for any* $x_i \in I$ $(i = 1; 2; \dots; n)$ *and* $p_i \in (0,1)$ *such that* $p_1 + p_2 + \dots + p_n = 1$ *we have:* $p_1 f(x_1) + p_2 f(x_2) + \dots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \dots + p_n x_n)$*. And if* $f(x)$ *is a concave function then the inequality is reversed.*

* *The classic inequality is a special case from the general one with* $p_1 = p_2 = \dots = p_n$

* *Let* a_i *and* b_i $(i = 1; 2; ..; n) \in I$ $(I \subset R)$ *such that:*

_ $a_1 \geq a_2 \geq \dots \geq a_n; b_1 \geq b_2 \geq \dots \geq b_n$

_ $a_1 \geq b_1; a_1 + a_2 \geq b_1 + b_2; \dots; a_1 + a_2 + \dots + a_{n-1} \geq b_1 + b_2 + \dots + b_{n-1}$

_ $a_1 + a_2 + \dots + a_{n-1} + a_n = b_1 + b_2 + \dots + b_{n-1} + b_n$

If $f(x)$ *is a convex function on* I *then we have:*

$f(a_1) + f(a_2) + \dots + f(a_n) \geq f(b_1) + f(b_2) + \dots + f(b_n)$ ***(Karamata's inequality)***

* *If* $f(x)$ *is a convex function on* $I \subset R$ *then for* $a_i \in I$ $(i = 1; 2; \dots; n)$ *we have:*

$$f(a_1) + f(a_2) + \dots + f(a_n) + n(n-2) f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) \geq$$

$\geq (n-1)[f(b_1) + f(b_2) + \dots + f(b_n)]$ *where* $b_i + \frac{a_i}{n-1} = \frac{a_1 + a_2 + \dots + a_n}{n-1}$ $(i = 1; 2; \dots; n)$

(Popoviciu's inequality)

3) *Define* $p = a + b + c, q = ab + bc + ca, r = abc$ *with* a, b, c *are any real numbers. If* $k = \sqrt{p^2 - 3q}$ *then we have:* $\frac{p^3 - 3pk^2 - 2k^3}{27} \leq r \leq \frac{p^3 - 3pk^2 + 2k^3}{27}$

$\to$ *The minimum and maximum happens iff* 2 *of* 3 *variables* $a; b; c$ *are equal.*

4) * *Let* $f(a; b; c)$ *be a symmetric polynomial of degree* 3 *with* $a, b, c \geq 0$*. Then:*

$f(a; b; c) \geq 0 \Leftrightarrow f(1; 1; 1); f(1; 1; 0); f(1; 0; 0) \geq 0$ ***(SD3 theorem)***

* *Let* $f(a; b; c)$ *be a cyclic homogeneous polynomial of degree* 3 *with* $a, b, c \geq 0$*. Then:*

$f(a; b; c) \geq 0 \Leftrightarrow f(1; 1; 1) \geq 0; f(a; b; 0) \geq 0$ ***(CD3 theorem)***

$\to$ *Let* $f_n(a; b; c)$ *be a cyclic homogeneous polynomial of degree* n $(n = 3; 4; 5)$ *with* $a, b, c \geq 0$*. Then* $f_n(a; b; c) \geq 0 \Leftrightarrow f_n(a; 1; 1) \geq 0$ *and* $f_n(0; b; c) \geq 0$*.*

5) ***(S.O.S technique)*** *Define* $S = S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2$*, where* $S_a; S_b; S_c$ *are functions with variables* $a; b; c$*. Then* $S \geq 0$ *iff:*

* $S_a; S_b; S_c \geq 0$

* $a \geq b \geq c; S_b \geq 0; S_b + S_a \geq 0; S_b + S_c \geq 0$

* $a \geq b \geq c; S_a \geq 0; S_c \geq 0; S_a + 2S_b \geq 0; S_c + 2S_b \geq 0$

* $a \geq b \geq c; S_b \geq 0; S_c \geq 0; a^2 S_b + b^2 S_a \geq 0$

* $S_a + S_b + S_c \geq 0; S_a S_b + S_b S_c + S_c S_a \geq 0$

→ *Consider* $f(a; b; c) = P(a-b)^2 + Q(a-c)(b-c) \geq 0$ *(*)*

** If* $f(a; b; c)$ *is symmetric then to prove (*) is true, we assume that* $a \geq b \geq c$ *or* $c = \min\{a; b; c\}$ *or* $c = \max\{a; b; c\}$ *and prove that* $P, Q \geq 0$.

** If* $f(a; b; c)$ *is cyclic then to prove (*) is true, we assume that* $c = \min\{a; b; c\}$ *or* $c = \max\{a; b; c\}$ *and prove that* $P, Q \geq 0$. ***(S.S technique)***

c) Some identities:

(*) Some useful identities and inequalities which can be proved by S.O.S, S.S technique:

1) $a^2 + b^2 - 2ab = (a-b)^2; \frac{a}{b} + \frac{b}{a} - 2 = \frac{(a-b)^2}{ab}$

$$\rightarrow \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)} - (a_1 + a_2 + \cdots + a_n) = \frac{\sum_{1 \leq i < j \leq n} (a_i - a_j)^2}{\sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)} + (a_1 + a_2 + \cdots + a_n)}$$

For $n = 2$: $\sqrt{2(a^2 + b^2)} - (a+b) = \frac{(a-b)^2}{\sqrt{2(a^2+b^2)} + a + b}$

For $n = 3$: $\sqrt{3(a^2 + b^2 + c^2)} - (a+b+c) = \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{\sqrt{3(a^2+b^2+c^2)} + (a+b+c)} = 2\frac{(a-b)^2 + (a-c)(b-c)}{\sqrt{3(a^2+b^2+c^2)} + (a+b+c)}$

$$\rightarrow (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 9 = \frac{(a-b)^2}{ab} + \frac{(b-c)^2}{bc} + \frac{(c-a)^2}{ca} = \frac{2(a-b)^2}{ab} + \left(\frac{1}{ac} + \frac{1}{bc}\right)(a-c)(b-c)$$

