DANIEL SITARU **LUCIAN TUȚESCU**

MATHEMATICAL ELEGANCE

FROM AUTHORS

In July 2016 was founded "Romanian Mathematical Magazine" (RMM) (www.ssmrmh.ro) as an Interactive Mathematical Journal. Same date was founded "Romanian Mathematical Magazine"-Online Mathematical Journal (ISSN-2501-0099) and "Romanian Mathematical Magazine"-Paper Variant (ISSN-1584-4897). In three years the website of RMM was visited by over 5,000,000 people from all over the world. With over 8,000 proposed problems posted, over 12,000 solutions and many math articles and math notes RMM is a big chance for young mathematicians from whole world to be known as great proposers and solvers. This book is a small part of RMM-Interactive Journal including only original problems and solutions by authors above and some outstanding solutions by RMM-Team:

Ravi Prakash,Marian Ursărescu,Daniel Baciu,Andrew Okukura,Adrian Popa,Florentin Vișescu,Remus Florin Stanca,Marian Dincă,Tran Hong, Soumava Chakraborty,Ruangkhaw Chaoka,Sudhir Jha,Igor Soposki,Lazaros Zachariadis,Michael Sterghiou,Abdilkadir Altintas,Mustafa Tarek,Avishek Mitra,Mokhtar Khassani,Jalil Hajimir,Samir HajAli,Ali Jaffal,Soumitra Mandal,Khaled Abd Imouti,Naren Bhandari

PREFACE

Solving problems is an integral and inseparable part of any Mathematical learning process. The present book 'Mathematical Elegance ' is aimed to be a step in this direction. The book contains over250 carefully crafted fully solved problems from Algebra, Analysis and Geometry. However, the Problems are neither calibrated nor arranged in any order of difficulty. The problems range from simple to very difficult.Some of these problems have already appeared in the online Romanian Mathematical Magazine (RMM). The RMM team consists of more than 7000 mathematics experts, lovers and enthusiasts. Whenever a problem is proposed in RMM, several group members put up their untiring efforts to provide different solutions to the problem. More than one solution to a problem shows the intrinsic beauty of mathematics - that we can reach the same result by following different approaches.The book 'Mathematical Elegance' provides a good opportunity for Mathematical lovers to learn some of the new techniques to solve problems. How a simple substitution, use of an algebraic identity or geometric visualisation reduces a daunting problem to a simple problem are very well illustrated through solutions to the problems in the book.It is hoped that the readers will enrich their mathematical knowledge by using the book. Regarding the misprints and errors in the book, we hope there is none but the experience of last several years suggests otherwise. Whenever you come across an error or misprint in the book, you are requested to bring it to our notice.

Current Position: Retired after serving as an Associate Professor in the Department of Mathematics, Rajdhani college – University of Delhi Served in the University for 40 years Educational Qualification B.A. (Hons.) Mathematics, University of Delhi First Position in the University (Was awarded 2 Gold Medals) M.A. (Mathematics), University of Delhi First Position in the University (Was awarded 3 Gold Medals) M.Phil. (Computer Science), JNU Ph.D. (Mathematics), University of Delhi Project Udaan of CBSE for JEE (Main) Delivered several lectures in the Udaan project of CBSE. Associated with CBSE for other supports in the project. Books authored and co-authored Authored and co-authored several books published by McGraw Hill, Oxford University Press, Pearson and IGNOU Books Published by McGraw Hill Educations 1. Complete Mathematics for JEE (Main) 3. Comprehensive Mathematics for IIT (Advanced) 4. Coordinate Geometry for Engineering Entrance Examinations 5. IIT Mathematics-Topic wise Solved Questions from 1978 5. Algebra I for JEE (Main) and JEE (Advanced) 6. Algebra II and Statistics for JEE (Main) and JEE (Advanced) 7. Trigonometry for JEE (Main) and JEE (Advanced). (Forthcoming) Books Published by Oxford University Press 1. Advantage Mathematics for Class 8 Books Published by Pearson 1. Mathematics for Class 9 (Forthcoming) 2. Mathematics for Class 10 (Forthcoming) IGONU Project Associated with IGNOU with development of course material Areas of Interest: Real analysis, Complex Analysis, Linear algebra, Probability and Statistics. Research Papers and Other Publications Published several research papers in reputed international Journals.

Dr. Ravi Prakash

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PROBLEMS AND SOLUTIONS

ALGEBRA

1.1. Solve for real numbers:

$$\begin{vmatrix} \frac{1}{x+2} & \frac{1}{x+3} & \frac{1}{x+4} \\ \frac{1}{y+2} & \frac{1}{y+3} & \frac{1}{y+4} \\ \frac{1}{\sin x+2} & \frac{1}{\sin x+3} & \frac{1}{\sin x+4} \end{vmatrix} = 0$$

Solution:

We notice that it is determinant of Cauchy type:

$$D_3 = \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \frac{1}{a_1+b_3} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \frac{1}{a_2+b_3} \\ \frac{1}{a_3+b_1} & \frac{1}{a_3+b_2} & \frac{1}{a_3+b_3} \end{vmatrix} \text{ with} \begin{cases} a_1 = x \\ a_2 = y \\ a_3 = \sin x \end{cases} \text{ and} \begin{cases} b_1 = 2 \\ b_2 = 3 \\ b_3 = 4 \end{cases}$$

$$D_3 = \frac{D_2}{a_3 + b_3} \cdot \prod_{k=1}^{2} \frac{(a_3 - a_k)(b_3 - b_k)}{(a_3 + a_k)(b_3 + b_k)}$$

$$\left.\begin{matrix} D_3 = \frac{D_2}{\sin x + 4} \cdot \frac{(\sin x - x)(4-2)}{(\sin x + x)(4+2)} \cdot \frac{(\sin x - y)(4-3)}{(\sin x + y)(4+3)} \\ D_2 = \begin{vmatrix} \frac{1}{x+2} & \frac{1}{x+3} \\ \frac{1}{y+2} & \frac{1}{y+3} \end{vmatrix} = \frac{y-x}{(x+2)(x+3)(y+2)(y+3)} \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow D_3 = \frac{(y-x)(\sin x - x)\cdot 2\cdot(\sin x - y)}{6(\sin x+4)(\sin x+x)(\sin x+y)\cdot 7(x+2)(x+3)(y+2)(y+3)} = 0 \Rightarrow y = x \text{ or}$$
$$\sin x = x \text{ or } \sin x = y$$

1.2. If $a, b, c, d > 1, abcd = 8$ then:
$$\log_b(ab) \cdot \log_c(bc) \cdot \log_d(cd) \cdot \log_a(da) \geq 3(\log_2 2 + \log_b 2 + \log_c 2 + \log_d 2)$$

Solution:

$$a, b, c, d > 1; abcd = 8$$

$$\underbrace{\log_b(ab)\cdot\log_c(bc)\cdot\log_d(cd)\cdot\log_a(da)}_{A} \geq \underbrace{3(\log_a 2+\log_b 2+\log_c 2+\log_d 2)}_{B}$$

$$B = \log_a 8+\log_b 8+\log_c 8+\log_d 8$$

$$= \log_a abcd+\log_a abcd+\log_c abcd+\log_d abcd$$

$$= 1+\log_a b+\log_a c+\log_a d+\frac{1+\log_a b+\log_a c+\log_a d}{\log_a b}+$$

$$+\frac{1+\log_a b+\log_a c+\log_a d}{\log_a c}+\frac{1+\log_a b+\log_a c+\log_a d}{\log_a d}$$

We denote $\log_a b = m; \log_a c = n; \log_a d = p$

$$B = 1+m+n+p+\frac{1}{m}+1+\frac{n}{m}+\frac{p}{m}+\frac{1}{n}+\frac{m}{n}+1+\frac{p}{n}+\frac{1}{p}+\frac{m}{p}+\frac{n}{p}$$
$$+1$$

$$A = \frac{1+\log_a b}{\log_a b}\cdot\frac{\log_a b+\log_a c}{\log_a c}\cdot\frac{\log_a c+\log_a d}{\log_a d}\cdot(\log_a d+1) \Rightarrow$$

$$\Rightarrow A = \left(\frac{1}{m}+1\right)\left(\frac{m}{n}+1\right)\left(\frac{n}{p}+1\right)(p+1)$$

$$= \left(\frac{1}{n}+\frac{1}{m}+\frac{m}{n}+1\right)\left(n+\frac{n}{p}+p+1\right) =$$

$$= 1+\frac{1}{p}+\frac{p}{n}+\frac{1}{n}+\frac{n}{m}+\frac{n}{mp}+\frac{p}{m}+\frac{1}{m}+m+\frac{m}{p}+\frac{mp}{n}+\frac{m}{n}+n+\frac{n}{p}+p+1$$

Being long expressions, in order to be able to compare A and B we will write them one under the other:

$$A = 1+\frac{1}{p}+\frac{p}{n}+\frac{1}{n}+\frac{n}{m}+\frac{n}{mp}+\frac{p}{m}+\frac{1}{m}+m+\frac{m}{p}+\frac{mp}{n}+\frac{m}{n}+n+\frac{n}{p}+p+1$$

$$B = 1+\frac{1}{p}+\frac{p}{n}+\frac{1}{n}+\frac{n}{m}+\frac{p}{m}+\frac{1}{m}+m+\frac{m}{p}\quad\frac{m}{n}+n+\frac{n}{p}+p+1+1+1$$

So, we have to prove that: $\frac{n}{mp}+\frac{mp}{n} \geq 2 \Rightarrow n^2+(mp)^2 > 2mnp$

$$n^2-2mnp+(mp)^2 \geq 0, (n-mp)^2 \geq 0 \quad (A)$$

1.3. $\boldsymbol{A = a+b+c, B = \sqrt[3]{(a+b)(b+c)(c+a)}}$**,**
$\boldsymbol{C = \sqrt[3]{abc}, a, b, c > 0}$**. If** $\boldsymbol{0 < x \leq y \leq z}$ **then:**
$\boldsymbol{\frac{6Ax+9By+18Cz}{2A+3B+6C} \leq x+y+z}$**.When does the equality holds?**

Solution:

$2A \geq 3B \Leftrightarrow 2(a+b+c) \geq 3\sqrt[3]{(a+b)(b+c)(c+a)}$*, true because*

$$\sqrt[3]{(a+b)(b+c)(c+a)} \leq \frac{2(a+b+c)}{3}$$

$$3B \geq 6C \Leftrightarrow B \geq 2C \Leftrightarrow \sqrt[3]{(a+b)(b+c)(c+a)} \geq 2\sqrt[3]{abc}$$
$$\Leftrightarrow (a+b)(b+c)(c+a) \geq 8abc, \text{ true because:}$$
$$a+b \geq 2\sqrt{ab}, b+c \geq 2\sqrt{bc}, c+a \geq 2\sqrt{ac} \Rightarrow$$
$$0 < x \leq y \leq z \text{ and } 2A \geq 3B \geq 6C \text{ from Cebyshev} \Rightarrow$$
$$(x+y+z)(2A+3B+6C) \geq 3(2Ax+3By+6Cz) \Leftrightarrow$$
$$(x+y+z)(2A+3B+6C) \geq 6Ax+9By+18Cz$$
$$\text{Equality for } a=b=c \Rightarrow A=3a, B=2a, C=a$$
$$(x+y+z)\cdot 18a = 18a(x+y+z) \Leftrightarrow x+y+z = x+y+z \Rightarrow$$
$$\Rightarrow 0 < x \leq y \leq z$$

1.4. Find $x, y, z, w \in \mathbb{R}$ such that:
$$\begin{pmatrix} \sin x & \cos y \\ \tan z & \cot w \end{pmatrix}^n = \begin{pmatrix} \sin^n x & \cos^n y \\ \tan^n z & \cot^n w \end{pmatrix}, \forall n \in \mathbb{N} - \{0\}$$

Solution:

For simplicity, we will note in $x = a, \cos y = b, \tan z = c, \cot w = d$.

Thus, the condition can be written as: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix}$,

$\forall n \in \mathbb{N} \setminus \{0\}$. *For $n = 2$ we have:* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix} \Leftrightarrow$

$$\begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix} = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix} \Rightarrow bc = 0 \Rightarrow b = 0 \text{ or } c = 0.$$

I. $b = 0$. That means the only equality is $(ca + d) = c^2$

If $c = 0$ then the matrix $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ satisfies the identity in the hypothesis.

(for any diagonal matrix $\begin{pmatrix} 0^n & 0 \\ 0 & d^n \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^n$)

If $c \neq 0 \Rightarrow c = a + d$. For $n = 3$ we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^3 = \begin{pmatrix} a^3 & b^3 \\ c^3 & d^3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a & 0 \\ a+d & d \end{pmatrix}^3 = \begin{pmatrix} a^3 & 0 \\ (a+d)^3 & d^3 \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} a^3 & 0 \\ & d^3 \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} a^3 & 0 \\ a(a+d)^2 + d^2(a+d) & d^3 \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ (a+d)^3 & d^3 \end{pmatrix}$$

Thus, we have: $a(a+d)^2 + d^2(a+d) = (a+d)^3 |: (a+d) \Rightarrow$

$$\Rightarrow a^2 + ad + d^2 = a^2 + 2ad + d^2 \Rightarrow ad = 0 \Rightarrow a = 0 \text{ or } d = 0$$

If $a = b = 0$ or $d = b = 0$ then the matrices $\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ satisfy the identity in the hypothesis.

Thus, in this case the matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ *and* $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ *satisfy the identity. By applying the same algorithm we obtain the solutions:*

$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ *(duplicate),* $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$

Thus, the matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ *are the only ones which satisfy the identity above,* $a, b, c, d \in \mathbb{R}$. *Thus, the solutions for* x, y, z, w *are:*

$$I.\ x, w \in \mathbb{R}, y = k\pi + \frac{\pi}{2} \wedge z = t\pi, k, z \in \mathbb{Z}, w \neq q\pi, \forall\, q \in \mathbb{Z}$$

$$II.\ z, w \in \mathbb{R}, x = k\pi \wedge y = t\pi + \frac{\pi}{2}, k, t \in \mathbb{Z}, w \neq q\pi \text{ and }$$
$$z \neq p\pi + \frac{\pi}{2}, \forall q, p \in \mathbb{Z}$$

$$III.\ x, z \in \mathbb{R}, y = t\pi + \frac{\pi}{2}, w = k\pi + \frac{\pi}{2}, k, t \in \mathbb{Z}, z \neq p\pi + \frac{\pi}{2}, \forall p \in \mathbb{Z}$$

$$IV.\ x, y \in \mathbb{R}, z = t\pi, w = k\pi + \frac{\pi}{2}, t, k \in \mathbb{Z}$$

$$V.\ y, w \in \mathbb{R}, x = k\pi, z = t\pi, t, k \in \mathbb{Z}, w \neq q\pi, \forall q \in \mathbb{Z}$$

1.5. If $x, y, z > 0$ then:

$$\frac{\sqrt[3]{xz^2}}{y} + \frac{\sqrt[3]{yx^2}}{z} + \frac{\sqrt[3]{zy^2}}{x} \leq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

Solution:

$$\sqrt[3]{xz^2} \overset{AM-GM}{\leq} \frac{x + z + z}{3} = \frac{x}{3} + \frac{2z}{3}$$

$$\sqrt[3]{yx^2} \overset{AM-GM}{\leq} \frac{y}{3} + \frac{2x}{3};\ \sqrt[3]{zy^2} \overset{AM-GM}{\leq} \frac{z}{3} + \frac{2y}{3}$$

$$\rightarrow LHS \leq \frac{1}{y}\left(\frac{x}{3} + \frac{2z}{x}\right) + \frac{1}{z}\left(\frac{y}{3} + \frac{2x}{3}\right) + \frac{1}{x}\left(\frac{z}{x} + \frac{2y}{3}\right) =$$

$$= \frac{x}{3y} + \frac{y}{3z} + \frac{z}{3x} + \frac{2y}{3x} + \frac{2z}{3y} + \frac{2x}{3z}$$

Suppose: $0 < x \leq y \leq z$. *We must show that:*

$$\frac{x}{3y} + \frac{y}{3z} + \frac{z}{3x} + \frac{2y}{3x} + \frac{2z}{3y} + \frac{2x}{3z} \leq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

$$\leftrightarrow \frac{2y}{3x} + \frac{2z}{3y} + \frac{2x}{3z} \leq \frac{2x}{3y} + \frac{2y}{3z} + \frac{2z}{3x} \leftrightarrow$$

$$\leftrightarrow zy^2 + xz^2 + yx^2 \leq yz^2 + zx^2 + xy^2$$

$\leftrightarrow (z - y)(y - x)(z - x) \geq 0$. *It is true because:* $0 < x \leq y \leq z$

$$Proved.\ Equality \leftrightarrow x = y = z.$$

1.6. If $x \in \mathbb{R}$ then:

$$\sqrt{4x^2+3}+\sqrt{x^2+x+1}+\sqrt{x^2-x+1} < 3\sqrt{(4x^2+3)(x^4+x^2+1)}$$

Solution:

$$For\ x \in \mathbb{R}, x^4+x^2+1 \geq 1 \Rightarrow \frac{1}{\sqrt{x^4+x^2+1}} \leq 1 \quad (1)$$

$$Also,\ (4x^2+3)(x^2-x+1) = (4x^2+3)\left[\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right] \geq \frac{9}{4}$$

$$\Rightarrow \frac{1}{\sqrt{(4x^2+3)(x^2-x+1)}} < \frac{2}{3}, \forall x \in \mathbb{R} \quad (2)$$

$$Next,\ (4x^2+3)(x^2+x+1) = (4x^2+3)\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right] > 3\left(\frac{3}{4}\right) = \frac{9}{4}$$

$$\Rightarrow \frac{1}{\sqrt{(4x^2+3)(x^2+x+1)}} < \frac{2}{3}; \forall x \in \mathbb{R} \quad (2)$$

From (1), (2) and (3) we get:

$$\frac{1}{\sqrt{x^4+x^2+1}}+\frac{1}{\sqrt{(4x^2+3)(x^2-x+1)}}+\frac{1}{\sqrt{(4x^2+3)(x^2+x+1)}} < 1+\frac{4}{3} < 3$$

$$\Rightarrow \sqrt{4x^2+3}+\sqrt{x^2+x+1}+\sqrt{x^2-x+1} <$$

$$< 3\sqrt{4x^2+3}\sqrt{x^4+x^2+1}$$

$$[\because x^4+x^2+1 = (x^2-x+1)(x^2+x+1)]$$

1.7. If $a, b \in [0,1]; a \leq b$ then:

$$4\sqrt{ab} \leq a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \leq 2(a+b)$$

Solution:

$$Let\ be\ f:[0,1] \to \mathbb{R}; f(x) = a\left(\frac{b}{a}\right)^x+b\left(\frac{a}{b}\right)^x$$

$$f'(x) = a\left(\frac{b}{a}\right)^x \ln\frac{b}{a}+b\left(\frac{a}{b}\right)^x \ln\frac{a}{b} = \ln\frac{b}{a}\left[a\left(\frac{b}{a}\right)^x-b\left(\frac{a}{b}\right)^x\right]$$

$$f'(x) = 0 \Rightarrow a\left(\frac{b}{a}\right)^x = b\left(\frac{a}{b}\right)^x \Rightarrow a\left(\frac{b}{a}\right)^{2x} = b$$

$$\left(\frac{b}{a}\right)^{2x} = \left(\frac{b}{a}\right)^1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$$

x	$0 \qquad\qquad\qquad\qquad \frac{1}{2} \qquad\qquad\qquad\qquad 1$
$f'(x)$	$- - - - - - - - - - - - 0 + + + + + + + +$
$f(x)$	$a+b \searrow 2\sqrt{ab} \nearrow a+b$

$$\Rightarrow 2\sqrt{ab} \le a\left(\frac{b}{a}\right)^x + b\left(\frac{a}{b}\right)^x \le a+b \quad (1)$$

For $x = \sqrt{ab} \in [a,b] \subseteq [0,1]$ *in (1):*

$$2\sqrt{ab} \le a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \le a+b \quad (2)$$

For $x = \frac{a+b}{2} \in [a,b] \subseteq [0,1]$ *in (1):*

$$2\sqrt{ab} \le a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \le a+b \quad (3)$$

By adding (2); (3):

$$4\sqrt{ab} \le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{b}{a}\right)^{\frac{a+b}{2}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \left(\frac{a}{b}\right)^{\frac{a+b}{2}}\right) \le 2(a+b)$$

$$4\sqrt{ab} \le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)$$
$$\le 2(a+b)$$

1.8. If $x, y, z \in C, x+y+z = 3+4i$ then:

$$|x-z| + |y-x| + |z-y| + 5 \ge 3\sqrt[3]{|xyz|}$$

Solution:

Note: $|x|$*: module of* $x \in \mathbb{C}$

$$x+y+z = 3+4i \to |x+y+z| = |3+4i| = \sqrt{3^2+4^2} = 5$$
$$|x-z| + |y-x| + |z-y| \ge 0 \to$$
$$\to LHS = |x-z| + |y-x| + |z-y| + 5$$
$$= |x-z| + |y-x| + |z-y| + |x+y+z| \overset{(1)}{\ge} |x| + |y| + |z|$$

We prove that (1) is true with for all $x, y, z \in \mathbb{C}$

In fact, by Hlawka's Inequality for complex a, b, c*:*

$$|a+b| + |b+c| + |c+a| \le |a| + |b| + |c| + |a+b+c| \ (*)$$

In (*) we choose: $x = a+b; y = b+c; z = c+a$

$$\to x+y+z = 2(a+b+c) \to a+b+c = \frac{x+y+z}{2} \to$$

$$\to a = \frac{x+z-y}{2}; b = \frac{x+y-z}{2}; c = \frac{y+z-x}{2}$$

$$(*) \leftrightarrow |x| + |y| + |z| \le \left|\frac{x+z-y}{2}\right| + \left|\frac{x+y-z}{2}\right| + \left|\frac{y+z-x}{2}\right| + \left|\frac{x+y+z}{2}\right|$$

$$\le \left|\frac{x-y}{2}\right| + \left|\frac{z}{2}\right| + \left|\frac{y-z}{2}\right| + \left|\frac{x}{2}\right| + \left|\frac{x-z}{2}\right| + \left|\frac{y}{2}\right| + \left|\frac{x+y+z}{2}\right| =$$

$$\frac{|x|+|y|+|z|}{2} + \left|\frac{x-y}{2}\right| + \left|\frac{y-z}{2}\right| + \left|\frac{x-z}{2}\right| + \left|\frac{x+y+z}{2}\right|$$

$$\leftrightarrow \frac{|x|+|y|+|z|}{2} \le \frac{|x-z|+|y-x|+|z-y|}{2} + \frac{|x+y+z|}{2}$$

$\leftrightarrow |x-z| + |y-x| + |z-y| + |x+y+z| \ge |x| + |y| + |z| \to$ *(1) is true. Because:* $|x|; |y|; |z| \ge 0$*. Using AM-GM we have:*

$$|x| + |y| + |z| \ge 3\sqrt[3]{|x| \cdot |y| \cdot |z|} = 3\sqrt[3]{|xyz|} = RHS. \text{ Proved.}$$

1.9. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} > 0$ then:

$$\frac{(\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2)^9}{2(\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca})(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})^7} \le \frac{\boldsymbol{a}^{10}}{\boldsymbol{b} + \boldsymbol{c}} + \frac{\boldsymbol{b}^{10}}{\boldsymbol{c} + \boldsymbol{a}} + \frac{\boldsymbol{c}^{10}}{\boldsymbol{a} + \boldsymbol{b}}$$

Solution:

Using Hölder's inequality:

$$(u^3 + v^3 + w^3)(x^3 + y^3 + z^3)(m^3 + n^3 + p^3) \ge (uxm + vyn + wzp)^3 \quad (*)$$

(with: $u, v, w, x, y, z, m, n, p > 0$*)*

Now, choose: $u^3 = x^3 = a; v^3 = y^3 = b, w^3 = z^3 = c; m^3 = a^4; n^3 = b^4; p^3 = c^4$

Then: $(*) \leftrightarrow (a+b+c)^2(a^4 + b^4 + c^4) \ge (a^2 + b^2 + c^2)^3 \quad (1)$

Lastly, choose: $u^3 = \frac{a^{10}}{b+c}; v^3 = \frac{b^{10}}{a+c}; w^3 = \frac{c^{10}}{a+b}; x^3 = a(b+c); y^3 = (b+c); z^3 = c(a+b); m^3 = a; n^3 = b; p^3 = c$

$$\text{Then: } (*) \leftrightarrow \left(\sum \frac{a^{10}}{b+c}\right)\left(\sum a(b+c)\right)\left(\sum a\right) \ge$$

$$\ge (a^4 + b^3 + c^4)^3 \overset{(1)}{\ge} \frac{(a^2 + b^2 + c^2)^9}{(a+b+c)^6}$$

$$\leftrightarrow \left(\sum \frac{a^{10}}{b+c}\right)\left(2\sum a\right)\left(\sum a\right)^7 \ge (a^2 + b^2 + c^2)^9.$$

Proved. Equality $\leftrightarrow a = b = c$*.*

1.10. $A=\begin{pmatrix}\sin^2 a & \cos^2 a\cdot\sin^2 b & \cos^2 a\cdot\cos^2 b\\ \cos^2 b\cdot\sin^2 c & \sin^2 b & \cos^2 b\cdot\cos^2 c\\ \cos^2 c\cdot\sin^2 a & \cos^2 c\cdot\cos^2 a & \sin^2 c\end{pmatrix}$

$A^{2019}=\begin{pmatrix}x_1 & x_2 & x_3\\ x_4 & x_5 & x_6\\ x_7 & x_8 & x_9\end{pmatrix}$**. If $a,b,c\in\mathbb{R}$ then find: $\Omega=\sum_{i=1}^{9}x_i$**

Solution:

$$Let\ x=\begin{pmatrix}1\\1\\1\end{pmatrix}, AX=\begin{pmatrix}\sin^2 a+\cos^2 a\sin^2 b+\cos^2 a\cos^2 b\\ \cos^2 b\sin^2 c+\sin^2 b+\cos^2 b\cos^2 c\\ \cos^2 c\sin^2 a+\cos^2 c\sin^2 a+\sin^2 c\end{pmatrix}=$$

$$=\begin{pmatrix}1\\1\\1\end{pmatrix}=x$$

Thus $A^2(x)=A(AX)=AX=x$*. Continuing in this way, we get*

$$A^{2019}X=X\Rightarrow\begin{pmatrix}x_1 & x_2 & x_3\\ x_4 & x_5 & x_6\\ x_7 & x_8 & x_9\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\begin{pmatrix}x_1+x_2+x_3\\ x_4+x_5+x_6\\ x_7+x_8+x_9\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\therefore\sum_{k=1}^{9}x_k=3$$

1.11. If $a,b,c>0,\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=6$ then:

$$\frac{3a}{4a^2+2a+1}+\frac{3b}{4b^2+2b+1}+\frac{3c}{4c^2+2c+1}\le a+b+c$$

Solution:

$$Let\ f(x)=\frac{3x}{4x^2+2x+1}, 0<x<\frac{1}{6}$$

$$f'(x)=\frac{3-12x^2}{(4x^2+2x+1)^2}; f''(x)=\frac{12(8x^3-6x-1)}{(4x^2+2x+1)^3}<-1<0,$$

$0<x<\frac{1}{6}$*. Using Jensen's Inequality with* $0<a,b,c<\frac{1}{6}$*:*

$$f(a)+f(b)+f(c)\le 3\cdot f\left(\frac{a+b+c}{3}\right)=\frac{9t}{4t^2+2t+1}$$

With: $t=\frac{a+b+c}{3},\ 6=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{9}{a+b+c}$

Then: $a+b+c\ge\frac{3}{2}$*. So,* $t\ge\frac{1}{2}$

We must show that: $\frac{9t}{4t^2+2t+1}\le 3t, 3\le 4t^2+2t+1$

$$2t^2+t-1\ge 0,(2t-1)(t+1)\ge 0\ \ \textit{(true with } t\ge \tfrac{1}{2}\textit{) Proved.}$$

1.12. If $\boldsymbol{a,b,c\ge 1,a,b,c\in\mathbb{N}}$ then:

$$\boldsymbol{a\left(\binom{2b}{b}+\binom{2c}{c}\right)+b\left(\binom{2c}{c}+\binom{2a}{a}\right)+c\left(\binom{2a}{a}+\binom{2b}{b}\right)\ge 2\left(4^{\sqrt{ab}}+4^{\sqrt{bc}}+4^{\sqrt{ca}}\right)}$$

Solution:

$$a\binom{2b}{b}+b\binom{2a}{a}\overset{AM-GM}{\ge}2\sqrt{a^b\binom{2b}{b}\cdot\binom{2a}{a}}=$$

$$=2\sqrt{ab\cdot\sum_{k=0}^{a}\binom{a}{k}^2\cdot\sum_{k=0}^{b}\binom{b}{k}^2}\overset{CBS}{\ge}2\sqrt{ab\cdot\frac{\left(\sum_{k=0}^{a}\binom{a}{k}\right)^2}{a}\cdot\frac{\left(\sum_{k=0}^{b}\binom{b}{k}\right)^2}{b}}=$$

$$=2\cdot\sum_{k=0}^{a}\binom{a}{k}\cdot\sum_{k=0}^{b}\binom{b}{k}=2\cdot 2^a\cdot 2^b=2\cdot 2^{a+b}\ge$$

$$\overset{AM-GM}{\ge}2\cdot 2^{2\sqrt{ab}}=2\cdot 4^{\sqrt{ab}}$$

$$a\binom{2b}{b}+b\binom{2a}{a}\ge 2\cdot 4^{\sqrt{ab}}\quad (1)$$

$$b\binom{2c}{c}+c\binom{2b}{b}\ge 2\cdot 4^{\sqrt{bc}}\quad (2)$$

$$c\binom{2a}{a}+a\binom{2c}{c}\ge 2\cdot 4^{\sqrt{ca}}\quad (3)$$

By adding (1); (2); (3):

$$a\left(\binom{2b}{b}+\binom{2c}{c}\right)+b\left(\binom{2c}{c}+\binom{2a}{a}\right)+c\left(\binom{2a}{a}+\binom{2b}{b}\right)\ge$$

$$\ge 2\left(4^{\sqrt{ab}}+4^{\sqrt{bc}}+4^{\sqrt{ca}}\right)$$

Equality holds for $a=b=c=1$.

1.13. Solve for real numbers:

$$\boldsymbol{\begin{cases}\sin x=\cos y\\ \begin{vmatrix}\sin(x+y) & \sin(y+\sqrt{xy}) & \sin(\sqrt{xy}+x)\\ \cos(x+y) & \cos(y+\sqrt{xy}) & \cos(\sqrt{xy}+x)\\ \cos(x-y) & \cos(y-\sqrt{xy}) & \cos(\sqrt{xy}-x)\end{vmatrix}=0\end{cases}}$$

Solution:

$$\Delta \to \begin{vmatrix} \sin(x+y) & \sin\left(y+\sqrt{xy}\right) & \sin\left(\sqrt{xy}+x\right) \\ \cos(x+y) & \cos\left(y+\sqrt{xy}\right) & \cos\left(\sqrt{xy}+x\right) \\ \cos(x-y) & \cos\left(y-\sqrt{xy}\right) & \cos\left(\sqrt{xy}-x\right) \end{vmatrix} = 0$$

$$\sin x = \cos y \Rightarrow \sin x = \sin\left(\frac{\pi}{2} - y\right)$$

$$\sin(x+y) = \sin x \cos y + \sin y \cos x = \cos^2 y + \sin^2 y = 1 \quad \textit{(1)}$$

$$\therefore \sin^2 x = \cos^2 y$$

$$1 - \cos^2 x = \cos^2 y \Rightarrow \cos^2 x = 1 - \cos^2 y \Rightarrow \cos x = \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y = \sin y \cos y - \cos y \sin y = 0$$

(2)

$$\cos(x-y) = \cos x \cos y + \sin x \sin y = \sin y \cos y + \cos y \sin y = 2 \sin y \cos y = \sin 2y \quad \textit{(3)}$$

$$\Delta = \begin{vmatrix} 1 & \sin\left(y+\sqrt{xy}\right) & \sin\left(\sqrt{xy}+x\right) \\ 0 & \cos\left(y+\sqrt{xy}\right) & \cos\left(\sqrt{xy}+x\right) \\ \sin 2y & \cos\left(y-\sqrt{xy}\right) & \cos\left(\sqrt{xy}-x\right) \end{vmatrix} = 0$$

We develop after the first column:

$$\cos\left(y+\sqrt{xy}\right) \cdot \cos\left(\sqrt{xy}-x\right) - \cos\left(\sqrt{xy}+x\right) \cdot \cos\left(y-\sqrt{xy}\right) +$$
$$+ \sin 2y \left(\sin\left(y+\sqrt{xy}\right) \cdot \cos\left(\sqrt{xy}+x\right) - \sin\left(\sqrt{xy}+x\right) \cdot \cos\left(y+\sqrt{xy}\right)\right) = 0 \Rightarrow$$

$$\Rightarrow \cos\left(y+\sqrt{xy}\right) \cdot \cos\left(\sqrt{xy} - \frac{\pi}{2} + y\right) - \cos\left(\sqrt{xy} + \frac{\pi}{2} - y\right) \cdot \cos\left(y-\sqrt{xy}\right)$$
$$+ \sin 2y \left(\sin\left(y+\sqrt{xy}\right) \cdot \cos\left(\sqrt{xy} + \frac{\pi}{2} - y\right)\right.$$
$$\left. - \sin\left(\sqrt{xy} + \frac{\pi}{2} - y\right) \cdot \cos\left(y+\sqrt{xy}\right)\right) = 0$$

$$\cos\left(y+\sqrt{xy}\right) \cdot \sin\left(y+\sqrt{xy}\right) - \sin\left(y-\sqrt{xy}\right) \cdot \cos\left(y-\sqrt{xy}\right) +$$
$$+ \sin 2y \left(\sin\left(y+\sqrt{xy}\right) \cdot \sin\left(y-\sqrt{xy}\right) - \cos\left(y-\sqrt{xy}\right) \cdot \cos\left(y+\sqrt{xy}\right)\right) = 0 \Rightarrow$$

$$\Rightarrow \frac{\sin 2\left(y+\sqrt{xy}\right)}{2} - \frac{\sin 2\left(y-\sqrt{xy}\right)}{2} - \sin 2y\,(\cos 2y) = 0 \Rightarrow$$

$$\Rightarrow \frac{2 \sin 2\sqrt{xy} \cos 2y}{2} - \sin 2y \cos 2y = 0$$

$$\cos 2y \left(\sin 2\sqrt{xy} - \sin 2y\right) = 0$$

Case I: $\cos 2y = 0 \Rightarrow 2y = \pm\frac{\pi}{2} + 2k\pi \Rightarrow \begin{cases} y = \pm\frac{\pi}{4} + k\pi \Rightarrow \\ x = \frac{\pi}{2} - y = \frac{\pi}{2} \mp \frac{\pi}{4} - k\pi \end{cases}; k \in \mathbb{Z}$

Case II: $\sin 2\sqrt{xy} - \sin 2y = 0 \Rightarrow \sin 2\sqrt{xy} = \sin 2y \Rightarrow$

$$\Rightarrow \sin 2\sqrt{xy} = \sin 2\sqrt{y \cdot y} \Rightarrow$$

$$\Rightarrow \begin{cases} y = 0 \\ x = \frac{\pi}{2} + 2k\pi \end{cases}; k \in \mathbb{N}, \textit{because } x \neq y; \begin{matrix} \sin x > 0 \\ y > 0 \end{matrix}$$

1.14. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} > 0, a + b + c = 3$ then:

$$\boldsymbol{a(a+c)e^{\frac{1}{a}} + b(b+a)e^{\frac{1}{b}} + c(c+b)e^{\frac{1}{c}} \geq 6e}$$

Solution:

$$\textit{Let } f(x) = xe^{\frac{1}{x}}, x > 0 \to f'(x) = \frac{\sqrt[x]{e}(x-1)}{x} \to f'(x) = 0 \overset{x>0}{\leftrightarrow} x = 1$$

x	0		1		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
$f(x)$	‖	$\searrow$	e	$\nearrow$	

$$\textit{So, } f(x) \geq f(1) = e, \forall x > 0 \to (a+c)f(a) + (b+a)f(b) +$$
$$+(c+b)f(c) \geq (a+c)f(1) + (b+a)f(1) + (c+b)f(1)$$
$$= (a+c+b+a+c+b)f(1) = 2(a+b+c)f(1) = 2 \cdot 3 \cdot e = 6e$$

1.15. If $\boldsymbol{0} < y < x < 2\boldsymbol{y}$ then:

$$\boldsymbol{x + y < 3y^2\sqrt{\frac{2}{4y^2 - x^2}}}$$

$$\boldsymbol{x(x+y) > 3(x-y)\sqrt{4y^2 - x^2}}$$

Solution:

Because: $0 < x < 2y \to x + y < 3y; 4y^2 - x^2 > 0$. *We need to prove:*

$$3y < 3y^2\sqrt{\frac{2}{4y^2 - x^2}} \leftrightarrow 1 < 3y\sqrt{\frac{2}{4y^2 - x^2}} \leftrightarrow (4y^2 - x^2) < 2 \cdot 9y^2$$

$$\leftrightarrow 14y^2 + x^2 > 0 \ \textit{(True).Hence: } x + y < 3y^2\sqrt{\frac{2}{4y^2-x^2}} \ \textit{(*)}$$

$$3(x-y)\sqrt{4y^2 - x^2} < x(x+y)$$

$$\overset{0<y<x}{\leftrightarrow} 9(x-y)^2(4y^2 - x^2) < \left(x(x+y)\right)^2$$

$$\leftrightarrow 5x^4 + 36xy^3 > 8yx^3 + 13x^2y^2 + 18y^4$$
$$\leftrightarrow 5u^4 + 36u > 8u^3 + 13u^2 + 18, \left(u = \frac{x}{y}; 1 < u < 2\right)$$
$$\leftrightarrow 5u^4 - 8u^3 - 13u^2 + 36u - 18 > 0$$
$$Let\ f(u) = 5u^4 - 8u^3 - 13u^2 + 36u - 18, (1 < u < 2)$$
$$\to f'(u) = 20u^3 - 24u^2 - 26u + 36 \to f''(u) =$$
$$= 60u^2 - 48u - 26 = 0 \to \left[\begin{array}{l} u = \dfrac{2 - \sqrt{\dfrac{89}{6}}}{5} \notin (1; 2) \\ u = \dfrac{2 + \sqrt{\dfrac{89}{6}}}{5} \in (1; 2) \end{array}\right.$$
$$\to f'(u) \geq f'\left(\frac{2 + \sqrt{\frac{89}{6}}}{5}\right) = \frac{2}{225}\left(2592 - 89\sqrt{534}\right) > 0$$
$$\to f \uparrow (1; 2) \to f(u) > f(1) = 2 > 0, 5u^4 - 8u^3 - 13u^2 + 36u - 18 > 0$$

1.16. $\boldsymbol{x * y = x\sqrt{1 + y^2} + y\sqrt{1 + x^2}, x \circ y = xy - 5x - 5y + 30}$, $\boldsymbol{G = (5, \infty)}$**. Prove that** $(\mathbb{R}, *) \cong (\boldsymbol{G}, \circ)$ **as abelian groups.**

Solution:

We first show that $(\mathbb{R}, *)$*, where* $x * y = x\sqrt{1 + y^2} + y\sqrt{1 + x^2}$
is an abelian group. Clearly, $x * y \in \mathbb{R}, \forall x, y \in \mathbb{R}$
$*$ *is associative suppose* $x, y, z \in \mathbb{R}$*. Let* $x = \tan\alpha, y = \tan\beta, z = \tan\gamma$
$$-\frac{\pi}{2} < \alpha, \beta, \gamma < \frac{\pi}{2}$$
$$x * y = (\tan\alpha)\sqrt{1 + \tan^2\beta} + \tan\beta\sqrt{1 + \tan^2\alpha} = \frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}$$
$$(x * y) * z = \frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\sqrt{1 + \tan^2\gamma} + \tan\alpha\sqrt{1 + \left(\frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\right)^2}$$
$$But\ 1 + \left(\frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\right)^2 = \frac{(1 - \sin^2\alpha)(1 - \sin^2\beta) + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta}{\cos^2\alpha\cos^2\beta} \underset{\oplus}{=}$$
$$= \frac{(1 + \sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta}. \textit{ Thus, } (x * y) * z = \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta\cos\gamma} + \frac{\sin\gamma(1 + \sin\alpha\sin\beta)}{\cos\alpha\cos\beta\cos\gamma} =$$
$$= \frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$$

Similarly, $x * (y * z) = \frac{\sin\alpha+\sin\beta+\sin\gamma+\sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$

Thus, $(x * y) * z = x * (y * z); \forall x, y, z \in \mathbb{R}$

$*$ *is commutative is obvious.*

Identity Element $= 0$, $x * 0 = x\sqrt{1+0^2} + 0\sqrt{1+x^2} = x; \forall x \in \mathbb{R}$

Inverse Element,For each $x \in \mathbb{R}, -x \in \mathbb{R}$ *is inverse of* x.

Indeed $x * (-x) = 0$

$\therefore$ $(\mathbb{R},*)$ *is an abelian group. Next, we show that if* $G = (5, \infty)$*, and* $a \circ b = ab - 5a - 5b + 30; \forall a, b \in \mathbb{G}$*, then* $(\mathbb{G}, 0)$ *is an abelian group.*

Note $a \circ b = (a-5)(b-5) + 5$

0 *is commutative and its identity element is* 6.

0 *is associative.Let* $a, b, c \in \mathbb{G}$,

$(a \circ b) \circ c = \big((a-5)(b-5)+5\big) \circ c =$

$= \big((a-5)(b-5)+5-5\big)(c-5)+5 = (a-5)(b-5)(c-5)+5$

Similarly, $a \circ (b \circ c) = (a-5)(b-5)(c-5)+5$

$\therefore (a \circ b) \circ c = a \circ (b \circ c); \forall a, b, c \in \mathbb{G}$

Finally, if $a \in 5$*, then* $a > 5$*, and* $b = 5 + \frac{1}{a-5}$ *is inverse of a. Indeed,*

$a \circ b = (a-5)(b-5)+5 = (a-5)\left(\frac{1}{a-5}\right) + 5 = 1 + 5 = 6 =$ *identity element.*

We now show that $\Phi: \mathbb{R} \to \mathbb{G}$ *defined by* $\Phi(x) = 5 + 5^{\sinh^{-1} x}$ *is the required isomorphism of* $\mathbb{R}$ *onto* $\mathbb{G}$

As $5^{\sinh^{-1} x} > 0, \forall x \in \mathbb{R}, \Phi(x) \in \mathbb{G}; \forall x \in \mathbb{R}$

For $x, y \in \mathbb{R}, (x * y) = 5^{\sinh^{-1}\left(x\sqrt{1+y^2}+y\sqrt{1+x^2}\right)} + 5$ *(1)*

and $\Phi(x) \circ \Phi(y) = 5^{\sinh^{-1} x} \cdot 5^{\sinh^{-1} y} + 5$ *(2)*

$= 5^{\sinh^{-1} x + \sinh^{-1} y} + 5$

But $\sinh^{-1} x + \sinh^{-1} y = \sinh^{-1}\left(x\sqrt{1+y^2} + y\sqrt{1+x^2}\right)$ *(3)*

$\therefore$ *from (1), (2), (3):* $\Phi(x * y) = \Phi(x) \circ \Phi(y)$

Thus, Φ *is a homomorphism from* $(\mathbb{R},*)$ *to* $(\mathbb{G}, a)$

Φ *is one – to – one .Let* $x, y \in \mathbb{R}$ *and* $\Phi(x) = \Phi(y)$

$\Rightarrow 5^{\sinh^{-1} x} + 5 = 5^{\sinh^{-1} y} + 5 \Rightarrow \sinh^{-1} x = \sinh^{-1} y \Rightarrow x = y$

$\therefore$ Φ *is one – to – one ,*Φ *is onto*

Let $y \in \mathbb{G} \Rightarrow y > 5 \Rightarrow y - 5 > 0$*.Let* $t = \log_5(y-5) \Rightarrow 5^t = y - 5$

As $t \in \mathbb{R}, \exists x \in \mathbb{R}$ *such that* $\sinh^{-1} x = t$ *or take* $x = \sinh t$.

Then $\Phi(x) = 5^{\sinh^{-1} x} + 5 = 5^t + 5 = y - 5 + 5 = y$.$\therefore$ Φ *is onto.*

Hence, $(\mathbb{R},*) \cong (\mathbb{G}, 0)$ *as abelian groups.*

1.17. Find $A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $t \in \mathbb{R}$**, such that:**

$$\boldsymbol{A^4 - 4A^3 + 6A^2 - 4A + I_2 = O_2}$$

Solution:

$$A^n = \begin{pmatrix} \cos nt & -\sin nt \\ \sin nt & \cos nt \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} \cos 4t & \sin 4t \\ \sin 4t & \cos 4t \end{pmatrix} - 4\begin{pmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{pmatrix} + 6\begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} -$$

$$-4\begin{pmatrix} \cos t & -\sin t \\ \sin 5 & \cos t \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} I & \cos 4t - 4\cos 3t + 6\cos 2t - 4\cos t + 1 = 0 \\ II & \sin 4t - 4\sin 3t + 6\sin 2t - 4\sin t + 0 = 0 \end{matrix}\right\}$$

$$I + iII: \cos 4t + i\sin 4t - 4(\cos 3t + i\sin 3t) + 6(\cos 2t + i\sin 2t) -$$

$$-4(\cos t + i\sin t) + 1 = 0 \Rightarrow$$

$$\Rightarrow (\cos t + i\sin t)^4 - 4(\cos t + i\sin t)^3 + 6(\cos t + i\sin t)^2$$

$$- 4(\cos t + i\sin t) + 1 = 0$$

$$\cos t + i\sin t = z \Rightarrow z^4 - 4z^3 + 6z^2 - 4z + 1 = 0$$

$$(z-1)^4 = 0 \Rightarrow z = 1$$

$$\cos t + i\sin t = 1 \Rightarrow \left.\begin{matrix} \cos t = 1 \\ \sin t = 0 \end{matrix}\right\} \Rightarrow t = 2k\pi; k \in \mathbb{Z}$$

$$A = \begin{pmatrix} \cos 2k\pi & -\sin 2k\pi \\ \sin 2k\pi & \cos 2k\pi \end{pmatrix}$$

1.18. If $\boldsymbol{a, b, c, d > 0, a + b + c + d = 1}$ then:

$$\boldsymbol{\sqrt[4]{\left(\frac{1}{a^2} - a^2\right)\left(\frac{1}{b^2} - b^2\right)\left(\frac{1}{c^2} - c^2\right)\left(\frac{1}{d^2} - d^2\right)} \geq \frac{255}{16}}$$

Solution:

$$\prod\left(\frac{1}{a^2} - a^2\right) = \prod\left(\frac{1-a^4}{a^2}\right) = \prod\left(\frac{(1-a)(1+a)(1+a^2)}{a^2}\right)$$

$$= \prod\left(\frac{(b+c+d)(2a+b+c+d)(1+a^2)}{a^2}\right) (\because 1 = a+b+c+d)$$

$$\Rightarrow LHS \overset{(1)}{=} \sqrt[4]{\frac{\{\prod(b+c+d)\,(2a+b+c+d)\}\{\prod(1+a^2)\}}{(abcd)^2}}$$

$$\textit{Now, } \prod(b+c+d) = (b+c+d)(c+d+a)(d+a+b)(a+b+c)$$

$$\overset{A-G}{\geq} \left(3\sqrt[3]{bcd}\right)\left(3\sqrt[3]{cda}\right)\left(3\sqrt[3]{dab}\right)\left(3\sqrt[3]{abc}\right) = 3^4(abcd)$$

$$\Rightarrow \prod(b+c+d) \overset{(i)}{\geq} 3^4(abcd). \ Again, \prod(2a+b+c+d) =$$
$$= (2a+b+c+d)(2b+c+d+a)(2c+d+a+b)(2d+a+b+c)$$
$$\overset{A-G}{\geq} \left(5\sqrt[5]{a^2bcd}\right)\left(5\sqrt[5]{b^2cda}\right)\left(5\sqrt[5]{c^2dab}\right)\left(5\sqrt[5]{d^2abc}\right) = 5abcd$$
$$\Rightarrow \prod(2a+b+c+d) \overset{(ii)}{\geq} 5^4(abcd)$$
$$(i)\cdot(ii) \Rightarrow \prod\{(b+c+d)(2a+b+c+d)\} \overset{(iii)}{\geq} 15^4(abcd)^2$$
$$(iii), (1) \Rightarrow LHS \geq \sqrt[4]{\frac{15^4(abcd)^2\{\prod(1+a^2)\}}{(abcd)^2}} = 15\sqrt[4]{\prod(1+a^2)} \overset{?}{\geq} \frac{255}{16}$$
$$\Rightarrow \frac{1}{4}\ln\left\{\prod(1+a^2)\right\} \overset{?}{\geq} \ln\frac{17}{16} \Leftrightarrow \sum \ln(1+a^2) \underset{(2)}{\overset{?}{\geq}} 4\ln\frac{17}{16}$$
$$Obviously, \because a,b,c,d > 0 \mid \sum a = 1 \therefore 0 < a,b,c,d < 1$$
$$Let\ f(x) = \ln(1+x^2)\ \forall x \in (0,1). \ Then,\ f''(x) = \frac{2(1-x^2)}{(1+x^2)^2} > 0$$
$$\Rightarrow f(x)\ is\ convex \therefore \sum \ln(1+a^2) \overset{Jensen}{\geq} 4\ln\left(1+\left(\frac{\sum a}{4}\right)^2\right) =$$
$$= 4\ln\left(1+\frac{1}{16}\right) = 4\ln\frac{17}{16} \Rightarrow (2)\ is\ true \Rightarrow given\ inequality\ is\ true$$

1.19. If $A, B, C \in M_n(\mathbb{C}); n \in \mathbb{N}; n \geq 2; 4A + B = 2AB;$ $9B + C = 3BC; 16C + A = 4CA$, then $ABC = CBA$.

Solution:

$$4A + B = 2AB \Rightarrow 2A + \frac{1}{2}B = AB \ \ (1)$$
$$(I_n - 2A)\left(I_n - \frac{1}{2}B\right) = I_n - 2A - \frac{1}{2}B + AB =$$
$$= I_n - \left(2A + \frac{1}{2}B\right) + AB \overset{(1)}{=} I_n - AB + AB = I_n$$
$$(I_n - 2A)\left(I_n - \frac{1}{2}B\right) = I_n \Rightarrow (I_n - 2A)^{-1} = I_n - \frac{1}{2}B \ \ \ (2)$$
$$I_n = (I_n - 2A)^{-1} \cdot (I_n - 2A) \overset{(2)}{=} \left(I_n - \frac{1}{2}B\right)(I_n - 2A) =$$
$$= I_n - 2A - \frac{1}{2}B + BA = I_n - \left(2A + \frac{1}{2}B\right) + BA \overset{(1)}{=} I_n - AB + BA$$
$$I_n = I_n - AB + BA \Rightarrow O_n = -AB + BA \Rightarrow AB = BA \ \ (3)$$
$$9B + C = 3BC \Rightarrow 3B + \frac{1}{3}C = BC \ \ \ (4)$$

$$(I_n-3B)\left(I_n-\frac{1}{3}C\right)=I_n-3B-\frac{1}{3}C+BC=$$
$$=I_n-\left(3B+\frac{1}{3}C\right)+BC\overset{(4)}{=}I_n-BC+BC=I_n$$
$$(I_n-3B)\left(I_n-\tfrac{1}{3}C\right)=I_n\Rightarrow(I_n-3B)^{-1}=I_n-\tfrac{1}{3}C\quad(5)$$
$$I_n=(I_n-3B)^{-1}\cdot(I_n-3B)\overset{(5)}{=}\left(I_n-\frac{1}{3}C\right)(I_n-3B)=$$
$$=I_n-3B-\frac{1}{3}C+CB=I_n-\left(3B+\frac{1}{3}C\right)+CB\overset{(4)}{=}I_n-BC+CB$$
$$I_n=I_n-BC+CB\Rightarrow O_n=-BC+CB\Rightarrow BC=CB\quad(6)$$
$$16C+A=4CA\Rightarrow 4C+\tfrac{1}{4}A=CA\quad(7)$$
$$(I_n-4C)\left(I_n-\frac{1}{4}A\right)=I_n-\frac{1}{4}A-4C+CA=$$
$$=I_n-\left(4C+\frac{1}{4}A\right)+CA\overset{(7)}{=}I_n-CA+CA=I_n$$
$$(I_n-4C)\left(I_n-\frac{1}{4}A\right)=I_n-4C-\frac{1}{4}A+CA=$$
$$=I_n-\left(4C+\frac{1}{4}A\right)+CA\overset{(7)}{=}I_n-CA+CA=I_n$$
$$(I_n-4C)\left(I_n-\tfrac{1}{4}A\right)=I_n\Rightarrow(I_n-4C)^{-1}=I_n-\tfrac{1}{4}A\quad(8)$$
$$I_n=(I_n-4C)^{-1}(I_n-4C)\overset{(8)}{=}\left(I_n-\frac{1}{4}A\right)(I_n-4C)=$$
$$=I_n-4C-\frac{1}{4}A+AC=I_n-\left(4C+\frac{1}{4}A\right)+AC=I_n-CA+AC$$
$$I_n=I_n-CA+AC\Rightarrow O_n=-CA+AC\Rightarrow AC=CA\quad(9)$$
$$ABC=A(BC)\overset{(6)}{=}A(CB)=(AC)B\overset{(9)}{=}(CA)B=C(AB)\overset{(3)}{=}C(BA)=CBA$$

1.20. If $x,y,z,t\in(0,1)$; $3\sqrt{3}(xyz+yzt+ztx+txy)=4$ then:

$$\frac{yzt}{x(1-x^2)}+\frac{ztx}{y(1-y^2)}+\frac{txy}{z(1-z^2)}+\frac{xyz}{t(1-t^2)}\geq 2$$

Solution:

$$2x^2\cdot(1-x^2)(1-x^2)\overset{AM-GM}{\leq}\left(\frac{2x^2+1-x^2+1-x^2}{3}\right)^3$$
$$2x^2(1-x^2)^2\leq\frac{8}{27}\Rightarrow x^2(1-x^2)^2\leq\frac{4}{3\sqrt{3}}$$

$$x(1-x^2) \le \frac{2}{3\sqrt{3}} \Rightarrow \frac{1}{x(1-x^2)} \ge \frac{3\sqrt{3}}{2}$$

$$\frac{yzt}{x(1-x^2)} \ge \frac{\sqrt{3}}{2} yzt \quad \textit{(1),Analogous:}$$

$$\frac{ztx}{y(1-y^2)} \ge \frac{3\sqrt{3}}{2} ztx \quad (2), \frac{txy}{z(1-z^2)} \ge \frac{3\sqrt{3}}{2} txy \quad (3), \frac{xyz}{t(1-t^2)} \ge \frac{3\sqrt{3}}{2} xyz \quad (4)$$

By adding (1); (2); (3); (4):

$$\frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} \ge$$

$$\ge \frac{3\sqrt{3}}{2}(yzt + ztx + txy + xyz) = \frac{3\sqrt{3}}{2} \cdot \frac{4}{3\sqrt{3}} = 2$$

Equality holds for $x = y = z = t = \frac{1}{\sqrt{3}}$.

1.21. If $a, b, c, x_1, x_2, x_3 > 0$ then:

$$(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})^5(\boldsymbol{a}\boldsymbol{x}_1^7 + \boldsymbol{b}\boldsymbol{x}_2^7 + \boldsymbol{c}\boldsymbol{x}_3^7) \ge (\boldsymbol{a}\boldsymbol{x}_1 + \boldsymbol{b}\boldsymbol{x}_2 + \boldsymbol{c}\boldsymbol{x}_3)^5(\boldsymbol{a}\boldsymbol{x}_1^2 + \boldsymbol{b}\boldsymbol{x}_2^2 + \boldsymbol{c}\boldsymbol{x}_3^2)$$

and prove that:

$$(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})^5 \left(\frac{\boldsymbol{a}}{\boldsymbol{b}^7} + \frac{\boldsymbol{b}}{\boldsymbol{c}^7} + \frac{\boldsymbol{c}}{\boldsymbol{a}^7}\right) \ge \left(\frac{\boldsymbol{a}}{\boldsymbol{b}} + \frac{\boldsymbol{b}}{\boldsymbol{c}} + \frac{\boldsymbol{c}}{\boldsymbol{a}}\right)^5 \left(\frac{\boldsymbol{a}}{\boldsymbol{b}^2} + \frac{\boldsymbol{b}}{\boldsymbol{c}^2} + \frac{\boldsymbol{c}}{\boldsymbol{a}^2}\right)$$

Solution:

$$ax_1^7 + bx_2^7 + cx_3^7 = \frac{a(x_1^2)^6}{x_1^5} + \frac{b(x_2^2)^6}{x_2^5} + \frac{c(x_2^2)^6}{x_3^5} =$$

$$= \frac{(ax_1^2)^6}{(ax_1)^5} + \frac{(bx_2^2)^6}{(bx_2)^5} + \frac{(cx_3^2)^6}{(cx_3)^5} \overset{RADON}{\ge} \frac{(ax_1^2 + bx_2^2 + cx_3^2)^6}{(ax_1 + bx_2 + cx_3)^5} \quad (1)$$

$$ax_1^2 + bx_2^2 + cx_3^2 = \frac{(ax_1)^2}{a} + \frac{(bx_2)^2}{b} + \frac{(cx_3)^2}{c} \overset{RADON}{\ge} \frac{(ax_1 + bx_2 + cx_3)^2}{a + b + c}$$

$$a + b + c \ge \frac{(ax_1 + bx_2 + cx_3)^2}{ax_1^2 + bx_2^2 + cx_3^2}, (a + b + c)^5 \ge \frac{(ax_1 + bx_2 + cx_3)^{10}}{(ax_1^2 + bx_2^2 + cx_3^2)^5} \quad (2)$$

By multiplying (1); (2):

$$(a + b + c)^5(ax_1^7 + bx_2^7 + cx_3^7) \ge$$

$$\ge \frac{(ax_1^2 + bx_2^2 + cx_3^2)^6}{(ax_1 + bx_2 + cx_3)^5} \cdot \frac{(ax_1 + bx_2 + cx_3)^{10}}{(ax_1^2 + bx_2^2 + cx_3^2)^5}$$

$$(a + b + c)^5(ax_1^7 + bx_2^7 + cx_3^7)$$

$$\ge (ax_1 + bx_2 + cx_3)^5(ax_1^2 + bx_2^2 + cx_3^2)$$

For $x_1 = \frac{1}{b}; x_2 = \frac{1}{c}; x_3 = \frac{1}{a}$

$$(a + b + c)^5 \left(\frac{a}{b^7} + \frac{b}{c^7} + \frac{c}{a^7}\right) \ge \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^5 \left(\frac{a}{b^2} + \frac{b}{c^2} + \frac{c}{a^2}\right)$$

1.22. Find $x, y, z > 0$ such that: $(1+x)^4(1+y)^3(1+z)^2 = 256xyz$

Solution:

$$\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{z}{(1+x)(1+y)(1+z)}+\frac{1}{(1+x)(1+y)(1+z)}=$$

$$=\frac{1+x-1}{1+x}+\frac{1+y-1}{(1+x)(1+y)}+\frac{1+z-1}{(1+x)(1+y)(1+z)}$$

$$+\frac{1}{(1+x)(1+y)(1+z)}=$$

$$=\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{1+x}-\frac{1}{(1+x)(1+y)}\right)+$$

$$+\left(\frac{1}{(1+x)(1+y)}-\frac{1}{(1+x)(1+y)(1+z)}\right)+$$

$$+\frac{1}{(1+x)(1+y)(1+z)}=1$$

$$1=\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{z}{(1+x)(1+y)(1+z)}$$

$$+\frac{1}{(1+x)(1+y)(1+z)}\geq$$

$$\overset{AM-GM}{\geq} 4\sqrt[4]{\frac{x}{1+x}\cdot\frac{y}{(1+x)(1+y)}\cdot\frac{z}{(1+x)(1+y)(1+z)}\cdot\frac{1}{(1+x)(1+y)(1+z)}}$$

$$=4\sqrt[4]{\frac{xyz}{(1+x)^4(1+y)^3(1+z)^2}}$$

$$\frac{1}{4^4}\geq\frac{xyz}{(1+x)^4(1+y)^3(1+z)^2},(1+x)^4(1+y)^3(1+z)^2\geq 256xyz$$

Equality holds in AM-GM if:

$$\frac{x}{1+x}=\frac{y}{(1+x)(1+y)}=\frac{z}{(1+x)(1+y)(1+z)}=\frac{1}{(1+x)(1+y)(1+z)}$$

$$x=\frac{y}{1+y};y=\frac{z}{1+z};z=1\Rightarrow y=\frac{1}{2};x=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}$$

Solution: $x=\frac{1}{3};y=\frac{1}{2};z=1$

1.23. If $a, b, c \geq 0$ then:

$$\left(\frac{ab}{1+a}+\frac{bc+a}{(1+a)(1+b)}\right)^6\leq\frac{ab^6}{1+a}+\frac{bc^6+a^6}{(1+a)(1+b)}$$

Solution:

$$\frac{a}{1+a}+\frac{b}{(1+a)(1+b)}+\frac{1}{(1+a)(1+b)}=$$

$$=\frac{1+a-1}{1+a}+\frac{1+b-1}{(1+a)(1+b)}+\frac{1}{(1+a)(1+b)}=$$

$$=\left(1-\frac{1}{1+a}\right)+\left(\frac{1}{1+a}-\frac{1}{(1+a)(1+b)}\right)+\frac{1}{(1+a)(1+b)}=1$$

Let be $f:[0,\infty)\to\mathbb{R}; f(x)=x^6$

$$f'(x)=6x^5; f''(x)=30x^4\geq 0$$

By Jensen's inequality for $\lambda_1,\lambda_2,\lambda_3>0$*; then:* $\lambda_1+\lambda_2+\lambda_3=1$

$$f(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)\leq\lambda_1f(x_1)+\lambda_2f(x_2)+\lambda_3f(x_3)$$

For $\lambda_1=\frac{a}{1+a};\ \lambda_2=\frac{b}{(1+a)(1+b)};\ \lambda_3=\frac{1}{(1+a)(1+b)}$

$$f\left(\frac{a}{1+a}x_1+\frac{b}{(1+a)(1+b)}x_2+\frac{1}{(1+a)(1+b)}x_3\right)\leq$$

$$\leq\frac{a}{1+a}f(x_1)+\frac{b}{(1+a)(1+b)}f(x_2)+\frac{1}{(1+a)(1+b)}f(x_3)$$

$$\left(\frac{a}{1+a}x_1+\frac{b}{(1+a)(1+b)}x_2+\frac{1}{(1+a)(1+b)}x_3\right)^6\leq$$

$$\leq\frac{a}{1+a}x_1^6+\frac{b}{(1+a)(1+b)}x_2^6+\frac{1}{(1+a)(1+b)}x_3^6$$

For $x_1=b; x_2=c; x_3=a$

$$\left(\frac{ab}{1+a}+\frac{bc}{(1+a)(1+b)}+\frac{a}{(1+a)(1+b)}\right)^6\leq\frac{ab^6}{1+a}+\frac{ba^6+c^6}{(1+a)(1+b)}$$

$$\left(\frac{ab}{1+a}+\frac{bc+a}{(1+a)(1+b)}\right)^6\leq\frac{ab^6}{1+a}+\frac{ba^6+c^6}{(1+a)(1+b)}$$

1.24. Find $x,y>0$ such that: $(1+x)^3(1+y)^2=27xy$

Solution:

$$\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{1}{(1+x)(1+y)}=$$

$$=\frac{1+x-1}{1+x}+\frac{1+y-1}{(1+x)(1+y)}+\frac{1}{(1+x)(1+y)}=$$

$$=\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{1+x}-\frac{1}{(1+x)(1+y)}\right)+\frac{1}{(1+x)(1+y)}=1$$

$$1=\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{1}{(1+x)(1+y)}\geq$$

$$\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{x}{1+x}\cdot\frac{y}{(1+x)(1+y)}\cdot\frac{1}{(1+x)(1+y)}}=$$

$$= 3\sqrt[3]{\frac{xy}{(1+x)^3(1+y)^2}}, \qquad 1 \geq 3^3 \cdot \frac{xy}{(1+x)^3(1+y)^2}$$

$(1+x)^3(1+y)^2 \geq 27xy$. *Equality holds in AM-GM if*

$$\frac{x}{1+x} = \frac{y}{(1+x)(1+y)} = \frac{1}{(1+x)(1+y)} \Rightarrow y = 1$$

$$\frac{x}{1+x} = \frac{1}{(1+x)\cdot 2} \Rightarrow x = \frac{1}{2}$$

1.25. Find $x, y, z, t > 0$ such that:

$$(\mathbf{1}+\boldsymbol{x})^5(\mathbf{1}+\boldsymbol{y})^4(\mathbf{1}+\boldsymbol{z})^3(\mathbf{1}+\boldsymbol{t})^2 = \mathbf{3125}\boldsymbol{xyzt}$$

Solution:

$$\frac{x}{1+x} + \frac{y}{(1+x)(1+y)} + \frac{z}{(1+x)(1+y)(1+z)} + \frac{t}{(1+x)(1+y)(1+z)(1+t)} +$$

$$+\frac{1}{(1+x)(1+y)(1+z)(1+t)} =$$

$$= \frac{1+x-1}{1+x} + \frac{1+y-1}{(1+x)(1+y)} + \frac{1+z-1}{(1+x)(1+y)(1+z)}$$

$$+\frac{1+t-1}{(1+x)(1+y)(1+z)(1+t)} -$$

$$+\frac{1}{(1+x)(1+y)(1+z)(1+t)} =$$

$$= \left(1 - \frac{1}{1+x}\right) + \left(\frac{1}{1+x} - \frac{1}{(1+x)(1+y)}\right)$$

$$+\left(\frac{1}{(1+x)(1+y)} - \frac{1}{(1+x)(1+y)(1+z)}\right) +$$

$$+\left(\frac{1}{(1+x)(1+y)(1+z)} - \frac{1}{(1+x)(1+y)(1+z)(1+t)}\right) +$$

$$+\frac{1}{(1+x)(1+y)(1+z)(1+t)} = 1$$

$$1 = \frac{x}{1+x} + \frac{y}{(1+x)(1+y)} + \frac{z}{(1+x)(1+y)(1+z)}$$

$$+\frac{t}{(1+x)(1+y)(1+z)(1+t)} +$$

$$+\frac{1}{(1+x)(1+y)(1+z)(1+t)} \geq$$

$$\overset{AM-GM}{\geq} 5 \cdot \sqrt[5]{\frac{x}{1+x} \cdot \frac{y}{(1+x)(1+y)} \cdot \frac{z}{(1+x)(1+y)(1+z)} \cdot \frac{t}{(1+x)(1+y)(1+z)(1+t)} \cdot \frac{1}{(1+x)(1+y)(1+z)(1+t)}} =$$

$$= 5 \cdot \sqrt[5]{\frac{xyzt}{(1+x)^5(1+y)^4(1+z)^3(1+t)^2}}$$

$$1\geq 5^5\cdot\frac{xyzt}{(1+x)^5(1+y)^4(1+z)^3(1+t)^2}$$
$$(1+x)^5(1+y)^4(1+z)^3(1+t)^2\geq 3125xyzt$$
Equality in AM-GM holds if:
$$\frac{x}{1+x}=\frac{y}{(1+x)(1+y)}=\frac{z}{(1+x)(1+y)(1+z)}=\frac{t}{(1+x)(1+y)(1+z)(1+t)}=$$
$$=\frac{1}{(1+x)(1+y)(1+z)(1+t)}$$
$$t=1;\frac{z}{(1+x)(1+y)(1+z)}=\frac{1}{(1+x)(1+y)(1+z)\cdot 2}\Rightarrow z=\frac{1}{2}$$
$$\frac{y}{(1+x)(1+y)}=\frac{\frac{1}{2}}{(1+x)(1+y)\left(1+\frac{1}{2}\right)}\Rightarrow y=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}$$
$$\frac{x}{1+x}=\frac{\frac{1}{3}}{(1+x)\left(1+\frac{1}{3}\right)}\Rightarrow x=\frac{\frac{1}{3}}{1+\frac{1}{3}}=\frac{\frac{1}{3}}{\frac{3+1}{3}}=\frac{\frac{1}{3}}{\frac{4}{3}}=\frac{1}{4}$$
Solution: $x=\frac{1}{4};y=\frac{1}{3};z=\frac{1}{2};t=1$

1.26. Let be $G=\left\{a+b\sqrt[3]{5}+c\sqrt[3]{25}\,|\,a,b,c\in\mathbb{Q}\right\}$. Prove that if $x\in G$ then, $x^{2019}\in G$.

Solution:

Let be $x=a+b\sqrt[3]{5}+c\sqrt[3]{25};a,b\in\mathbb{Q},y=d+e\sqrt[3]{5}+f\sqrt[3]{25};d,e,f\in\mathbb{Q}$
$$xy=\left(a+b\sqrt[3]{5}+c\sqrt[3]{25}\right)\left(d+e\sqrt[3]{5}+f\sqrt[3]{25}\right)=$$
$$=ad+ae\sqrt[3]{5}+af\sqrt[3]{25}+bd\sqrt[3]{5}++be\sqrt[3]{25}+5bf+dc\sqrt[3]{25}+5ce$$
$$+5fc\sqrt[3]{5}=$$
$$=ad+5bf+5ce+(ae+bd+5fc)\sqrt[3]{5}++(af+be+dc)\sqrt[3]{25}\in G$$
because: $ad+5bf+5ce;ae+bd+5fc;af+be+dc\in\mathbb{Q}$
$$x,y\in G\Rightarrow xy\in G\quad (1)$$
$$x\in G\overset{(1)}{\Rightarrow}x\cdot x\in G\overset{(2)}{\Rightarrow}x^2\cdot x\in G\Rightarrow x^n\in G;n\in\mathbb{N}^*$$
Inductively. For $n=2019\Rightarrow x^{2019}\in G.$

1.27. If $a,b,c\in\mathbb{N};ab+bc+ca=27$ then:

$$135+\sqrt[a+b]{a^{2a}\cdot b^{2b}}+\sqrt[b+c]{b^{2b}\cdot c^{2c}}+\sqrt[c+a]{c^{2c}\cdot a^{2a}}\leq 2(a+b+c)^2$$

Solution:

$$\sqrt[a+b]{a^{2a}\cdot b^{2b}}=\sqrt[a+b]{(a^2)^a\cdot(b^2)^b}\overset{AM-GM}{\le}\frac{\underbrace{a^2+a^2+\cdots+a^2}_{for\ "a"\ times}+\underbrace{b^2+b^2+\cdots+b^2}_{for\ "b"\ times}}{a+b}=$$

$$=\frac{a\cdot a^2+b\cdot b^2}{a+b}=\frac{a^3+b^3}{a+b}=\frac{(a+b)(a^2-ab+b^2)}{a+b}=a^2-ab+b^2$$

$$\sum_{cyc}\sqrt[a+b]{a^{2a}\cdot b^{2b}}\le\sum_{cyc}(a^2-ab+b^2)=$$

$$=2\sum_{cyc}a^2-\sum_{cyc}ab=2(a+b+c)^2-4\sum_{cyc}ab-\sum_{cyc}ab=$$

$$=2(a+b+c)^2-5\sum_{cyc}ab=2(a+b+c)^2-135$$

$$135+\sum_{cyc}\sqrt[a+b]{a^{2a}\cdot b^{2b}}\le 2(a+b+c)^2$$

Equality holds for $a=b=c=3$.