2) $(a+b+c)^2 - 3(ab+bc+ca) = a^2 + b^2 + c^2 - (ab+bc+ca) =$

$$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] = (a-b)^2 + (a-c)(b-c)$$

3) $a^3 + b^3 - ab(a+b) = (a+b)(a-b)^2$

$$\rightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} = \frac{(b-c)^2}{2(a+b)(a+c)} + \frac{(c-a)^2}{2(b+c)(b+a)} + \frac{(a-b)^2}{2(c+a)(c+b)} =$$

$$= \frac{(a-b)^2}{(c+a)(c+b)} + \frac{(a+b+2c)(a-c)(b-c)}{2(a+b)(b+c)(c+a)}$$

$$\rightarrow 3(a^3 + b^3 + c^3) - (a+b+c)(a^2 + b^2 + c^2) = (a+b)(a-b)^2 + (b+c)(b-c)^2 + (c+a)(c-a)^2$$

$$= 2(a+b)(a-b)^2 + (a+b+2c)(a-c)(b-c)$$

4) $(a+b+c)(ab+bc+ca) - 9abc = (a+b)(b+c)(c+a) - 8abc =$

$$= a(b-c)^2 + b(c-a)^2 + c(a-b)^2 = 2c(a-b)^2 + (a+b)(a-c)(b-c)$$

5) $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] =$

$$=(a+b+c)[(a-b)^2+(a-c)(b-c)]$$

6) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3=\frac{1}{6abc}[(a-b)^2(3c+a-b)+(b-c)^2(3a+b-c)+(c-a)^2(3b+c-a)]=$

$$=\frac{(a-b)^2}{ab}+\frac{(a-c)(b-c)}{bc}$$

7) $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-(a+b+c)=\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}=\left(\frac{1}{a}+\frac{1}{b}\right)(a-b)^2+\frac{b+c}{ac}(a-c)(b-c)$

8) $a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)=$

$$=\frac{1}{2}[(b+c-a)(b-c)^2+(c+a-b)(c-a)^2+(a+b-c)(a-b)^2]$$

$$=(a+b-c)(a-b)^2+c(a-c)(b-c)$$

9) $\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}-(a+b+c)=$

$$=\frac{1}{2abc}\{(b-c)^2[(b+c)^2+a^2]+(c-a)^2[(c+a)^2+b^2]+(a-b)^2[(a+b)^2+c^2]\}$$

$$=\frac{(a+b)^2+c^2}{abc}(a-b)^2+\left[\frac{1}{c}+\frac{(a+c)(b+c)}{abc}\right](a-c)(b-c)$$

10) $\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}-(a^2+b^2+c^2)=\left(\frac{a}{b}+\frac{1}{2}\right)(a-b)^2+\left(\frac{b}{c}+\frac{1}{2}\right)(b-c)^2+\left(\frac{c}{a}+\frac{1}{2}\right)(c-a)^2$

11) $a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2-a(b^3+c^3)-b(c^3+a^3)-c(a^3+b^3)=$

$$=\frac{1}{2}[(a^2+b^2)(a-b)^2+(b^2+c^2)(b-c)^2+(c^2+a^2)(c-a)^2]$$

$$=(a^2+b^2)(a-b)^2+(a^2+ab+b^2+c^2)(a-c)(b-c)$$

(**) More identities:

1) * $\frac{a+bc}{b-c}\cdot\frac{b+ca}{c-a}+\frac{b+ca}{c-a}\cdot\frac{c+ab}{a-b}+\frac{c+ab}{a-b}\cdot\frac{a+bc}{b-c}=a+b+c-1$ *with* $a\neq b\neq c$

* $\frac{a-bc}{b-c}\cdot\frac{b-ca}{c-a}+\frac{b-ca}{c-a}\cdot\frac{c-ab}{a-b}+\frac{c-ab}{a-b}\cdot\frac{a-bc}{b-c}=-a-b-c-1$ *with* $a\neq b\neq c$

2) * $\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}=\frac{(a-b)(b-c)(a-c)}{(a+b)(b+c)(a+c)}$

* $\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}=\frac{a(b-c)^2+b(c-a)^2+c(a-b)^2}{(a-b)(b-c)(a-c)}$ *with* $a\neq b\neq c$

3) * $\frac{a+b}{a-b}\cdot\frac{b+c}{b-c}+\frac{b+c}{b-c}\cdot\frac{c+a}{c-a}+\frac{c+a}{c-a}\cdot\frac{a+b}{a-b}=-1$ *with* $a\neq b\neq c$

* $\frac{a-b}{a+b}\cdot\frac{b-c}{b+c}+\frac{b-c}{b+c}\cdot\frac{c-a}{c+a}+\frac{c-a}{c+a}\cdot\frac{a-b}{a+b}=\frac{-[a(b-c)^2+b(c-a)^2+c(a-b)^2]}{(a+b)(b+c)(c+a)}$

4) * $\frac{a^2+bc}{b+c}\cdot\frac{b^2+ca}{c+a}+\frac{b^2+ca}{c+a}\cdot\frac{c^2+ab}{a+b}+\frac{c^2+ab}{a+b}\cdot\frac{a^2+bc}{b+c}=a^2+b^2+c^2$

$*\frac{a^2-bc}{b-c}\cdot\frac{b^2-ca}{c-a}+\frac{b^2-ca}{c-a}\cdot\frac{c^2-ab}{a-b}+\frac{c^2-ab}{a-b}\cdot\frac{a^2-bc}{b-c}=-(a+b+c)^2$ *with* $a\neq b\neq c$

5) $(a^2-bc)(b+c)+(b^2-ca)(c+a)+(c^2-ab)(a+b)=0$

$(a^2+bc)(b-c)+(b^2+ca)(c-a)+(c^2+ab)(a-b)=-2(a-b)(b-c)(c-a)$

6) $\frac{(b+c)^2}{(a-b)(a-c)}+\frac{(c+a)^2}{(b-c)(b-a)}+\frac{(a+b)^2}{(c-a)(c-b)}=1$ *with* $a\neq b\neq c$

7) $\frac{1-ab}{a-b}\cdot\frac{1-bc}{b-c}+\frac{1-bc}{b-c}\cdot\frac{1-ca}{c-a}+\frac{1-ca}{c-a}\cdot\frac{1-ab}{a-b}=-1$ *with* $a\neq b\neq c$

8) $*\ \frac{a^2+bc}{b-c}\cdot\frac{b^2+ca}{c-a}+\frac{b^2+ca}{c-a}\cdot\frac{c^2+ab}{a-b}+\frac{c^2+ab}{a-b}\cdot\frac{a^2+bc}{b-c}=a^2+b^2+c^2$ *with* $a\neq b\neq c$

$*\frac{a^2-bc}{b+c}\cdot\frac{b^2-ca}{c+a}+\frac{b^2-ca}{c+a}\cdot\frac{c^2-ab}{a+b}+\frac{c^2-ab}{a+b}\cdot\frac{a^2-bc}{b+c}=-(a+b+c)[ab(a-b)^2+bc(b-c)^2+ca(c-a)^2]$

9) $\frac{a^2+bc}{a^2-bc}\cdot\frac{b^2+ca}{b^2-ca}+\frac{b^2+ca}{b^2-ca}\cdot\frac{c^2+ab}{c^2-ab}+\frac{c^2+ab}{c^2-ab}\cdot\frac{a^2+bc}{a^2-bc}=-1$

10) $(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

11) $2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$

12) $(ab^n+bc^n+ca^n)-(a^nb+b^nc+c^na)=(a-b)(b-c)(c-a)\sum_{sym}a^pb^qc^r$ *with* $p,q,r\in N$ *and* $n\geq 2$ *such that* $p+q+r=n-2$ *(?)*

E.g: for $n=2$: $(ab^2+bc^2+ca^2)-(a^2b+b^2c+c^2a)=(a-b)(b-c)(c-a)$

$n=3$: $(ab^3+bc^3+ca^3)-(a^3b+b^3c+c^3a)=(a-b)(b-c)(c-a)(a+b+c)$

$n=4$: $(ab^3+bc^3+ca^3)-(a^3b+b^3c+c^3a)=(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$

13) $a^n(a-b)(a-c)+b^n(b-c)(b-a)+c^n(c-a)(c-b)=$

$$=\frac{1}{2}[(a^n+b^n-c^n)(a-b)^2+(b^n+c^n-a^n)(b-c)^2+(c^n+a^n-b^n)(c-a)^2]$$

14) *If* $a,b,c\neq 0$ *such that* $abc=1$ *then:* $\frac{1}{ab+b+1}+\frac{1}{bc+c+1}+\frac{1}{ca+a+1}=1$

15) $a^2+b^2+c^2+abc=4\Leftrightarrow 2a+bc=\sqrt{(4-a^2)(4-b^2)}$, *etc* $\Leftrightarrow$

$$\Leftrightarrow\frac{2a+bc}{(2+b)(2+c)}+\frac{2b+ca}{(2+c)(2+a)}+\frac{2c+ab}{(2+a)(2+b)}=$$

$$=\frac{(2-b)(2-c)}{2a+bc}+\frac{(2-c)(2-a)}{2b+ca}+\frac{(2-a)(2-b)}{2c+ab}=\frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab}=$$

$$=\frac{bc}{2a+bc}+\frac{ca}{2b+ca}+\frac{ab}{2c+ab}=1\Leftrightarrow\frac{1}{2a+bc}+\frac{1}{2b+ca}+\frac{1}{2c+ab}=\frac{1}{a+b+c-2}=$$

$$=\frac{a+b+c+2}{2(ab+bc+ca)-abc}\Leftrightarrow(a+b+c-2)^2=(2-a)(2-b)(2-c)$$

→ *From the identity, there exists* $x,y,z>0$ *such that:*

$a = 2\sqrt{\frac{xy}{(z+x)(z+y)}}; b = 2\sqrt{\frac{yz}{(x+y)(x+z)}}; c = 2\sqrt{\frac{xz}{(y+z)(y+x)}}.$

And there exists triangle ABC such that: $a = 2\cos A; b = 2\cos B; c = 2\cos C$

16) $a^2 + b^2 + c^2 + 2abc = 1 \Leftrightarrow a + bc = \sqrt{(1-a^2)(1-b^2)}$, *etc* $\Leftrightarrow$

$$\Leftrightarrow \frac{a+bc}{(1+b)(1+c)} + \frac{b+ca}{(1+c)(1+a)} + \frac{c+ab}{(1+a)(1+b)} =$$

$$= \frac{(1-b)(1-c)}{a+bc} + \frac{(1-c)(1-a)}{b+ca} + \frac{(1-a)(1-b)}{c+ab} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{a}{a+bc} + \frac{b}{b+ca} + \frac{c}{c+ab} = 2 \Leftrightarrow \frac{bc}{a+bc} + \frac{ca}{b+ca} + \frac{ab}{c+ab} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{a+bc} + \frac{1}{b+ca} + \frac{1}{c+ab} = \frac{2}{a+b+c-1} = \frac{a+b+c+1}{ab+bc+ca-abc} \Leftrightarrow$$

$$\Leftrightarrow (a+b+c-1)^2 = 2(1-a)(1-b)(1-c)$$

→ *If we substitute* $a \to \frac{a}{2}; b \to \frac{b}{2}; c \to \frac{c}{2}$ *we will get identity 15, so:*

From the identity, there exists $x, y, z > 0$ *such that:* $a = \sqrt{\frac{xy}{(z+x)(z+y)}}$; $b = \sqrt{\frac{yz}{(x+y)(x+z)}}$;

$c = \sqrt{\frac{zx}{(y+z)(y+x)}}$. *And there exists triangle ABC such that:* $a = \cos A; b = \cos B; c = \cos C$.