1.28. If $x>0$ then:

$$\{x\}+\left\{x+\frac{1}{2}\right\}+\left\{2x+\frac{1}{2}\right\}>2\sqrt{\{4x\}};$$

$\{x\}=x-[x]$; $[*]$ **- great integer function**

Solution:

$$\{2x\}-\{x\}-\left\{x+\frac{1}{2}\right\}=$$

$$=2x-[2x]-(x-[x])-\left(x+\frac{1}{2}-\left[x+\frac{1}{2}\right]\right)=$$

$$=2x-[2x]-x+[x]-x-\frac{1}{2}+\left[x+\frac{1}{2}\right]=$$

$$=[x]+\left[x+\frac{1}{2}\right]-[2x]-\frac{1}{2}\overset{HERMITE}{=}[2x]-[2x]-\frac{1}{2}=-\frac{1}{2}$$

$$\{2x\}-\{x\}-\left\{x+\tfrac{1}{2}\right\}=-\tfrac{1}{2}\quad (1)$$

Replacing x with $2x$ *in (1):* $\{4x\}-\{2x\}-\left\{2x+\frac{1}{2}\right\}=-\frac{1}{2}$ *(2)*

By adding (1); (2): $\{4x\}-\{x\}-\left\{x+\frac{1}{2}\right\}-\left\{2x+\frac{1}{2}\right\}=-1$

$$\{x\}+\left\{x+\frac{1}{2}\right\}+\left\{2x+\frac{1}{2}\right\}=1+\{4x\}\overset{AM-GM}{>}2\sqrt{1\cdot\{4x\}}=2\sqrt{\{4x\}}$$

1.29. If $\boldsymbol{a,b,c > 1, abc = e^3}$ then:

$$\left(\log\sqrt{ab}\right)^{\log\sqrt{ab}}\cdot\left(\log\sqrt{bc}\right)^{\log\sqrt{bc}}\cdot\left(\log\sqrt{ca}\right)^{\log\sqrt{ca}}\geq \log a\cdot\log b\cdot\log c$$

Solution:

$$\underbrace{\left(\ln\sqrt{ab}\right)^{\ln\sqrt{ab}}\cdot\left(\ln\sqrt{bc}\right)^{\ln\sqrt{bc}}\cdot\left(\ln\sqrt{ac}\right)^{\ln\sqrt{ac}}}_{A} > \underbrace{\ln a\ln b\ln c}_{B}$$

$$\sqrt[\sum\ln\sqrt{ab}]{\left(\ln\sqrt{ab}\right)^{\ln\sqrt{ab}}\cdot\left(\ln\sqrt{bc}\right)^{\ln\sqrt{bc}}\cdot\left(\ln\sqrt{ac}\right)^{\ln\sqrt{ac}}}\overset{MG\geq MH}{\geq}$$

$$\geq\frac{\ln\sqrt{ab}+\ln\sqrt{ac}+\ln\sqrt{bc}}{\frac{\ln\sqrt{ab}}{\ln\sqrt{ab}}+\frac{\ln\sqrt{ac}}{\ln\sqrt{ac}}+\frac{\ln\sqrt{bc}}{\ln\sqrt{bc}}}=\frac{\ln\sqrt{a^2b^2c^2}}{3}=\frac{\ln abc}{3}=\frac{\ln e^3}{3}=\frac{3\ln e}{3}=1$$

$$\ln a\ln b\ln c\overset{MA<MG}{<}\left(\frac{\ln a+\ln b+\ln c}{3}\right)^3=\left(\frac{\ln abc}{3}\right)^3=\left(\frac{3}{3}\right)^3$$

$$\text{So, }\left.\begin{matrix}A>1\\B<1\end{matrix}\right\}\Rightarrow A>B$$

1.30. If $\boldsymbol{a,b,c,d} > 0, ac = bd$ then:

$$\frac{(a+b)(b+c)(c+d)(d+a)}{ab(c+d)+cd(a+b)}\geq 4\sqrt{ac}$$

Solution:

$$ab(c+d)+cd(a+b)=abc+abd+cda+cdb\overset{ac=bd}{=}abc+aac+$$
$$+cda+cac=ac(b+a+d+c)=ac(a+b+c+d)$$

$$(a+b)(b+c)(c+d)(d+a)=(ab+b^2+bc+ac)(ca+cd+d^2+da)\overset{ac=bd}{=}$$
$$=(ab+b^2+bc+bd)(bd+cd+d^2+da)=b(a+b+c+d)\cdot d(b+c+d+a)=$$
$$=bd(a+b+c+d)^2\overset{ac=bd}{=}ac(a+b+c+d)^2$$

$$\rightarrow\frac{(a+b)(b+c)(c+d)(d+a)}{ab(c+d)+cd(a+b)}=\frac{ac(a+b+c+d)^2}{ac(a+b+c+d)}$$

$$=a+b+c+d\overset{AM-GM}{\geq}4\sqrt[4]{abcd}\overset{ac=bd}{=}4\sqrt[4]{(ac)^2}=4\sqrt{ac}$$

Proved. Equality if and only if $a=b=c=d$

1.31. $\Delta_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2^3 & 3^3 & \cdots & n^3 \\ 1 & 2^5 & 3^5 & \cdots & n^5 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-1} & 3^{2n-1} & \cdots & n^{2n-1} \end{vmatrix}, n \geq 2$

Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{\Delta_{n+1}}{n^{2n+1} \cdot \Delta_n} \right)$$

Solution:

First, we will prove $\Delta_n = 1! \cdot 3! \cdot \ldots \cdot (2n-1)!$ (1) *by mathematical induction. For* $P(2)$: $\Delta_2 = \begin{vmatrix} 1 & 2 \\ 1 & 2^3 \end{vmatrix} = 6$ *true.*

Let $P(n)$ *true and we will prove* $P(n+1)$

$$\text{Let } f(x) = \begin{vmatrix} 1 & 2 & \cdots & n & x \\ 1^3 & 2^3 & \cdots & n^3 & x^3 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1^{2n+1} & 2^{2n+1} & \cdots & n^{2n+1} & x^{2n+1} \end{vmatrix}$$

$f \in \mathbb{R}[x]$ *and grad* $f = 2n+1$ *and have* $0, \pm 1, \pm 2, \ldots, \pm n$ *roots* $\Rightarrow$

$$f(x) = a_{2n+1}(x-x_1)(x-x_2)\ldots(x-x_{2n+1}) \Rightarrow$$

$$f(x) = 1!\,3! \cdots (2n-1)!\, x(x^2-1)(x^2-2^2)\cdots(x^2-n^2) \Rightarrow$$

$$\Delta_{n+1} = 1!\,3! \cdots (2n-1)!\,(2n+1)! \Rightarrow P(n+1) \textit{ true.}$$

From (1)$\Rightarrow \Omega = \lim_{n\to\infty} \frac{\Delta_{n+1}}{n^{2n+1}\Delta_n} = \lim_{n\to\infty} \frac{(2n+1)!}{n^{2n+1}}$ *(2)*

Let $x_n = \frac{(2n+1)!}{n^{2n+1}}$, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(2n+3)!}{(n+1)^{2n+3}} \cdot \frac{n^{2n+1}}{(2n+1)!} =$

$$= \lim_{n\to\infty} \frac{(2n+3)(2n+2)}{(n+1)^2} \cdot \frac{n^{2n+1}}{(n+1)^{2n+1}} =$$

$$= \lim_{n\to\infty} 4 \cdot \left(\frac{n}{n+1}\right)^{2n+1} = 4 \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^n\right]^{\frac{2n+1}{4}} = \frac{4}{e^2} < 1 \quad \textit{(3)}$$

From (2)+(3)$\Rightarrow \Omega = 0.$

1.32. $X, Y \in M_2(\mathbb{R}), X^{19} + X^{17} = Y^{21} + Y^{19} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{Tr\,(X^{n+1})}{Tr\,(Y^{n+2})}}$$

Solution:

$$Z = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{matrix} XZ = XZ \\ YZ = ZY \end{matrix}$$

$$XZ=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}a-b & -a+b\\ c-d & -c+d\end{pmatrix}$$
$$ZX=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a-c & b-d\\ -a+c & -b+d\end{pmatrix}$$
$$\Rightarrow \begin{matrix}a-b=+a-c\Rightarrow b=c\\ -a+b=b-d\Rightarrow a=d\\ c-d=-a+c\Rightarrow a=d\\ -c+d=-b+d\Rightarrow c=b\end{matrix}$$
$$X=\begin{pmatrix}a & b\\ b & a\end{pmatrix};Y=\begin{pmatrix}m & n\\ n & m\end{pmatrix}$$
$$We\ know\ that\text{: }X^n=\begin{pmatrix}\frac{(a+b)^n+(a-b)^n}{2} & \frac{(a+b)^n-(a-b)^n}{2}\\ \frac{(a+b)^n-(a-b)^n}{2} & \frac{(a+b)^n+(a-b)^n}{2}\end{pmatrix}$$
$$X^{19}+X^{17}=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\Rightarrow$$
$$\Rightarrow \frac{(a+b)^{19}+(a-b)^{19}}{2}+\frac{(a+b)^{17}+(a-b)^{17}}{2}=1$$
$$\frac{(a+b)^{19}-(a-b)^{19}}{2}+\frac{(a+b)^{17}-(a-b)^{17}}{2}=-1$$
$$\begin{cases}(a+b)^{19}+(a+b)^{17}=0\\ (a-b)^{19}+(a-b)^{17}=2\end{cases}\begin{cases}(a+b)^{17}((a+b)^2+1)=0\\ (a-b)^{17}((a-b)^2+1)=2\end{cases}\Rightarrow b=-a$$
$$(2a)^{17}((2a)^2+1)2 \quad t^{19}+t^{17}-2=0$$
$$t^{19}-1+t^{17}-1=0$$
$$(t-1)(t^{18}+t^{17}+\cdots+t+1)+(t-1)(t^{16}+t^{15}+\cdots+t+1)=0$$
$$t-1=0 \quad t=1\Rightarrow 2a=1\,;a=\frac{1}{2};b=-\frac{1}{2}$$
$$\underbrace{t^{18}+t^{17}+\cdots+t+1}_{>0}+\underbrace{t^{16}+t^{15}+\cdots+t+1}_{>0\ \ im\mathbb{R}}=0$$
$$So,\ x=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}$$
$$Analog\ Y=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}\Rightarrow X^{n+1}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}\ Tr\,X^{n+1}=1$$
$$Y^{n+2}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}\ Tr\,Y^{n+2}=1$$
$$\Rightarrow \sqrt[n^2]{\frac{Tr\,(X^{n+1})}{Tr\,(Y^{n+2})}}=\sqrt[n^2]{\frac{1}{1}}=1\ constant\ sequence$$

1.33. If $a, b, c, x, y, z > 0; a + b + c = 3$ then:

$$\boldsymbol{a^a \cdot b^b \cdot c^c (x + y + z)^3 \geq 27x^a y^b z^c}$$

When does the equality holds?

Solution:

By weighted GM-HM:

$$\sqrt[a+b+c]{\left(\frac{a}{x}\right)^a \cdot \left(\frac{b}{y}\right)^b \cdot \left(\frac{c}{z}\right)^c} \geq \frac{a+b+c}{\left(\frac{a}{\frac{a}{x}}\right) + \left(\frac{b}{\frac{b}{y}}\right) + \left(\frac{c}{\frac{c}{z}}\right)} = \frac{a+b+c}{x+y+z}$$

$$\left(\frac{a}{x}\right)^a \cdot \left(\frac{b}{y}\right)^b \cdot \left(\frac{c}{z}\right)^c \geq \left(\frac{a+b+c}{x+y+z}\right)^{a+b+c} = \left(\frac{3}{x+y+z}\right)^3$$

$$\frac{a^a}{x^a} \cdot \frac{b^b}{y^b} \cdot \frac{c^c}{z^c} \geq \frac{27}{(x+y+z)^3}$$

$$a^a \cdot b^b \cdot c^c (x+y+z)^3 \geq 27x^a y^b z^c$$

Equality holds for $a = b = c = 1$ and $x = y = z$.

1.34. If $a, b, c, x, y, z > 0; a + b + c \geq x + y + z$ then:

$$\boldsymbol{a^a b^b c^c \geq x^a y^b z^c}$$

Solution:

By weighted GM-HM:

$$\sqrt[a+b+c]{\left(\frac{a}{x}\right)^a \cdot \left(\frac{b}{y}\right)^b \cdot \left(\frac{c}{z}\right)^c} \geq \frac{a+b+c}{\left(\frac{a}{\frac{a}{x}}\right) + \left(\frac{b}{\frac{b}{y}}\right) + \left(\frac{c}{\frac{c}{z}}\right)} = \frac{a+b+c}{x+y+z} \geq$$

$$\geq \frac{x+y+z}{x+y+z} \geq 1, \sqrt[a+b+c]{\left(\frac{a}{x}\right)^a \cdot \left(\frac{b}{y}\right)^b \cdot \left(\frac{c}{z}\right)^c} \geq 1$$

$$\left(\frac{a}{x}\right)^a \cdot \left(\frac{b}{y}\right)^b \cdot \left(\frac{c}{z}\right)^c \geq 1, a^a \cdot b^b \cdot c^c \geq x^a \cdot y^b \cdot z^c$$

Equality holds for $x = a; y = b; z = c$.

1.35. If $a, b, c, d, x, y, z, t > 0$ then:

$$\frac{(\boldsymbol{ax})^{\boldsymbol{a}} \cdot (\boldsymbol{by})^{\boldsymbol{b}} \cdot (\boldsymbol{cz})^{\boldsymbol{c}} \cdot (\boldsymbol{dt})^{\boldsymbol{d}}}{(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})^{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}}} \geq \left(\frac{\boldsymbol{xyzt}}{\boldsymbol{xyz}+\boldsymbol{xyt}+\boldsymbol{xzt}+\boldsymbol{yzt}}\right)^{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}}$$

Solution:

If $a, b, c, d, u, w, s > 0$ by weighted GM-HM:

$$\sqrt[a+b+c]{\left(\frac{a}{u}\right)^a \cdot \left(\frac{b}{v}\right)^b \cdot \left(\frac{c}{w}\right)^c \cdot \left(\frac{d}{s}\right)^d} \geq \frac{a+b+c+d}{\frac{a}{\frac{a}{u}}+\frac{b}{\frac{b}{v}}+\frac{c}{\frac{c}{w}}+\frac{d}{\frac{d}{s}}} = \frac{a+b+c+d}{u+v+w+s}$$

$$\left(\frac{a}{u}\right)^a \cdot \left(\frac{b}{v}\right)^b \cdot \left(\frac{c}{w}\right)^c \cdot \left(\frac{d}{s}\right)^d \geq \left(\frac{a+b+c+d}{u+v+w+s}\right)^{a+b+c+d}$$

Let be $u = \frac{1}{x}; v = \frac{1}{y}; w = \frac{1}{z}; s = \frac{1}{t}$

$$(ax)^a \cdot (by)^b \cdot (cz)^c \cdot (dt)^d \geq \frac{(a+b+c+d)^{a+b+c+d}}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)^{a+b+c+d}}$$

$$\frac{(ax)^a \cdot (by)^b \cdot (cz)^c \cdot (dt)^d}{(a+b+c+d)^{a+b+c+d}} \geq \left(\frac{1}{\frac{xyz+xyt+xzt+yzt}{xyzt}}\right)^{a+b+c+d} =$$

$$= \left(\frac{xyzt}{xyz+xyt+xzt+yzt}\right)^{a+b+c+d}$$

1.36. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} > 0; a+b+c=3$ then:

$$(\boldsymbol{a}+\boldsymbol{c})\boldsymbol{e}^{\frac{1}{\boldsymbol{a}}} + \boldsymbol{b}(\boldsymbol{b}+\boldsymbol{a})\boldsymbol{e}^{\frac{1}{\boldsymbol{b}}} + \boldsymbol{c}(\boldsymbol{c}+\boldsymbol{b})\boldsymbol{e}^{\frac{1}{\boldsymbol{c}}} \geq 9\boldsymbol{e}$$

Solution:

Let be $f: (0, \infty) \to \mathbb{R}; f(x) = e^x; f'(x) = e^x; f''(x) = e^x > 0; f$ *convexe. By Jensen's inequality:*

$$e^{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3} \leq \lambda_1 e^{x_1} + \lambda_2 e^{x_2} + \lambda_3 e^{x_3} \quad (1)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1; \lambda_1, \lambda_2, \lambda_3 > 0$$

We take:

$$\lambda_1 = \frac{a^2+2ac}{(a+b+c)^2}; \lambda_2 = \frac{b^2+2ba}{(a+b+c)^2}; \lambda_3 = \frac{c^2+2cb}{(a+b+c)^2}$$

$$\lambda_1+\lambda_2+\lambda_3=\frac{a^2+2ac+b^2+2ba+c^2+2cb}{(a+b+c)^2}=\frac{(a+b+c)^2}{(a+b+c)^2}=1$$

$$x_1=\frac{1}{a}; x_2=\frac{1}{b}; x_3=\frac{1}{c}$$

Replace in (1):

$$e^{\frac{a^2+2ac}{(a+b+c)^2}\cdot\frac{1}{a}+\frac{b^2+2ba}{(a+b+c)^2}\cdot\frac{1}{b}+\frac{c^2+2cb}{(a+b+c)^2}\cdot\frac{1}{c}}\le$$

$$\le\frac{a^2+2ac}{(a+b+c)^2}e^{\frac{1}{a}}+\frac{b^2+2ba}{(a+b+c)^2}e^{\frac{1}{b}}+\frac{c^2+2cb}{(a+b+c)^2}e^{\frac{1}{c}}$$

$$e^{\frac{a+2c+b+2a+c+2b}{(a+b+c)^2}}\le\frac{a^2+2ac}{(a+b+c)^2}e^{\frac{1}{a}}+\frac{b^2+2ba}{(a+b+c)^2}e^{\frac{1}{b}}+\frac{c^2+2cb}{(a+b+c)^2}e^{\frac{1}{c}}$$

$$e^{\frac{3}{a+b+c}}\cdot(a+b+c)^2\le(a^2+2ac)e^{\frac{1}{a}}+(b^2+2ba)e^{\frac{1}{b}}+(c^2+2cb)e^{\frac{1}{c}}$$

$$e^{\frac{3}{3}}\cdot 3^2\le a(a+c)e^{\frac{1}{a}}+b(b+a)e^{\frac{1}{b}}+c(c+b)e^{\frac{1}{c}}$$

$$a(a+c)e^{\frac{1}{a}}+b(b+a)e^{\frac{1}{b}}+c(c+b)e^{\frac{1}{c}}\le 9e$$

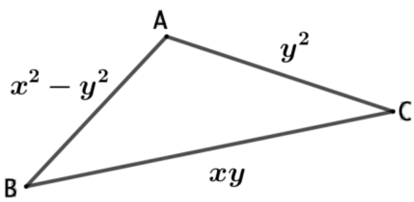
Equality holds for $a=b=c=1$.

1.37. If $0<y<x<2y$ then:

$$\boldsymbol{x(x+y)\sqrt{4y^2-x^2}<3y^3\sqrt{3}}$$

Solution:

Let's consider ΔABC; $BC=a=xy$; $AC=b=y^2$; $AB=c=x^2-y^2$



$$a<b+c\Leftrightarrow xy<x^2-y^2+y^2\Leftrightarrow y<x\quad (true)$$

$$b<a+c\Leftrightarrow y^2<xy+x^2-y^2\Leftrightarrow x(x+y)>2y^2$$

But: $x(x+y)>y(y+y)=2y^2$

$$c<a+b\Leftrightarrow x^2-y^2<xy+y^2\Leftrightarrow x^2<y(x+2y)$$

$$\Leftrightarrow 2y^2+xy-x^2>0\Leftrightarrow 2t^2-t-1>0; t=\frac{y}{x}$$

$$\Delta=9; t_1=2; t_2=-1; \min(2t^2-t-1)=-\frac{9}{8}$$

$$t>1\Rightarrow 2t^2-t-1>2\cdot 1^2-1-1=0\quad (true)$$

Denote s – semiperimeter; R – circumradii

$$s = \frac{a+b+c}{2} = \frac{xy+y^2+x^2-y^2}{2} = \frac{x^2+xy}{2} \quad (1)$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{x^2y^2+(x^2-y^2)^2-y^4}{2xy(x^2-y^2)} =$$

$$= \frac{x^4-x^2y^2+y^4-y^4}{2xy(x^2-y^2)} = \frac{x^2(x^2-y^2)}{2xy(x^2-y^2)} = \frac{x}{2y} < 1 \quad (true)$$

$$\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\frac{x^2}{4y^2}} = \frac{\sqrt{4y^2-x^2}}{2y}$$

$$R = \frac{b}{2\sin B} = \frac{y^2}{2\cdot\frac{\sqrt{4y^2-x^2}}{2y}} = \frac{y^3}{\sqrt{4y^2-x^2}}$$

By Mitrinovic's inequality: $s \le \frac{3\sqrt{3}}{2}R$

By (1); (2): $\frac{x(x+y)}{2} < \frac{3\sqrt{3}}{2}\cdot\frac{y^3}{\sqrt{4y^2-x^2}}$

$$x(x+y)\sqrt{4y^2-x^2} < 3y^3\sqrt{3}$$

(Equality doesn't hold because ΔABC can't be an equilateral one:
$x < y \Rightarrow xy < y^2 \Rightarrow a < b$*)*

1.38. If $0 < y < x < 2y$ then:

$$\boldsymbol{x(x+y) > 3(x-y)\sqrt{3(4y^2-x^2)}}$$

Solution:

Let's consider $\Delta ABC; BC = a = xy; AC = b = y^2; AB = c = x^2-y^2$.

$$a < b+c \Leftrightarrow xy < x^2-y^2+y^2 \Leftrightarrow y < x \ (true)$$

$$b < a+c \Leftrightarrow y^2 < xy+x^2-y^2 \Leftrightarrow x(x+y) > 2y^2$$

But: $x(x+y) > y(y+y) = 2y^2$

$$c < a+b \Leftrightarrow x^2-y^2 < xy+y^2 \Leftrightarrow x^2 < y(x+2y)$$

$$\Leftrightarrow 2y^2+xy-x^2 > 0 \Leftrightarrow 2t^2-t-1 > 0; t = \frac{y}{x}$$

$$\Delta = 9; t_1 = \frac{1+3}{2} = 2; t_2 = \frac{1-3}{2} = -1$$

$$\min(2t^2-t-1) = -\frac{9}{8}$$

$$t > 1 \Rightarrow 2t^2-t-1 > 2\cdot 1^2-1-1 = 0 \quad (true)$$

Denote s – semiperimeter; F – area; r – inradii

$$s = \frac{a+b+c}{2} = \frac{xy+y^2+x^2-y^2}{2} = \frac{x^2+xy}{2} \quad (1)$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{x^2y^2+(x^2-y^2)^2-y^4}{2xy(x^2-y^2)} =$$

$$= \frac{x^4-x^2y^2+y^4-y^4}{2xy(x^2-y^2)} = \frac{x^2(x^2-y^2)}{2xy(x^2-y^2)} = \frac{x}{2y} < 1 \text{ (true)}$$

$$\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\frac{x^2}{4y^2}} = \frac{\sqrt{4y^2-x^2}}{2y} \quad (2)$$

$$F = \frac{1}{2}ac\sin B = \frac{1}{2}\cdot xy\cdot(x^2-y^2)\cdot\frac{\sqrt{4y^2-x^2}}{2y}$$

$$F = \frac{x(x^2-y^2)\sqrt{4y^2-x^2}}{4} \quad (2)$$

By Mitrinovic's inequality:

$s > 3\sqrt{3}r$ *(Equality doesn't hold because* ΔABC *can't be an equilateral one* $x < y \Rightarrow xy < y^2 \Rightarrow a < b$*)*

$$s > 3\sqrt{3}\cdot\frac{F}{s} \Rightarrow s^2 > 3\sqrt{3}F$$

By (1); (2): $\frac{x^2(x+y)^2}{4} > 3\sqrt{3}\cdot\frac{x(x-y)(x+y)\sqrt{4y^2-x^2}}{4}$

$$x(x+y) > 3\sqrt{3}(x-y)\sqrt{4y^2-x^2}, \frac{x(x+y)}{3(x-y)} > \sqrt{3(4y^2-x^2)}$$

$$x(x+y) > 3(x-y)\sqrt{3(4y^2-x^2)}$$

1.39. If $x, y, z > 1; xyz = 8$ then:

$$\left(\frac{x}{2}\right)^x + \left(\frac{y}{2}\right)^y + \left(\frac{z}{2}\right)^z \geq 3$$

Solution:

$$\left(\frac{x}{2}\right)^x = \left(1+\left(\frac{x}{2}-1\right)\right)^x \overset{BERNOULLI}{\geq} 1 + x\left(\frac{x}{2}-1\right) \geq x-1$$

$$\Leftrightarrow 1 + \frac{x^2}{2} - x \geq x - 1$$

$$\frac{x^2}{2} - 2x + 2 \geq 0 \Leftrightarrow x^2 - 4x + 4 \geq 0 \Leftrightarrow (x-2)^2 \geq 0$$

$$\left(\frac{x}{2}\right)^x \geq x-1, \left(\frac{y}{2}\right)^y \geq y-1, \left(\frac{z}{2}\right)^z \geq z-1$$

By adding: $\left(\frac{x}{2}\right)^x + \left(\frac{y}{2}\right)^y + \left(\frac{z}{2}\right)^z \geq x+y+z-3 \geq$

$$\overset{AM-GM}{\geq} 3\sqrt[3]{xyz} - 3 = 3\sqrt[3]{8} - 3 = 3\cdot 2 - 3 = 3$$

Equality holds for $a = b = c = 2$*.*

1.40. If $a, b, c \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ then:

$$b^2 \cdot \sqrt[a]{a} + c^2 \cdot \sqrt[b]{b} + a^2 \cdot \sqrt[c]{c} \geq 48\sqrt{2}$$

Solution:

$$\text{By induction we prove } 4^n \geq n^4; n \geq 4$$
$$\text{For } n = 4 \Rightarrow 4^4 \geq 4^4 \quad (\text{true})$$
$$P(n): 4^n \geq n^4 \quad (\text{true})$$
$$P(n+1): 4^{n+1} \geq (n+1)^4 \quad (\text{to prove})$$
$$4^{n+1} = 4^n \cdot 4 \geq 4n^4 \geq (n+1)^4 \quad (\text{to prove})$$
$$\sqrt[4]{4n^4} \geq \sqrt[4]{(n+1)^4} \Rightarrow n\sqrt{2} \geq n+1$$
$$n(\sqrt{2}-1) \geq 4(\sqrt{2}-1) > 1 \Leftrightarrow 4\sqrt{2} > 5 \Leftrightarrow 32 > 25$$
$$P(n) \to P(n+1)$$
$$\text{If } a, b, c \geq 4 \Rightarrow \sqrt[a]{a} \geq \sqrt[4]{4} = \sqrt{2};\ \sqrt[b]{b} \geq \sqrt{2};\ \sqrt[c]{c} \geq \sqrt{2} \Rightarrow$$
$$\Rightarrow \frac{1}{\sqrt[a]{a}} \leq \frac{1}{\sqrt{2}}; \frac{1}{\sqrt[b]{b}} \leq \frac{1}{\sqrt{2}}; \frac{1}{\sqrt[c]{c}} \leq \frac{1}{\sqrt{2}}$$
$$\frac{1}{\sqrt[a]{a}} + \frac{1}{\sqrt[b]{b}} + \frac{1}{\sqrt[c]{c}} \leq \frac{3}{\sqrt{2}} \quad (1)$$
$$b^2 \cdot \sqrt[a]{a} + c^2 \cdot \sqrt[b]{b} + a^2 \cdot \sqrt[c]{c} = \frac{b^2}{\frac{1}{\sqrt[a]{a}}} + \frac{c^2}{\frac{1}{\sqrt[b]{b}}} + \frac{a^2}{\frac{1}{\sqrt[c]{c}}} \overset{BERGSTROM}{\geq}$$
$$\geq \frac{(a+b+c)^2}{\frac{1}{\sqrt[a]{a}} + \frac{1}{\sqrt[b]{b}} + \frac{1}{\sqrt[c]{c}}} \overset{(1)}{\geq} \frac{(4+4+4)^2}{\frac{3}{\sqrt{2}}} = \frac{9 \cdot 16 \cdot \sqrt{2}}{3}$$
$$b^2 \cdot \sqrt[a]{a} + c^2 \cdot \sqrt[b]{b} + a^2 \cdot \sqrt[c]{c} \geq 48\sqrt{2}$$

1.41. If $m, n, p, q \in \mathbb{N}; m, n, p, q \geq 4$ then:

$$4^n(4^n+1) + 4^m(4^m+1) + 4^p(4^p+1) + 4^q(4^q+1) \geq 4mnpq(mnpq+1)$$

Solution:

$$\text{We prove by induction: } 4^n \geq n^4; n \geq 4$$
$$\text{For } n = 4 \Rightarrow 4^4 \geq 4^4 \quad (\text{True})$$
$$P(n): 4^n \geq n^4 \quad (\text{True})$$
$$P(n+1): 4^{n+1} \geq (n+1)^4 \quad (\text{to prove})$$
$$4^{n+1} = 4^n \cdot 4 \geq 4n^4 \geq (n+1)^4 \quad (\text{to prove})$$
$$\sqrt[4]{n^4 \cdot 4} \geq \sqrt[4]{(n+1)^4} \Leftrightarrow n\sqrt{2} \geq n+1 \Leftrightarrow n(\sqrt{2}-1) \geq 1$$
$$\text{But } n \geq 4 \Rightarrow n(\sqrt{2}-1) \geq 4(\sqrt{2}-1) \geq 1 \Leftrightarrow$$
$$4\sqrt{2} \geq 5 \Leftrightarrow 32 > 25; P(n) \to P(n+1)$$

$$m,n,p,q\geq 4\Rightarrow 4^n\geq n^4;4^m\geq m^4;4^p\geq p^4;4^q\geq q^4. \text{ By adding:}$$

$$4^n+4^m+4^p+4^q\geq n^4+m^4+p^4+q^4\overset{AM-GM}{\geq}$$

$$\geq 4\sqrt[4]{n^4\cdot m^4\cdot p^4\cdot q^4}=4mnpq\quad (1)$$

$$\sqrt{\frac{16^n+16^m+16^p+16^q}{4}}\overset{QM-AM}{\geq}\frac{4^n+4^m+4^p+4^q}{4}\overset{(1)}{\geq}mnpq$$

By squaring: $16^n+16^m+16^p+16^q\geq 4m^2n^2p^2q^2\quad (2)$

By adding (1); (2):

$$16^n+4^n+16^m+4^m+16^p+4^p+16^q+4^q\geq 4m^2n^2p^2q^2+4mnpq$$

$$4^n(4^n+1)+4^m(4^m+1)+4^p(4^p+1)+4^q(4^q+1)\geq 4mnpq(mnpq+1)$$

1.42. If $a,b\in\mathbb{R}$; $A=\begin{pmatrix}\sin^2 a & \cos^2 a\sin^2 b & \cos^2 a\cos^2 b\\ \cos^2 b\sin^2 c & \sin^2 b & \cos^2 b\cos^2 c\\ \cos^2 c\sin^2 a & \cos^2 c\cos^2 a & \sin^2 c\end{pmatrix}$

$A^n=\begin{pmatrix}x_1 & x_2 & x_3\\ x_4 & x_5 & x_6\\ x_7 & x_8 & x_9\end{pmatrix}$; $n\in\mathbb{N}$; $n\geq 2$; $x_i\in\mathbb{R}$; $i\in\overline{1,9}$

then find $\Omega=\sum_{i=1}^{9}x_i$

Solution:

$$A\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}\sin^2 a & \cos^2 a\sin^2 b & \cos^2 a\cos^2 b\\ \cos^2 b\sin^2 c & \sin^2 b & \cos^2 b\cos^2 c\\ \cos^2 c\sin^2 a & \cos^2 c\cos^2 a & \sin^2 c\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=$$

$$=\begin{pmatrix}\sin^2 a+\cos^2 a\sin^2 b+\cos^2 a\cos^2 b\\ \cos^2 b\sin^2 c+\sin^2 b+\cos^2 b\cos^2 c\\ \cos^2 c\sin^2 a+\cos^2 c\cos^2 a+\sin^2 c\end{pmatrix}=\begin{pmatrix}\sin^2 a+\cos^2 a\,(\sin^2 b+\cos^2 b)\\ \sin^2 b+\cos^2 b\,(\sin^2 c+\cos^2 c)\\ \sin^2 c+\cos^2 c\,(\sin^2 a+\cos^2 a)\end{pmatrix}=$$

$$=\begin{pmatrix}\sin^2 a+\cos^2 a\\ \sin^2 b+\cos^2 b\\ \sin^2 c+\cos^2 c\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$A^2\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=A\cdot A\begin{pmatrix}1\\1\\1\end{pmatrix}=A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$P(n): A^n\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\ \textit{(suppose true)}$$

$$P(n+1): A^{n+1}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\ \textit{(to prove)}$$

$$A^{n+1}\begin{pmatrix}1\\1\\1\end{pmatrix} = A \cdot A^n\begin{pmatrix}1\\1\\1\end{pmatrix} = A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$P(n) \to P(n+1)$$

$$A^n \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \begin{pmatrix}x_1 & x_2 & x_3\\x_4 & x_5 & x_6\\x_7 & x_8 & x_9\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x_1+x_2+x_3\\x_4+x_5+x_6\\x_7+x_8+x_9\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \sum_{i=1}^{9} x_i = 9 \Rightarrow \Omega = 9$$

1.43. If $\boldsymbol{a,b,c} > 0; n \in \mathbb{N}; \boldsymbol{n \geq 2}; \boldsymbol{x_i} \in \mathbb{R}; \boldsymbol{i} \in \overline{\boldsymbol{1,9}}$

$$\boldsymbol{A} = \begin{pmatrix}\frac{\boldsymbol{a^2}}{(\boldsymbol{a+b})^2} & \frac{\boldsymbol{2ab}}{(\boldsymbol{a+b})^2} & \frac{\boldsymbol{b^2}}{(\boldsymbol{a+b})^2}\\ \frac{\boldsymbol{c^2}}{(\boldsymbol{b+c})^2} & \frac{\boldsymbol{b^2}}{(\boldsymbol{b+c})^2} & \frac{\boldsymbol{2bc}}{(\boldsymbol{b+c})^2}\\ \frac{\boldsymbol{2ca}}{(\boldsymbol{c+a})^2} & \frac{\boldsymbol{a^2}}{(\boldsymbol{c+a})^2} & \frac{\boldsymbol{c^2}}{(\boldsymbol{c+a})^2}\end{pmatrix}; \boldsymbol{A^n} = \begin{pmatrix}\boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3}\\\boldsymbol{x_4} & \boldsymbol{x_5} & \boldsymbol{x_6}\\\boldsymbol{x_7} & \boldsymbol{x_8} & \boldsymbol{x_9}\end{pmatrix}$$

then find: $\boldsymbol{\Omega = \sum_{i=1}^{9} x_i}$

Solution:

$$A \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\frac{a^2}{(a+b)^2} & \frac{2ab}{(a+b)^2} & \frac{b^2}{(a+b)^2}\\ \frac{c^2}{(b+c)^2} & \frac{b^2}{(b+c)^2} & \frac{2bc}{(b+c)^2}\\ \frac{2ca}{(c+a)^2} & \frac{a^2}{(c+a)^2} & \frac{c^2}{(c+a)^2}\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} =$$

$$= \begin{pmatrix}\frac{a^2+2ab+b^2}{(a+b)^2}\\ \frac{c^2+b^2+2bc}{(b+c)^2}\\ \frac{2ca+a^2+c^2}{(c+a)^2}\end{pmatrix} = \begin{pmatrix}\frac{(a+b)^2}{(a+b)^2}\\ \frac{(c+b)^2}{(c+b)^2}\\ \frac{(c+a)^2}{(c+a)^2}\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$A^2 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = A \cdot A\begin{pmatrix}1\\1\\1\end{pmatrix} = A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$By\ induction:$$
$$P(n): A^n\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}\ \ (suppose\ true)$$
$$P(n+1): A^{n+1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}\ \ (to\ prove)$$
$$A^{n+1}\begin{pmatrix}1\\1\\1\end{pmatrix} = A\cdot A^n\begin{pmatrix}1\\1\\1\end{pmatrix} = A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$P(n)\to P(n+1)$$
$$A^n\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \begin{pmatrix}x_1 & x_2 & x_3\\x_4 & x_5 & x_6\\x_7 & x_8 & x_9\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$\Rightarrow \begin{pmatrix}x_1+x_2+x_3\\x_4+x_5+x_6\\x_7+x_8+x_9\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \begin{matrix}x_1+x_2+x_3=1\\x_4+x_5+x_6=1\\x_7+x_8+x_9=1\end{matrix}$$
$$\Rightarrow \Omega = \sum_{i=1}^{9} x_i = 1+1+1 = 3$$

1.44. If $\boldsymbol{a,b,c,d} > 0$; $\boldsymbol{a^2cd + b^2da + c^2ab + d^2bc = 4abcd}$ then:
$$\boldsymbol{\frac{a^2}{b^2}\left(\frac{a}{b}-1\right) + \frac{b^2}{c^2}\left(\frac{b}{c}-1\right) + \frac{c^2}{d^2}\left(\frac{c}{d}-1\right) + \frac{d^2}{a^2}\left(\frac{d}{a}-1\right) = 0}$$

Solution:

$$If\ u > 0 \Rightarrow (u-1)^2(u+1) \ge 0$$
$$(u^2-2u+1)(u+1) \ge 0$$
$$u^3+u^2-2u^2-2u+u+1 \ge 0$$
$$u^3-u^2-u+1 \ge 0 \Rightarrow u^3 \ge u^2+u-1$$
$$For\ u = \frac{a}{b}\ and\ successively\ u = \frac{b}{c}; u = \frac{c}{d}; u = \frac{a}{b}\ then:$$
$$\left(\frac{a}{b}\right)^3 \ge \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) - 1\ \ (1)$$
$$\left(\frac{b}{c}\right)^3 \ge \left(\frac{b}{c}\right)^2 + \left(\frac{b}{c}\right) - 1\ \ (2)$$
$$\left(\frac{c}{d}\right)^3 \ge \left(\frac{c}{d}\right)^2 + \left(\frac{c}{d}\right) - 1\ \ (3)$$
$$\left(\frac{d}{a}\right)^3 \ge \left(\frac{d}{a}\right)^2 + \left(\frac{d}{a}\right) - 1\ \ (4)$$
$$By\ adding\ (1); (2); (3); (4):$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^3 \ge \sum_{cyc}\left(\frac{c}{d}\right)^2 + \sum_{cyc}\frac{a}{b} - 4 =$$
$$= \sum_{cyc}\left(\frac{c}{d}\right)^2 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} - 4 =$$
$$= \sum_{cyc}\left(\frac{a}{b}\right)^2 + \frac{a^2cd + b^2da + c^2ab + d^2bc - 4abcd}{abcd} =$$
$$= \sum_{cyc}\left(\frac{a}{b}\right)^2 + \frac{4abcd - 4abcd}{abcd} = \sum_{cyc}\left(\frac{a}{b}\right)^2$$

1.45. If $a, b \in \mathbb{C}$ then for any $z_1, z_2, z_3 \in \mathbb{C}$ the following relationship holds:

$$\boldsymbol{|z_1 + a + b| + |z_2 - a| + |z_3 - b| \le}$$
$$\boldsymbol{\le |z_1 + z_2 + 2b - z_3| + |z_1 + z_3 + a - z_2 + a| +}$$
$$\boldsymbol{+|-z_1 + z_2 + z_3 - 2a - 2b|}$$

Solution:

$$|z_1 + z_2 - z_3 + 2b| + |z_1 - z_2 + z_3 + 2a| \ge |2z_1 + 2a + 2b| \quad (1)$$
$$|z_1 + z_2 - z_3 + 2b| + |-z_1 + z_2 + z_3 - 2a - 2b| \ge |2z_2 - 2a| \quad (2)$$
$$|z_1 + z_3 - z_2 + 2a| + |-z_1 + z_2 + z_3 - 2a - 2b| \ge |2z_3 - 2b| \quad (3)$$

By adding (1); (2); (3): $2RHS \ge 2LHS \Rightarrow LHS \le RHS$

1.46. If $a, b, c \ge 1$ then:

$$\boldsymbol{\frac{\left(a^{\frac{1}{a}} + b^{\frac{1}{b}} - c^{\frac{1}{c}}\right)\left(b^{\frac{1}{b}} + c^{\frac{1}{c}} - a^{\frac{1}{a}}\right)\left(c^{\frac{1}{c}} + a^{\frac{1}{a}} - b^{\frac{1}{b}}\right)}{a^{\frac{1}{a}} \cdot b^{\frac{1}{b}} \cdot c^{\frac{1}{c}}} \le 8}$$

Solution:

$$[a] \le a < [a] + 1; [*] - \textit{great integer function}$$
$$2^a \ge 2^{[a]} = (1+1)^{[a]} \overset{BERNOULLI}{>} 1 + [a] > a$$
$$2^a > a \Rightarrow (2^a)^{\frac{1}{a}} > a^{\frac{1}{a}} \ge 1 \Rightarrow 2 > a^{\frac{1}{a}} \ge 1$$
Analogous: $2 > b^{\frac{1}{b}} \ge 1; 2 > c^{\frac{1}{c}} \ge 1$
$$a^{\frac{1}{a}} + b^{\frac{1}{b}} \ge 1 + 1 = 2 > c^{\frac{1}{c}}$$
$a^{\frac{1}{a}} + b^{\frac{1}{b}} > c^{\frac{1}{c}}$. *Analogous:* $b^{\frac{1}{b}} + c^{\frac{1}{c}} > a^{\frac{1}{a}}; c^{\frac{1}{c}} + a^{\frac{1}{a}} > b^{\frac{1}{b}}$

$$a^{\frac{1}{a}}; b^{\frac{1}{b}}; c^{\frac{1}{c}} \text{ can be sides in a triangle.}$$

In a triangle with sides x, y, z by Padoa's inequality:

$$(x+y-z)(y+z-x)(z+x-y) \le 8xyz$$

$$\frac{(x+y-z)(y+z-x)(z+x-y)}{xyz} \le 8$$

$$\text{For } x = a^{\frac{1}{a}}; y = b^{\frac{1}{b}}; z = c^{\frac{1}{c}}$$

$$\frac{\left(a^{\frac{1}{a}} + b^{\frac{1}{b}} - c^{\frac{1}{c}}\right)\left(b^{\frac{1}{b}} + c^{\frac{1}{c}} - a^{\frac{1}{a}}\right)\left(c^{\frac{1}{c}} + a^{\frac{1}{a}} - b^{\frac{1}{b}}\right)}{a^{\frac{1}{a}} \cdot b^{\frac{1}{b}} \cdot c^{\frac{1}{c}}} \le 8$$

Equality holds for $a = b = c = 1$.