Also if we let $\boldsymbol{a^2 = yz}; \boldsymbol{b^2 = zx}; \boldsymbol{c^2 = xy}$ *with* $x, y, z > 0$ *then* $abc = xyz$, *so we have 2 identities 17-18:*

17) $xy + yz + zx + xyz = 4 \Leftrightarrow \frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} = \frac{x}{x+2} + \frac{y}{y+2} + \frac{z}{z+2} = 1 \Leftrightarrow$

$$\Leftrightarrow \frac{\sqrt{x}}{2+x} + \frac{\sqrt{y}}{2+y} + \frac{\sqrt{z}}{2+z} = \frac{\sqrt{xyz}}{\sqrt{xy} + \sqrt{yz} + \sqrt{zx} - 2}$$

18) $xy + yz + zx + 2xyz = 1 \Leftrightarrow \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 2 \Leftrightarrow \frac{x}{x+1} + \frac{y}{y+1} + \frac{z}{z+1} = 1 \Leftrightarrow$

$$\Leftrightarrow \frac{\sqrt{x}}{1+x} + \frac{\sqrt{y}}{1+y} + \frac{\sqrt{z}}{1+z} = \frac{2\sqrt{xyz}}{\sqrt{xy} + \sqrt{yz} + \sqrt{zx} - 1}$$

→ *From identity 17, there also exists,* $m, n, p > 0$ *such that* $x = \frac{2m}{n+p}; y = \frac{2n}{p+m}; z = \frac{2p}{m+n}$,

similarly for identity 18.

19) $x(x^2 + xy + y^2) + y(y^2 + yz + z^2) + z(z^2 + zx + x^2) = y(x^2 + xy + y^2) +$

$$+z(y^2 + yz + z^2) + x(z^2 + zx + x^2) = (x+y+z)(x^2+y^2+z^2)$$

$$z(x^2 + xy + y^2) + x(y^2 + yz + z^2) + y(z^2 + zx + x^2) = (x+y+z)(xy+yz+zx)$$

20) $(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)=p^2+pq+q^2$, *where*

$$p=x^2y+y^2z+z^2x;q=xy^2+yz^2+zx^2$$

$$=3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)+[(x-y)(y-z)(z-x)]^2$$

$$=\frac{3}{4}[xy(x+y)+yz(y+z)+zx(z+x)]^2+\frac{1}{4}[(x-y)(y-z)(z-x)]^2$$

$$=\frac{1}{2}[(x+y+z)^2(x^2y^2+y^2z^2+z^2x^2)+(xy+yz+zx)^2(x^2+y^2+z^2)]$$

$$\rightarrow\left\langle\begin{array}{l}*\ (xy+yz+zx)^2(x^2+y^2+z^2)=(x^2+2yz)(y^2+2zx)(z^2+2xy)+[(x-y)(y-z)(z-x)]^2\\ *\ (x+y+z)^2(x^2y^2+y^2z^2+z^2x^2)=(2x^2+yz)(2y^2+zx)(2z^2+xy)+[(x-y)(y-z)(z-x)]^2\end{array}\right.$$

21) $2(x^2+y^2)(y^2+z^2)(z^2+x^2)=[xy(x+y)+yz(y+z)+zx(z+x)-2xyz]^2+$

$$+[(x-y)(y-z)(z-x)]^2$$

22) $2[x^2(y-z)^4+y^2(z-x)^4+z^2(x-y)^4]=[x(y-z)^2+y(z-x)^2+z(x-y)^2]^2+$

$$+[(x-y)(y-z)(z-x)]^2$$

23) $\frac{a^2}{a^2+bc}+\frac{b^2}{b^2+ca}+\frac{c^2}{c^2+ab}=\frac{3}{2}\Leftrightarrow\left(\frac{ab+bc+ca}{a+b+c}\right)^3=abc\Leftrightarrow a^2=bc\ or\ b^2=ca\ or\ c^2=ab$

d) Useful lemmas:

d. 1) If $a,b,c>0$ such that $abc=1$ then:

1) $a^m+b^m+c^m\ge a^n+b^n+c^n\ (m,n\in Z^+;m>n)$

2) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c;\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{3}{2}(a+b+c-1)\rightarrow$

$$\rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\ge a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

3) $\frac{1}{a^{2k}+a^k+1}+\frac{1}{b^{2k}+b^k+1}+\frac{1}{c^{2k}+c^k+1}\ge 1$ *with* $k\in Z^+$

4) $\frac{1}{a^k+b^k+1}+\frac{1}{b^k+c^k+1}+\frac{1}{c^k+a^k+1}\le 1$ *with* $k\in Z^+$

5) * $\frac{1}{a(b+1)}+\frac{1}{b(c+1)}+\frac{1}{c(a+1)}\ge\frac{3}{2}$

* $\frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+a)}\ge\frac{3}{2}\rightarrow\frac{1}{a(ma+nb)}+\frac{1}{b(mb+nc)}+\frac{1}{c(mc+na)}\ge\frac{3}{m+n}$ *with* $m,n>0$ *(*)*

6) $\frac{1}{a^2+2b^2+3}+\frac{1}{b^2+2c^2+3}+\frac{1}{c^2+2a^2+3}\le\frac{1}{2}$

7) $\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1$

8) *Let* $p=a+b+c;q=ab+bc+ca$ *then:*

$* p^2 + 3 \geq 4q \to \begin{cases} p + \frac{3}{q} \geq 4 \\ p + \frac{3}{q} \geq 4 \cdot \frac{q}{p} \end{cases} \quad * pq \geq 5p - 6$

9) $\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} \geq 1 \to \frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} \geq \frac{a}{a+2} + \frac{b}{b+2} + \frac{c}{c+2} \geq 1 \geq$

$$\geq \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} \to \frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} \geq 1 \geq \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2}$$

$$\left(Extra: \frac{1}{3a+1} + \frac{1}{3b+1} + \frac{1}{3c+1} \geq \frac{3}{4} \geq \frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}\right)$$

10) $(a^2+1)(b^2+1)(c^2+1) \geq (a+b)(b+c)(c+a) \geq (a+1)(b+1)(c+1) \geq 8$

11) $\sqrt{(n^2-1)a^2+1} + \sqrt{(n^2-1)b^2+1} + \sqrt{(n^2-1)c^2+1} \leq n(a+b+c)$ *with*

$$a + b + c \geq ab + bc + ca$$

$\to$ *Special case:* $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ *and without condition* $abc = 1$.

12) $\frac{1}{(a+1)^2} + \frac{1}{(b+1)^2} + \frac{1}{(c+1)^2} \geq \frac{3}{4}$

13) $(a+b+c)^5 \geq 81(a^2+b^2+c^2)$

d. 2) *If* $a, b, c > 0$ *such that* $a + b + c = 3$ *then:*

1) $a^2 + b^2 + c^2 \geq 3 \geq ab + bc + ca$

2) $a^m + b^m + c^m \geq 3$ *for* $m \in Z^+$

3) $\frac{a^m}{b^n} + \frac{b^m}{c^n} + \frac{c^m}{a^n} \geq 3$ *for* $m, n \in Z^+$ *such that* $m \geq n$

4) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2$

5) $a^{n-1} + b^{n-1} + c^{n-1} \leq \frac{a^n+b^n}{a+b} + \frac{b^n+c^n}{b+c} + \frac{c^n+a^n}{c+a} \leq a^n + b^n + c^n$ *with all* $n \in Z^+$

6) $\frac{1}{a^2+b^2+n} + \frac{1}{b^2+c^2+n} + \frac{1}{c^2+a^2+n} \leq \frac{3}{n+2}$ *with all* $n \geq 2$

7) $\frac{a^m}{b^n+c^n} + \frac{b^m}{c^n+a^n} + \frac{c^m}{a^n+b^n} \geq \frac{3}{2}$ *with* $m, n \in Z^+$ *such that* $m > n$

8) $abc(a^2+b^2+c^2) \leq 3 \to (abc)^n \cdot (a^2+b^2+c^2) \leq 3$

with $n \in Z^+ \to \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq a^2 + b^2 + c^2$

9) $\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \geq 1$

10) $a^2b + b^2c + c^2a + abc \leq 4$

11) $a^2 + b^2 + c^2 \geq a^2b + b^2c + c^2a$

12*) $\frac{a}{b^2+pc}+\frac{b}{c^2+pa}+\frac{c}{a^2+pb}\geq\frac{3}{1+p}$ *with* $p\geq 1$

d.3) If $\boldsymbol{a,b,c>0: ab+bc+ca=3}$ ***then:***

1) $a+b+c\geq 3abc\rightarrow(a+b+c)^5\geq 243abc$

2) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a+b+c$

3) $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\geq\frac{3}{2}\rightarrow\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}\geq\frac{3}{4}$

4) $a^2b+b^2c+c^2a\leq a^2+b^2+c^2$

5) $\frac{1}{(a+b)^2+m}+\frac{1}{(b+c)^2+m}+\frac{1}{(c+a)^2+m}\leq\frac{3}{4+m}$ *with* $m\geq 2$

$\rightarrow\frac{1}{(xa+yb)^2+m}+\frac{1}{(xb+yc)^2+m}+\frac{1}{(xc+ya)^2+m}\leq\frac{3}{(x+y)^2+m}$

with $x,y>0$ *and* $m\geq 2(x^2-xy+y^2)$

6) $\frac{1}{a^2+b^2+k}+\frac{1}{b^2+c^2+k}+\frac{1}{c^2+a^2+k}\leq\frac{3}{2+k}$ *with* $k\geq 1$

7) $\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\leq 1$

8*) $\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\leq 1$

d.4) If $\boldsymbol{a,b,c>0}$ ***such that*** $\boldsymbol{a^2+b^2+c^2=3}$ ***then:***

1) $a+b+c\geq ab+bc+ca\rightarrow\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}\geq ab+bc+ca$ *with* $n\in Z^+$

2) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{9}{a+b+c}\rightarrow(a+b+c)^3\geq 9(ab+bc+ca)$

3) $\frac{a}{b+2}+\frac{b}{c+2}+\frac{c}{a+2}\leq 1;\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}\geq\frac{3}{2}$

4) $a^3b+b^3c+c^3a\leq 3$

5) $a^2b+b^2c+c^2a\leq 2+abc\rightarrow$ *In case* $a,b,c\geq 0$: *the equality happens*

if $\left[\begin{array}{c} a=b=c=1 \\ a=\sqrt{2};b=1;c=0\text{ or any cyclic permutation}\end{array}\right.$

d.5) If $\boldsymbol{a,b,c>0}$ ***such that*** $\boldsymbol{a+b+c=ab+bc+ca}$ ***then:***

1) $\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}\leq 1$

2) $a+b+c+abc\geq 4\rightarrow\begin{cases} *\ a+b+c+1\geq 4abc \\ *\ (a+1)(b+1)(c+1)\geq 8\end{cases}$

3) $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\leq\frac{3}{2}$

4) $(a+b)(b+c)(c+a)\geq 8$

5) $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a^2+b^2+c^2\to\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{3n}{a^2+b^2+c^2}\geq 3+n$ *with* $n\leq 3$

6) $a+b+c\geq abc+2$

***d.* 6) If $a,b,c>0$ such that $ab+bc+ca+abc=4$ then:**

1) $a+b+c\geq ab+bc+ca\geq 3\to\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{a+b+c}{2}\to\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a+b+c$