1.47. If $a, b, c > 0, a+b+c=1$ then:

$$\left(\frac{a+b}{2}\right)^{2ab} \cdot \left(\frac{b+c}{2}\right)^{2bc} \cdot \left(\frac{c+a}{2}\right)^{2ca} \ge a^{a-a^2} \cdot b^{b-b^2} \cdot c^{c-c^2}$$

Solution:

$$\frac{a+b}{2} \overset{AM-GM}{\ge} \sqrt{ab} \Rightarrow \left(\frac{a+b}{2}\right)^{2ab} \ge \left(\sqrt{ab}\right)^{2ab}$$

$$\prod_{cyc}\left(\frac{a+b}{2}\right)^{2ab} \ge \prod_{cyc}\left(\sqrt{ab}\right)^{2ab} = \prod_{cyc}(ab)^{ab} =$$

$$= (ab)^{ab} \cdot (bc)^{bc} \cdot (ca)^{ca} = a^{ab} \cdot b^{ab} \cdot b^{bc} \cdot c^{bc} \cdot c^{ca} \cdot a^{ca} =$$

$$= a^{ab+ca} \cdot b^{ab+bc} \cdot c^{bc+ca} = a^{(b+c)a} \cdot b^{(a+c)b} \cdot c^{(b+a)c} =$$

$$= a^{(1-a)a} \cdot b^{(1-b)b} \cdot c^{(1-c)c} = a^{a-a^2} \cdot b^{b-b^2} \cdot c^{c-c^2}$$

1.48. If $x, y, z \in \mathbb{R}$ then:

$$\sqrt{3}(x^2+y^2+z^2) \ge \left|x^2+y^2-z^2\right| + 2|z|(|x|+|y|)$$

Solution:

$$(x^2+y^2+z^2)^2 = (x^2+y^2-z^2)^2 + (2yz)^2 + (2xz)^2 =$$

$$= \frac{(|x^2+y^2-z^2|)^2}{1} + \frac{(2|yz|)^2}{1} + \frac{(2|xz|)^2}{1} \ge$$

$$\overset{CBS}{\geq} \frac{\left(|x^2+y^2-z^2|+2|z|(|x|+|y|)\right)^2}{1+1+1} \overset{AM-GM}{\geq}$$

$$\geq \frac{(|x^2+y^2-z^2|+4|xyz|)^2}{3}$$

$$3(x^2+y^2+z^2)^2 \geq (|x^2+y^2-z^2|+4|xyz|)^2$$

$$\sqrt{3}(x^2+y^2+z^2) \geq |x^2+y^2-z^2|+4|xyz|$$

1.49. Solve for real numbers:

$$\begin{cases} \mathbf{1 \leq x, y, z \leq 3} \\ \mathbf{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{35}{3}} \\ \mathbf{3^y + \log_2 z = 3} \end{cases}$$

Solution:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{35}{3} \quad (1)$$

$$3^y + \log_2 z = 3 \quad (2)$$

$$We'll\ show\ that\ (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \leq \frac{35}{3}; 1 \leq x, y, z \leq 3$$

$$Or\ \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z} \overset{??}{\leq} \frac{26}{3}$$

$$WLOG\ x \geq y \geq z \Rightarrow 0 \leq (x-y)(y-z) \Leftrightarrow zx + y^2 \leq xy + yz$$

$$\begin{cases} \frac{z}{y}+\frac{y}{x} \leq 1+\frac{z}{x} \\ \frac{x}{y}+\frac{y}{z} \leq \frac{x}{z}+1 \end{cases} \overset{+}{\Rightarrow} \frac{x}{y}+\frac{y}{z}+\frac{z}{y}+\frac{y}{x} \leq \frac{x}{z}+\frac{z}{x}+2$$

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z} \leq 2\left(\frac{x}{z}+\frac{z}{x}\right)+2$$

$$It\ remains\ to\ show\ that\ 2\left(\frac{x}{z}+\frac{z}{x}\right)+2 \leq \frac{26}{3} \ (3)$$

$$Let\ t = \frac{x}{z}; 1 \leq t \leq 3 \Leftrightarrow (3); t + \frac{1}{t} \leq \frac{10}{3} \Leftrightarrow (3t-1)(t-3) \leq 0$$

$$Which\ is\ true\ \therefore (1)\ is\ the\ hold\ point\ of\ inequality\ at\ t = \frac{x}{z} = 3$$

$$\begin{cases} x = 3, z = 1 \Rightarrow y = 1 \\ x = 3, z = 1 \Rightarrow y = 3 \end{cases} and\ permutations \rightarrow (2)$$

$$\therefore (x, y, z) = (3,1,1)$$

1.50. If $a, b > 0$ then:

$$\frac{a+b}{\sqrt{ab}}+\frac{4ab}{(a+b)^2}\le\frac{4\sqrt{ab}}{a+b}+\frac{(a+b)^2}{4ab}$$

Solution:

Inequality can be written: $\frac{\frac{a+b}{\sqrt{ab}}}{}+\frac{4}{\left(\frac{a+b}{\sqrt{ab}}\right)^2}\le\frac{4}{\frac{a+b}{\sqrt{ab}}}+\frac{\left(\frac{a+b}{\sqrt{ab}}\right)^2}{4}$

Denote $x=\frac{a+b}{\sqrt{ab}}$*. By AM-GM;* $\frac{a+b}{2}\ge\sqrt{ab}\Rightarrow\frac{a+b}{\sqrt{ab}}\ge 2\Rightarrow x\ge 2$

$$x+\frac{4}{x^2}\le\frac{4}{x}+\frac{x^2}{4}$$
$$4x^3+16\le 16x+x^4$$
$$x^4-4x^3+16x-16\ge 0$$
$$x^4-2x^3-2x^3+4x^2-4x^2+8x+8x-16\ge 0$$
$$x^3(x-2)-2x^2(x-2)-4x(x-2)+8(x-2)\ge 0$$
$$(x-2)(x^3-2x^2-4x+8)\ge 0$$
$$(x-2)\big(x^2(x-2)-4(x-2)\big)\ge 0$$
$$(x-2)^2(x^2-4)\ge 0$$
$$(x-2)^3(x+2)\ge 0$$

Which is true because: $x\ge 2\Rightarrow x-2\ge 0; x+2>0$

1.51. If $a, b, c, x, y, z > 0; a^x\cdot b^y\cdot c^z=1$ then:

$$a^{x^2}\cdot b^{y^2}\cdot c^{z^2}\cdot(a+b)^{2xy}\cdot(b+c)^{2yz}\cdot(c+a)^{2zx}\ge 4^{xy+yz+zx}$$

Solution:

$$\frac{a+b}{2}\ge\sqrt{ab}\Rightarrow a+b\ge 2\sqrt{ab}\quad \textit{(AM-GM)}$$
$$(a+b)^{2xy}\ge\left(2\sqrt{ab}\right)^{2xy}=4^{xy}\cdot(ab)^{xy}\quad (1)$$
$$(b+c)^{2yz}\ge 4^{yz}\cdot(bc)^{yz}\quad (2)$$
$$(c+a)^{2zx}\ge 4^{zx}\cdot(ca)^{zx}\quad (3)$$

By multiplying (1); (2); (3):

$$(a+b)^{2xy}\cdot(b+c)^{2yz}\cdot(c+a)^{2zx}\ge 4^{xy+yz+zx}\cdot(ab)^{xy}\cdot(bc)^{yz}\cdot(ca)^{zx}$$
$$a^{x^2}\cdot b^{y^2}\cdot c^{z^2}\cdot(a+b)^{2xy}\cdot(b+c)^{2yz}\cdot(c+a)^{2zx}\ge$$
$$\ge 4^{xy+yz+zx}\cdot(ab)^{xy}\cdot(bc)^{yz}\cdot(ca)^{zx}\cdot a^{x^2}\cdot b^{y^2}\cdot c^{z^2}=$$
$$=4^{xy+yz+zx}\cdot(a^x)^{x+y+z}\cdot(b^y)^{x+y+z}\cdot(c^z)^{x+y+z}=$$
$$=4^{xy+yz+zx}\cdot(a^x\cdot b^y\cdot c^z)^{x+y+z}=4^{xy+yz+zx}\cdot 1^{x+y+z}=4^{xy+yz+zx}$$

Equality holds for $a=b=c=x=y=z=1$

1.52. Solve for real numbers:

$$\left(\log\left(\frac{x}{x^2+1}\right)+x\right)^3=$$
$$=\left(\log\left(\frac{x}{x^2+1}\right)-x\right)^3+(x-\log(x^3+x))^3+(x+\log(x^3+x))^3$$

Solution:

$$\text{Denote } a=\log\left(\frac{x}{x^2+1}\right);b=\log(x^3+x)$$
$$(a+x)^3=(a-x)^3+(x-b)^3+(x+b)^3$$
$$a^3+3a^2x+3ax^2+x^3=a^3-3a^2x+3ax^2-x^3+$$
$$+x^3-3x^2b+3xb^2-b^3+x^3+3x^2b+3xb^2+b^3$$
$$6a^2x-6xb^2=0,6x(a^2-b^2)=0$$
$$a^2-b^2=0\Rightarrow\left(\log\frac{x}{x^2+1}\right)^2-(\log(x^3+x))^2=0$$
$$\left(\log\frac{x}{x^2+1}-\log(x^3+x)\right)\left(\log\frac{x}{x^2+1}+\log(x^3+x)\right)=0$$
$$\log\frac{x}{x^2+1}=\log(x^3+x)\Rightarrow\frac{x}{x^2+1}=x^3+x\Rightarrow$$
$$\Rightarrow 1=(x^2+1)^2\Rightarrow x^4+2x^2=0.\text{ No solution.}$$
$$\log\frac{x}{x^2+1}=-\log(x^3+x)$$
$$\frac{x}{x^2+1}=\frac{1}{x(x^2+1)}\Rightarrow x^2=1\Rightarrow x=1$$

1.53. If $a,b,c>0,a+b+c=1$ then:

$$a^{a-a^2}\cdot b^{b-b^2}\cdot c^{c-c^2}\le\left(\frac{a+b}{2}\right)^{2ab}\cdot\left(\frac{b+c}{2}\right)^{2bc}\cdot\left(\frac{c+a}{2}\right)^{2ca}$$

Solution:

We have AM ≥ GM: $ab\le\left(\frac{a+b}{2}\right)^2\Rightarrow(ab)^{ab}\le\left(\frac{a+b}{2}\right)^{2ab}$ *(1)*

Similarly, $(bc)^{bc}\le\left(\frac{b+c}{2}\right)^{2bc}$ *(2) and* $(ca)^{ca}\le\left(\frac{c+a}{2}\right)^{2ca}$ *(3)*

Then (1)×(2)×(3):

$$a^{ab+ca}\cdot b^{ab+bc}\cdot c^{bc+ca}\le\left(\frac{a+b}{2}\right)^{2ab}\cdot\left(\frac{b+c}{2}\right)^{2bc}\cdot\left(\frac{c+a}{2}\right)^{2ca}$$
$$\Rightarrow a^{a(b+c)}\cdot b^{b(a+c)}\cdot c^{c(a+b)}\le\left(\frac{a+b}{2}\right)^{2ab}\cdot\left(\frac{b+c}{2}\right)^{2bc}\cdot\left(\frac{c+a}{2}\right)^{2ca}$$

$$\Rightarrow a^{a(1-a)}b^{b(1-b)}c^{c(1-c)} \le \left(\frac{a+b}{2}\right)^{2ab} \cdot \left(\frac{b+c}{2}\right)^{2bc} \cdot \left(\frac{c+a}{2}\right)^{2ca}$$

$$\Rightarrow a^{a-a^2} \cdot b^{b-b^2} \cdot c^{c-c^2} \le \left(\frac{a+b}{2}\right)^{2ab} \cdot \left(\frac{b+c}{2}\right)^{2bc} \cdot \left(\frac{c+a}{2}\right)^{2ca}$$

1.54. $\boldsymbol{\Omega_1 = \frac{1}{(1-a^2)^7} + \frac{1}{(1-b^2)^7} + \frac{1}{(1-c^2)^7}, \Omega_2 = \frac{1}{(1-x^2)^7} + \frac{1}{(1-y^2)^7} + \frac{1}{(1-z^2)^7}}$

$$\boldsymbol{\Omega_3 = \frac{1}{(1-ax)^7} + \frac{1}{(1-by)^7} + \frac{1}{(1-cz)^7}, a, b, c, x, y, z \in (-1, 1)}$$

Prove that: $\boldsymbol{\Omega_1\Omega_2 \ge \Omega_3^2}$

Solution:

$$(1-a^2)(1-x^2) \le (1-ax)^2 \Leftrightarrow 1 - a^2 - x^2 + a^2x^2 \le 1 + a^2x^2 - 2ax$$

$$\Leftrightarrow (a-x)^2 \ge 0 \to true \Rightarrow \frac{1}{\sqrt{(1-a^2)(1-x^2)}} \overset{(1)}{\ge} \frac{1}{|1-a^x|}$$

$$Similarly, \frac{1}{\sqrt{(1-b^2)(1-y^2)}} \overset{(2)}{\ge} \frac{1}{|1-by|} \text{ and } \frac{1}{\sqrt{(1-c^2)(1-z^2)}} \overset{(3)}{\ge} \frac{1}{|1-cz|}$$

$$Let \frac{1}{\sqrt{1-a^2}} = A, \frac{1}{\sqrt{1-b^2}} = B, \frac{1}{\sqrt{1-c^2}} = C$$

$$\frac{1}{\sqrt{1-x^2}} = X, \frac{1}{\sqrt{1-y^2}} = Y, \frac{1}{\sqrt{1-z^2}} = Z$$

$$\therefore \Omega_1\Omega_2 = \left(\sum A^{14}\right)\left(\sum x^{14}\right) \overset{CBS}{\ge} \left\{\sum (AX)^7\right\}^2 \ge \left(\sum \left|\frac{1}{1-ax}\right|^7\right)^2$$

$$\textit{(using (1), (2), (3))} = \left(\sum \left|\left(\frac{1}{1-ax}\right)^7\right|\right)^2$$

$$\ge \left(\sum \left(\frac{1}{1-ax}\right)^7\right)^2 = \Omega_3^2 \ (\because |u| \ge u) \quad \textit{(Proved)}$$

1.55. $\boldsymbol{\Omega_1 = |z_1 + z_2 + z_3|, z_1, z_2, z_3 \in \mathbb{C}}$

$$\boldsymbol{\Omega_2 = |z_1 + z_2 - z_3 + 4i| + |z_1 - z_2 + z_3 + 2i| + |-z_1 + z_2 + z_3 - 6i|}$$

Prove that: $\boldsymbol{\Omega_1 \le \Omega_2}$

Solution:

$$\Omega_1 = |z_1 + z_2 + z_3|$$

$$\Omega_2 = |z_1 + z_2 - z_3 + 4i| + |z_1 - z_2 + z_3 + 2i| + |-z_1 + z_2 + z_3 - 6i|$$

$$\ge |z_1 + z_2 - z_3 + 4i + z_1 - z_2 + z_3 + 2i - z_1 + z_2 + z_3 - 6i| =$$

$$= |z_1 + z_2 + z_3| = \Omega_1 \therefore \Omega_2 \ge \Omega_1$$

1.56. If $a, b, c > 0$ then:

$$\frac{(2a+b+c)(a+2b+c)(a+b+2c)}{(a+b)(b+c)(c+a)} \geq 8$$

Solution:

$$Let\ be\ f:(0,1)\to\mathbb{R}; f(x)=\log\left(\frac{1+x}{1-x}\right)$$

$$f(x)=\log(1+x)-\log(1-x)$$

$$f'(x)=\frac{1}{1+x}+\frac{1}{1-x}$$

$$f''(x)=\frac{-1}{(1+x)^2}+\frac{1}{(1-x)^2}=\frac{-1+2x-x^2+1+2x+x^2}{(1-x^2)^2}$$

$f''(x)=\frac{4x}{(1-x^2)^2}>0; (\forall)x\in(0,1)\Rightarrow f$ *convexe. By Jensen's inequality:*

$$f\left(\frac{\frac{a}{a+b+c}+\frac{b}{a+b+c}+\frac{c}{a+b+c}}{3}\right)\leq\frac{1}{3}\sum_{cyc}f\left(\frac{a}{a+b+c}\right)$$

$$f\left(\frac{1}{3}\right)\leq\frac{1}{3}\sum_{cyc}\log\left(\frac{1+\frac{a}{a+b+c}}{1-\frac{a}{a+b+c}}\right), 3f\left(\frac{1}{3}\right)\leq\log\left(\prod_{cyc}\left(\frac{2a+b+c}{b+c}\right)\right)$$

$$3\log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)\leq\log\left(\prod_{cyc}\left(\frac{2a+b+c}{b+c}\right)\right)$$

$$\log 8\leq\log\left(\prod_{cyc}\left(\frac{2a+b+c}{b+c}\right)\right), \prod_{cyc}\left(\frac{2a+b+c}{b+c}\right)\geq 8$$

1.57. If $a, b, c > 0$ then:

$$\frac{abc(a+b)(b+c)(c+a)}{8}\leq\left(\frac{a+b+c}{3}\right)^6$$

Solution:

$$abc\overset{AM-GM}{\leq}\left(\frac{a+b+c}{3}\right)^3\ (1)$$

$$(a+b)(b+c)(c+a)\overset{AM-GM}{\leq}\left(\frac{a+b+b+c+c+a}{3}\right)^3=$$

$$=\left(\frac{2(a+b+c)}{3}\right)^3=\frac{8(a+b+c)^3}{3^3}=8\left(\frac{a+b+c}{3}\right)^3$$

$$(a+b)(b+c)(c+a)\le 8\left(\frac{a+b+c}{3}\right)^3 \quad (2)$$

By multiplying (2); (3):

$$abc(a+b)(b+c)(c+a)\le 8\left(\frac{a+b+c}{3}\right)^6$$

$$\frac{abc(a+b)(b+c)(c+a)}{8}\le\left(\frac{a+b+c}{3}\right)^6$$

1.58. If $a,b,c\ge 1$; $a,b,c\in\mathbb{N}$ then:

$$\frac{1}{\binom{2a}{a}^2}\sum_{k=0}^{a}\binom{a}{k}^3+\frac{1}{\binom{2b}{b}^2}\sum_{k=0}^{b}\binom{b}{k}^3+\frac{1}{\binom{2c}{c}^2}\sum_{k=0}^{c}\binom{c}{k}^3\ge\frac{9}{2^a+2^b+2^c}$$

Solution:

$$\binom{a}{0}^3+\binom{a}{1}^3+\cdots+\binom{a}{a}^3=\frac{\binom{a}{0}^4}{\binom{a}{0}}+\frac{\binom{a}{1}^4}{\binom{a}{1}}+\cdots+\frac{\binom{a}{a}^4}{\binom{a}{a}}\ge$$

$$\overset{BERGSTROM}{\ge}\frac{\left(\binom{a}{0}^2+\binom{a}{1}^2+\cdots+\binom{a}{a}^2\right)^2}{\binom{a}{0}+\binom{a}{1}+\cdots+\binom{a}{a}}=\frac{1}{2^a}\binom{2a}{a}^2$$

$$\frac{1}{\binom{2a}{a}^2}\sum_{k=0}^{a}\binom{a}{k}^3\ge\frac{1}{2^a} \quad (1)$$

$$\frac{1}{\binom{2b}{b}^2}\sum_{k=0}^{b}\binom{b}{k}^3\ge\frac{1}{2^b} \quad (2)$$

$$\frac{1}{\binom{2c}{c}^2}\sum_{k=0}^{c}\binom{c}{k}^3\ge\frac{1}{2^c} \quad (3)$$

By adding (1); (2); (3):

$$\frac{1}{\binom{2a}{a}^2}\sum_{k=0}^{a}\binom{a}{k}^3+\frac{1}{\binom{2b}{b}^2}\sum_{k=0}^{b}\binom{b}{k}^3+\frac{1}{\binom{2c}{c}^2}\sum_{k=0}^{c}\binom{c}{k}^3\ge$$

$$\ge\frac{1}{2^a}+\frac{1}{2^b}+\frac{1}{2^c}\overset{BERGSTROM}{\ge}\frac{(1+1+1)^2}{2^a+2^b+2^c}=\frac{9}{2^a+2^b+2^c}$$

$$Equality\ holds\ for\ a=b=c=1.$$

1.59. If $2\le a,b,c\le 3$ then:

$$\mathbf{2\left(\sqrt{a^a}+\sqrt{b^b}+\sqrt{c^c}\right)+a+b+c\ge a^2+b^2+c^2+6}$$

Solution:

$$Let\ be\ f:[2,3]\to\mathbb{R}, f(x)=\sqrt{x^x}+\tfrac{x}{2}-\tfrac{x^2}{2}-1$$

$$f'(x)=\left(x^{\frac{x}{2}}-\frac{x^2}{2}+\frac{x}{2}-1\right)'=\frac{1}{2}\cdot x^{\frac{x}{2}}\cdot\log x+\frac{x}{2}\cdot x^{\frac{x}{2}-1}-x+\frac{1}{2}=$$

$$=\frac{1}{2}x^{\frac{x}{2}}(\log x+1)-x+\frac{1}{2}=\frac{x^{\frac{x}{2}}(\log x+1)-2x+1}{2}$$

$$Let\ be\ g:[2,3]\to\mathbb{R}; g(x)=x^{\frac{x}{2}}(\log x+1)-2x+1$$

$$g'(x)=\left(x^{\frac{x}{2}}\right)'(1+\log x)+\frac{1}{x}x^{\frac{x}{2}}-2$$

$$g'(x)=\left(\frac{1}{2}\sqrt{x^x}\log x+\frac{1}{2}\sqrt{x^x}\right)(1+\log x)+\frac{1}{x}\sqrt{x^x}-2$$

$$g'(x)=\sqrt{x^x}(\log x+1)^2+\frac{1}{x}\sqrt{x^x}-2,$$

$$g'(x)=\sqrt{x^x}\left(\frac{1}{2}(\log x+1)^2+\frac{1}{x}\right)-2$$

$$x\ge 2\Rightarrow x^{\frac{x}{2}}\ge 2^{\frac{2}{2}}=2, x\le 3\Rightarrow\frac{1}{x}\ge\frac{1}{3}; 1+\log x\ge 1+\log 2$$

$$g'(x)=x^{\frac{x}{2}}\left(\frac{1}{2}(\log x+1)^2+\frac{1}{x}\right)-2\ge$$

$$\ge 2\left(\frac{1}{2}(1+\log 2)^2+\frac{1}{3}\right)-2=(1+\log 2)^2+\frac{2}{3}-2=$$

$$=1+2\log 2+\log^2 2-\frac{4}{3}=\log^2 2+2\log 2-\frac{1}{3}>0$$

$$g'(x) > 0 \Rightarrow f'(x) > 0 \Rightarrow f \ increasing$$

$$\Rightarrow f(x) \geq f(2) = \sqrt{2^2} + \frac{2}{2} - \frac{2^2}{2} - 1 = 0$$

$$f(x) \geq 0 \Rightarrow$$

$$\sqrt{a^a} + \frac{a}{2} - \frac{a^2}{2} - 1 \geq 0 \quad (1)$$

$$\sqrt{b^b} + \frac{b}{2} - \frac{b^2}{2} - 1 \geq 0 \quad (2)$$

$$\sqrt{c^c} + \frac{c}{2} - \frac{c^2}{2} - 1 \geq 0 \quad (3)$$

By adding (1); (2); (3):

$$2\left(\sqrt{a^a} + \sqrt{b^b} + \sqrt{c^c}\right) + a + b + c - (a^2 + b^2 + c^2) - 6 \geq 0$$

$$2\left(\sqrt{a^a} + \sqrt{b^b} + \sqrt{c^c}\right) + a + b + c \geq a^2 + b^2 + c^2 + 6$$

ANALYSIS

2.1. Find:

$$\boldsymbol{\Omega(a,b)} = \int \left(\tan(ax)\tan(bx)\tan((a+b)x)\right) dx, a, b > 0, 0 < x < \frac{\pi}{2(a+b)}$$

Solution:

$$\Leftrightarrow \tan(ax+bx) = \frac{\tan(ax)+\tan(bx)}{1-\tan(ax)\cdot\tan(bx)}$$

$$\Rightarrow \tan(a+b)x - \tan(ax)\cdot\tan(bx)\cdot\tan(a+b)x = \tan(ax)+\tan(bx)$$

$$\Rightarrow \tan(ax)\cdot\tan(bx)\cdot\tan(a+b)x = \tan(a+b)x - \tan(ax) - \tan(bx)$$

$$\Omega(a,b) = \int (\tan(ax)+\tan(bx)+\tan(a+b)\,x)\,dx$$

$$= \int \tan(a+b)\,x\,dx - \int \tan(ax)\,dx - \int \tan(bx)\,dx$$

$$= \frac{1}{(a+b)}\log|(\sec(a+b)\,x)| + \frac{1}{a}\cdot\log|(\cos(ax))| + \frac{1}{b}\cdot\log|\cos(bx)| + c$$

$$\left[\because \int \tan(mx)\,dx = \frac{1}{m}\log|\sec(mx)| + c = -\frac{1}{m}\log|(\cos mx)|\right]$$

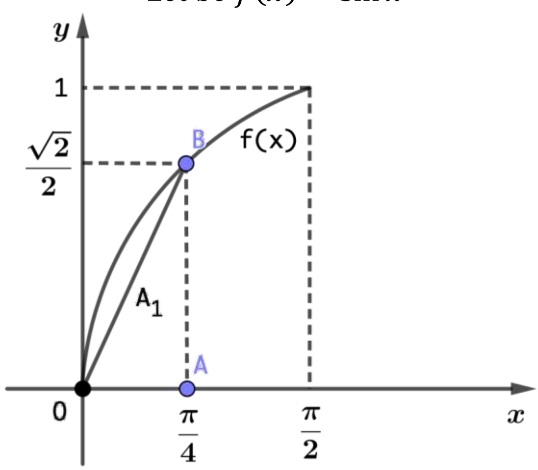
$$[c = \textit{integration constant}]$$

2.2. If $\boldsymbol{0 \le x, y, z \le \frac{\pi}{4}}$ then:

$$\boldsymbol{\pi(\sin x + \sin y + \sin z) \ge 6\sqrt[6]{8x^2y^2z^2}}$$

Solution:

$$\textit{Let be } f(x) = \sin x$$



$$S_{A_1} > S_{OAB} \Rightarrow \int_0^{\frac{\pi}{4}} \sin x\, dx > \int_0^{\frac{\pi}{4}} \frac{2\sqrt{2}}{\pi} x\, dx \quad (1)$$

$\therefore$ *The equation of the line OB:* $\left.\begin{matrix} O(0,0) \\ B\left(\frac{\pi}{4}; \frac{\sqrt{2}}{2}\right) \end{matrix}\right\} \Rightarrow \frac{x}{\frac{\pi}{4}} = \frac{y}{\frac{\sqrt{2}}{2}} \Rightarrow y = \frac{\frac{\sqrt{2}}{2}x}{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} \cdot$

$$\frac{4}{\pi}x \Rightarrow y = \frac{2\sqrt{2}}{\pi} \therefore$$

$$\textit{From (1)} \Rightarrow \sin x \geq \frac{2\sqrt{2}x}{\pi}\ (\forall) x \in \left[0, \frac{\pi}{4}\right] \Rightarrow \pi \sin x \geq 2\sqrt{2}x$$

$$\textit{So, } \pi(\sin x + \sin y + \sin z) \geq 2\sqrt{2}(x + y + z) \Rightarrow$$

$$\Rightarrow \pi(\sin x + \sin y + \sin z) \geq$$

$$\geq \sqrt{2}x + \sqrt{2}x + \sqrt{2}y + \sqrt{2}y + \sqrt{2}z + \sqrt{2}z \overset{MA \geq MG}{\geq}$$

$$\geq 6\sqrt[6]{\sqrt{2}^6 x^2 y^2 z^2} = 6\sqrt[6]{8x^2y^2z^2}$$

2.3. If $\mathbf{0} < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b \left(\frac{\tan x}{x \cos x}\right) dx \geq \log\left(\frac{1 + \tan b}{1 + \tan a}\right)$$

Solution:

$$\therefore f(x) = \sin x + \sin x \tan x - x; f'(x) = \cos x + \sin x + \frac{\sin x}{\cos^2 x} - 1 =$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4} - x\right) + \frac{\sin x}{\cos^2 x} - 1 > 0, f(0) = 0, f(x) \nearrow \forall x \in \left]0, \frac{\pi}{2}\right[\Rightarrow$$

$$\Rightarrow \frac{\sin x}{x} \geq \frac{1}{1 + \tan x}$$

$$\therefore \int_a^b \frac{\tan x}{x \cos x} dx = \int_a^b \frac{\sin x}{x \cos^2 x} dx \geq \int_a^b \frac{d(\tan x)}{1 + \tan x} = \log\left(\frac{1 + \tan b}{1 + \tan a}\right)$$

2.4. $\boldsymbol{\rho(x)} = \left|\left[\boldsymbol{x} + \frac{1}{2}\right] - \boldsymbol{x}\right|, [*]$ - great integer function. Prove that:

$$\left(\int_0^1 \left(\sqrt[3]{\rho(x)}\right) dx\right)\left(\int_0^1 \left(\sqrt[5]{\rho(x)}\right) dx\right)\left(\int_0^1 \left(\sqrt[7]{\rho(x)}\right) dx\right) < 1$$

Solution:

Using the following facts:

$$1.\ 0 \le x \le 1 \text{ then } \int_0^1 \sqrt[n]{x} < 1$$

$$2.\ 0 \le \rho(x) \le 1$$

We have:

$$\int_0^1 \sqrt[n]{\rho(x)}\,dx < 1$$

$$\therefore \left(\int_0^1 \sqrt[3]{\rho(x)}\,dx\right)\left(\int_0^1 \sqrt[5]{\rho(x)}\,dx\right)\left(\int_0^1 \sqrt[7]{\rho(x)}\,dx\right) < 1$$

2.5. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}\left(\log\left(\frac{2n+2k+1}{n+k}\right)\right) - n\log 2 - \frac{\log 2}{2}\right)\right)}$$

Solution:

$$\textit{First: } \sum_{k=1}^{n} \ln\left(\frac{2n+2k+1}{n+k}\right) = \ln\left(\prod_{k=1}^{n} \frac{2\left(n+k+\frac{1}{2}\right)}{n+k}\right) =$$

$$= n\cdot\ln 2 + \ln\left(\prod_{k=1}^{n} \frac{n+k+\frac{1}{2}}{n+k}\right)$$

$$\textit{But: } \prod_{k=1}^{n}\left(n+k+\frac{1}{2}\right) = \frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)},\ \prod_{k=1}^{n}(n+k) = \frac{\Gamma(2n+1)}{\Gamma(n+1)}$$

$$\textit{Then: } \sum_{k=1}^{n} \ln\left(\frac{2n+2k+1}{n+k}\right) = n\cdot\ln 2 + \underbrace{\ln\left(\frac{\Gamma\left(2n+\frac{3}{2}\right)\times\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)\times\Gamma(2n+1)}\right)}_{=\theta_n}$$

$$\textit{Therefore: } \Omega = \lim\left(n\left(\ln(\theta_n) + n\ln 2 - n\ln 2 - \frac{\ln 2}{2}\right)\right)$$

$$\Omega = \lim\left(n\left(\ln(\theta_n) - \frac{\ln 2}{2}\right)\right)$$

$$\textit{But, by Laurent series: } \ln(\theta_n) = \frac{\ln 2}{2} - \frac{3}{16n} + \theta\left(\left(\frac{1}{n}\right)^{\frac{3}{2}}\right)$$

$$\Leftrightarrow \Omega = \lim\left[n\left(\frac{\ln 2}{2} - \frac{3}{16n} - \frac{\ln 2}{2}\right)\right] = \lim_{n\to\infty}\left[-\frac{3}{16}\right] = -\frac{3}{16}$$

2.6.

$$\boldsymbol{\Omega_n + H_n = \sum_{k=3}^{n}\left(\left(\frac{\sqrt[k+1]{(k+1)!}}{\sqrt[k]{k!}}\right)^{k+1}\cdot\left(\frac{\sqrt[k-1]{(k-1)!}}{\sqrt[k]{k!}}\right)^{k-1}\right), n\in\mathbb{N}, n\geq 3}$$

Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\sqrt[n]{\Omega_n}}$$

Solution:

$$\Omega_n + H_n = \sum_{k=3}^{k=n}\left(\left(\frac{\sqrt[k+1]{(k+1)!}}{\sqrt[k]{k!}}\right)^{k+1}\left(\frac{\sqrt[k-1]{(k-1)!}}{\sqrt[k]{k!}}\right)^{k-1}\right)$$

$$= \sum_{k=3}^{k=n}\frac{(k+1)!}{(k!)^{1+\frac{1}{k}}}\cdot\frac{(k-1)!}{(k!)^{1-\frac{1}{k}}} = \sum_{k=3}^{k=n}\frac{(k+1)!\cdot(k-1)!}{k!\cdot k!} =$$

$$= \sum_{k=3}^{k=n}\frac{(k+1)!}{k!}\cdot\frac{(k-1)!}{k!} = \sum_{k=3}^{k=n}(k+1)\times\frac{1}{k}$$

$$\Omega_n + H_n = \sum_{k=3}^{k=n}k + \sum_{k=3}^{k=n}\frac{1}{k} = (n-2) + H_n - 1 - \frac{1}{2}$$

$$\Omega_n = n - 2 - \frac{3}{2} = n - \frac{7}{2}, \textit{ so, } \lim_{n\to\infty}\sqrt{\Omega_n} = 1$$

2.7. $\boldsymbol{\pi(x)}$ **– number of prime numbers less than** $\boldsymbol{x, x\geq 2}$**. Find:**

$$\boldsymbol{\Omega = \lim_{x\to\infty}\left(\frac{\pi(1+\log(1+3x))}{\pi(1+\log(1+2x))}\right)}$$

Solution:

$$\lim_{x\to\infty}\frac{\pi(x)}{\frac{x}{\ln x}} = 1$$

$$\Omega = \lim_{x\to\infty}\left(\frac{\pi(1+\ln(1+3x))}{\pi(1+\ln(1+2x))}\right) = \lim_{x\to\infty}\frac{\frac{1+\ln(1+3x)}{\ln(1+\ln(1+3x))}}{\frac{1+\ln(1+2x)}{\ln(1+\ln(1+2x))}} =$$

$$= \lim_{x\to\infty} \frac{\dfrac{1+\dfrac{\ln(1+3x)}{3x}\cdot 3x}{\dfrac{\ln(1+\ln(1+3x))}{\ln(1+3x)}\cdot\ln(1+3x)}}{\dfrac{1+\dfrac{\ln(1+2x)}{2x}\cdot 2x}{\dfrac{\ln(1+\ln(1+2x))}{\ln(1+2x)}\cdot\ln(1+2x)}} = \lim_{x\to\infty}\frac{\dfrac{1+3x}{\ln(1+3x)}}{\dfrac{1+2x}{\ln(1+2x)}} =$$

$$= \lim_{x\to\infty} \frac{1+3x}{\dfrac{\ln(1+3x)}{3x}\cdot 3x}\cdot\frac{\dfrac{\ln(1+2x)}{2x}\cdot 2x}{1+2x} =$$

$$= \lim_{x\to\infty}\frac{1+3x}{3x}\cdot\frac{2x}{1+2x} = \lim_{x\to\infty}\frac{6x+2}{6x+3} = 1$$

2.8. If $\mathbf{0} < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b\int_a^b\left(\frac{\sqrt{xy}\cdot\sin\left(\frac{x+y}{2}\right)}{\sin\left(\sqrt{xy}\right)}\right)dx\,dy \le \frac{(b+a)(b-a)^2}{2}$$

Solution:

We know that: $\tan x \ge x \ge \sin x$ *for all* $x \ge 0$. *Let* $f(x) = \frac{x}{\sin x}$ *for all* $x \ge 0$. *Now,* $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} = \frac{\cos x}{\sin^2 x}(\tan x - x) \ge 0$ *for all* $x \ge 0$

Hence f is an increasing function. For increasing functions, we have:

$f(m) \ge f(n)$ *for all* $m, n \in \mathbb{R}$ *where* $m \ge n$

so, for $\sqrt{xy} \le \frac{x+y}{2}$, *we have* $\frac{\sqrt{xy}}{\sin(\sqrt{xy})} \le \frac{\frac{x+y}{2}}{\sin\left(\frac{x+y}{2}\right)}$

$$\therefore \int_a^b\int_a^b\frac{\sqrt{xy}\sin\left(\frac{x+y}{2}\right)}{\sin\left(\sqrt{xy}\right)}dx\,dy \le \frac{1}{2}\int_a^b x\,dx\int_a^b dy + \frac{1}{2}\int_a^b y\,dy\int_a^b dx$$

$$\le \frac{(a-b)(a^2-b^2)}{4} + \frac{(a-b)(a^2-b^2)}{4} = \frac{(a+b)(a-b)^2}{2}$$

2.9. If $\mathbf{0} < a, b, c < 2\boldsymbol{\pi}$ then:

$$\frac{1}{(a+b+c)^3}\int_{\frac{a}{2}}^{a}\int_{\frac{b}{2}}^{b}\int_{\frac{c}{2}}^{c}\left(\frac{\sin\frac{x}{2}\sin\frac{y}{2}+\sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)}{\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)}\right)dxdydz \le \frac{1}{216}$$

Solution:

$$\sin\frac{x}{2}\sin\frac{y}{2}+\sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)=$$

$$=\frac{1}{2}\left(\cos\frac{x-y}{2}-\cos\frac{x+y}{2}+\cos\frac{x+y}{2}-\cos\frac{x+y+2z}{2}\right)$$

$$=\frac{1}{2}\cdot 2\sin\left(\frac{\frac{x-y}{2}+\frac{x+y+2z}{2}}{2}\right)\sin\left(\frac{\frac{x+y+2z}{2}-\frac{x-y}{2}}{2}\right)=$$

$$=\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)$$

$$\therefore LHS=\frac{1}{(a+b+c)^3}\int_{\frac{a}{2}}^{a}\int_{\frac{b}{2}}^{b}\int_{\frac{c}{2}}^{c}\frac{\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)}{\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)}dxdydz=$$

$$=\frac{abc}{8(a+b+c)^3}\overset{A-G}{\leq}\frac{abc}{8\cdot 27abc}=\frac{1}{216}\ \textit{(Proved)}$$

2.10.