2) $\frac{a+b+c}{2}\geq\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\geq\frac{3}{2}\to\frac{a+b+c}{2}\geq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$

3) $\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\leq 1$

4) $\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\leq 3$

5) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq a+b+c$

6) $a^2b+b^2c+c^2a\leq a^2+b^2+c^2$

***d.*7) If $a,b,c>0$ such that $a^2+b^2+c^2+abc=4$ then:**

1) $ab+bc+ca\leq a+b+c\leq 3\to ab+bc+ca\leq abc+2\leq a+b+c\leq 3$

2) $\frac{3}{2}\leq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\leq\frac{9}{2(ab+bc+ca)}$

3) $a(a^2+b^2)+b(c^2+a^2)+c(a^2+b^2)\leq 6$

4) $\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\geq 3\to\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\geq a^2+b^2+c^2$

5) $a+bc\leq a+\left(\frac{b+c}{2}\right)^2\leq 2$; *etc*

6) $a^2+b^2+c^2+ab+bc+ca\geq 2(a+b+c)\to a+b+c\geq\sqrt{a}+\sqrt{b}+\sqrt{c}\geq ab+bc+ca$

***d.*8) If $a,b,c>0$ such that $a^2+b^2+c^2=2(ab+bc+ca)$ then: $\frac{a+b+c}{3}\geq\sqrt[3]{2abc}$**

→ the equality happens if $(a;b;c)\sim(k;k;4k)$ with $k>0$.

***d.*9) If $a,b,c>0$ such that $ab+bc+ca=abc+2$ then:**

1) *Assume that* $(b-1)(c-1)\geq 0$ *then:* $c+ab\geq 2\to a^2+b^2+c^2+abc\geq 4$

2) $\max\{ab;bc;ca\}\geq 1;\max\{a;b;c\}\geq 1$

• Inequalities with classic condition (like $a,b,c>0$):

*d.*10) *If* $a,b,c>0$ *then:* $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b+c}{\sqrt[3]{abc}}$

*d.*11) *If* $a_1;a_2;\dots;a_n>0$ *then:* $\frac{1}{1+a_1^n}+\frac{1}{1+a_2^n}+\dots+\frac{1}{1+a_n^n}\geq\frac{n}{1+a_1a_2\dots a_n}$

d.12) If $a,b,c \geq 0$, no 2 of which are 0 then: $\frac{a^n+b^n}{a+b}+\frac{b^n+c^n}{b+c}+\frac{c^n+a^n}{c+a} \leq 3\cdot\frac{a^n+b^n+c^n}{a+b+c}$ *with* $\forall n \in Z^+$

d.13) If $a,b,c > 0$ then: $\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{a^2+b^2+c^2}{2(ab+bc+ca)}+1$

d.14) If $a,b,c,x,y,z > 0$ then:

$$a(y+z)+b(z+x)+c(x+y) \geq 2\sqrt{(ab+bc+ca)(xy+yz+zx)}$$

d.15) If $a,b,c,d \geq 0$ then:

$a^4+b^4+c^4+d^4+2abcd \geq a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2$ *(Turkevich's inequality)→ The equality happens if $a=b=c=d$ and $a=b=c=k>0, d=0$ or any cyclic permutation.*

d.16) If $a,b,c > 0$ then: $\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b} \geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$

d.17) If a,b,c are sides of a triangle then:

$$\sqrt{3(a+b+c)} \geq \sqrt{a}+\sqrt{b}+\sqrt{c} \geq \sqrt{b+c-a}+\sqrt{c+a-b}+\sqrt{a+b-c}$$

d.18,19) If $a,b,c > 0$ then:

* $\frac{a^3}{b^2+bc+c^2}+\frac{b^3}{c^2+ca+a^2}+\frac{c^3}{a^2+ab+b^2} \geq \frac{a^2+b^2+c^2}{a+b+c} \geq \frac{a+b+c}{3}$

* $\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \geq a+b+c \geq \frac{3(ab+bc+ca)}{a+b+c}$

d.20) If $a,b,c > 0$ then:

$$\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2} \geq \frac{a+b+c}{ab+bc+ca} \geq \frac{a}{a^2+2bc}+\frac{b}{b^2+2ca}+\frac{c}{c^2+2ab}$$

d.21) If $a,b,c > 0$ then: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$

d.22) If $a,b,x,y,z > 0$ then: $\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by} \geq \frac{3}{a+b}$

d.23) If $x \geq 0$ then:* $e^x \geq 1+x+\frac{x^2}{2} \geq 1+x \rightarrow e^x \geq \left(1+\frac{x}{n}\right)^n$

d.24) If $a_i > 0\ \forall i = 1;2;\dots;n$ then: $\sum_{i=1}^{n}\frac{a_i^3}{a_i^2+a_i\cdot a_{i+1}+a_{i+1}^2} \geq \frac{a_1+a_2+\dots+a_n}{3}$ $(a_{n+1}=a_1)$

d.25) If $a,b,c > 0$ then: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{2ab}{(a+b)^2} \geq 2$

d.26) If $a,b,c > 0$ then:

$$a^2+b^2+c^2+2abc+1 \geq 2(ab+bc+ca) \rightarrow a^2+b^2+c^2+abc+5 \geq 3(a+b+c)$$

d.27) If a, b, c are sides of a triangle then: $a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0$ (IMO 1983)*

d.28) If $a, b, c > 0$ and $m\in Z^+$ then: $\frac{a^m}{b^{m-1}}+\frac{b^m}{c^{m-1}}+\frac{c^m}{a^{m-1}}\geq\frac{a^{m-1}}{b^{m-2}}+\frac{b^{m-1}}{c^{m-2}}+\frac{c^{m-1}}{a^{m-2}}\geq\cdots\geq$

$$\geq\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\geq\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c$$

d.29) If $a, b, c > 0$ then: $\frac{a}{a+\sqrt{(a+b)(a+c)}}+\frac{b}{b+\sqrt{(b+c)(b+a)}}+\frac{c}{c+\sqrt{(c+a)(c+b)}}\leq 1$

d.30) If $a, b, c > 0$ then: $\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\leq\frac{3}{\sqrt{2}}$

d.31) If a, b, c are sides of a triangle then: $a^4+b^4+c^4\leq 2(a^2b^2+b^2c^2+c^2a^2)$ (the equality happens iff a, b, c are sides of a degenerate triangle.)

d.32) If $a, b, c > 0$ and $k\geq 0$ then:* $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{ka+b}{ka+c}+\frac{kb+c}{kb+a}+\frac{kc+a}{kc+b}$

d.33) If $a, b, c > 0$ then: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{8abc}{(a+b)(b+c)(c+a)}\geq 4$

d.34) If $a, b, c\geq 0$, no 2 of which are 0 then: $\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)}\geq 2$ ***(Jack Garfunkel's inequality)***

• Inequalities with classic condition – part 2: (denote $p=a+b+c; q=ab+bc+ca; r=abc$)

Firstly, we have some identities about p, q, r:

1) $a^2+b^2+c^2=p^2-2q$

2) $a^2b^2+b^2c^2+c^2a^2=q^2-2pr$

3) $(a+b)(b+c)(c+a)=pq-r\rightarrow ab(a+b)+bc(b+c)+ca(c+a)=pq-3r$

4) $(a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b)=p^2q$

5) $a^3+b^3+c^3=p^3-3pq+3r$

6) $a^3b^3+b^3c^3+c^3a^3=q^3-3pqr+3r^2$

7) $a^4+b^4+c^4=p^4-4p^2q+2q^2+4pr$

8) $ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)=p^2q-2q^2-pr$

9) *Denote $S=-4p^3r+p^2q^2+18pqr-4q^3-27r^2$, then:*

* $[(a-b)(b-c)(c-a)]^2=S$

* $a^2b+b^2c+c^2a=\frac{pq-3r-\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)<0;\frac{pq-3r+\sqrt{S}}{2}$ *if*

$$(a-b)(b-c)(c-a)\geq 0$$

$* \ a^3b+b^3c+c^3a=\frac{p^2q-2q^2-pr-p\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)<0;\frac{p^2q-2q^2-pr+p\sqrt{S}}{2}$ *if*

$$(a-b)(b-c)(c-a)\geq 0$$

10) $a^2b^2(a^2+b^2)+b^2c^2(b^2+c^2)+c^2a^2(c^2+a^2)=-2p^3r+p^2q^2+4pqr-2q^3-3r^2$

11) $ab(a^4+b^4)+bc(b^4+c^4)+ca(c^4+a^4)=p^4q-p^3r-4p^2q^2+7pqr+2q^3-3r^2$

12) $a^6+b^6+c^6=p^6-6p^4q+6p^3r+9p^2q^2-12pqr-2q^3+3r^2$

• ***Some inequalities about the relation of*** $\boldsymbol{p,q,r}$**:**

1) $p^2\geq 3q; q^2\geq 3pr$

2) $pq\geq 9r\rightarrow p^3\geq\frac{27}{8}(pq-r)\geq 3pq\geq\frac{9q^2}{p}\geq 27r$

3) $p^3+9r\geq 4pq\rightarrow r\geq\frac{p(4q-p^2)}{9}\rightarrow r\geq\max\left\{0;\frac{p(4q-p^2)}{9}\right\}$

4) $2p^3+9r\geq 7pq$

5) $p^2q+3pr\geq 4q^2$

6) $p^4+4q^2+6pr\geq 5p^2q$

7) $r\geq\frac{(4q-p^2)(p^2-q)}{6p}\rightarrow r\geq\max\left\{0;\frac{(4q-p^2)(p^2-q)}{6p}\right\}$

8) $r\leq\frac{p(5q-p^2)}{18}; r\leq\frac{p^4-7p^2q+13q^2}{9p}$

Combining inequality 3, 6, 8 *and we get:*

$$\min\left\{\frac{p(5q-p^2)}{18};\frac{p^4-7p^2q+13q^2}{9p}\right\}\geq r\geq\max\left\{0;\frac{p(4q-p^2)}{9};\frac{(4q-p^2)(p^2-q)}{6p}\right\}$$

9*) $\frac{9pq-2p^3-2k\sqrt{k}}{27}\leq r\leq\frac{9pq-2p^3+2k\sqrt{k}}{27}$, *with* $k=p^2-3q$. *This result comes from solving the inequality* $S=-4p^3r+p^2q^2+18pqr-4q^3-27r^2\geq 0$ *with variable* r.

From this result, by AM-GM we have:

$* \ 27r\geq 9pq-2p^3-2k\sqrt{k}=9pq-2p^3-\frac{2(p^2-3q)(p^2-2q)\sqrt{p^2(p^2-3q)}}{p(p^2-2q)}$

$$\geq\frac{p^2(9q-2p^2)(p^2-2q)-(p^2-3q)[(p^2-2q)^2+p^2(p^2-3q)]}{p(p^2-2q)}=$$

$$=\frac{(p^2-2q)(-3p^4+14p^2q-6q^2)-p^2(p^2-3q)^2}{p(p^2-2q)}=\frac{(4q-p^2)(4p^2-10p^2q+3q^2)}{p(p^2-2q)}$$

$$*\ 27r \le 9pq - 2p^3 + 2k\sqrt{k} = 9pq - 2p^3 + \frac{2(p^2-3q)\left(p^2-\frac{3}{2}q\right)\sqrt{p^2(p^2-3q)}}{p\left(p^2-\frac{3}{2}q\right)}$$

$$\le \frac{p(9pq-2p^3)(2p^2-3q)+2(p^2-3q)\left[\left(p^2-\frac{3}{2}q\right)^2+p^2(p^2-3q)\right]}{p(2p^2-3q)} =$$

$$= \frac{(2p^2-3q)[2(9pq-2p^3)+(p^2-3q)(2p^2-3q)]+4p^2(p^2-3q)^2}{2p(2p^2-3q)}$$

$$= \frac{(2p^2-3q)(-2p^4+9p^2q+9q^2)+4p^2(p^4-6p^2q+9q^2)}{2p(2p^2-3q)} = \frac{27q^2(p^2-q)}{2p(2p^2-3q)}$$