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(a^{\frac{1}{n+1}}+a^{\frac{1}{n+2}}+\cdots+a^{\frac{1}{2n}}-n\right)}$$

Prove that:

$$\boldsymbol{\frac{1}{\log 2}\big(a\Omega(b)+b\Omega(a)\big)>\log\left(\frac{ab}{(a+b)^2}\right),a,b\in(0,1)}$$

Solution:

$$u_n=\underbrace{a^{\frac{1}{n+1}}+a^{\frac{1}{n+2}}+\cdots+a^{\frac{1}{2n}}}_{t_n}-n$$

$$n\cdot a^{\frac{1}{2n}}\leq t_n\leq n\cdot a^{\frac{1}{n+1}}$$

$$n\cdot a^{\frac{1}{2n}}-n\leq t_n-n\leq n\cdot a^{\frac{1}{n+1}}-n$$

$$n\left(a^{\frac{1}{2n}}-1\right)\leq u_n\leq n\left(a^{\frac{1}{n+1}}-1\right)$$

$$\frac{a^{\frac{1}{2n}}-1}{\frac{1}{n}}\leq u_n\leq\frac{a^{\frac{1}{n+1}}-1}{\frac{1}{n}}$$

$$\frac{e^{\frac{1}{2n}\ln(a)}-1}{\frac{1}{n}}\leq u_n\leq\frac{e^{\frac{1}{n+1}\ln(a)}-1}{\frac{1}{n}}$$

$$Suppose\text{: } f(x) = \frac{e^{\frac{1}{2n}\ln(a)}-1}{\frac{1}{n}}, \lim_{n\to+\infty} f(x) = \lim_{n\to+\infty}\left[\frac{\frac{-\ln(a)}{2x^2}e^{\frac{1}{2x}\ln(a)}}{-\frac{1}{x^2}}\right] = \frac{1}{2}\ln(a)$$

$$Suppose\text{: } g(x) = \frac{e^{\frac{1}{n+1}\ln(a)}-1}{\frac{1}{n}}, \lim_{n\to+\infty} g(x) = \lim_{n\to+\infty}\left[\frac{\frac{-\ln(a)}{(x+1)^2}\cdot e^{\frac{1}{n+1}\ln(a)}}{-\frac{1}{n^2}}\right] = \ln(a)$$

$$So\text{: } \frac{1}{2}\ln(a) \le \lim_{n\to+\infty} u_n \le \ln(a)$$

$$\frac{1}{2}\ln(a) \le \Omega(a) \le \ln(a)$$

$$\frac{1}{\log 2}\big(a\cdot\Omega(b) + b\Omega(a)\big) \overset{?}{>} \log\left(\frac{a\cdot b}{(a+b)^2}\right)$$

$$\Omega(b) \ge \frac{1}{2}\ln(b), \Omega(a) \ge \frac{1}{2}\ln(b)$$

$$a\cdot\Omega(b) + b\Omega(a) \ge \frac{1}{2}a\ln(b) + \frac{1}{2}b\ln(a)$$

$$Let\ us\ prove\text{: } \frac{1}{\log(2)}\big(a\cdot\Omega(b) + b\Omega(a)\big) \overset{?}{>} \log\left(\frac{a\cdot b}{(a+b)^2}\right)$$

$$\underbrace{\frac{1}{\log(2)}\big(a\cdot\Omega(b) + b\Omega(a)\big)}_{L_1} \ge \frac{1}{\log 2}\left(\frac{1}{2}a\cdot\ln(b) + \frac{1}{2}b\ln a\right)$$

$$L_1 \ge \frac{1}{2}\cdot\frac{1}{\log 2}[a\cdot\ln b + b\ln a] \overset{?}{>} \log\left(\frac{a\cdot b}{(a+b)^2}\right)$$

$$\begin{matrix} 0 < b < 1 \\ 0 < a < 1 \end{matrix} \Rightarrow b\cdot\log a > \log a \Rightarrow \frac{1}{2}b\cdot\log a > \frac{1}{2}\log a$$

$$Similarly\text{: } \frac{1}{2}a\cdot\log b > \frac{1}{2}\log b$$

$$\frac{1}{2}(a\cdot\log b + b\cdot\log a) > \frac{1}{2}\log(a\cdot b)$$

$$So\text{: } \frac{1}{2\log 2}[a\cdot\log b + b\cdot\log a] > \frac{1}{2\log 2}\cdot\log(a\cdot b)$$

$$Let\ us\ prove\text{: } \frac{1}{2\cdot\log 2}\cdot\log(a\cdot b) \overset{?}{>} \log\left(\frac{a\cdot b}{(a+b)^2}\right)$$

$$\frac{1}{2\log 2}\cdot\log(a\cdot b) \overset{?}{>} \log(a\cdot b) - \log(a+b)^2$$

$$\log(a+b)^2 \overset{?}{>} \log(a\cdot b) - \frac{1}{2\log 2}\cdot\log(a\cdot b)$$

$$\log(a+b)^2 \overset{?}{>} \log(a\cdot b)\left[1-\frac{1}{2\log 2}\right]$$

$$\frac{\log(a+b)^2}{\log(a\cdot b)} \overset{?}{<} 1-\frac{1}{2\log 2} \quad (*)$$

Because: $a\cdot b < (a+b)^2 \Rightarrow \log(a\cdot b) < \log(a+b)^2$

$a, b \in]0,1[$

$$1 > \frac{\log(a+b)^2}{\log(a\cdot b)} \Rightarrow \frac{\log(a+b)^2}{\log(a\cdot b)} - 1 < 0 < -\frac{1}{2\log 2}$$

Relation (*) is true.

So: $\frac{1}{\log 2}\left(a\cdot\Omega(b)+b\Omega(a)\right) > \log\left(\frac{a\cdot b}{(a+b)^2}\right)$

2.11. If $a, b, c > 0, a+b+c=3$ then:

$$\int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx + \int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx + \int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx > 3e-6$$

Solution:

$$\int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx = \int_0^a \left(1+\frac{e^x}{2}-1\right)^{e^x} dx \overset{Bernoulli}{\geq} \int_0^a 1+e^x\left(\frac{e^x}{2}-1\right)dx =$$

$$= \int_0^a \left(1-e^x+\frac{e^{2x}}{2}\right)dx = x\Big|_0^a - e^x\Big|_0^a + \frac{e^{2x}}{4}\Big|_0^a =$$

$$= \left.\begin{matrix} a-e^a+1+\frac{e^{2a}}{4}-\frac{1}{4} = a-\frac{1}{4}+\left(\frac{e^a}{2}-1\right)^2 \\ a>0 \Rightarrow e^a>1 \Rightarrow \left(\frac{e^a}{2}-1\right)^2 > \left(\frac{1}{2}-1\right)^2 = \frac{1}{4} \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx > a-\frac{1}{4}+\frac{1}{4} = a$$

Similar: $\int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx > b$ *and* $\int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx > c$

$$\int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx + \int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx + \int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx >$$

$$> a+b+c = 3 \overset{?}{>} 3e-6 \Leftrightarrow 9 > 3e \text{ (true)}$$

$$\text{Note: } \begin{cases} \frac{e^x}{2} - 1 > -1 \text{ because } x \in (0; a) \text{ and } a > 0 \\ e^x > 1 \text{ because } x > 0 \end{cases}$$

2.12. If $0 < a \le b < \frac{\pi}{2}, a + b = \frac{\pi}{2}$ then:

$$\int_a^b \left(\frac{\sin x}{x}\right) dx \ge \frac{b-a}{2}$$

Solution:

Use Hermite – Hadamard inequality:

$f\left(\frac{a+b}{2}\right) \le \frac{\int_a^b f(x)dx}{b-a} \le \frac{f(a)+f(b)}{2}$, *for any convex function because* $f(x) = \frac{\sin x}{x}$, *is concave,* $x \in \left(0, \frac{\pi}{2}\right)$; *result -* $f(x)$, *convex. We obtain:*

$$\frac{\sin\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \ge \frac{\int_a^b \frac{\sin x}{x} dx}{b-a} \ge \frac{\frac{\sin a}{a} + \frac{\sin b}{b}}{2}$$

and use the well-known Jordan inequality: $\frac{\sin x}{x} \ge \frac{2}{\pi}$, *it follows:*

$$\frac{\sin a}{a} + \frac{\sin b}{b} > \frac{2}{\pi} + \frac{2}{\pi} = \frac{4}{\pi} > 1, \text{ it follows:}$$

$$\int_a^b \frac{\sin x}{x} dx \ge (b-a)\left(\frac{\frac{\sin a}{a} + \frac{\sin b}{b}}{2}\right) > \frac{b-a}{2}$$

2.13. If $f: [a,b] \to (0,1), a, b \in \mathbb{R}, a \le b, f$ – continuous then:

$$\int_a^b \int_a^b \int_a^b \left(\prod_{cyc} (\sin f(x) \cdot \sin^{-1} f(x))\right) dx\, dy\, dz \ge \left(\int_a^b f^2(x)\, dx\right)^3$$

Solution:

Because: $f(x)$: *continuous on* $[a; b], t = f(x) \in (0,1)$, *we let:*

$$g(t) = (\sin t \cdot \sin^{-1} t) - t^2, 0 < t < 1$$

$$\to g'(t) = \frac{\sin t}{\sqrt{1-t^2}} - 2t + \sin t \cdot \sin^{-1} t =$$

$$= \frac{\sin t - 2t\sqrt{1-t^2} + \sin t \cdot \sin^{-1} t \cdot \sqrt{1-t^2}}{\sqrt{1-t^2}} = 0 \leftrightarrow t = 0 \notin (0,1)$$

t	0		1
$g'(t)$	0	$+$	‖
$g(t)$	0	↗	$+\infty$

$$\text{Hence: } g(t) > 0 \leftrightarrow \sin t \cdot \sin^{-1} t > t^2, 0 < t < 1$$

$$\to \int_a^b \int_a^b \int_a^b \left(\prod_{cyc} \sin f(x) \cdot \sin^{-1} f(x) \right) dx\, dy\, dz$$

$$\geq \int_a^b \int_a^b \int_a^b \left(f^2(x) \cdot f^2(y) \cdot f^2(z) \right) dx\, dy\, dz$$

$$= \left(\int_a^b f^2(x)\, dx \right) \left(\int_a^b f^2(y)\, dy \right) \left(\int_a^b f^2(z) \right) = \left(\int_a^b f^2(x)\, dx \right)^3$$

Proved. Equality if and only if $a = b$.

2.14. Find:

$$\boldsymbol{\Omega(a) = \int_0^1 \left(\frac{a^x + 3 + (3-a)^x}{(3 + a^x - a)(a + (3-a)^x)} \right) dx, 1} < a < 2$$

Solution:

$$Let\ a = 2x; b = 3y; c = 6z\ (x, y, z > 0)$$

$$\text{We have: Inequality} \leftrightarrow \frac{(2x)^7}{2} \cdot \frac{1}{3y+6z} + \frac{(3y)^7}{3} \cdot \frac{1}{6z+2x} + \frac{(6z)^7}{6} \cdot \frac{1}{2x+3y} \geq$$

$$\geq \frac{(x+y+z)^7}{x+2y+5z}. \text{ Suppose: } a \leq b \leq c \to 2x \leq 3y \leq 6z.$$

By Chebyshev's inequality:

$$\leftrightarrow \frac{(2x)^7}{2} \cdot \frac{1}{3y+6z} + \frac{(3y)^7}{3} \cdot \frac{1}{6z+2x} + \frac{(6z)^7}{6} \cdot \frac{1}{2x+3y} \geq$$

$$\geq \frac{1}{3} \left(\frac{(2x)^7}{2} + \frac{(3y)^7}{3} + \frac{(6z)^7}{6} \right) \left(\frac{1}{3y+6z} + \frac{1}{6z+2x} + \frac{1}{2x+3y} \right) \overset{Chebysev}{\geq}$$

$$\geq \frac{1}{3} \cdot \frac{1}{3} \cdot \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \right) \left((2x)^7 + (3y)^7 + (6z)^7 \right) \cdot \frac{9}{4x + 6y + 12z} \geq$$

$$\ge \frac{1}{9}\cdot\frac{(2x+3y+6z)^7}{3^6}\cdot\frac{9}{4x+6y+12z}$$

(Because: $\frac{1}{3y+6z}+\frac{1}{6z+2x}+\frac{1}{2x+3y}\overset{Schwarz}{\ge}\frac{9}{(4x+6y+12)}$*)*

Now, we must show that: $\frac{(2x+3y+6z)^7}{3^7}\cdot\frac{3}{4x+6y+12z}\ge\frac{(x+y+z)^7}{x+2y+5z}$

$$\leftrightarrow \frac{(2x+3y+6z)^7}{3^7}\cdot\frac{3(x+2y+5z)}{4x+6y+12z}\ge(x+y+z)^7 \quad (*)$$

$$6z\ge 2x\leftrightarrow 3z\ge x\leftrightarrow 2x+3y+6z\ge 3x+3y+3z\leftrightarrow$$

$$\leftrightarrow\frac{(2x+3y+6z)^7}{3^7}\ge(x+y+z)^7$$

$$\frac{3(x+2y+5z)}{4x+6y+12z}\ge 1\leftrightarrow 3x+6y+15z\ge 4x+6y+12z\leftrightarrow 3z\ge x\leftrightarrow$$

$\leftrightarrow 6z\ge 2x$*. True. Hence, (*) true. We proved. Equality* $\leftrightarrow a=b=c$.

2.15. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\int_{e^{H_n}}^{e^{H_{n+1}}}\left(\tan^{-1}\left(\frac{x}{n}\right)\right)^5dx}$$

Solution:

Let $f(x)=\left(\tan^{-1}\left(\frac{x}{n}\right)\right)^5$*, f is continuous on* $I=[e^{H_n};e^{H_{n+1}}]\Rightarrow\exists\zeta\in I$ *such that*

$$\int_{e^{H_n}}^{e^{H_{n+1}}}\left(\tan^{-1}\left(\frac{x}{n}\right)\right)^5=(e^{H_{n+1}}-e^{H_n})\cdot\left(\tan^{-1}\left(\frac{\zeta}{n}\right)\right)^5\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}\int_{e^{H_n}}^{e^{H_{n+1}}}\left(\tan^{-1}\left(\frac{x}{n}\right)\right)^5dx=\lim_{n\to\infty}(e^{H_{n+1}}-e^{H_n})\left(\tan^{-1}\left(\frac{\zeta}{n}\right)\right)^5=$$

$$=\lim_{n\to\infty}e^{H_n}(e^{H_{n+1}-H_n}-1)\left(\tan^{-1}\left(\frac{\zeta}{n}\right)\right)^5=$$

$$=\lim_{n\to\infty}e^{H_n-\ln n}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\cdot\left(\tan^{-1}\left(\frac{\zeta}{n}\right)\right)^5=$$

$$=e^{\gamma}\cdot\lim_{n\to\infty}\left(\tan^{-1}\left(\frac{\zeta}{n}\right)^5\right)\quad (1)$$

$$e^{H_n} \le \zeta \le e^{H_{n+1}} \Rightarrow \frac{e^{H_n}}{n} \le \frac{\zeta}{n} \le \frac{e^{H_{n+1}}}{n} \quad (2)$$

$$\lim_{n\to\infty} \frac{e^{H_n}}{n} = \lim_{n\to\infty} e^{H_n - \ln n} = e^{\gamma} \quad (3)$$

$$\lim_{n\to\infty} \frac{e^{H_{n+1}}}{n} = \lim_{n\to\infty} e^{H_{n+1} - \ln(n+1)} = e^{\gamma} \quad (4)$$

$$f \text{ is continuous} \overset{(2);(3);(4)}{\Rightarrow} \lim_{n\to\infty} \frac{\zeta}{n} = e^{\gamma} \overset{(1)}{\Rightarrow} e^{\gamma} \cdot (\tan^{-1}(e^{\gamma}))^5$$

2.16. If $n \in \mathbb{N}, n \ge 2$ then:

$$\mathbf{3^{n-2} \cdot \left(\prod_{k=2}^{n}\left(\int_0^1 \left(x^k\sqrt{x^2+1}\right)dx\right)\right)^2 < 4^{n-2} \cdot \prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right)}$$

Solution:

$$\Leftrightarrow \int_a^b f(x) \cdot g(x)dx \overset{Holder}{\le} \sqrt{\int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx}$$

$$\Leftrightarrow \int_0^1 \left(x^k\sqrt{x^2+1}dx\right) \overset{Holder}{\le} \int_0^1 x^{2k}\,dx \cdot \int_0^1 (x^2+1)\,dx = \frac{2}{\sqrt{3(2k+1)}}$$

$$\Rightarrow \prod_{k=2}^{n}\left(\int_0^1 x^k\sqrt{x^2+1}dx\right) \le \prod_{k=2}^{n} \frac{2}{\sqrt{3(2k+1)}}$$

$$\Rightarrow \left(\prod_{k=2}^{n}\left(\int_0^1 x^k\sqrt{x^2+1}\,dx\right)\right)^2 \le \left(\prod_{k=2}^{n} \frac{2}{\sqrt{3(2k+1)}}\right)^2 =$$

$$= \prod_{k=2}^{n} \frac{2}{\sqrt{3(2k+1)}} \cdot \prod_{k=2}^{n} \frac{2}{\sqrt{3(2k+1)}}$$

$$= \prod_{k=2}^{n} \frac{4}{3(2k+1)} = \prod_{k=2}^{n}\left(\frac{4}{3}\right) \cdot \left(\int_0^1 x^{2k}\,dx\right) = \left(\frac{4}{3}\right)^{n-1} \prod_{k=2}^{n}\left(\int_0^1 x^{2k}\,dx\right)$$

$$\Rightarrow 3^{n-1}\left(\prod_{k=2}^{n}\left(\int_0^1 x^k\sqrt{x^2+1}\,dx\right)\right)^2 \le 4^{n-1} \prod_{k=2}^{n}\left(\int_0^1 x^{2k}\,dx\right)$$

$$\Rightarrow \frac{\left(\prod_{k=2}^{n}\left(\int_0^1 x^k\sqrt{x^2+1}\right)\right)^2}{3} < \frac{\prod_{k=2}^{n}\left(\int_0^1 x^{2k} dx\right)}{4} \ldots \quad \mathit{(1)} \quad [\because n \geq 2 \Rightarrow 3^n \leq 4^n]$$

$$\Leftrightarrow RHS = 4^{n-2}\prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right) = 4^{n-2}\prod_{k=2}^{n}\left[\frac{1}{4}\left(\frac{1}{2k+1}+\frac{1}{2k-1}\right)\right]$$

$$= \frac{4^{n-2}}{4^{n-1}}\prod_{k=2}^{n}\left(\frac{1}{2k-1}+\frac{1}{2k+1}\right) = \frac{1}{4}\prod_{k=2}^{n}\left(\int_0^1 x^{2k}\,dx + \int_0^1 x^{2k-2}\,dx\right)$$

$$\Leftrightarrow \int_0^1 x^{2k}\,dx - \left(\int_0^1 x^{2k}\,dx + \int_0^1 x^{2k-2}\,dx\right) = -\int_0^1 x^{2k-2}\,dx < 0$$

$$\Rightarrow \int_0^1 x^{2k}\,dx < \int_0^1 x^{2k}\,dx + \int_0^1 x^{2k-2}\,dx$$

$$\Rightarrow \frac{1}{4}\prod_{k=2}^{n}\left(\int_0^1 x^{2k}\,dx\right) < \frac{1}{4}\left[\prod_{k=2}^{n}\left(\int_0^1 x^{2k}\,dx + \int_0^1 x^{2k-2}\,dx\right)\right]$$

$$\Rightarrow \frac{1}{4}\prod_{k=2}^{n}\left(\int_0^1 x^{2k}\,dx\right) < 4^{n-2}\prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right)$$

$$\Leftrightarrow \mathit{From\ (1)} \ldots \Leftrightarrow$$

$$\Rightarrow \frac{1}{3}\left(\prod_{k=2}^{n}\left(\int_0^1 x^k\sqrt{x^2+1}\right)\right)^2 < \frac{1}{4}\prod_{k=2}^{n}\left(\int_0^1 x^{2k}\,dx\right) <$$

$$< 4^{n-2}\prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right)$$

$$\Rightarrow 3^{n-1}\left(\prod_{k=2}^{n}\left(\int_0^1 x^k\sqrt{x^2+1}\,dx\right)\right)^2 < 4^{n-1}\prod_{k=2}^{n}\left(\int_0^1 x^{2k}\,dx\right) <$$

$$< 4^{n-2}\prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right) \quad [n \geq 2 \Leftrightarrow 3^n \leq 4^n]$$

$$\Leftrightarrow 3^{n-1}\left(\prod_{k=2}^{n}\left(\int_0^1 x^k\sqrt{x^2+1}\,dx\right)\right)^2 < 4^{n-2}\prod_{k=2}^{n}\left(\frac{1}{4k^2-1}\right)$$

$$\Leftrightarrow \frac{3^{n-1}\left(\prod_{k=2}^{n}\left(\int_0^1 x^k\sqrt{x^2+1}\right)dx\right)^2}{3} < 4^{n-2}\prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right)$$

$$\Leftrightarrow 3^{n-2}\left(\prod_{k=2}^{n}\left(\int_0^1 x^k\sqrt{x^2+1}\right)dx\right)^2 < 4^{n-2}\prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right)$$

2.17. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(H_n + \log_2\frac{1}{3} + \log_3\frac{1}{5} + \cdots + \log_n\frac{1}{2n-1}\right)}$$

Solution:

$$2k-1 = k + \underbrace{1+1+\cdots+1}_{for\ "k-1"\ times} \overset{AM-GM}{>} k\cdot\sqrt[k]{k}$$

$$\log_k(2k-1) > \log_k\left(k\cdot\sqrt[k]{k}\right) = 1 + \log_k\sqrt[k]{k}$$

$$\log_k(2k-1) > 1 + \frac{1}{k}$$

$$\sum_{k=2}^{n}\log_k(2k-1) > n-1+\sum_{k=2}^{n}\frac{1}{k} =$$

$$= n-2+H_n - \sum_{k=2}^{n}\log_k\left(\frac{1}{2k-1}\right) > n-2+H_n$$

$$H_n + \sum_{k=2}^{n}\log_k\left(\frac{1}{2k-1}\right) < 2-n$$

$$\Omega \le \lim_{n\to\infty}(2-n) = -\infty \Rightarrow \Omega = -\infty$$

2.18. $\boldsymbol{x_1 = 3, n(x_1+x_2+\cdots+x_n) = x_n, n\in\mathbb{N}, n\ge 1}$

Find:

$$\boldsymbol{\Omega = \sum_{n=1}^{\infty}(-1)^{n+1}x_n}$$

Solution:

We have: $n(x_1+x_2+\cdots+x_n) = x_n, n\ge 1$ *then*

$$x_1+x_2+\cdots+x_n = \frac{x_n}{n}$$

$x_1+x_2+\cdots+x_{n-1} = \frac{x_{n-1}}{n-1}$ *for* $n\ge 2$ *then* $\frac{x_{n-1}}{n-1} + x_n = \frac{x_n}{n}$*. So,*

$x_n = -\frac{n}{(n-1)^2} x_{n-1}$ *for* $n \geq 2$. *So,* $x_{n+1} = -\frac{n+1}{n^2} x_n$ *for* $n \geq 1$. *We will prove that* $x_n = +\frac{3n(-1)^{n-1} \cdot n}{(n-1)!}$ *For* $n \geq 1$

By using mathematical induction: For $n = 1,2$ *we have* $x_1 = +3$ *and* $x_2 = -6$. *So, it's true. Suppose that's true for* $n = k \geq 1$. *So,* $x_k = \frac{3(-1)^{k-1} \cdot k}{(k-1)!}$. *We will prove that is true for* $n = k + 1$

$x_{k+1} = -\frac{k+1}{k^2}, x_k = -\frac{k+1}{k^2} \cdot \frac{3(-1)^{k-1}}{(k-1)!} k$. *So,* $x_{k+1} = \frac{3(-1)^k (k+1)}{k!}$ *(true)*

Then, $x_n = \frac{3n(-1)^{n-1}}{(n-1)!}$ *for* $n \geq 1$. *Let*

$$S_n = \sum_{k=1}^{k=n} (-1)^{k+1} x_k = \sum_{k=1}^{k=n} \frac{3k(-1)^{k+1} \cdot (-1)^{k-1}}{(k-1)!}$$

$$= \sum_{k=1}^{k=n} \frac{3k}{(k-1)!} = 3 \sum_{k=0}^{k=n-1} \frac{k+1}{k!} = 3 \sum_{k=0}^{k=n-1} \frac{k}{k!} + 3 \sum_{k=0}^{k=n-1} \frac{1}{k!}$$

$$\lim_{n \to +\infty} S_n = 3 \sum_{0}^{\infty} \frac{k}{k!} + 3 \sum_{k=0}^{\infty} \frac{1}{k!} = 3 + 3 \sum_{k=1}^{\infty} \frac{k}{k!} + 3e$$

Since $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$ *for all n.*

$$= 3 + 3 \sum_{k=1}^{\infty} \frac{1}{(k-1)!} + 3e = 3 + 3 \sum_{k=1}^{\infty} \frac{1}{k!} + 3e = 3 + 6e$$

2.19. Prove that:

$$\frac{\boldsymbol{\pi + 4}}{\boldsymbol{\pi - 4}} + \int_{1}^{a} \frac{(\tan^{-1} x)^2}{(x - \tan^{-1} x)^2} dx > \frac{1 + \sin a \cdot \tan^{-1} a}{\tan^{-1} a - a}, a > 1$$

Solution:

First, we compute: $I(a) = \int_1^a \frac{(\tan^{-1} x)^2}{(x - \tan^{-1} x)^2} dx$

We have: $f(x) = \frac{(\tan^{-1} x)^2}{(x - \tan^{-1} x)^2} = \frac{(\tan^{-1} x)^2}{x^2 - 2x \tan^{-1} x + (\tan^{-1} x)^2} =$

$$= 1 - \frac{x(x - 2\tan^{-1} x)}{(x - \tan^{-1} x)^2}$$

More, $g(x) = \frac{x^2+1}{\tan^{-1} x - x} \rightarrow$

$$\to g'(x) = \frac{2x(\tan^{-1} x - x) + (x^2+1) \cdot \frac{x^2}{x^2+1}}{(x - \tan^{-1} x)^2} = -\frac{x(x - 2\tan^{-1} x)}{(x - \tan^{-1} x)^2}$$

$$\to \int \left(-\frac{x(x-2\tan^{-1} x)}{(x-\tan^{-1} x)^2}\right) dx = g(x) + C = \frac{x^2+1}{\tan^{-1} x - x} + C \ \ (C: const).$$

Hence:

$$I(a) = \int_1^a \frac{(\tan^{-1} x)^2}{(x - \tan^{-1} x)^2} dx = \int_1^a f(x)\, dx =$$

$$= \int_1^a \left(1 - \frac{x(x - 2\tan^{-1} x)}{(x - \tan^{-1} x)^2}\right) dx$$

$$= \left(x + \frac{x^2+1}{\tan^{-1} x - x}\right)\Bigg|_1^a = \left(a + \frac{a^2+1}{\tan^{-1} a - a}\right) - \left(1 + \frac{2}{\tan^{-1} 1 - 1}\right)$$

$$= \frac{a\tan^{-1} a + 1}{\tan^{-1} a - a} - \frac{\tan^{-1} 1 + 1}{\tan^{-1} 1 - 1} = \frac{a\tan^{-1} a + 1}{\tan^{-1} a - a} - \frac{\pi+4}{\pi-4}$$

$$\to LHS = \frac{\pi+4}{\pi-4} + I(a) = \frac{\pi+4}{\pi-4} + \frac{a\tan^{-1} a + 1}{\tan^{-1} a - a} - \frac{\pi+4}{\pi-4} =$$

$$= \frac{a\tan^{-1} a + 1}{\tan^{-1} a - a} = \frac{1 + a\tan^{-1} a}{\tan^{-1} a - a} \overset{a > \sin a, a > 1}{>} \frac{1 + \sin a \tan^{-1} a}{\tan^{-1} a - a}. \textit{ Proved.}$$

2.20. If $x, y, z > 0$, $\Omega(x,y) = \lim_{n\to\infty} \left(\frac{\frac{n}{1^{\frac{x}{x+y}}} + \frac{n}{2^{\frac{x}{x+y}}} + \cdots + \frac{n}{n^{\frac{x}{x+y}}}}{1^{\frac{y}{x+y}} + 2^{\frac{y}{x+y}} + \cdots + n^{\frac{y}{x+y}}}\right)$

then: $\Omega(x,y) \cdot \Omega(y,z) \cdot \Omega(z,x) \geq 16\sqrt{2}$

Solution:

$$\Omega(x,y) \overset{Stolz\ Cesaro}{=} \lim_{n\to\infty} \frac{\frac{n+1}{1^{\frac{x}{x+y}}} + \frac{n+1}{2^{\frac{x}{3+y}}} + \cdots + \frac{n+1}{n^{\frac{x}{x+y}}} + \frac{n+1}{(n+1)^{\frac{x}{x+y}}} - \frac{n}{1^{\frac{x}{x+y}}} - \frac{n}{2^{\frac{x}{x+y}}} - \cdots - \frac{n}{n^{\frac{x}{x+y}}}}{1^{\frac{y}{x+y}} + 2^{\frac{y}{x+y}} + \cdots + (n+1)^{\frac{y}{x+y}} - 1^{\frac{y}{x+y}} - \cdots - n^{\frac{y}{x+y}}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1^{\frac{x}{x+y}}} + \frac{1}{2^{\frac{x}{x+y}}} + \cdots + \frac{1}{n^{\frac{x}{x+y}}} + \frac{n+1}{(n+1)^{\frac{x}{x+y}}}}{(n+1)^{\frac{y}{x+y}}} \overset{Stolz\ Cesaro}{=}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{\frac{x}{x+y}}} + \frac{n+2}{(n+2)^{\frac{x}{x+y}}} - \frac{n+1}{(n+1)^{\frac{x}{x+y}}}}{(n+2)^{\frac{y}{x+y}} - (n+1)^{\frac{y}{x+y}}} =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{\frac{x}{x+y}}} + (n+2)^{\frac{y}{x+y}} - (n+1)^{\frac{y}{x+y}}}{(n+2)^{\frac{y}{x+y}} - (n+1)^{\frac{y}{x+y}}}$$

$$= 1 + \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{\frac{x}{x+y}}}}{(n+1)^{\frac{y}{x+y}}\left(\left(\frac{n+2}{n+1}\right)^{\frac{y}{x+y}} - 1\right)} = 1 + \lim_{n\to\infty} \frac{1}{(n+1)\left(\left(\frac{n+2}{n+1}\right)^{\frac{y}{x+y}} - 1\right)} =$$

$$= 1 + \lim_{n\to\infty} \frac{\ln\left(\left(\frac{n+2}{n+1}\right)^{\frac{y}{x+y}}\right)}{\frac{y}{x+y}\ln\left(\frac{n+2}{n+1}\right)(n+1)\left(e^{\ln\left(\left(\frac{n+2}{n+1}\right)^{\frac{y}{x+y}}\right)} - 1\right)} =$$

$$= 1 + \frac{x+y}{y}\lim_{n\to\infty} \ln\left(\left(1+\frac{1}{n+1}\right)^{n+1}\right) = 1 + \frac{x+y}{y} = 2 + \frac{x}{y} \Rightarrow$$

$$\Rightarrow \Omega(x,y) = 2 + \frac{x}{y}; \Omega(y,z) = 2 + \frac{y}{z}; \Omega(z,x) = 2 + \frac{z}{x}$$

$$\Rightarrow \Omega(x,y)\cdot\Omega(y,z)\cdot\Omega(z,x) = \left(2+\frac{x}{y}\right)\left(2+\frac{y}{z}\right)\left(2+\frac{z}{x}\right) \quad \textit{(1)}$$

$$2 + \frac{x}{y} \geq 2\sqrt{\frac{2x}{y}} \quad \textit{(2)};\ 2 + \frac{y}{z} \geq 2\sqrt{\frac{2y}{z}} \quad \textit{(3)};\ 2 + \frac{z}{x} \geq 2\sqrt{\frac{2z}{x}} \quad \textit{(4)}$$

$$\overset{(2);(3);(4)}{\Rightarrow} \left(2+\frac{x}{y}\right)\left(2+\frac{y}{z}\right)\left(2+\frac{z}{x}\right) \geq 8\cdot 2\sqrt{2} = 16\sqrt{2} \quad \textit{(5)}$$

$$\overset{(1);(5)}{\Rightarrow} \Omega(x,y)\cdot\Omega(y,z)\cdot\Omega(z,x) \geq 16\sqrt{2}\ \textit{(Q.E.D.)}$$

2.21. Find:

$$\boldsymbol{\Omega = \sum_{n=0}^{\infty}\left(\left(\frac{1!!}{0!} + \frac{3!!}{1!} + \frac{5!!}{2!} + \cdots + \frac{(2n+1)!!}{n!}\right)\frac{2n+2}{(2n+3)!!}\right)}$$

Solution:

$$\Omega=\sum_{n=0}^{\infty}\left(\left(\frac{1!!}{0!}+\frac{3!!}{1!}+\frac{5!!}{2!}+\cdots+\frac{(2n+1)!!}{n!}\right)\frac{2n+2}{(2n+3)!!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\left(\sum_{n=0}^{\infty}\frac{(2k+1)!!}{k!}\right)\frac{2n+2}{(2n+3)!!}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(2k+1)!!}{k!}\left(\sum_{n=0}^{\infty}\frac{(2n+2)}{(2n+3)!!}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(2k+1)!!}{k!}\times\frac{1}{(2k+1)!!}=\sum_{n=0}^{\infty}\frac{1}{k!}=e$$

2.22. $\boldsymbol{x_n=6\cdot 3^n\cdot n!,y_n=12\cdot 4^n\cdot n!,n\geq 1}$

Find:

$$\boldsymbol{\Omega=\lim_{t\to 0}\left(\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{x_{n+1}^{\cosh^2 t}}{y_{n+1}^{\sinh^2 t}}}-\sqrt[n]{\frac{x_n^{\cosh^2 t}}{y_n^{\sinh^2 t}}}\right)\right)}$$

Solution:

$$\lim_{n\to\infty}\sqrt[n]{\frac{x_{n+1}^{\cosh^2 t}}{y_{n+1}^{\sinh^2 t}}}-\sqrt[n]{\frac{x_n^{\cosh^2 t}}{y_n^{\sinh^2 t}}}=\lim_{n\to\infty}\sqrt[n]{\frac{x_n^{\cosh^2 t}}{y_n^{\sinh^2 t}}}\left(\sqrt[n+1]{\frac{x_{n+1}^{\cosh^2 t}}{y_{n+1}^{\sinh^2 t}}}\cdot\sqrt[n]{\frac{y_n^{\sinh^2 t}}{x_n^{\cosh^2 t}}}-1\right)$$

(a)

$$\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\lim_{n\to\infty}e^{\frac{\ln\frac{x_n}{n^n}}{n}}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{x_n}=$$

$$=\lim_{n\to\infty}3(n+1)\cdot\frac{1}{e}(n+1)=\frac{3}{e}\Rightarrow\lim_{n\to\infty}\frac{\sqrt[n]{x_n^{\cosh^2 t}}}{n^{\cosh^2 t}}\left(\frac{3}{e}\right)^{\cosh^2 t}\quad (1)$$

$$\Rightarrow\lim_{n\to\infty}\frac{\sqrt[n+1]{x_{n+1}^{\cosh^2 t}}}{(n+1)^{\cosh^2 t}}=\left(\frac{3}{e}\right)^{\cosh^2 t}\quad (2)$$

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$$\underset{(1);(2)}{\overset{":"}{\Rightarrow}}\lim_{n\to\infty}\left(\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^{\cosh^2 t}\cdot\left(\frac{n+1}{n}\right)^{\cosh^2 t}=1$$

$$\Rightarrow\lim_{n\to\infty}\left(\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^{\cosh^2 t}=1$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{y_n}}{n}=\lim_{n\to\infty}e^{\frac{\ln\frac{y_n}{n^n}}{n}}\overset{Stolz\ Cesaro}{=}\lim_{n\to\infty}\frac{y_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{y_n}=\lim_{n\to\infty}4\cdot$$

$$\cdot(n+1)\cdot\frac{1}{e}\cdot\frac{1}{n+1}=\frac{4}{e}\Rightarrow\lim_{n\to\infty}\frac{n^{\sinh^2 t}}{\sqrt[n]{y_n^{\sinh^2 t}}}=\left(\frac{e}{4}\right)^{\sinh^2 t}\underset{(a)}{\Rightarrow}$$

$$\underset{(a)}{\Rightarrow}\lim_{n\to\infty}\sqrt[n+1]{\frac{x_{n+1}^{\cosh^2 t}}{y_{n+1}^{\sinh^2 t}}}-\sqrt[n]{\frac{x_n^{\cosh^2 t}}{y_n^{\sinh^2 t}}}=\lim_{n\to\infty}\frac{\sqrt[n]{x_n^{\cosh^2 t}}}{n^{\cosh^2 t}}\cdot\frac{n^{\cosh^2 t}}{\sqrt[n]{y_n^{\sinh^2 t}}}\cdot\frac{n^{\sinh^2 t}}{n^{\sinh^2 t}}\cdot$$

$$\cdot\left(\frac{\sqrt[n+1]{x_{n+1}^{\cosh^2 t}}}{\sqrt[n]{x_n^{\cosh^2 t}}}\cdot\frac{\sqrt[n]{y_n^{\sinh^2 t}}}{\sqrt[n+1]{y_{n+1}^{\sinh^2 t}}}-1\right)$$

$$=\left(\frac{3}{e}\right)^{\cosh^2 t}\cdot\left(\frac{e}{4}\right)^{\sinh^2 t}\cdot$$

$$\cdot\lim_{n\to\infty}n\left(\left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}\right)^{\cosh^2 t}\cdot\left(\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^{\cosh^2 t}\cdot\left(\frac{\sqrt[n]{y_n}}{\sqrt[n+1]{y_{n+1}}}\right)^{\sinh^2 t}-\left(\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^{\cosh^2 t}\right)=$$

$$=\frac{1}{e}\cdot\frac{3^{\cosh^2 t}}{4^{\sinh^2 t}}\cdot\lim_{n\to\infty}n\left(\left(\frac{\sqrt[n]{y_n}}{\sqrt[n+1]{y_{n+1}}}\right)^{\sinh^2 t}-1+1-\left(\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^{\cosh^2 t}\right)=$$

$$=\frac{1}{e}\cdot\frac{3^{\cosh^2 t}}{4^{\sinh^2 t}}\cdot\left(\lim_{n\to\infty}n\left(\left(\frac{\sqrt[n]{y_n}}{\sqrt[n+1]{y_{n+1}}}\right)^{\sinh^2 t}-1\right)-\lim_{n\to\infty}n\left(\left(\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^{\cosh^2 t}-1\right)\right)$$

(3)

$$\lim_{n\to\infty}n\left(\left(\frac{\sqrt[n]{y_n}}{\sqrt[n+1]{y_{n+1}}}\right)^{\sinh^2 t}-1\right)=$$

$$=\lim_{n\to\infty}n\cdot\frac{\left(\frac{\sqrt[n]{y_n}}{\sqrt[n+1]{y_{n+1}}}\right)^{\sinh^2 t}-1}{\frac{\sqrt[n]{y_n}}{\sqrt[n+1]{y_{n+1}}}-1}\cdot\left(\frac{\sqrt[n]{y_n}}{\sqrt[n+1]{y_{n+1}}}-1\right)=$$

$$=\sinh^2 t\cdot\lim_{n\to\infty}n\cdot\left(\frac{\sqrt[n]{y_n}}{\sqrt[n+1]{y_{n+1}}}-1\right)=\sinh^2 t\cdot\lim_{n\to\infty}\ln\left(\frac{y_n}{y_{n+1}}\cdot\sqrt[n+1]{y_{n+1}}\right)=$$

$$= \sinh^2 t \cdot \lim_{n\to\infty} \ln\left(\frac{\sqrt[n+1]{y_{n+1}}}{n+1} \cdot (n+1) \cdot \frac{y_n}{y_{n+1}}\right) =$$

$$= \sinh^2 t \cdot \lim_{n\to\infty} \ln\left(\frac{4}{e} \cdot (n+1) \cdot \frac{1}{(n+1)\cdot 4}\right) = -\sinh^2 t \quad (4)$$

$$\lim_{n\to\infty} n \cdot \frac{\left(\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^{\cosh^2 t} - 1}{\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}} - 1} \cdot \left(\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}} - 1\right) =$$

$$= \cosh^2 t \cdot \lim_{n\to\infty} n\left(\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}} - 1\right) =$$

$$= \cosh^2 t \cdot \lim_{n\to\infty} \ln\left(\frac{x_n}{x_{n+1}} \cdot \sqrt[n+1]{x_{n+1}}\right) =$$

$$= \cosh^2 t \cdot \lim_{n\to\infty} \ln\left(\frac{\sqrt[n+1]{x_{n+1}}}{n+1} \cdot (n+1) \cdot \frac{1}{3(n+1)}\right)$$

$$= \cosh^2 t \ln\left(\frac{3}{e} \cdot \frac{1}{3}\right) = -\cosh^2 t \quad (5)$$

$$\overset{(3);(4);(5)}{\Rightarrow} \lim_{n\to\infty} \sqrt[n+1]{\frac{x_{n+1}^{\cosh^2 t}}{y_{n+1}^{\sinh^2 t}}} - \sqrt[n]{\frac{x_n^{\cosh^2 t}}{y_n^{\sinh^2 t}}} = \frac{1}{e} \cdot \frac{3^{\cosh^2 t}}{4^{\sinh^2 t}}(\cosh^2 t - \sinh^2 t) =$$

$$= \frac{1}{e} \cdot \frac{3^{\cosh^2 t}}{4^{\sinh^2 t}} \Rightarrow \Omega = \lim_{t\to 0} \frac{1}{e} \cdot \frac{3^{\cosh^2 t}}{3^{\sinh^2 t}} \cdot \left(\frac{3}{4}\right)^{\sinh^2 t} = \frac{3}{e} \cdot \lim_{t\to 0} \left(\frac{3}{4}\right)^{\frac{e^{2t}+e^{-2t}-2}{4}} = \frac{3}{e} \Rightarrow$$

$$\Rightarrow \Omega = \frac{3}{e}.$$

2.23. GENERALIZATION OF MARIAN URSĂRESCU'S LIMIT

Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\frac{1}{n^p \cdot \sqrt[q]{n^p}} \sum_{i_1=1}^{n}\left(\sum_{i_2=1}^{n}\left(\cdots \sum_{i_p=1}^{n} \sqrt[q]{i_1 \cdot i_2 \cdot \ldots \cdot i_p}\right)\right)\right),}$$

$$\boldsymbol{p, q \ge 2, p, q \in \mathbb{N}}$$

Solution:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^p \cdot \sqrt[q]{n^p}} \sum_{i_1=1}^{n}\left(\sum_{i_2=1}^{n}\left(\cdots \sum_{i_n=1}^{n} \sqrt[q]{i_1 \cdot i_2 \cdot \ldots \cdot i_p}\right)\right)\right)$$

Note that sums are independent to each other thus

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i_1=1}^{n}\left(\frac{i_1}{n}\right)^{\frac{1}{q}}\cdot\frac{1}{n}\sum_{i_2=1}^{n}\left(\frac{i_2}{n}\right)^{\frac{1}{q}}\cdots\frac{1}{n}\sum_{i_1=1}^{n}\left(\frac{i_p}{n}\right)^{\frac{1}{q}}\right)$$

Applying the product rule of limit, we observe The Riemann sums giving us

$$\Omega = \underbrace{\int_0^1\cdot\int_0^1\cdots\int_0^1}_{p\ times}\left(\prod_{k=1}^{p}x_k\right)^{\frac{1}{q}} = \int_0^1\sqrt[q]{x_1}\cdot\int_0^1\sqrt[q]{x_2}\cdots\int_0^1\sqrt[q]{x_p}$$

$$and\ gives\ us\ \Omega = \underbrace{\frac{q}{q+1}\cdot\frac{q}{q+1}\cdots\frac{q}{q+1}}_{p\ times} = \frac{q^p}{(q+1)^p}$$

2.24. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{1\le i<j<k\le n}\frac{i\cdot j\cdot k}{(1+i^2)(1+j^2)(1+k^2)}\right)}$$

Solution:

$$For\ i,j,k\ge 1\ we\ have:$$

$$(1+i^2)(1+j^2)(1+k^2)\overset{AM-GM}{\ge} 2i\cdot 2j\cdot 2k = 8ijk$$

$$\Rightarrow \frac{ijk}{(1+i^2)(1+j^2)(1+k^2)}\le\frac{1}{8}$$

$$\Rightarrow 0<\Omega_n = \frac{1}{n^3}\sum_{1\le i<j<k\le n}\frac{ijk}{(1+i^2)(1+j^2)(1+k^2)}\le$$

$$\le\frac{1}{n^3}\sum_{1\le i<j<k\le n}\frac{1}{8} = \frac{1}{n^3}\cdot\frac{1}{8}\cdot n = \frac{1}{8n^2}\to 0\ (n\to\infty)\Rightarrow \Omega = \lim_{n\to\infty}\Omega_n = 0$$

2.25. Find:

$$\boldsymbol{\Omega = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8(n-k+1)^2-4n+4k-5}\right)\right)\right)}$$

Solution:

$$\Omega = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8(n-k+1)^2-4n+4k-5}\right)\right)\right)$$

We can observe that for $k = n - k + 1$ *the sum becomes*

$$\Omega = \sum_{k=1}^{\infty}\left(\sum_{k=n-k+1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)\right)\right)$$

which further can be decomposed into two infinite arctangent sum le.

$$\Omega = \left(\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{1}{k^2-k-1}\right)\right)\left(\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)\right)$$

Now, note that:

$$\lim_{M\to\infty}\sum_{k=1}^{M}\tan^{-1}\left(\frac{3}{k^2-k-1}\right) = \lim_{M\to\infty}\sum_{k=1}^{M}\tan^{-1}\left(\frac{(k+1)+(2-k)}{1+(2-k)(k+1)}\right)$$

$$= \lim_{M\to\infty}\sum_{k=1}^{M}\left(\tan^{-1}(k+1)+\tan^{-1}(2-k)\right)$$

Since sum is telescoping sum and gives us the partial sum as

$$\Omega_1 = \lim_{M\to\infty}\left(\tan^{-1}(M+4)+\tan^{-1}(M+5)+\tan^{-1}(M+6)\right) -$$

$$-\tan^{-1}(0) = \tfrac{3\pi}{2} \textit{ as } k+1+2-k = 3$$

Since $8k^2 - 4k - 1$ *cannot be factored into two linear factors so that sum becomes telescoping to make it multiple and divide by any number now*

$$\Omega_2 = \sum_{k=1}^{\infty}\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right) = \lim_{N\to\infty}\sum_{k=1}^{N}\tan^{-1}\left(\frac{10}{40k^2-20k-5}\right)$$

As $40k^2 - 20k - 5 = 4(k+1) + (6k+1)(6k-5)$

$$10 = \tfrac{6k+1}{2(k+1)} - \tfrac{6k-5}{2k} \textit{ thus}$$

$$\Omega_2 = \sum_{k=1}^{N}\tan^{-1}\left(\frac{\frac{6k+1}{2(k+1)}-\frac{6k-5}{2k}}{1+\frac{(6k+1)(6k-5)}{4k(k+1)}}\right)$$

$$= \sum_{k=1}^{N}\left(\tan^{-1}\left(\frac{6k+1}{2(k+1)}\right)-\tan^{-1}\left(\frac{6k-5}{2k}\right)\right)$$

Observe that Ω_2 *is telescoping sum giving us the partial sum as*

$$\Omega_2 = \lim_{N\to\infty}\left(\left(\tan^{-1}\left(\frac{6N+1}{2(N+1)}\right)\right)-\tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$= \tan^{-1}(3) - \tan^{-1}\left(\frac{1}{2}\right)$$

Therefore,

$$\Omega = \frac{\pi}{2}\left(\tan^{-1}(3) - \tan^{-1}\left(\frac{1}{2}\right)\right) = \frac{\pi}{2}\tan^{-1}(1) = \frac{3\pi^2}{8}$$

Note: The principle branch

$-\frac{\pi}{2} \leq \tan^{-1} x \leq \frac{\pi}{2}$ *thus*

$-\pi \leq \tan^{-1} x + \tan^{-1} y \leq \pi$ *. As*

$\Omega_1 = \frac{3\pi}{2} > \pi$*. Therefore,*

$\Omega_1 = \frac{3\pi}{2} - \pi = \frac{\pi}{2}$*. So, the answer is* $\frac{\pi^2}{8}$

2.26. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(n^3 \sum_{1\leq i<j<k\leq n} \frac{i \cdot j \cdot k}{(n^2+i^2)(n^2+j^2)(n^2+k^2)}\right)}$$

Solution:

$$\Omega = \lim_{n\to\infty}\left(\sum_{1\leq i<j<k\leq n} \frac{n^3 \cdot \frac{i}{n} \cdot \frac{j}{n} \cdot \frac{k}{n}}{n^6\left(1+\left(\frac{i}{n}\right)^2\right)\left(1+\left(\frac{j}{n}\right)^2\right)\left(1+\left(\frac{k}{n}\right)^2\right)}\right) =$$

$$= \lim_{n\to\infty}\left(\sum_{1\leq i<j<k\leq n} \frac{1}{n} \cdot \frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} \cdot \frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2} \cdot \frac{1}{n} \cdot \frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{1}{6}\left(\sum_{i=1}^n\left(\sum_{j=1}^n\left(\sum_{k=1}^n \frac{1}{n} \cdot \frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} \cdot \frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2} \cdot \frac{1}{n} \cdot \frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\right)\right) - \right.\right.$$

$$\left.\left. - \frac{\sum_{1\leq i<j<k\leq n}\left(\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2} + \frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\right)^3}{n^3}\right) + (n-2) \cdot \frac{\left(\frac{\frac{1}{n}}{1+\left(\frac{1}{n}\right)^2}\right)^3 + \cdots + \left(\frac{\frac{n}{n}}{1+\left(\frac{n}{n}\right)^2}\right)^3}{n^3}\right) \quad (1)$$

Let $a_i = \frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}$ *and* $f: (0;1] \to \mathbb{R}$ *such that* $f(x) = \frac{x}{x^2+1} \Rightarrow$

$$\Rightarrow f'(x)=\frac{1-x^2}{(x^2+1)^2} \quad (2)$$

$$0<x\le 1\Rightarrow x^2\le 1\Rightarrow 1-x^2\ge 0\Rightarrow \frac{1-x^2}{(x^2+1)^2}\ge 0\overset{(2)}{\Rightarrow}$$

$$\Rightarrow f'(x)\ge 0\ \forall x\in(0;1]\Rightarrow$$

$$\Rightarrow f \text{ is an increasing function} \Rightarrow a_1\le a_2\le\cdots\le a_n$$

$$i<j\Rightarrow a_i<a_j\Rightarrow a_i+a_j<2a_j\Rightarrow$$

$$\Rightarrow \sum_{1\le i<j<k\le n}\frac{\left(\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}+\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\right)^3}{n^3}=\sum_{1\le i<j<k\le n}\frac{\left(a_i+a_j\right)^3}{n^3}\le$$

$$\le 8\cdot\frac{a_2^3+a_3^3+\cdots+a_n^3+a_3^3+a_4^3+\cdots+a_n^3+\cdots+a_n^3}{n^3}=8\cdot\frac{a_2^3+2a_3^3+\cdots+(n-1)a_n^3}{n^3} \quad (3)$$

$$f \text{ is increasing and } a_n=\frac{1}{2}<1\Rightarrow f(x)<1\Rightarrow f^3(x)<1\Rightarrow$$

$$\Rightarrow f^3(x)<1\Rightarrow a_i^3<1$$

$$\Rightarrow a_2^3+2a_3^3+\cdots+(n-1)a_n^3<1+2+\cdots+(n-1)=\frac{n(n-1)}{2}\Rightarrow$$

$$\Rightarrow 8\cdot\frac{a_2^3+2a_3^3+\cdots+(n-1)a_n^3}{n^3}<8\cdot\frac{n(n-1)}{2n^3}, \text{ but } \lim_{n\to\infty}\frac{n(n-1)}{2n^3}=0\overset{(3)}{\Rightarrow}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\sum_{1\le i<j\le n}\left(\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}+\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\right)^3}{n^3}=0\overset{(1)}{\Rightarrow}$$

$$\Rightarrow \Omega=\frac{1}{6}\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{\frac{j}{n}}{1+\left(\frac{j}{n}\right)^2}\cdot\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\right)\right)\right)=$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+\left(\frac{i}{n}\right)^2}\right)^3=\frac{1}{6}\cdot\left(\int_0^1\frac{x}{x^2+1}dx\right)^3=$$

$$=\frac{1}{6}\cdot\left(\frac{1}{2}\int_0^1\frac{2x}{x^2+1}dx\right)^3=\frac{1}{6}\left(\frac{\ln(2)}{2}\right)^3=\frac{\ln^3\left(\sqrt{2}\right)}{6}\Rightarrow\Omega=\frac{\ln^3\left(\sqrt{2}\right)}{6}$$

2.27. $\boldsymbol{f:[0,1]\to\mathbb{R}}$**, continuous,** $\boldsymbol{\int_0^1 f(x)\,dx=1}$

Find:

$$\boldsymbol{\Omega=\sum_{p=1}^{\infty}\left(\lim_{n\to\infty}\left(\frac{1}{n^p}\sum_{1\le i_1<i_2<\dots<i_p\le n} f\left(\frac{i_1}{n}\right)f\left(\frac{i_2}{n}\right)\cdot\ldots\cdot f\left(\frac{i_p}{n}\right)\right)\right)}$$

Solution:

As f is continuous on $[0,1]$, there exist $m>0$ such that:

$$|f(x)|\le m\ \forall x\in[0,1]$$

If two of $i_1,i_2,\dots,i_p$ are equal, then there are at most $n_{c_{p-1}}$ expressions of the type, $f\left(\frac{i_1}{n}\right)f\left(\frac{i_2}{n}\right)\dots f\left(\frac{i_p}{n}\right)$. Thus,

$$\lim_{n\to\infty}\left|\frac{1}{n^p}\sum f\left(\frac{i_1}{n}\right)f\left(\frac{i_2}{n}\right)\dots f\left(\frac{i_p}{n}\right)\right|\le\lim_{n\to\infty}\frac{m^p}{n^p}\left(n_{c_{p-1}}\right)=0$$

$$\Rightarrow\lim_{n\to\infty}\frac{1}{n^p}\sum f\left(\frac{i_1}{n}\right)f\left(\frac{i_2}{n}\right)\dots f\left(\frac{i_p}{n}\right)=0$$

if at least two of $i_1,i_2,\dots,i_p$ are equal. Now,

$$\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\right)^p=$$

$$=p!\left(\sum_{1\le i_1<i_2\dots<i_p\le n}f\left(\frac{i_1}{n}\right)f\left(\frac{i_2}{n}\right)\dots f\left(\frac{i_p}{n}\right)\right)\frac{1}{n^p}+\frac{1}{n^p}(n^p-p!)$$

term in which at least $i_1,i_2,\dots,i_p$ are equal)

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\right)^p=p!\,a_p+0=p!\,a_p$$

where $a_p=\lim_{n\to\infty}\frac{1}{n^p}\sum_{1\le i_1<i_2<\dots\le i_p\le n}f\left(\frac{i_1}{n}\right)f\left(\frac{i_2}{n}\right)\dots f\left(\frac{i_p}{n}\right)$

$$\Rightarrow\left(\int_0^1 f(x)dx\right)^p=p!\,a_p\Rightarrow a_p=\frac{1}{p!}$$

Thus,

$$\Omega = \sum_{p=1}^{\infty} a_p = \sum_{p=1}^{\infty} \frac{1}{p!} = e - 1$$

2.28. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\frac{1}{n^6}\sum_{1\le i<j<k\le n}\frac{i\cdot j\cdot k}{(n^2+i^2)(n^2+j^2)(n^2+k^2)}\right)}$$

Solution:

$$\text{For } i,j,k \ge 1 \text{ we have:}$$

$$(1+i^2)(1+j^2)(1+k^2) \overset{AM-GM}{\ge} 2i\cdot 2j\cdot 2k = 8ijk$$

$$\Rightarrow \frac{ijk}{(1+i^2)(1+j^2)(1+k^2)} \le \frac{1}{8}$$

$$\Rightarrow 0 < \Omega_n = \frac{1}{n^6}\sum_{1\le i<j<k\le n}\frac{ijk}{(n+i^2)(n+j^2)(n+k^2)} \le$$

$$\le \frac{1}{n^6}\sum_{1\le i<j\le k\le n}\frac{ijk}{(1+i^2)(1+j^2)(1+k^2)} \le$$

$$\le \frac{1}{n^6}\cdot\frac{1}{8}\cdot n = \frac{1}{8n^5} \to 0\ (n\to\infty) \Rightarrow \Omega = \lim_{n\to\infty}\Omega_n = 0$$

2.29. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\left(\frac{k}{2k-1}\right)^k\cdot\left(\frac{n+k}{n}\right)^{\frac{1}{n+k}}\right)\right)}$$

Solution:

$$\text{We define: } f:[1,+\infty[\ \to \mathbb{R};\ x\to f(x) = \frac{x}{2x-1}$$

$$\text{Therefore: } f'(x) = \frac{2x-1-2x}{(2x-1)^2} = \frac{-1}{(2x-1)^2} < 0, \text{ then:}$$

$$\forall k\ge 2, \frac{k}{2k-1} \le \frac{2}{3} \Rightarrow \left(\frac{k}{2k-1}\right)^k \le \left(\frac{2}{3}\right)^k$$

$$\text{Hence: } \prod_{k=1}^{n}\left(\frac{k}{2k-1}\right)^k \le \prod_{k=1}^{n}\left(\frac{2}{3}\right)^k \Rightarrow 0 < \Omega \le \lim\left[\prod_{k=1}^{n}\left(\frac{2}{3}\right)^k\cdot\left(\frac{n+k}{n}\right)^{\frac{1}{n+k}}\right]$$

$$=\left(\lim\prod_{k=1}^{n}\left(\frac{2}{3}\right)^{k}\right)\left(\lim\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{\frac{1}{n+k}}\right)$$

$$=\left(\lim_{n\to\infty}\left(\frac{2}{3}\right)^{\frac{n(n+1)}{2}}\right)\times L\ \ldots\ (*)$$

$$L=\lim\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{\frac{1}{n+k}}\Rightarrow \ln(L)=\lim\sum_{k=1}^{n}\frac{\ln\left(1+\frac{k}{n}\right)}{n+k}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\ln\left(1+\frac{k}{n}\right)}{1+\frac{k}{n}}=\int_0^1\frac{\ln(1+t)}{1+t}dt=\frac{\ln^2(1+t)}{2}\Big|_0^1=\frac{\ln^2 2}{2}\Rightarrow$$

$$\Rightarrow L=\sqrt{e^{\ln^2 2}}$$

from (*): $0\le\Omega\le 0\times L=0\ \therefore\ \Omega=0$

2.30. If $0<a\le b,[*]$ - great integer function, then:

$$\frac{1}{4}\int_a^b\left(\frac{2x-[x]}{[x]^2+2(x-[x])^2}\right)dx+\frac{1}{4}\int_a^b\left(\frac{x+[x]}{2[x]^2+(x-[x])^2}\right)dx\le\log\left(\frac{b}{a}\right)$$

Solution:

$$LHS\overset{(a)}{=}\frac{1}{4}\int_a^b\left(\frac{2x-[x]}{[x]^2+2(x-[x])^2}+\frac{x+[x]}{2[x]^2+(x-[x])^2}\right)dx$$

$$\because 0<a\le x\le b,\therefore x>0\Rightarrow[x]\ge 0$$

$$Let\ [x]=I\ and\ x-[x]=f.\ Then,\ I\ge 0\ and\ 0<f<1.$$

$$Now,\ \frac{1}{4}\left(\frac{2x-[x]}{[x]^2+2(x-[x])^2}+\frac{x+[x]}{2[x]^2+(x-[x])^2}\right)\overset{(1)}{\le}\frac{1}{x}$$

$$\Leftrightarrow\frac{2(I+f)-I}{I^2+2f^2}+\frac{I+f+I}{2I^2+f^2}\le\frac{4}{I+f}\Leftrightarrow\frac{I+2f}{I^2+2f^2}+\frac{2I+f}{2I^2+f^2}\le\frac{4}{I+f}$$

$$\Leftrightarrow 4(I^2+2f^2)(2I^2+f^2)-$$

$$-(I+f)[(I+2f)(2I^2+f^2)+(2I+f)(I^2+2f^2)]\ge 0$$

$$\Leftrightarrow 4I^4+4f^4+10I^2f^2-9fI(I^2+f^2)\ge 0\Leftrightarrow$$

$$\Leftrightarrow 4(I^2+f^2)^2-9fI(I^2+f^2)+2I^2f^2\ge 0$$

$$\Leftrightarrow 4m^2-9mn+2n^2\ge 0\ (m=I^2+f^2\ \&n=If)\Leftrightarrow$$

$$\Leftrightarrow(m-2n)(4m-n)\ge 0$$

$$\Leftrightarrow(I-f)^2\{4(I-f)^2+7If\}\ge 0\rightarrow true\because I\ge 0\ and\ f>0$$

$\therefore$ *(1) is true* $\therefore$ *(a), (1)* $\Rightarrow LHS\le\int_a^b\frac{1}{x}dx=\ln\left(\frac{b}{a}\right)$ *(Proved)*

2.31.

$$\mathbf{erf}(\boldsymbol{x})=\frac{\mathbf{2}}{\sqrt{\boldsymbol{\pi}}}\int_{\mathbf{0}}^{\boldsymbol{x}}\boldsymbol{e}^{-\boldsymbol{t}^{2}}\boldsymbol{dt}$$

Prove without softs:

$$\int_{\mathbf{0}}^{\mathbf{1}}\left(\boldsymbol{x}^{3}\,\mathbf{erf}(\boldsymbol{x}^{3})+\boldsymbol{x}^{5}\,\mathbf{erf}\left(\boldsymbol{x}^{5}\right)+\boldsymbol{x}^{7}\mathbf{erf}(\boldsymbol{x}^{7})\right)\boldsymbol{dx}>\frac{\mathbf{3}}{\mathbf{2}}\int_{\mathbf{0}}^{\mathbf{1}}\boldsymbol{x}\mathbf{erf}(\boldsymbol{x})\,\boldsymbol{dx}$$

Solution:

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_{0}^{x}e^{-t^{2}}\,dt\Rightarrow\operatorname{erf}'(x)=\frac{2}{\sqrt{\pi}}\cdot e^{-x^{2}}$$

$$\textit{Now, } I_{1}=\int_{0}^{1}x^{3}\cdot\operatorname{erf}(x^{3})\,dx\Rightarrow$$

$$\Rightarrow \textit{let } x^{3}=u\Rightarrow 3x^{2}dx=du\Rightarrow x^{3}dx=\frac{1}{3}\cdot u^{\frac{1}{3}}du$$

$$=\frac{1}{3}\int_{0}^{1}u^{\frac{1}{3}}\operatorname{erf}(u)\,du=\left[\frac{1}{3}\cdot\operatorname{erf}(u)\cdot\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\right]_{0}^{1}-\frac{1}{3}\int_{0}^{1}\frac{2}{\sqrt{\pi}}\cdot e^{-u^{2}}\cdot\frac{u^{\frac{4}{3}}}{\frac{4}{3}}du$$

$$=\frac{1}{4}\cdot\operatorname{erf}(1)-\frac{1}{2\sqrt{\pi}}\int_{0}^{1}u^{\frac{4}{3}}\cdot e^{-u^{2}}du$$

$$\Leftrightarrow LHS=\frac{1}{4}\operatorname{erf}(1)+\frac{1}{6}\operatorname{erf}(1)+\frac{1}{8}\operatorname{erf}(1)-\frac{1}{2\sqrt{\pi}}\int_{0}^{1}u^{\frac{4}{3}}\cdot u^{-u^{2}}du-$$

$$-\frac{1}{3\sqrt{\pi}}\int_{0}^{1}u^{\frac{6}{5}}\cdot e^{-u^{2}}du-\frac{1}{4\sqrt{\pi}}\int_{0}^{1}u^{\frac{8}{7}}\cdot e^{-u^{2}}du$$

$$RHS=\int_{0}^{1}x\operatorname{erf}(x)\,dx=\left[\frac{x^{2}}{2}\cdot\operatorname{erf}(x)\right]_{0}^{1}-\int_{0}^{1}\frac{2}{\sqrt{\pi}}\cdot e^{-x^{2}}\cdot\frac{x^{2}}{2}dx$$

$$=\frac{1}{2}\cdot\operatorname{erf}(1)-\frac{1}{\sqrt{\pi}}\cdot\int_{0}^{1}x^{2}\cdot e^{-x^{2}}dx$$

$$RHS\Rightarrow\frac{3}{2}\int_{0}^{1}x\operatorname{erf}(x)\,dx=\frac{3}{4}\operatorname{erf}(1)-\frac{3}{2\sqrt{\pi}}\int_{0}^{1}x^{2}\cdot e^{-x^{2}}\,dx$$

$$\textit{Need to prove}\Rightarrow I=LHS-RHS<0$$

$$I=-\frac{5}{24}\operatorname{erf}(1)+\frac{1}{\sqrt{\pi}}\int_{0}^{1}e^{-x^{2}}\left[\frac{3x^{2}}{2}-\frac{x^{\frac{4}{3}}}{2}-\frac{x^{\frac{6}{5}}}{3}-\frac{x^{\frac{8}{7}}}{4}\right]dx$$

$$\Leftrightarrow \frac{x^{\frac{4}{3}}}{2}+\frac{x^{\frac{6}{5}}}{3}+\frac{x^{\frac{8}{7}}}{4} \overset{AM-GM}{\geq} 3\left(\frac{x^{\frac{386}{105}}}{24}\right)^{\frac{1}{3}} \geq \frac{3x^2}{2}\cdot\frac{1}{3^{\frac{1}{3}}}\cdot\left(\frac{1}{x}\right)^{\frac{244}{315}}$$

$$\Leftrightarrow p > \frac{3x^2}{2} \Rightarrow \frac{3x^2}{2} - p < 0$$

$$I = -\frac{5}{24}\operatorname{erf}(1) - \frac{1}{\sqrt{\pi}}\int_0^1 e^{-x^2}\left(\frac{3x^2}{2}-p\right)dx < 0$$

$$\Leftrightarrow \int_0^1 (x^3\operatorname{erf}(x^3)+x^5\operatorname{erf}(x^5)+x^7\operatorname{erf}(x^7)\,dx) < \frac{3}{2}\int_0^1 x\operatorname{erf}(x)\,dx$$

2.32. Find in terms of $a, b, 0 < a \leq b < 2\pi$:

$$\boldsymbol{\Omega(a,b)} = \int_a^b\int_a^b\int_a^b \left(\sin(x+y+z) - 4\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{z+x}{2}\right)\right)dx\,dy\,dz$$

Solution:

$$\int_a^b\int_a^b\int_a^b \left(\sin(x+y+z) - 4\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{z+y}{2}\right)\right)dx\,dy\,dz =$$

$$= \int_a^b\int_a^b\int_a^b (2\sin(x+y+z) - \sin x - \sin y - \sin z)\,dx\,dy\,dz$$

$$= \int_a^b\int_a^b (\cos(a+y+z) - \cos(b+y+z))dy\,dz -$$

$$-\int_a^b\int_a^b (\cos a - \cos b + (b-a)\sin y + (b-a)\sin z)\,dy\,dz$$

$$= 2\int_a^b (\sin(a+b+z) - \sin(2a+z) + \sin(a+b+z) - \sin(2b+z))\,dz -$$

$$-\int_a^b \left((b-a)\cos a - (b-a)\cos b + (b-a)(\cos a - \cos b) + (b-a)^2\sin z\right)dz$$

$$= 2\begin{pmatrix}\cos(2a+b)+\cos(a+2b)+\cos(2a+b)-\cos(3a)+\cos(2a+b)- \\ -\cos(a+2b)+\cos(3b)-\cos(a+2b)\end{pmatrix} -$$

$$-\left((b-a)^2(\cos b - \cos a) + (b-a)^2(\cos b - \cos a) + (b-a)^2(\cos b - \cos a)\right)$$

$$= \frac{1}{16}\sin^3\left(\frac{b-a}{2}\right)\sin\left(3\frac{a+b}{2}\right) + 3(b-a)^2(\cos b - \cos a)$$

2.33. If $0 < a \le b < 2\pi$ then:

$$\frac{1}{6(b-a)^2}\int_a^b\int_a^b\int_a^b\left(\frac{\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{z+x}{2}\right)}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)} + \cot\left(\frac{x+y+z}{2}\right)\right)dx\,dy\,dz \le$$

$$\le \log\left|\frac{b}{2\sin\frac{a}{2}}\right|$$

Solution:

$$With\ 0 < x, y, z < 2\pi$$

$$Let\ \Omega(x,y,z) = \frac{\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{x+z}{2}\right)}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\frac{x+y+z}{2}} + \cot\frac{x+y+z}{2} =$$

$$= \frac{\sin\left(\frac{x+y}{2}\right) + \sin\left(\frac{y+z}{2}\right) + \sin\left(\frac{x+z}{2}\right)}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\frac{x+y+z}{2}} + \frac{\cos\frac{x+y+z}{2}}{\sin\frac{x+y+z}{2}} =$$

$$= \frac{\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{x+z}{2}\right) + \cos\left(\frac{x+y+z}{2}\right)\cdot\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\frac{x+y+z}{2}}$$

$$= \frac{\left(\sin\frac{x+y+z}{2}\right)\left(\sin\frac{y}{2}\sin\frac{z}{2}\cos\frac{x}{2} + \sin\frac{x}{2}\sin\frac{z}{2}\cos\frac{y}{2} + \sin\frac{x}{2}\sin\frac{y}{2}\cos\frac{z}{2}\right)}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\frac{x+y+z}{2}}$$

$$= \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} + \frac{\cos\frac{y}{2}}{\sin\frac{y}{2}} + \frac{\cos\frac{z}{2}}{\sin\frac{z}{2}} \to \frac{1}{6(b-a)^2}\int_a^b\int_a^b\int_a^b \Omega(x,y,z)\,dx\,dy\,dz$$

$$= \frac{1}{6(b-a)^2}\int_a^b\int_a^b\int_a^b\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} + \frac{\cos\frac{y}{2}}{\sin\frac{y}{2}} + \frac{\cos\frac{z}{2}}{\sin\frac{z}{2}}\right)dx\,dy\,dz = \omega$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)dx\,dy\,dz = \left(\int_a^b\int_a^b dx\,dy\right)\left(\int_a^b \frac{\cos\frac{z}{2}}{\sin\frac{z}{2}}dz\right) =$$

$$= 2(b-a)^2\left(\ln\sin\frac{b}{2} - \ln\sin\frac{a}{2}\right) = 2(b-a)^2\left(\ln\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}\right)$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}\right)dx\,dy\,dz=\left(\int_a^b\int_a^b dx\,dy\right)\left(\int_a^b\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}dy\right)=$$

$$=2(b-a)^2\left(\ln\sin\frac{b}{2}-\ln\sin\frac{a}{2}\right)=2(b-a)^2\left(\ln\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}\right)$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)dx\,dy\,dz=\left(\int_a^b\int_a^b dy\,dz\right)\left(\int_a^b\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}dx\right)=$$

$$=2(b-a)^2\left(\ln\sin\frac{b}{2}-\ln\sin\frac{a}{2}\right)=2(b-a)^2\left(\ln\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}\right)$$

$$\to\omega=\ln\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}\overset{\sin\frac{b}{2}\le\frac{b}{2};b>0}{\le}\ln\frac{\frac{b}{2}}{\sin\frac{a}{2}}=\ln\frac{b}{2\sin\frac{a}{2}}$$

2.34. Find:

$$\boldsymbol{\Omega=\lim_{\substack{x\to 0\\ x>0}}\left(x\left(1-\sum_{n=0}^{\infty}\frac{(-x)^n}{(2n+1)!}\right)\right)}$$

Solution:

Denote the sum

$$S=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n+1)!}$$

Expanding the sum, we observe: $S=1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\cdots$

Since: (*) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$

Putting x in (*) by $\sqrt{x}$ *we observe that:*

(**) $\sin\sqrt{x}=\sqrt{x}-\frac{x\sqrt{x}}{3!}+\frac{x^2\sqrt{x}}{5!}-\frac{x^3\sqrt{x}}{7!}+\cdots$

and dividing by $\sqrt{x}$ *in (**) we have:* $\frac{\sin\sqrt{x}}{\sqrt{x}}=1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\cdots$ *and hence the limit*

$$\lim_{\substack{x\to 0\\ x>0}}x\left(1-\frac{\sin\sqrt{x}}{\sqrt{x}}\right)=0$$

2.35. Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n]{(n!)^2\sin^2\left(\frac{n}{n+1}\right)}\left(\frac{1}{(n+1)^2}\sqrt[n+1]{(n+1)!\sin\left(\frac{n+1}{n+2}\right)}-\frac{1}{n^2}\sqrt[n]{n!\sin\left(\frac{n}{n+1}\right)}\right)\right)$$

Solution:

$$\Omega = \lim_{n\to\infty}\frac{1}{n^2}\sqrt[n]{(n!)^3\cdot\sin^3\left(\frac{n}{n+1}\right)}\left(\frac{n^2}{(n+1)^2}\cdot\frac{\sqrt[n+1]{(n+1)!\sin\left(\frac{n+1}{n+2}\right)}}{\sqrt[n]{n!\sin\left(\frac{n}{n+1}\right)}}-1\right)=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{(n!)^3\cdot\sin^3\left(\frac{n}{n+1}\right)}}{n^3}\cdot n\left(\left(\frac{n}{n+1}\right)^2\cdot\frac{\sqrt[n+1]{(n+1)!\sin\left(\frac{n+1}{n+2}\right)}}{\sqrt[n]{n!\cdot\sin\left(\frac{n}{n+1}\right)}}-1\right)\quad (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(n!)^3\sin^3\left(\frac{n}{n+1}\right)}}{n^3}=\left(\lim_{n\to\infty}\sqrt[n]{\frac{n!\sin\left(\frac{n}{n+1}\right)}{n^n}}\right)^3\overset{C.D.}{=}$$

$$=\left(\lim_{n\to\infty}\frac{(n+1)!\sin\left(\frac{n+1}{n+2}\right)}{(n+1)^{n+1}}\cdot\frac{n^n}{n!\sin\left(\frac{n}{n+1}\right)}\right)^3=$$

$$=\left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{\sin\left(\frac{n+1}{n+2}\right)}{\sin\left(\frac{n}{n+1}\right)}\right)^3=\left(\frac{1}{e}\cdot\frac{\sin 1}{\sin 1}\right)^3=\frac{1}{e^3}\quad (2)$$

$$Let\ x_n=\left(\frac{n}{n+1}\right)^2\frac{\sqrt[n+1]{(n+1)!\sin\left(\frac{n+1}{n+2}\right)}}{\sqrt[n]{n!\sin\left(\frac{n}{n+1}\right)}}\Rightarrow$$

$$\lim_{n\to\infty}n(x_n-1)=\lim_{n\to\infty}n\left(e^{\ln x_n}-1\right)=\lim_{n\to\infty}\frac{\left(e^{\ln x_n}-1\right)}{\ln x_n}\cdot\ln x_n=$$

$$=\lim_{n\to\infty}n\ln x_n=\lim_{n\to\infty}\ln x_n^n=\ln\left(\lim_{n\to\infty}x_n^n\right)=$$

$$=\ln\left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n}\cdot\frac{\left(\sqrt[n+1]{(n+1)!\sin\left(\frac{n+1}{n+2}\right)}\right)^n}{n!\sin\left(\frac{n}{n+1}\right)}\right)=$$

$$=\ln\lim_{n\to\infty}\left(\left(\left(\frac{n}{n+1}\right)^n\right)^2\cdot\frac{(n+1)!\sin\left(\frac{n+1}{n+2}\right)}{n!\sin\left(\frac{n}{n+1}\right)\sqrt[n+1]{(n+1)!\sin\left(\frac{n+1}{n+2}\right)}}\right)=$$

$$= \ln\left(\lim_{n\to\infty}\frac{1}{e^2}\cdot\frac{\sin\left(\frac{n+1}{n+2}\right)}{\sin\left(\frac{n}{n+1}\right)}\cdot\frac{(n+1)}{\sqrt[n+1]{(n+1)!\sin\left(\frac{n+1}{n+2}\right)}}\right) =$$

$$= \ln\left(\lim_{n\to\infty}\frac{1}{e^2}\cdot\frac{n}{\sqrt[n]{n!\sin\left(\frac{n}{n+1}\right)}}\right) \overset{(2)}{=} \ln\left(\frac{1}{e^2}\cdot e\right) =$$

$$= \ln\frac{1}{e} = -1 \quad (3)$$

From (1)+(2)+(3)$\Rightarrow \Omega = -\frac{1}{e^3}$

2.36. Find:

$$\boldsymbol{\Omega_1 = \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\cos\left(\frac{i}{n}\right)\cos\left(\frac{j}{n}\right)\cos\left(\frac{k}{n}\right)\right)\right)\right)}$$

$$\boldsymbol{\Omega_2 = \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{1\le i<j<k\le n}\cos\left(\frac{i}{n}\right)\cos\left(\frac{j}{n}\right)\cos\left(\frac{k}{n}\right)\right)}$$