So we have the chain: $\frac{q^2}{3p} \ge \frac{q^2(p^2-q)}{2p(2p^2-3q)} \ge r \ge \frac{(4q-p^2)(4p^2-10p^2q+3q^2)}{27p(p^2-2q)}$*, more interesting, the third inequality is stronger than Schur deg* 3 *and* 4*, since:*

$\frac{(4q-p^2)(4p^4-10p^2q+3q^2)}{27p(p^2-2q)} - \frac{p(4q-p^2)}{9} = \frac{(4q-p^2)(p^2-3q)(p^2-q)}{27p(p^2-2q)} \ge 0$ *in case* $4q \ge p^2$

$$\frac{(4q-p^2)(4p^2-10p^2q+3q^2)}{27p(p^2-2q)} - \frac{(4q-p^2)(p^2-q)}{6p} = \frac{(4q-p^2)^2(p^2-3q)}{54p(p^2-2q)} \ge 0$$

Hence we can also conclude that: $r \ge \max\left\{0; \frac{(4p-p^2)(4p^4-10p^2q+3q^2)}{27p(p^2-2q)}\right\}$

Further more, we can write inequality 9 *as:* $\frac{p^3-3pk-2k\sqrt{k}}{27} \le r \le \frac{p^3-3pk+2k\sqrt{k}}{27}$

$$\Leftrightarrow \frac{\left(p+\sqrt{k}\right)^2\left(p-2\sqrt{k}\right)}{27} \le r \le \frac{\left(p-\sqrt{k}\right)^2\left(p+2\sqrt{k}\right)}{27}$$

BIBLIOGRAPHY

1.Mihaly Bencze,Daniel Sitaru,Marian Ursărescu:"*Olympic Mathematical Energy"-Studis-Publishing House*-Iasi-2018

2. Daniel Sitaru: "*Algebraic Phenomenon*", Publishing House Paralela 45, Pitesti, 2017, ISBN 978-973-47-2523-6

3. Daniel Sitaru: *"Murray Klamkin's Duality Principle for Triangle Inequalities*",

The Pentagon Journal-Volume 75 NO 2, Spring 2016

4. Daniel Sitaru, Claudia Nănuți: "*Generating Inequalities using Schweitzer's Theorem*"-CRUX MATHEMATICORUM-Volume 42, NO1-January 2016

5. Daniel Sitaru, Claudia Nănuți: "*A "probabilistic" method for proving inequalities*", -CRUX MATHEMATICORUM-Volume 43,NO7-September 2017

6.Daniel Sitaru, Mihaly Bencze:"*699 Olympic Mathematical Challenges*"-Publishing House Studis, Iasi-2017

7.Daniel Sitaru:"*Analitycal Phenomenon*"-Publishing House Cartea Romaneasca-Pitesti-2018

8.Daniel Sitaru, George Apostolopoulos:"*The Olympic Mathematical Marathon*"-Publishing House Cartea Romaneasca-Piesti-2018

9.Daniel Sitaru-"*Contest Problems*"-Publishing House Cartea Romaneasca-Pitesti-2018

10. Mihaly Bencze,Daniel Sitaru:"*Quantum Mathematical Power*" -Publishing House Studis, Iasi-2018

11.Daniel Sitaru:"*A Class of Inequalities in triangles with Cevians*"- The Pentagon Journal-Volume 77 NO 2, Fall 2017

12.Daniel Sitaru,Marian Ursarescu:"*Calculus Marathon"-Studis-Publishing House*-Iasi-2018

13. Popescu M., Sitaru D.: *""Traian Lalescu" Constest. Geometry problems",* Lithography University of Craiova Publishing, Craiova, 1985

14. Daniel Sitaru, Claudia Nănuţi: "*National contest of applied mathematics - "Adolf Haimovici"- the county stage"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2011.

15. Daniel Sitaru, Claudia Nănuţi: "*National contest of applied mathematics - "Adolf Haimovici"- the national stage"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2011.

16. Daniel Sitaru, Claudia Nănuți: *"Contest problems",* Ecko – Print Publishing, Drobeta Turnu Severin, 2011

17. Daniel Sitaru, Claudia Nănuţi: "*Baccalaureate – Problems – Solutions –Topics -Scales"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2011

18. Daniel Sitaru: *„Affine and euclidiane geometry problems"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2012

19. Daniel Sitaru, Claudia Nănuţi, "*Baccalaureate – Problems – Tests – Topics – 2010 – 2013"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2012.

20. Daniel Sitaru, "*Hipercomplex and quaternion geometry"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2013.

21. Daniel Sitaru, Claudia Nănuţi: "*Algebra Basis"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2013.

21. Daniel Sitaru, Claudia Nănuţi: "*Mathematical Lessons"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2013.

22. Daniel Sitaru, Claudia Nănuţi: "*Basics of mathematical analysis"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2014.

23. Daniel Sitaru, Claudia Nănuți: *"Mathematics Olympics"*, Ecko – Print Publishing, Drobeta Turnu Severin, 2014

24. Do Huu Duc Thinh:"*The beauty of inequalities*" "*Romanian Mathematical Magazine*"-Interactive Journal-*www.ssmrmh.ro*

25.Daniel Sitaru,Marian Ursărescu:"Ice Math*"-Studis-Publishing House*-Iasi-2019

26. "*Romanian Mathematical Magazine*"-Interactive Journal-*www.ssmrmh.ro*