Solution:

$$\Omega_1 = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\frac{1}{n}\cos\left(\frac{i}{n}\right)\cdot\frac{1}{n}\cos\left(\frac{j}{n}\right)\cdot\frac{1}{n}\cos\left(\frac{k}{n}\right)\right)\right)\right) =$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{n}\cos\frac{i}{n}\right)^3 = \left(\int_0^1 \cos x\,dx\right)^3 = \sin^3(1) \Rightarrow \Omega_1 = \sin^3(1)$$

$$\Omega_2 = \lim_{n\to\infty}\left(\frac{1}{6}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\frac{1}{n}\cos\left(\frac{i}{n}\right)\cdot\frac{1}{n}\cos\left(\frac{j}{n}\right)\cdot\frac{1}{n}\cos\left(\frac{k}{n}\right)\right)\right)-\right.\right.$$

$$\left.\left.-\frac{1}{n^3}\sum_{1\le i<j\le n}\left(\cos\left(\frac{i}{n}\right)+\cos\left(\frac{j}{n}\right)\right)^3 + (n-2)\cdot\frac{\cos^3\left(\frac{1}{n}\right)+\ldots+\cos^3\left(\frac{n}{n}\right)}{n^3}\right)\right) \; (1)$$

Let $f:\left[0;\frac{\pi}{2}\right]\to\mathbb{R}; f(x) = \cos(x) \Rightarrow f$ *is decreasing on* $(0,1)\subset\left[0,\frac{\pi}{2}\right]$;

for $\frac{i}{n} < 1 \Rightarrow \frac{i}{n}\in(0,1)$, *so* $\cos\left(\frac{i}{n}\right) > \cos\left(\frac{j}{n}\right) \Rightarrow$

$$\Rightarrow \cos\left(\frac{i}{n}\right)+\cos\left(\frac{j}{n}\right) < 2\cos\left(\frac{i}{n}\right) \Rightarrow$$

$$\Rightarrow\left(\cos\left(\frac{i}{n}\right)+\cos\left(\frac{j}{n}\right)\right)^3<8\cos^3\left(\frac{i}{n}\right)\Rightarrow$$

$$\frac{1}{n^3}\sum_{1\le i<j\le n}\left(\cos\left(\frac{i}{n}\right)+\cos\left(\frac{j}{n}\right)\right)^3<$$

$$<8\cdot\frac{(n-1)\cos^3\left(\frac{1}{n}\right)+(n-2)\cos^3\left(\frac{2}{n}\right)+\cdots+\left(n-(n-1)\right)\cos^3\left(\frac{n-1}{n}\right)}{n^3}=$$

$$=8\cdot\frac{n\left(\cos^3\left(\frac{1}{n}\right)+\cdots+\cos^2\left(\frac{n-1}{n}\right)\right)-\left(\cos^3\left(\frac{1}{n}\right)+\cdots+(n-1)\cos^3\left(\frac{n-1}{n}\right)\right)}{n^3}\quad (1)$$

$$\lim_{n\to\infty}\frac{n\left(\cos^3\left(\frac{1}{n}\right)+\cdots+\cos^3\left(\frac{n-1}{n}\right)\right)-\left(\cos^3\left(\frac{1}{n}\right)+\cdots+(n-1)\cos^3\left(\frac{n-1}{n}\right)\right)}{n^3}=$$

$$=\lim_{n\to\infty}\frac{\frac{1}{n}\left(\cos^3\left(\frac{1}{n}\right)+\cdots+\cos^3\left(\frac{n-1}{n}\right)\right)}{n}-$$

$$-\lim_{n\to\infty}\frac{\cos^2\frac{1}{n}+\cdots+(n-1)\cos^3\left(\frac{n-1}{n}\right)}{n^3}=\lim_{n\to\infty}\left(\int_0^1\cos^3(x)\,dx\right)\cdot\frac{1}{n}-$$

$$-\lim_{n\to\infty}\frac{\cos^3\left(\frac{1}{n}\right)+\cdots+(n-1)\cos^3\left(\frac{n-1}{n}\right)}{n^3}=-\lim_{n\to\infty}\frac{\cos^2\left(\frac{1}{n}\right)+\cdots+(n-1)\cos^3\left(\frac{n-1}{n}\right)}{n^3}\quad (2)$$

$$\cos x\le 1\Rightarrow\cos^3(x)\le 1\Rightarrow\cos^3\left(\frac{1}{n}\right)+\cdots+(n-1)\cos^3\left(\frac{n-1}{n}\right)\le$$

$$\le 1+2+\cdots+(n-1)=\frac{n(n-1)}{2}\Rightarrow$$

$$\Rightarrow\frac{\cos^3\left(\frac{1}{n}\right)+\cdots+(n-1)\cos^3\left(\frac{n-1}{n}\right)}{n^3}\le\frac{n(n-1)}{2n^3}$$

$$\lim_{n\to\infty}\frac{n(n-1)}{2n^3}=0\overset{(2)}{\Rightarrow}\lim_{n\to\infty}\frac{\cos^3\left(\frac{1}{n}\right)+\cdots+(n-1)\cos^3\left(\frac{n-1}{n}\right)}{n^3}=0\overset{(1)}{\Rightarrow}$$

$$\Rightarrow\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}\left(\cos\left(\frac{i}{n}\right)+\cos\left(\frac{j}{n}\right)\right)^3=0\Rightarrow$$

$$\Rightarrow\Omega_2=\frac{1}{6}\cdot\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\cos\frac{i}{n}\right)^3=\frac{1}{6}\left(\int_0^1\cos x\,dx\right)^3=\frac{\sin^3(1)}{6}\Rightarrow\Omega_2=\frac{\sin^3(1)}{6}$$

2.37. Find:

$$\boldsymbol{\Omega=\sum_{n=1}^{\infty}\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}}\frac{\cos 7x+7\cos 5x+21\cos 3x+35\cos x}{(4+\sin^2 x)(\cos 6x+6\cos 4x+15\cos 2x+10)}dx\right)}$$

Solution:

$$Let\ z = \cos x + i\sin x \Rightarrow \frac{1}{z} = \cos x - i\sin x\ then\ z + \frac{1}{z} = 2\cos x$$

$$\because z^n = \cos n\,x + i\sin nx \ \ (By\ De\ Moivre\ theorem)$$

$$\frac{1}{z^n} = \cos nx - i\sin nx$$

$$Adding \Rightarrow z^n + \frac{1}{z^n} = 2\cos nx.\ Putting\ n = 1,2,\dots,7,\ we\ get:$$

$$z + \frac{1}{z} = 2\cos x\,, z^2 + \frac{1}{z^2} = 2\cos 2x\,, z^3 + \frac{1}{z^3} = 2\cos 3x$$

$$\dots z^7 + \frac{1}{z^7} = 2\cos 7x,\ respectively.$$

$$Now, \left(z + \frac{1}{z}\right)^7 = z^7 + 7z^5 + 21z^3 + 35z + 35\frac{1}{z} + 21\frac{1}{z^3} + 7\frac{1}{z^5} + \frac{1}{z^7}$$

$$\Rightarrow (2\cos x)^7 = \left(z^7 + \frac{1}{z^7}\right) + 7\left(z^5 + \frac{1}{z^5}\right) + 21\left(z^3 + \frac{1}{z^3}\right) + 35\left(z + \frac{1}{z}\right)$$

$$\Rightarrow 2^7\cos^7 x = 2\cos 7x + 7(2\cos 5x) + 21(2\cos 3x) + 35(2\cos x)$$

$$\Rightarrow 2^6\cos^7 x = \cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x \ \ (1)$$

$$and \left(z + \frac{1}{2}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + 15\frac{1}{z^2} + 6\frac{1}{z^4} + \frac{1}{z^6}$$

$$\Rightarrow (2\cos x)^6 = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$$\Rightarrow 2^6\cos^6 x = 2\cos 6x + 6(2\cos 4x) + 15(2\cos 2x) + 20$$

$$\Rightarrow 2^5\cos^6 x = \cos 6x + 6\cos 4x + 15\cos 2x + 10 \ \ \ (2)$$

$$(1)\div(2)\Rightarrow \frac{\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x}{\cos 6x + 6\cos 4x + 15\cos 2x + 10} = 2\cos x$$

$$Then,\ \Omega = \sum_{n=1}^{\infty}\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{2\cos x\,dx}{4 + \sin^2 x}\right)$$

$$\Omega = \sum_{n=1}^{\infty}\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{2\cos x\ dx}{4 + \sin^2 x}\right)$$

$$Let\ I = \int \frac{2\cos x dx}{4 + \sin^2 x}\ putting\ \sin x = z; \cos x\,dx = dz$$

$$= 2\int \frac{dz}{(2)^2 + z^2} = \left[2 \cdot \frac{1}{2}\tan^{-1}\left(\frac{z}{2}\right)\right] = \tan^{-1}\left(\frac{\sin x}{2}\right)$$

$$\therefore \sum_{n=1}^{\infty}\left[\tan^{-1}\left(\frac{\sin x}{2}\right)\right]_{\frac{1}{n+1}}^{\frac{1}{n}} = \sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{\sin\frac{1}{n}}{2}\right) - \tan^{-1}\left(\frac{\sin\frac{1}{n+1}}{2}\right)\right) =$$

$$= \tan^{-1}\left(\frac{1}{2}\sin 1\right)$$

2.38. If $0 < x < \frac{\pi}{2}$ then:

$$\left(\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^3 + \left(\frac{\tan x}{x}\right)^4\right)\left(\left(\frac{\sin x}{x}\right)^4 + \left(\frac{\tan x}{x}\right)^5\right) > 8$$

Solution:

$$\textit{For } x > 0 \textit{ we have: } \tan x > x \Rightarrow \frac{\tan x}{x} > 1$$

$$\because \left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^3 > \left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^2 > \frac{1}{2}\left[\frac{\sin x}{x} + \frac{\tan x}{x}\right]^2 \overset{(1)}{>} 2$$

$$(1) \Leftrightarrow \left(\frac{\sin x}{x} + \frac{\tan x}{x}\right)^2 > 4$$

$$\Leftrightarrow (\sin x + \tan x) > 4x^2 \Leftrightarrow \sin x + \tan x > 2x$$

$$\Leftrightarrow \sin x + \tan x - 2x > 0 \quad (2)$$

$$\textit{Let } f(x) = \sin x + \tan x - 2x \quad \left(0 < x < \frac{\pi}{2}\right) \Rightarrow$$

$$\Rightarrow f'(t) = \cos x + \frac{1}{\cos^2 x} - 2$$

$$= \frac{(\cos x - 1)(\cos^2 x - \cos x - 1)}{\cos^2 x} > 0 \ \ (\because 0 < \cos x < 1)$$

$$f(x) \nearrow \left(0, \frac{\pi}{2}\right) \Rightarrow f(x) > f(0) \Rightarrow \textit{(2) true} \Rightarrow \textit{(1) true.}$$

$$\left(\frac{\sin x}{x}\right)^3 + \left(\frac{\tan x}{x}\right)^4 > \left(\frac{\sin x}{x}\right)^3 + \left(\frac{\tan x}{x}\right)^3 > \frac{1}{2^2}\left[\frac{\sin x}{x} + \frac{\tan x}{x}\right]^3 \overset{(3)}{>} 2$$

$$(3) \Leftrightarrow \left(\frac{\sin x}{x} + \frac{\tan x}{x}\right)^3 > 2^3 \Leftrightarrow (\sin x + \tan x)^3 > (2x)^3$$

$$\Leftrightarrow \sin x + \tan x > 2x \Leftrightarrow \sin x + \tan x - 2x > 0$$

(It is true by (2))⇒ (3) true.

$$\left(\frac{\sin x}{x}\right)^4 + \left(\frac{\tan x}{x}\right)^5 > \left(\frac{\sin x}{x}\right)^4 + \left(\frac{\tan x}{x}\right)^4 > \frac{1}{2^3}\left[\frac{\sin x}{x} + \frac{\tan x}{x}\right]^4 \overset{(4)}{>} 2$$

$$\Leftrightarrow \left[\frac{\sin x}{x} + \frac{\tan x}{x}\right]^4 > 2^4 \Leftrightarrow (\sin x + \tan x)^4 > (2x)^4 \Leftrightarrow \sin x + \tan x > 2x$$

$$\Leftrightarrow \sin x + \tan x - 2x > 0 \ \textit{(It is true by (2))} \overset{(1).(3).(4)}{\Rightarrow} \ LHS > 2 \cdot 2 \cdot 2 = 8$$

2.39. Find:

$$\boldsymbol{\Omega = 3\sqrt{3}\left(\log \phi + \sum_{n=1}^{\infty} \frac{\left((2n-1)!\right)^2}{(4n-1)!}\right) - 4\int_0^1 \sqrt{1 - x^2}\, dx}$$

Solution:

Given integral:

$$I = 4\int_0^1 \sqrt{1-x^2}\,dx$$

represent the total area of circle of radius $r = 1$. So, $I = \pi$. Evaluating the sum as:

$$\sum_{n=1}^{\infty} \frac{\left((2n-1)!\right)^2}{(4n-1)!} = \sum_{n=1}^{\infty} \int_0^1 t^{2n-1}(1-t)^{2n-1}\,dt =$$

$$= \int_0^1 \frac{t^2-t}{(t^2-t-1)(t^2-t+1)}\,dt = \frac{1}{2}\int_0^1 \left(\frac{1}{t^2-t-1} + \frac{1}{t^2-t+1}\right)dt =$$

$$= \frac{1}{2}\int_0^1 \left(\frac{1}{\left(t-\frac{1}{2}\right)^2 - \frac{5}{4}} + \frac{1}{\left(t-\frac{1}{2}\right)^2 + \frac{3}{4}}\right)dt$$

Using the standard integration formula, we have:

$$= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2\sqrt{3}}\ln\left[\frac{\left|2x-1-\sqrt{5}\right|}{\left|2x-1+\sqrt{5}\right|}\right]$$

Now, setting limits, we deduce: $= \frac{\pi}{3\sqrt{3}} + \frac{1}{2\sqrt{5}}\ln\left[\frac{3-\sqrt{5}}{3+\sqrt{5}}\right]$

Thus, $\Omega = 3\sqrt{3}\left(\log\phi + \frac{\pi}{3\sqrt{3}} + \frac{1}{2\sqrt{5}}\ln\left[\frac{3-\sqrt{5}}{3+\sqrt{5}}\right]\right) - \pi =$

$$= 3\sqrt{3}\log\phi + \frac{3\sqrt{5}}{2\sqrt{5}}\ln\left[\frac{3-\sqrt{5}}{3+\sqrt{5}}\right]$$

2.40. If $0 < d < c < b < a < \frac{\pi}{2}$ then:

$$\csc\left(\frac{\pi b}{2a}\right)\cdot\csc\left(\frac{\pi c}{2b}\right)\cdot\csc\left(\frac{\pi d}{2c}\right) < \frac{\sin a}{\sin d}$$

Solution:

$$\csc\left(\frac{\pi a}{2b}\right) = \frac{1}{\sin\left(\frac{\pi a}{2b}\right)} \le \frac{1}{\frac{2}{\pi}\left(\frac{\pi a}{2b}\right)} = \frac{b}{a}$$

$$\csc\left(\frac{\pi b}{2c}\right) = \frac{1}{\sin\left(\frac{\pi b}{2c}\right)} \le \frac{1}{\frac{2}{\pi}\cdot\left(\frac{\pi}{2c}\right)} = \frac{c}{b}$$

$$\csc\left(\frac{\pi c}{2d}\right)=\frac{1}{\sin\left(\frac{\pi c}{2d}\right)}\le\frac{1}{\frac{2}{\pi}\cdot\left(\frac{\pi c}{2d}\right)}=\frac{d}{c}$$

$$\to \csc\left(\frac{\pi b}{2a}\right)\cdot\csc\left(\frac{\pi c}{2b}\right)\cdot\csc\left(\frac{\pi d}{2c}\right)\le\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{d}{c}=\frac{d}{a}$$

Now, we must show that: $\frac{d}{a}<\frac{\sin a}{\sin d}$

$\leftrightarrow d\sin d<a\sin a$ *(*)*

Let $f(x)=x\sin x\quad\left(0<x<\frac{\pi}{2}\right)\to$

$$\to f'(x)=\sin x+x\cos x>0\quad\left(0<x<\frac{\pi}{2}\right)$$

Hence, $f(x)\nearrow\left(0;\frac{\pi}{2}\right)$

Because $0<d<a<\frac{\pi}{2}\to f(d)<f(a)\to d\sin d<a\sin a\to$ *(*) true. Proved.*

2.41. If $\mathbf{0}<a,b<\frac{\pi}{2}$ then:

$$(\mathbf{4}\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}+\mathbf{4}\boldsymbol{b})\cdot\sin\boldsymbol{a}\cdot\sin\boldsymbol{b}\le\mathbf{25}\boldsymbol{ab}\cdot\sin\left(\frac{\mathbf{4}\boldsymbol{a}+\boldsymbol{b}}{\mathbf{5}}\right)\sin\left(\frac{\boldsymbol{a}+\mathbf{4}\boldsymbol{b}}{\mathbf{5}}\right)$$

Solution:

$$\textit{Inequality}\Leftrightarrow\frac{\sin a}{a}\cdot\frac{\sin b}{b}\le\frac{\sin\left(\frac{4a+b}{5}\right)}{\frac{4a+b}{5}}\cdot\frac{\sin\left(\frac{a+4b}{5}\right)}{\frac{a+4b}{5}}$$

$$\Leftrightarrow\ln\left(\frac{\sin a}{a}\right)+\ln\left(\frac{\sin b}{b}\right)\le\ln\left(\frac{\sin\frac{4a+b}{5}}{\frac{4a+b}{5}}\right)+\ln\left(\frac{\sin\frac{a+4b}{5}}{\frac{a+4b}{5}}\right)$$

$$\Leftrightarrow\ln\left(\frac{\sin a}{a}\right)-\ln\left(\frac{\sin\frac{4a+b}{5}}{\frac{4a+b}{5}}\right)\le\ln\left(\frac{\sin\frac{a+4b}{5}}{\frac{a+4b}{5}}\right)-\ln\left(\frac{\sin b}{b}\right)$$

Let $f(x)=\frac{\sin x}{x},x\in\left(0;\frac{\pi}{2}\right)\Rightarrow f'(x)=\cot x-\frac{1}{x}\Rightarrow$

$$\Rightarrow f''(x)=\frac{\sin^2 x-x^2}{(x\sin x)^2}<0,x\in\left(0;\frac{\pi}{2}\right)\Rightarrow f'(x)\searrow\left(0;\frac{\pi}{2}\right)$$

Suppose $0<a\le b\le\frac{\pi}{2}\Rightarrow 0<a\le\frac{4a+b}{5}\le\frac{a+4b}{5}\le b<\frac{\pi}{2}$

Now, using Lagrange's theorem: $\exists c_1\in\left(a;\frac{4a+b}{5}\right)$ *such that:*

$$f\left(\frac{4a+b}{5}\right)-f(a)=f'(c_1)\cdot\frac{b-a}{5}$$

$$\exists c_2 \in \left(\frac{a+4b}{5}; b\right) \text{ such that: } f(b) - f\left(\frac{a+4b}{5}\right) = f'(c_2) \cdot \frac{b-a}{5}$$

$$\text{We must show that: } -f'(c_1) \le -f'(c_2) \Leftrightarrow f'(c_1) \ge f'(c_2)$$

$$\text{It is true because: } c_1 \le c_2 \Rightarrow f'(c_1) \ge f'(c_2) \left(\because f' \searrow \left(0; \frac{\pi}{2}\right)\right)$$

2.42. If $0 < a \le b < \frac{\pi}{6}$ then:

$$\sin(5\sqrt{ab}) \cdot \sin\left(\frac{12ab}{a+b}\right) \ge \sin(6\sqrt{ab}) \cdot \sin\left(\frac{10ab}{a+b}\right)$$

Solution:

$$\sin(5\sqrt{ab}) \cdot \sin\left(\frac{12ab}{a+b}\right) =$$

$$= \frac{1}{2}\left[\cos\left(5\sqrt{ab} - \frac{12ab}{a+b}\right) - \cos\left(5\sqrt{ab} + \frac{12ab}{a+b}\right)\right] \sin(6\sqrt{ab}) \cdot \sin\left(\frac{10ab}{a+b}\right) =$$

$$= \frac{1}{2}\left[\cos\left(6\sqrt{ab} - \frac{10ab}{a+b}\right) - \cos\left(6\sqrt{ab} + \frac{10ab}{a+b}\right)\right]$$

$$\cos\left(5\sqrt{ab} - \frac{12ab}{a+b}\right) + \cos\left(6\sqrt{ab} + \frac{10ab}{a+b}\right) \ge$$

$$\ge \cos\left(6\sqrt{ab} - \frac{10ab}{a+b}\right) + \cos\left(5\sqrt{ab} + \frac{12ab}{a+b}\right)$$

$$\Leftrightarrow 2\left\{\cos\left[\frac{ab}{a+b} - \frac{11\sqrt{ab}}{2}\right] \cdot \cos\left[\frac{\sqrt{ab}}{2} + \frac{11ab}{a+b}\right]\right\} \ge 2\left\{\cos\left[\frac{11\sqrt{ab}}{2} + \frac{ab}{a+b}\right] \cdot \cos\left[\frac{\sqrt{ab}}{2} - \frac{11ab}{a+b}\right]\right\} \quad (*)$$

$$\cos\left[\frac{11ab}{a+b} + \frac{\sqrt{ab}}{2}\right] \overset{(1)}{\ge} \cos\left[\frac{11\sqrt{ab}}{2} + \frac{ab}{a+b}\right] \Leftrightarrow \frac{11\sqrt{ab}}{2} + \frac{ab}{a+b} \ge$$

$$\ge \frac{11ab}{a+b} + \frac{\sqrt{ab}}{2} \Leftrightarrow 5\sqrt{ab} \ge 10 \cdot \frac{ab}{a+b} \Leftrightarrow a+b \ge 2\sqrt{ab} \quad \textit{(true)}$$

$$\cos\left(\frac{ab}{a+b} - \frac{11\sqrt{ab}}{2}\right) \overset{(2)}{\ge} \cos\left(\frac{\sqrt{ab}}{2} - \frac{11ab}{a+b}\right) \Leftrightarrow \frac{\sqrt{ab}}{2} - \frac{11ab}{a+b} \ge$$

$$\ge \frac{ab}{a+b} - \frac{11\sqrt{ab}}{2} \Leftrightarrow 6\sqrt{ab} \ge \frac{12ab}{a+b} \Leftrightarrow a+b \ge 2\sqrt{ab} \quad \textit{(true)}$$

From (1) and (2) we have: (*) true. Proved.

2.43. Find:

$$\boldsymbol{\Omega = \sum_{n=0}^{\infty}\left(\left(\frac{1!!}{0!} + \frac{3!!}{1!} + \frac{5!!}{2!} + \cdots + \frac{(2n+1)!!}{n!}\right) \cdot \frac{2n+2}{(2n+3)!!}\right)}$$

Solution:

$$\Omega=\sum_{n=0}^{\infty}\left(\left(\sum_{k=0}^{n}\frac{(2k+1)!!}{k!}\right)\cdot\frac{2n+2}{(2n+3)!!}\right)=\sum_{n=0}^{\infty}\left(\frac{(2k+1)!!}{k!}\left(\sum_{n=k}^{\infty}\frac{2n+2}{(2n+3)!!}\right)\right)=$$

$$=\sum_{k=0}^{\infty}\left(\frac{(2k+1)!!}{k!}\cdot\frac{1}{(2k+1)!!}\right)=\sum_{k=0}^{\infty}\frac{1}{k!}=e$$

2.44. If $f,g,h:[a,b]\to(0,\infty)$; $a\le b$; continuous then:

$$\int_a^b\left(\sum_{cyc}\left(\frac{\left(f(x)+g(x)\right)^3}{f(x)g(x)h(x)+h^3(x)}\right)\right)dx\ge 12(b-a)$$

Solution:

First, we prove that if $x\in(0,3)$ then: $\frac{(3-x)^3}{1+x^3}\ge -12x+16$ *(1)*

$$(3-x)^3\ge(-12x+16)(1+x^3)$$

$$27-27x+9x^2-x^3\ge -12x-12x^4+16+16x^3$$

$$12x^4-17x^3+9x^2-15x+11\ge 0$$

$$12x^4-12x^3-5x^3+5x^2+4x^2-4x-11x+11\ge 0$$

$$12x^3(x-1)-5x^2(x-1)+4x(x-1)-11(x-1)\ge 0$$

$$(x-1)(12x^3-5x^3+4x-11)\ge 0$$

$$(x-1)^2(12x^2+7x+11)\ge 0\quad \textit{(true)}$$

Now, we prove: $\sum_{cyc}\frac{(a+b)^3}{abc+c^3}\ge 12$ *(2)*

Due the homogeneity we can suppose that $a+b+c=3$.

$$\sum_{cyc}\frac{(a+b)^3}{abc+c^3}\overset{AM-GM}{\ge}\sum_{cyc}\frac{(a+b)^3}{\frac{(a+b+c)^3}{27}+c^3}=$$

$$=\sum_{cyc}\frac{(a+b)^3}{\frac{27}{27}+c^3}=\sum_{cyc}\frac{(3-c)^3}{1+c^3}\overset{(1)}{\ge}$$

$$\ge\sum_{cyc}(-12c+16)=-12(a+b+c)+3\cdot 16=-12\cdot 3+48=12$$

In (2) we take $a=f(x);b=g(x);c=h(x)$

$\sum_{cyc}\frac{\left(f(x)+g(x)\right)^3}{f(x)g(x)h(x)+h^3(x)}\ge 12$ *(3)*

Integrating (3):

$$\int_a^b\left(\sum_{cyc}\left(\frac{\left(f(x)+g(x)\right)^3}{f(x)g(x)h(x)+h^3(x)}\right)\right)dx\geq\int_a^b 12dx=12(b-a)$$

2.45. Let be:

$$\boldsymbol{\Omega(a)=\sum_{n=1}^{\infty}\log\left(\cos\left(\frac{a}{2^n}\right)\right);a\in\left(0,\frac{\pi}{2}\right)}$$

Prove that:

$$\boldsymbol{\Omega\left(\frac{a^2+b^2+c^2}{a+b+c}\right)\geq\frac{a\Omega(b)+b\Omega(c)+c\Omega(a)}{a+b+c}}$$

Solution:

$$\Omega(a)=\log\left(\prod_{k=1}^{\infty}\cos\frac{a}{2^k}\right)=\log\left(\lim_{n\to\infty}\left(\cos\frac{a}{2}\cos\frac{a}{2^2}\cdot\ldots\cdot\cos\frac{a}{2^n}\right)\right)=$$

$$=\log\left(\lim_{n\to\infty}\left(\frac{2^n\sin\frac{a}{2^n}\cos\frac{a}{2^n}\cos\frac{a}{2^{n-1}}\cdot\ldots\cdot\cos\frac{a}{2}}{2^n\sin\frac{a}{2^n}}\right)\right)=$$

$$=\log\left(\lim_{n\to\infty}\frac{\sin a}{2^n\sin\frac{a}{2^n}}\right)=\log\left(\lim_{n\to\infty}\frac{\sin a}{a}\cdot\frac{\frac{a}{2^n}}{\sin\frac{a}{2^n}}\right)=\log\left(\frac{\sin a}{a}\right)$$

$$\Omega(a)=\log(\sin a)-\log(a),\Omega'(a)=\frac{\cos a}{\sin a}-\frac{1}{a}=\cot a-\frac{1}{a}$$

$$\Omega''(a)=-\frac{1}{\sin^2 a}+\frac{1}{a^2}=\frac{\sin^2 a-a^2}{a^2\sin^2 a}<0$$

because $a\in\left(0,\frac{\pi}{2}\right);\sin a<a,\Omega$ *– concave. By Jensen's inequality:*

$$\frac{a}{a+b+c}\Omega(a)+\frac{b}{a+b+c}\Omega(b)+\frac{c}{a+b+c}\Omega(c)\leq$$

$$\leq\Omega\left(\frac{a}{a+b+c}a+\frac{b}{a+b+c}b+\frac{c}{a+b+c}c\right)=\Omega\left(\frac{a^2+b^2+c^2}{a+b+c}\right)$$

2.46. Prove without softs:

$$\boldsymbol{\int_0^{10}2^{x^2}dx+\frac{1}{4}\int_0^{20}2^{x^2}dx+\frac{1}{2}\int_0^{30}2^{x^2}dx>\frac{1}{3}\int_0^{15}2^{x^2}dx+\frac{1}{5}\int_0^{25}2^{x^2}dx}$$

Solution:

$$f:[0,\infty)\to\mathbb{R}; f(x)=2^{x^2}>0$$

$$LHS=\int_0^{10} 2^{x^2}\,dx+\frac{1}{4}\int_0^{10} 2^{x^2}\,dx+\frac{1}{4}\int_{10}^{20} 2^{x^2}dx+$$

$$+\frac{1}{2}\int_0^{10} 2^{x^2}\,dx+\frac{1}{2}\int_{10}^{20} 2^{x^2}\,dx+\frac{1}{2}\int_{20}^{30} 2^{x^2}\,dx$$

$$LHS=\left(1+\frac{1}{4}+\frac{1}{2}\right)\int_0^{10} 2^{x^2}dx+\left(\frac{1}{4}+\frac{1}{2}\right)\int_{10}^{20} 2^{x^2}\,dx+\frac{1}{2}\int_{20}^{30} 2^{x^2}\,dx$$

$$LHS=\frac{7}{4}\int_0^{10} 2^{x^2}\,dx+\frac{3}{4}\int_{10}^{20} 2^{x^2}\,dx+\frac{1}{2}\int_{20}^{30} 2^{x^2}\,dx$$

$$RHS=\frac{1}{3}\left(\int_0^{10} 2^{x^2}\,dx+\int_{10}^{15} 2^{x^2}\,dx\right)+\frac{1}{5}\left(\int_0^{10} 2^{x^2}\,dx+\int_{10}^{20} 2^{x^2}dx+\int_{20}^{25} 2^{x^2}dx\right)$$

$$RHS=\left(\frac{1}{3}+\frac{1}{5}\right)\int_0^{10} 2^{x^2}\,dx+\frac{1}{3}\int_{10}^{20} 2^{x^2}\,dx-\frac{1}{3}\int_{15}^{20} 2^{x^2}dx+$$

$$+\frac{1}{5}\int_{10}^{20} 2^{x^2}\,dx+\frac{1}{5}\int_{20}^{30} 2^{x^2}\,dx-\frac{1}{5}\int_{25}^{30} 2^{x^2}\,dx$$

$$RHS=\frac{8}{15}\int_0^{10} 2^{x^2}\,dx+\frac{8}{15}\int_{10}^{20} 2^{x^2}\,dx+\frac{1}{5}\int_0^{30} 2^{x^2}\,dx-$$

$$-\frac{1}{3}\int_{15}^{20} 2^{x^2}\,dx-\frac{1}{5}\int_{25}^{30} 2^{x^2}\,dx$$

$$LHS-RHS=\left(\frac{7}{4}-\frac{8}{15}\right)\int_0^{10} 2^{x^2}\,dx+$$

$$+\left(\frac{3}{4}-\frac{8}{15}\right)\int_{10}^{20} 2^{x^2}\,dx+\left(\frac{1}{2}-\frac{1}{5}\right)\int_{20}^{30} 2^{x^2}\,dx+$$

$$+\frac{1}{3}\int_{15}^{20} 2^{x^2}\,dx+\frac{1}{5}\int_{25}^{30} 2^{x^2}\,dx$$

$$LHS-RHS=\frac{73}{60}\int_{0}^{10} 2^{x^2}\,dx+\frac{13}{60}\int_{10}^{20} 2^{x^2}\,dx+\frac{3}{10}\int_{20}^{30} 2^{x^2}\,dx+$$

$$+\frac{1}{3}\int_{15}^{20} 2^{x^2}dx+\frac{1}{5}\int_{25}^{30} 2^{x^2}\,dx>0$$

$$LHS-RHS>0\Rightarrow LHS>RHS$$

2.47. If $x,y,z>0$ then:

$$\frac{e^{x^3+y^3}}{e^{2(x+y)}}+\frac{e^{y^3+z^3}}{e^{2(y+z)}}+\frac{e^{z^3+x^3}}{e^{2(z+x)}}\geq\frac{1}{e^2}\left(x^xy^y+y^yz^z+z^zx^x\right)$$

Solution:

It's known that $\log x\leq x-1;(\forall)x>0$*. Let be* $f:(0,\infty)\to\mathbb{R}$*;*

$$f(x)=x^3-2x-x\log x+1\geq x^3-2x-x(x-1)+1=$$
$$=x^3-2x-x^2+x+1=x^3-x^2-x+1=$$
$$=x^2(x-1)-(x-1)=(x-1)(x^2-1)=$$
$$=(x-1)^2(x+1)\geq 0;(\forall)x\geq 1$$
$$x^3-2x-x\log x+1\geq 0$$
$$x^3-2x\geq x\log x-1$$
$$e^{x^3-2x}\geq e^{x\log x-1}$$
$$\frac{e^{x^2}}{e^{2x}}\geq\frac{e^{\log x^x}}{e}=\frac{x^x}{e}\quad (1)$$
$$\text{Analogous: }\frac{e^{y^2}}{e^{2y}}\geq\frac{y^y}{e}\quad (2)$$

$$\text{By multiplying (1); (2): }\frac{e^{x^2+y^2}}{e^{2x}\cdot e^{2y}}\geq\frac{x^x\cdot y^y}{e^2}$$
$$\frac{e^{x^2+y^2}}{e^{2(x+y)}}\geq\frac{x^x\cdot y^y}{e^2}$$

$$\sum_{cyc}\frac{e^{x^2+y^2}}{e^{2(x+y)}}\geq\frac{1}{e^2}\sum_{cyc}(x^x\cdot y^y)$$

2.48. If $x,y,z\geq 0$ then:

$$e^{x^2(x^2+1)}+e^{y^2(y^2+1)}+e^{z^2(z^2+1)}\geq e^{xy(xy+1)}+e^{yz(yz+1)}+e^{zx(zx+1)}$$

Solution:

$$e^{x^2(x^2+1)}+e^{y^2(y^2+1)}\overset{AM-GM}{\geq}2\sqrt{e^{x^2(x^2+1)+y^2(y^2+1)}}\geq$$
$$\overset{AM-GM}{\geq}2\sqrt{e^{2\sqrt{x^2(x^2+1)y^2(y^2+1)}}}=$$
$$=2e^{\sqrt{x^2y^2(x^2+1)(y^2+1)}}=2e^{xy\sqrt{(x^2+1)(y^2+1)}}\geq$$
$$\overset{CBS}{\geq}2e^{xy(xy+1\cdot 1)}=2e^{xy(xy+1)}\quad (1)$$

Analogous: $e^{y^2(y^2+1)}+e^{z^2(z^2+1)}\geq 2e^{yz(yz+1)}$ *(2)*

$e^{z^2(z^2+1)}+e^{x^2(x^2+1)}\geq 2e^{zx(zx+1)}$ *(3)*

By adding (1); (2); (3): $2\left(e^{x^2(x^2+1)}+e^{y^2(y^2+1)}+e^{z^2(z^2+1)}\right)\geq$

$$\geq 2\left(e^{xy(xy+1)}+e^{yz(yz+1)}+e^{zx}(zx+1)\right)$$

$$e^{x^2(x^2+1)}+e^{y^2(y^2+1)}+e^{z^2(z^2+1)}\geq e^{xy(xy+1)}+e^{yz(yz+1)}+e^{zx(zx+1)}$$

2.49. Find:

$$L=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{1\leq i<j<k<l\leq n}\frac{i\cdot j\cdot k\cdot l}{(n^2+i^2)(n^2+j^2)(n^2+k^2)(n^2+l^2)}\right)$$

Solution:

By AM-GM: $(n^2+i^2)(n^2+j^2)(n^2+k^2)(n^2+l^2)\geq$

$$\geq 2ni\cdot 2nj\cdot 2nk\cdot 2nl=16n^4ijkl$$

$$\frac{1}{(n^2+i^2)(n^2+j^2)(n^2+k^2)(n^2+l^2)}\leq\frac{1}{16n^4ijkl}$$

$$\frac{ijkl}{(n^2+i^2)(n^2+j^2)(n^2+k^2)(n^2+l^2)}\leq\frac{1}{16n^4}$$

$$\sum_{1\leq i<j<k<l\leq n}\frac{ijkl}{(n^2+i^2)(n^2+j^2)(n^2+k^2)(n^2+l^2)}\leq$$
$$\leq\sum_{1\leq i<j<k<l\leq n}\frac{1}{16n^4}\leq\frac{1}{16n^4}\cdot n^4=\frac{1}{16}$$

$$0<\frac{1}{n}\sum_{1\leq i<j<k<l\leq n}\frac{ijkl}{(n^2+i^2)(n^2+j^2)(n^2+k^2)(n^2+l^2)}\leq\frac{1}{16n}$$

$$0 \le L \le \lim_{n\to\infty} \frac{1}{16n} = 0;\ L = 0$$

2.50. Prove that if $\mathbf{0} < a \le b < \frac{\boldsymbol{\pi}}{\mathbf{5}}$ then:

$$\mathbf{\sin\left(5\sqrt{ab}\right)\sin(2a+2b) \ge \sin\left(4\sqrt{ab}\right)\sin\left(\frac{5a+5b}{2}\right)}$$

Solution:

$$Let\ be\ f:\left(0,\frac{\pi}{5}\right) \to \mathbb{R}; f(x) = \frac{\sin 5x}{\sin 4x}$$

$$f'(x) = \frac{5\cos 5x \sin 4x - 4\cos 4x \sin 5x}{\sin^2(4x)}$$

$$Let\ be\ g:\left(0,\frac{\pi}{5}\right) \to \mathbb{R}; g(x) = 5\cos 5x \sin 4x - 4\cos 4x \sin 5x$$

$$g'(x) = 5\cdot(-5)\sin 5x \sin 4x + 5\cos 5x \cdot 4\cos 4x -$$
$$+(4\cdot 4\sin 4x \sin 5x - 4\cos 4x \cdot 5\cos 5x)$$
$$g'(x) = -25\sin 5x \sin 4x + 16\sin 4x \sin 5x$$
$$g'(x) = -9\sin 5x \sin 4x < 0; (\forall)x \in \left(0,\frac{\pi}{5}\right)$$
$$g\ decreasing\ on\ \left(0,\frac{\pi}{5}\right)$$
$$\sup_{x\in\left(0,\frac{\pi}{2}\right)} g(x) = 0 \Rightarrow g(x) < 0, (\forall)x \in \left(0,\frac{\pi}{3}\right)$$
$$\Rightarrow f'(x) < 0, (\forall)x \in \left(0,\frac{\pi}{5}\right) \Rightarrow \sup_{x\in\left(0,\frac{\pi}{5}\right)} f(x) = 0$$
$$\Rightarrow f(x)\ decreasing\ on\ \left(0,\frac{\pi}{5}\right)$$
$$0 < a \le \sqrt{ab} \le \frac{a+b}{2} \le b < \frac{\pi}{5}$$
$$f - decreasing \Rightarrow f\left(\sqrt{ab}\right) \ge f\left(\frac{a+b}{2}\right)$$
$$\frac{\sin\left(5\sqrt{ab}\right)}{\sin\left(4\sqrt{ab}\right)} \ge \frac{\sin\left(5\cdot\frac{a+b}{2}\right)}{\sin\left(4\cdot\frac{a+b}{2}\right)}$$
$$\sin\left(5\sqrt{ab}\right)\sin(2a+2b) \ge \sin\left(4\sqrt{ab}\right)\sin\left(\frac{5a+5b}{2}\right)$$

2.51. Find:

$$\boldsymbol{\Omega} = \int \frac{(\log x - 1)(x^2 - \log^2 x)}{(x^2 + 4\log^2 x)(4x^2 + \log^2 x)}\,dx; x > 0$$

Solution:

$$\Omega=\frac{1}{3}\int\frac{(\log x-1)(3x^2-3\log^2 x)}{(x^2+4\log^2 x)(4x^2+\log^2 x)}dx=$$

$$=\frac{1}{3}\int\frac{(\log x-1)(4x^2+\log^2 x-x^2-4\log^2 x)}{(x^2+4\log^2 x)(4x^2+\log^2 x)}dx=$$

$$=\frac{1}{3}\int\frac{(\log x-1)(4x^2+\log^2 x)}{(x^2+4\log^2 x)(4x^2+\log^2 x)}dx-$$

$$-\frac{1}{3}\int\frac{(\log x-1)(x^2+4\log^2 x)}{(x^2+4\log^2 x)(4x^2+\log^2 x)}dx=$$

$$=\frac{1}{3}\int\frac{\log x-1}{x^2+4\log^2 x}dx-\frac{1}{3}\int\frac{\log x-1}{4x^2+\log^2 x}=$$

$$=\frac{1}{3}\int\frac{\frac{\log x-1}{\log^2 x}}{\left(\frac{x}{\log x}\right)^2+4}dx+\frac{1}{3}\int\frac{\frac{1-\log x}{x^2}}{\left(\frac{\log x}{x}\right)^2+4}dx=$$

$$=\frac{1}{3}\int\frac{\left(\frac{x}{\log x}\right)'}{\left(\frac{x}{\log x}\right)^2+4}dx+\frac{1}{3}\int\frac{\left(\frac{\log x}{x}\right)'}{\left(\frac{\log x}{x}\right)^2+4}dx=$$

$$=\frac{1}{6}\tan^{-1}\left(\frac{x}{\log x}\right)+\frac{1}{6}\tan^{-1}\left(\frac{\log x}{x}\right)+C$$

2.52. If $a,b,c\in(0,\pi)$ then:

$$\prod_{cyc}\left((2a+b)\sin c\right)\le 27\prod_{cyc}\left(a\sin\left(\frac{2b+c}{3}\right)\right)$$

Solution:

$$\textit{Let be } f:(0,\pi)\to\mathbb{R};\ f(x)=\log x-\log(\sin x)$$

$$f'(x)=\frac{1}{x}-\cot x\,;\ f''(x)=-\frac{1}{x^2}+\frac{1}{\sin^2 x}$$

$$f''(x)=\frac{x^2-\sin^2 x}{x^2\sin^2 x}>0 \textit{ because } x>\sin x$$

$$f \textit{ convexe. By Jensen's inequality:}$$

$$f\left(\frac{2a+b}{3}\right)\le\frac{2}{3}f(a)+\frac{1}{3}f(b)$$

$$\log\left(\frac{2a+b}{3}\right)-\log\left(\sin\left(\frac{2a+b}{3}\right)\right)\le\frac{2}{3}(\log a-\log(\sin a))+$$

$$+\frac{1}{3}(\log b-\log(\sin b))$$

$$\log\left(\frac{2a+b}{3}\right)-\left(\frac{2}{3}\log a+\frac{1}{3}\log b\right)\leq$$

$$\leq\log\left(\sin\left(\frac{2a+b}{3}\right)\right)-\frac{1}{3}(2\log(\sin a)+\log(\sin b))$$

$$\log\left(\frac{2a+b}{3}\right)-\log\sqrt[3]{a^2b}\leq\log\left(\sin\left(\frac{2a+b}{3}\right)\right)-\log\sqrt[3]{\sin^2 a\sin b}$$

$$\log\left(\frac{2a+b}{3\sqrt[3]{a^2b}}\right)\leq\log\left(\frac{\sin\left(\frac{2a+b}{3}\right)}{\sqrt[3]{\sin^2 a\sin b}}\right)$$

$$\sum_{cyc}\log\left(\frac{2a+b}{3\sqrt[3]{a^2b}}\right)\leq\sum_{cyc}\log\left(\frac{\sin\left(\frac{2a+b}{3}\right)}{\sqrt[3]{\sin^2 a\sin b}}\right)$$

$$\log\left(\frac{(2a+b)(2b+c)(2c+a)}{27abc}\right)\leq$$

$$\leq\log\left(\frac{\sin\left(\frac{2a+b}{3}\right)\sin\left(\frac{2b+c}{3}\right)\sin\left(\frac{2c+a}{3}\right)}{\sin a\sin b\sin c}\right)$$

$$\frac{(2a+b)(2b+c)(2c+a)}{27abc}\leq\frac{\sin\left(\frac{2a+b}{3}\right)\sin\left(\frac{2b+c}{3}\right)\sin\left(\frac{2c+a}{3}\right)}{\sin a\sin b\sin c}$$

$$\prod_{cyc}\left((2a+b)\sin c\right)\leq 27\prod_{cyc}\left(a\sin\left(\frac{2b+c}{3}\right)\right)$$

Equality holds for $a=b=c$.

2.53. If $0<a,b,c\leq 16$ then:

$$\mathbf{125\exp\left(\sum_{cyc}\left(\sqrt{\frac{2a+3b}{5}}-\sqrt{a}\right)\right)\leq\frac{(2a+3b)(2b+3c)(2c+3a)}{abc}}$$

Solution:

Let be $f:(0,16]\to\mathbb{R};\ f(x)=\sqrt{x}-\log x$

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{x};\ f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}+\frac{1}{x^2}$$

$$f''(x) = -\frac{1}{4x\sqrt{x}} + \frac{1}{x^2} = \frac{\sqrt{x}-4}{4x^2} \ge 0; (\forall)x \in (0,16]$$

f convexe. By Jensen's inequality: $f\left(\frac{2a+3b}{5}\right) \le \frac{2}{5}f(a) + \frac{3}{5}f(b)$

$$\sqrt{\frac{2a+3b}{5}} - \log\left(\frac{2a+3b}{5}\right) \le \frac{2}{5}\left(\sqrt{a} - \log a\right) + \frac{3}{5}\left(\sqrt{b} - \log b\right)$$

$$\sqrt{\frac{2a+3b}{5}} - \frac{2\sqrt{a}+3\sqrt{b}}{5} \le \log\left(\frac{2a+3b}{5}\right) - \log\left(\sqrt[5]{a^2b^3}\right)$$

$$\sum_{cyc}\left(\sqrt{\frac{2a+3b}{5}} - \frac{2\sqrt{a}+3\sqrt{b}}{5}\right) \le \sum_{cyc}\log\left(\frac{2a+3b}{5\sqrt[5]{a^2b^3}}\right)$$

$$\sum_{cyc}\left(\sqrt{\frac{2a+3b}{5}}\right) - \frac{1}{5}\sum_{cyc}\left(2\sqrt{a}+3\sqrt{b}\right) \le \log\left(\prod_{cyc}\frac{2a+3b}{5\sqrt[5]{a^2b^3}}\right)$$

$$\sum_{cyc}\left(\sqrt{\frac{2a+3b}{5}}\right) - \frac{1}{5}\sum_{cyc}5\sqrt{a} \le \log\left(\frac{(2a+3b)(2b+3c)(2c+3a)}{125abc}\right)$$

$$\sum_{cyc}\left(\sqrt{\frac{2a+3b}{5}} - \sqrt{a}\right) \le \log\left(\frac{(2a+3b)(2b+3c)(2c+3a)}{125abc}\right)$$

$$125\exp\left(\sum_{cyc}\left(\sqrt{\frac{2a+3b}{5}} - \sqrt{a}\right)\right) \le \frac{(2a+3b)(2b+3c)(2c+3a)}{abc}$$

2.54. If $0 < a,b,c \le 16$ then:

$$\mathbf{27}\exp\left(\sum_{cyc}\left(\sqrt{\frac{\boldsymbol{a}+2\boldsymbol{b}}{3}} - \sqrt{\boldsymbol{a}}\right)\right) \le \frac{(\boldsymbol{a}+2\boldsymbol{b})(\boldsymbol{b}+2\boldsymbol{c})(\boldsymbol{c}+2\boldsymbol{a})}{\boldsymbol{abc}}$$

Solution:

Let be $f:(0,16] \to \mathbb{R}; f(x) = \sqrt{x} - \log x$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{x}; f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{1}{x^2}$$

$$f''(x) = -\frac{1}{4x\sqrt{x}} + \frac{1}{x^2} = \frac{\sqrt{x}-4}{4x^2} \ge 0; (\forall)x \in (0,16]$$

f convexe. By Jensen's inequality:

$$\sqrt{\frac{a+2b}{3}}-\log\left(\frac{a+2b}{3}\right)\le\frac{1}{3}\left(\sqrt{a}-\log a\right)+\frac{2}{3}\left(\sqrt{b}-\log b\right)$$

$$f\left(\frac{a+2b}{3}\right)\le\frac{1}{3}f(a)+\frac{2}{3}f(b)$$

$$\sqrt{\frac{a+2b}{3}}-\frac{\sqrt{a}+2\sqrt{b}}{3}\le\log\left(\frac{a+2b}{3}\right)-\log\sqrt[3]{ab^2}$$

$$\sqrt{\frac{a+2b}{3}}-\frac{\sqrt{a}+2\sqrt{b}}{3}\le\log\left(\frac{a+2b}{3\sqrt[3]{ab^2}}\right)$$

$$\sum_{cyc}\sqrt{\frac{a+2b}{3}}-\frac{1}{3}\sum_{cyc}\left(\sqrt{a}+2\sqrt{b}\right)\le\log\left(\prod_{cyc}\frac{a+2b}{3\sqrt[3]{ab^2}}\right)$$

$$\sum_{cyc}\sqrt{\frac{a+2b}{3}}-\sum_{cyc}\sqrt{a}\le\log\left(\frac{(a+2b)(b+2c)(c+2a)}{27abc}\right)$$

$$27\exp\left(\sum_{cyc}\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right)\le\frac{(a+2b)(b+2c)(c+2a)}{abc}$$

Equality holds if $a=b=c$.

2.55. If $n\in\mathbb{N}; n\neq 0$ then:

$$\boldsymbol{n\int_0^1(\tan^{-1}x)\,dx<\frac{\pi}{4}+\sum_{k=0}^{n-1}\tan^{-1}\left(\frac{k}{n}\right)}$$

Solution:

$$\int_0^1(\tan^{-1}x)\,dx-\frac{1}{n}\sum_{k=0}^{n-1}\tan^{-1}\left(\frac{k}{n}\right)=$$

$$=\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(\tan^{-1}x)\,dx-\left(\frac{k}{n}-\frac{k-1}{n}\right)\sum_{k=0}^{n-1}\tan^{-1}\left(\frac{k}{n}\right)=$$

$$= \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} (\tan^{-1} x)\, dx - \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \tan^{-1}\left(\frac{k}{n}\right) dx =$$

$$= \sum_{k=1}^{n} \left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left((\tan^{-1} x) - \tan^{-1}\left(\frac{k}{n}\right) \right) dx \right) \quad \textit{(1)}$$

$$\tan^{-1} x = \arctan x\, ;\, (\tan^{-1} x)' = \frac{1}{1+x^2};\ \textit{increasing}$$

$$x \le \frac{k}{n} \Rightarrow \tan^{-1} x \le \tan^{-1}\left(\frac{k}{n}\right)$$

$$\tan^{-1}\left(\frac{k}{n}\right) \ge \tan^{-1} x \Rightarrow \tan^{-1}\left(\frac{k}{n}\right) - \tan^{-1}\left(\frac{k-1}{n}\right) \ge \tan^{-1} x - \tan^{-1}\left(\frac{k-1}{n}\right)$$

$$\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\tan^{-1}\left(\frac{k}{n}\right) - \tan^{-1}\left(\frac{k-1}{n}\right) \right) dx \ge \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\tan^{-1} x - \tan^{-1}\left(\frac{k-1}{n}\right) \right) dx \quad \textit{(2)}$$

By (1); (2):

$$\int_{0}^{1} (\tan^{-1} x)\, dx - \frac{1}{n} \sum_{k=0}^{n-1} \tan^{-1}\left(\frac{k}{n}\right) = \sum_{k=1}^{n} \left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left((\tan^{-1} x) - \tan^{-1}\left(\frac{k}{n}\right) \right) \right) dx$$

$$< \sum_{k=1}^{n} \left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\tan^{-1}\left(\frac{k}{n}\right) - \tan^{-1}\left(\frac{k-1}{n}\right) \right) dx \right) =$$

$$= \sum_{k=1}^{n} \left(\left(\frac{k}{n} - \frac{k-1}{n} \right) \left(\tan^{-1}\left(\frac{k}{n}\right) - \tan^{-1}\left(\frac{k-1}{n}\right) \right) \right) =$$

$$= \frac{1}{n} \sum_{k=1}^{n} \left(\tan^{-1}\left(\frac{k}{n}\right) - \tan^{-1}\left(\frac{k-1}{n}\right) \right) = \frac{1}{n}\left(\tan^{-1}\left(\frac{n}{n}\right) - \tan^{-1}\left(\frac{0}{n}\right) \right) =$$

$$= \frac{1}{n}(\tan^{-1} 1 - \tan^{-1} 0) = \frac{1}{n}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4n}$$

$$\int_{0}^{1} (\tan^{-1} x)\, dx - \frac{1}{n} \sum_{k=0}^{n-1} \tan^{-1}\left(\frac{k}{n}\right) < \frac{\pi}{4n}, n \int_{0}^{1} (\tan^{-1} x)\, dx - \sum_{k=0}^{n-1} \tan^{-1}\left(\frac{k}{n}\right) < \frac{\pi}{4}$$

$$n \int_{0}^{1} (\tan^{-1} x)\, dx < \frac{\pi}{4} + \sum_{k=0}^{n-1} \tan^{-1}\left(\frac{k}{n}\right)$$

2.56. If $n \in \mathbb{N}; n \geq 2$ then:

$$3^{n-2}\left(\prod_{k=2}^{n}\left(\int_0^1 \left(x^k\sqrt{x^2+1}\right)dx\right)\right)^2 < 4^{n-2}\cdot\prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right)$$

Solution:

$$\left(\int_0^1 \left(x^k\sqrt{x^2+1}\right)dx\right)^2 = \left(\int_0^1 \left(x\cdot x^{k-1}\sqrt{x^2+1}\right)dx\right)^2 \leq$$

$$\overset{CBS}{\leq}\left(\int_0^1 x^2dx\right)\left(\int_0^1 \left(x^{2k-2}(x^2+1)\right)dx\right) = \frac{1}{3}\int_0^1 (x^{2k}+x^{2k-2})dx =$$

$$= \frac{1}{3}\left(\int_0^1 x^{2k}dx + \int_0^1 x^{2k-2}\,dx\right) = \frac{1}{3}\left(\frac{1}{2k+1}+\frac{1}{2k-1}\right) =$$

$$= \frac{1}{3}\cdot\frac{2k+1+2k-1}{4k^2-1} = \frac{4}{3}\cdot\frac{k}{4k^2-1}$$

$$\prod_{k=2}^{n}\left(\int_0^1 \left(x^k\sqrt{x^2+1}\right)dx\right)^2 \leq \frac{4^{n-2}}{3^{n-2}}\cdot\prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right)$$

$$3^{n-2}\left(\prod_{k=2}^{n}\left(\int_0^1 \left(x^k\sqrt{x^2+1}\right)dx\right)\right)^2 \leq 4^{n-2}\cdot\prod_{k=2}^{n}\left(\frac{k}{4k^2-1}\right)$$

2.57. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1}}{e^n}\right)$$

Solution:

We prove by induction:

$$P(n): 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1} \leq n; n \geq 1$$

$$P(1): \frac{1}{2^1-1} \leq 1 \Leftrightarrow 1 \leq 1 \ (True)$$

$$P(k): 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k-1} \leq k \ \ (suppose\ true)$$

$$P(k+1): 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{k+1}-1} \le k+1 \text{ (to prove)}$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{k+1}-1} \le$$

$$\le k+\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\cdots+\frac{1}{2^{k+1}-1} \le$$

$$\le k+\underbrace{\frac{1}{2^k}+\frac{1}{2^k}+\cdots+\frac{1}{2^k}}_{for\ "2^k"\ times} = k+2^k\cdot\frac{1}{2^k} = k+1$$

$$P(k) \to P(k+1)$$

$$0 < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1} \le n$$

$$0 < \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1}}{e^n} \le \frac{n}{e^n}$$

$$\lim_{n\to\infty}\frac{n}{e^n} \overset{TCS}{=} \lim_{n\to\infty}\frac{n+1-n}{e^{n+1}-e^n} = \lim_{n\to\infty}\frac{1}{e^n(e-1)} = \frac{1}{\infty} = 0$$

$$0 \le \lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1}}{e^n}\right) \le \lim_{n\to\infty}\frac{n}{e^n} = 0$$

$$\lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1}}{e^n}\right) = 0$$

2.58. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b\int_a^b\left(\frac{\cot x+\cot y+\tan(x+y)}{\cot x\cot y\tan(x+y)}\right)dx\,dy \le \frac{\prod(b-a)}{2}$$

Solution:

$$\frac{\cot x+\cot y+\tan(x+y)}{\cot x\cot y\tan(x+y)} = \frac{\frac{1}{\tan x}+\frac{1}{\tan y}+\frac{\tan x+\tan y}{1-\tan x\tan y}}{\frac{1}{\tan x}\cdot\frac{1}{\tan y}\cdot\frac{\tan x+\tan y}{1-\tan x\tan y}} =$$

$$= \frac{(1-\tan x\tan y)(\tan x+\tan y)+(\tan x+\tan y)\tan x\tan y}{\tan x+\tan y} =$$

$$= \frac{(\tan x+\tan y)(1-\tan x\tan y+\tan x\tan y)}{\tan x+\tan y} = 1$$

$$\int_a^b \int_a^b \left(\frac{\cot x + \cot y + \tan(x+y)}{\cot x \cot y \tan(x+y)}\right) dx\, dy =$$

$$= \int_a^b \int_a^b dx\, dy = (b-a)^2 = (b-a)(b-a) \le (b-a) \cdot \frac{\pi}{2} = \frac{\pi(b-a)}{2}$$

2.59. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \int_a^b \frac{(1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+\tan x \tan y \tan\left(\frac{\pi}{4}-x-y\right)} dx\, dy \le \pi(b-a)$$

Solution:

$$\frac{(1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+\tan x \tan y \tan\left(\frac{\pi}{4}-x-y\right)} =$$

$$= \frac{(1+\tan x)(1+\tan y)\left(1+\frac{1-\tan(x+y)}{1+\tan(x+y)}\right)}{1+\tan x \tan y \cdot \frac{1-\tan(x+y)}{1+\tan(x+y)}} =$$

$$= \frac{(1+\tan x)(1+\tan y)(1+\tan(x+y)+1-\tan(x+y))}{1+\tan(x+y)+\tan x \tan y\,(1-\tan(x+y))} =$$

$$= \frac{2(1+\tan x)(1+\tan y)}{1+\tan(x+y)+\tan x \tan y - \tan x \tan y \tan(x+y)} =$$

$$= \frac{2(1+\tan x)(1+\tan y)}{1+(1-\tan x \tan y)\tan(x+y)+\tan x \tan y} =$$

$$= \frac{2(1+\tan x)(1+\tan y)}{1+(1-\tan x \tan y)\cdot \frac{\tan x + \tan y}{1-\tan x \tan y}+\tan x \tan y} =$$

$$= \frac{2(1+\tan x)(1+\tan y)}{1+\tan x + \tan y + \tan x \tan y} =$$

$$= \frac{2(1+\tan x + \tan y + \tan x \tan y)}{1+\tan x + \tan y + \tan x \tan y} = 2$$

$$\int_a^b \int_a^b \frac{(1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+\tan x \tan y \tan\left(\frac{\pi}{4}-x-y\right)} dx\, dy =$$

$$= \int_a^b \int_a^b 2\,dx\,dy = 2(b-a)^2 = 2(b-a)(b-a) \le 2(b-a)\cdot\frac{\pi}{2} = (b-a)\pi$$

2.60. Find:

$$\boldsymbol{\Omega = \int \left(\tan\left(\frac{\pi-9x}{3}\right)\tan\left(\frac{\pi-3x}{3}\right)\tan x\tan\left(\frac{\pi+3x}{3}\right)\tan\left(\frac{\pi+9x}{3}\right)\right)dx}$$

Solution:

$$\Omega = \int\left(\frac{\left(\tan\frac{\pi}{3}\right)^2-(\tan 3x)^2}{1-\left(\tan\frac{\pi}{3}\tan 3x\right)^2}\cdot\frac{\left(\tan\frac{\pi}{3}\right)^2-(\tan x)^2}{1-\left(\tan\frac{\pi}{3}\tan x\right)^2}\cdot\tan x\right)dx$$

$$\Omega = \int\left(\frac{3-\tan^2 3x}{1-3\tan^2 3x}\cdot\frac{(3-\tan^2 x)\tan x}{1-3\tan^2 x}\right)dx$$

$$\Omega = \int\frac{(3-\tan^2 3x)\tan 3x}{1-3\tan^2 3x}dx = \int \tan 9x\,dx$$

$$\Omega = \int\frac{\sin 9x}{\cos 9x}dx = -\frac{1}{9}\int\frac{(\cos 9x)'}{\cos 9x} = -\frac{1}{9}\ln|\cos 9x| + C$$

2.61. If $\boldsymbol{a, b \in \mathbb{R}}$ then:

$$\boldsymbol{8\int_a^b\int_a^b (\cos x\cos y\cos(x+y))\,dx\,dy + (b-a)^2 \ge 0}$$

Solution:

$$\cos x\cos y\cos(x+y) = \frac{1}{2}[\cos(x+y)+\cos(x-y)]\cos(x+y) =$$

$$= \frac{1}{2}(\cos u+\cos v)\cos v = \frac{1}{2}(\cos u\cos v+\cos^2 v) =$$

$$(u = x+y; v = x-y)$$

$$= \frac{1}{2}\left(\cos^2 v+\cos u\cos v+\frac{\cos^2 u}{4}-\frac{\cos^2 u}{4}\right) =$$

$$= \frac{1}{2}\left[\left(\cos v+\frac{\cos u}{2}\right)^2-\frac{\cos^2 u}{4}\right] \ge -\frac{\cos^2 u}{8} \ge -\frac{1}{8}$$

$$8\cos x\cos y\cos(x+y)\geq 1$$

$$8\int_a^b\int_a^b(\cos x\cos y\cos(x+y))\,dx\,dy+\int_a^b\int_a^b dx\,dy\geq 0$$

$$8\int_a^b\int_a^b(\cos x\cos y\cos(x+y))\,dx\,dy+(b-a)^2\geq 0$$

2.62. Prove without softs:

$$\left|\int_0^2\left(\sqrt{x}\sin\pi x\right)dx\right|<\int_0^1\left(\left|\sqrt{x}-\sqrt{x+1}\right|\sin\pi x\right)dx$$

Solution:

$$\left|\int_0^2\left(\sqrt{x}\sin\pi x\right)dx\right|=\left|\int_0^1\left(\sqrt{x}\sin\pi x\right)dx+\int_1^2\left(\sqrt{x}\sin\pi x\right)dx\right|=$$

$$=\left|\int_0^1\left(\sqrt{x}\sin\pi x\right)dx+\int_0^1\left(\sqrt{y+1}\sin\left(\pi(y+1)\right)\right)d(y+1)\right|=$$

$$=\left|\int_0^1\left(\sqrt{x}\sin\pi x\right)dx-\int_0^1\left(\sqrt{y+1}\sin\pi y\right)dy\right|=$$

$$=\left|\int_0^1\left(\sqrt{x}\sin\pi x\right)dx-\int_0^1\left(\sqrt{x+1}\sin\pi x\right)dx\right|=$$

$$=\left|\int_0^1\left(\left(\sqrt{x}-\sqrt{x+1}\right)\sin\pi x\right)dx\right|<\int_0^1\left(\left|\sqrt{x}-\sqrt{x+1}\right|\sin\pi x\right)dx$$

2.63. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{10k-7}{10k-2}\right)}$$

Solution:

$$\prod_{k=1}^{n}\left(\frac{10k-7}{10k-2}\right)=\prod_{k=1}^{n}\left(\frac{10k-10+3}{10k-5+3}\right)=$$

$$=\prod_{k=1}^{n}\frac{3+(10k-10)}{3+(10k-3)}=\prod_{k=1}^{n}\frac{3+(2k-2)\cdot 5}{3+(2k-1)\cdot 5}$$

We prove by induction:

$$P(n):\prod_{k=1}^{n}\left(\frac{3+(2k-2)\cdot 5}{3+(2k-1)\cdot 5}\right)<\sqrt{\frac{3}{3+10n}};n\ge 1$$

$$P(1):\frac{10-7}{10-2}<\sqrt{\frac{3}{10}}\Leftrightarrow\frac{3}{8}<\sqrt{\frac{3}{13}}\Leftrightarrow\frac{9}{64}<\frac{3}{13}\Leftrightarrow 39<64$$

Suppose that $P(n)$ is true. We prove $P(n+1)$.

$$P(n+1):\prod_{k=1}^{n+1}\left(\frac{3+(2k-2)\cdot 5}{3+(2k-1)\cdot 5}\right)<\sqrt{\frac{3}{3+10(n+1)}}\ \textit{(to prove)}$$

$$\prod_{k=1}^{n+1}\frac{3+(2k-2)\cdot 5}{3+(2k-1)\cdot 5}=\prod_{k=1}^{n}\left(\frac{3+(2k-2)\cdot 5}{3+(2k-1)\cdot 5}\right)\cdot\frac{3+10n}{8+10n}<$$

$$\overset{P(n)}{<}\sqrt{\frac{3}{3+10n}}\cdot\frac{3+10n}{8+10n}<\sqrt{\frac{3}{13+10n}}\ \textit{(to prove)}$$

$$\frac{3}{3+10n}\cdot\frac{(3+10n)^2}{(8+10n)^2}<\frac{3}{13+10n}$$

$$(3+10n)(13+10n)<(8+10n)^2$$

$$39+30n+130n+100n^2<64+160n+100n^2$$

$$0<25$$

$$P(n)\to P(n+1)$$

$$0<\prod_{k=1}^{n}\left(\frac{10k-7}{10k-2}\right)<\sqrt{\frac{3}{3+10n}}$$

$$0\le\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{10k-7}{10k-2}\right)\le\lim_{n\to\infty}\sqrt{\frac{3}{3+10n}}=0$$

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{10k-7}{10k-2}\right)=0$$

2.64. If $\mathbf{0} < a \leq b < \frac{\pi}{2}$ then:

$$\mathbf{2\int_{a}^{b}(x^2 \tan x)\,dx + 6\log\left|\frac{\cos b}{\cos a}\right| + 3(b^2 - a^2) \leq 0}$$

Solution:

$$\text{Let be } f:\left[0,\frac{\pi}{2}\right) \to \mathbb{R};\ f(x) = 3\sin x - 3x\cos x - x^2\sin x$$

$$f'(x) = 3\cos x - 3\cos x + 3x\sin x - 2x\sin x - x^2\cos x$$

$$f'(x) = x\sin x - x^2\cos x = x(\sin x - x\cos x)$$

$$\tan x \geq x;\ (\forall)x \in \left[0,\frac{\pi}{2}\right) \Rightarrow \frac{\sin x}{\cos x} \geq x \Rightarrow \sin x \geq x\cos x$$

$$\Rightarrow \sin x - x\cos x \geq 0 \Rightarrow f'(x) \geq 0;\ (\forall)x \in \left[0,\frac{\pi}{2}\right)$$

$$f \text{ increasing, } \min f(x) = f(0) = 0 \Rightarrow f(x) \geq 0$$

$$3\sin x - 3x\cos x - x^2\sin x \geq 0,\ x^2\sin x \leq 3\sin x - 3x\cos x$$

$$\frac{x^2\sin x}{\cos x} \leq 3\frac{\sin x}{\cos x} - 3x,\ x^2\tan x \leq 3\tan x - 3x$$

$$\int_{a}^{b}(x^2\tan x)\,dx \leq 3\int_{a}^{b}\tan x\,dx - 3\int_{a}^{b}x\,dx =$$

$$= 3\int_{a}^{b}\frac{\sin x}{\cos x}dx - 3\cdot\frac{b^2 - a^2}{2} =$$

$$= -3(\log|\cos b| - \log|\cos a|) - \frac{3(b^2 - a^2)}{2}$$

$$2\int_{a}^{b}(x^2\tan x)\,dx \leq -6\log\left|\frac{\cos b}{\cos a}\right| - 3(b^2 - a^2)$$

$$2\int_{a}^{b}(x^2\tan x)\,dx + 6\log\left|\frac{\cos b}{\cos a}\right| + 3(b^2 - a^2) \leq 0$$

Equality holds for $a = b$.

2.65. If $\boldsymbol{f}:[\mathbf{1},\infty) \to [\mathbf{1},\infty)$ continuous; $\boldsymbol{a} \geq \mathbf{1}$ then:

$$\mathbf{2(a-1)\int_{1}^{a} f(x)\,dx + \int_{1}^{a}\int_{1}^{a}\frac{dx\,dy}{f(x)f(y)} \leq \left(\int_{1}^{a} f(x)dx\right)^2 + 2(a-1)\int_{1}^{a}\frac{dx}{f(x)}}$$

Solution:

$$f(x)\geq 1; f(y)\geq 1\Rightarrow f(x)f(y)\geq 1\Rightarrow \frac{1}{f(x)f(y)}\leq 1$$

$$\frac{1}{f(x)f(y)}-1\leq 0; f(x)-1\geq 0; f(y)-1\geq 0; (\forall)x,y\in[1,\infty)$$

$$(f(x)-1)(f(y)-1)\left(\frac{1}{f(x)f(y)}-1\right)\leq 0$$

$$(f(x)f(y)-f(x)-f(y)+1)\ \left(\frac{1}{f(x)f(y)}-1\right)\leq 0$$

$$1-f(x)f(y)-\frac{f(x)+f(y)}{f(x)f(y)}+f(x)+f(y)+\frac{1}{f(x)f(y)}-1\leq 0$$

$$f(x)+f(y)+\frac{1}{f(x)f(y)}\leq f(x)f(y)+\frac{1}{f(x)}+\frac{1}{f(y)}$$

$$\int_1^a\int_1^a\big(f(x)+f(y)\big)\,dxdy+\int_1^a\int_1^a\frac{dx\,dy}{f(x)f(y)}\leq\int_1^a\int_1^a f(x)f(y)\,dxdy+$$

$$+\int_1^a\int_1^a\left(\frac{1}{f(x)}+\frac{1}{f(y)}\right)dx\,dy$$

$$2(a-1)\int_1^a f(x)\,dx+\int_1^a\int_1^a\frac{dx\,dy}{f(x)f(y)}\leq$$

$$\leq\left(\int_1^a f(x)\,dx\right)^2+2(a-1)\int_1^a\frac{dx}{f(x)}$$

2.66. Let be $\boldsymbol{A},\boldsymbol{B}\in\boldsymbol{M_2}(\mathbb{R})$ such that $\boldsymbol{Tr\ A}\cdot\boldsymbol{Tr\ B}=\boldsymbol{Tr\ (AB)}$. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(n\det A+\ln n\det B-\sum_{k=1}^{n}\det\left(A+\frac{1}{\sqrt{k}}B\right)\right)}$$

Solution:

$$Let\ be\ A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}; B=\begin{pmatrix}e & f\\ g & h\end{pmatrix}; a,b,c,d,e,f,g,h\in\mathbb{R}$$

$$\det A=ad-bc;\det B=eh-gf$$

$$Tr\ A=a+d; Tr\ B=e+h$$

$$AB=\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}; Tr\,(AB)=ae+bg+cf+dh$$

$$Tr\,A\cdot Tr\,B=Tr\,(AB)\Rightarrow (a+d)(e+h)=ae+bg+cf+dh$$

$$ae+ah+de+dh=ae+bg+cf+dh$$

$$ah+de-bg-cf=0 \quad (1)$$

$$\det\left(A+\frac{1}{\sqrt{k}}B\right)=\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\frac{1}{\sqrt{k}}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)=$$

$$=\det\begin{pmatrix} a+\dfrac{e}{\sqrt{k}} & b+\dfrac{f}{\sqrt{k}} \\ c+\dfrac{g}{\sqrt{k}} & d+\dfrac{h}{\sqrt{k}} \end{pmatrix}=$$

$$=\left(a+\frac{e}{\sqrt{k}}\right)\left(d+\frac{h}{\sqrt{k}}\right)-\left(b+\frac{f}{\sqrt{k}}\right)\left(c+\frac{g}{\sqrt{k}}\right)=$$

$$=ad+\frac{ah}{\sqrt{k}}+\frac{de}{\sqrt{k}}+\frac{eh}{k}-bc-\frac{bg}{\sqrt{k}}-\frac{fc}{\sqrt{k}}-\frac{bg}{k}=$$

$$=ad-bc+\frac{1}{\sqrt{k}}(ah+de-bg-fc)+\frac{1}{k}(eh-bg)=$$

$$\overset{(1)}{=}\det A+\frac{1}{k}\det B$$

$$\sum_{k=1}^{n}\det\left(A+\frac{1}{\sqrt{k}}B\right)=\sum_{k=1}^{n}\left(\det A+\frac{1}{k}\det B\right)=$$

$$=n\det A+\det B\cdot\sum_{k=1}^{n}\frac{1}{k}$$

$$\Omega=\lim_{n\to\infty}\left(n\det A+\ln n\cdot\det B-\sum_{k=1}^{n}\det\left(A+\frac{1}{\sqrt{k}}B\right)\right)=$$

$$=\lim_{n\to\infty}\left(n\det A+\ln n\cdot\det B-n\det A-\det B\sum_{k=1}^{n}\frac{1}{k}\right)=$$

$$=\lim_{n\to\infty}\left(\det B\left(\ln n-\sum_{k=1}^{n}\frac{1}{k}\right)\right)=\det B\cdot(-\gamma)=-\gamma\det B$$

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)$$

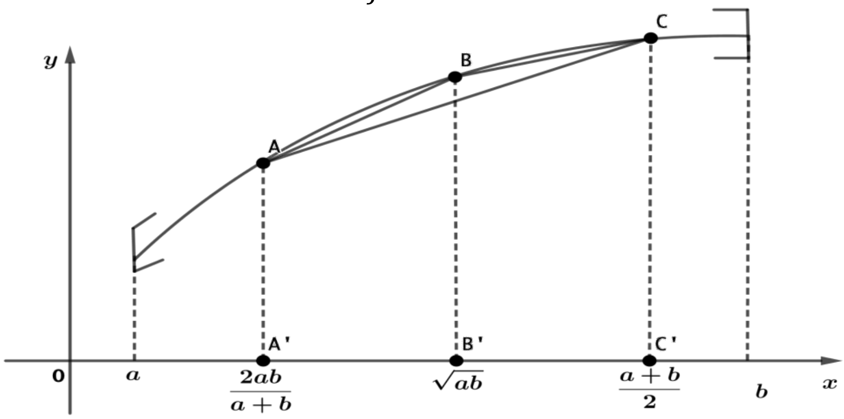
2.67. If $a, b > 0$ then:

$$\left(\sqrt{ab}-\frac{a+b}{2}\right)\arctan\left(\frac{2ab}{a+b}\right)+\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)\arctan\left(\sqrt{ab}\right)+$$
$$+\left(\frac{2ab}{a+b}-\sqrt{ab}\right)\arctan\left(\frac{a+b}{2}\right)\geq 0$$

Solution:

If $a=b$ then, the equality holds. WLOG $a<b$. Let be $f:[a,b]\to\mathbb{R}$

$f(x)=\arctan x$; $f'(x)=\frac{1}{1+x^2}$; $f''(x)=\frac{-2x}{(1+x^2)^2}<0$; *$f$ concave function*



$$A\left(\frac{2ab}{a+b},\arctan\left(\frac{2ab}{a+b}\right)\right);B\left(\sqrt{ab},\arctan\left(\sqrt{ab}\right)\right);C\left(\frac{a+b}{2},\arctan\left(\frac{a+b}{2}\right)\right)$$
$$A'\left(\frac{2ab}{a+b},0\right);B'\left(\sqrt{ab},0\right);C'\left(\frac{a+b}{2},0\right)$$
$$S[ABB'A']+S[BCC'B']>S[ACC'A']$$
$$\left(\arctan\left(\frac{2ab}{a+b}\right)+\arctan\left(\sqrt{ab}\right)\right)\left(\sqrt{ab}-\frac{2ab}{a+b}\right)+$$
$$+\left(\arctan\left(\sqrt{ab}\right)+\arctan\left(\frac{a+b}{2}\right)\right)\left(\frac{a+b}{2}-\sqrt{ab}\right)>$$
$$>\left(\arctan\left(\frac{2ab}{a+b}\right)+\arctan\left(\frac{a+b}{2}\right)\right)\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)$$
$$\arctan\left(\frac{2ab}{a+b}\right)\cdot\left(\sqrt{ab}-\frac{a+b}{2}\right)+\arctan\left(\sqrt{ab}\right)\cdot\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)+$$
$$+\arctan\left(\frac{a+b}{2}\right)\cdot\left(\frac{2ab}{a+b}-\sqrt{ab}\right)\geq 0$$

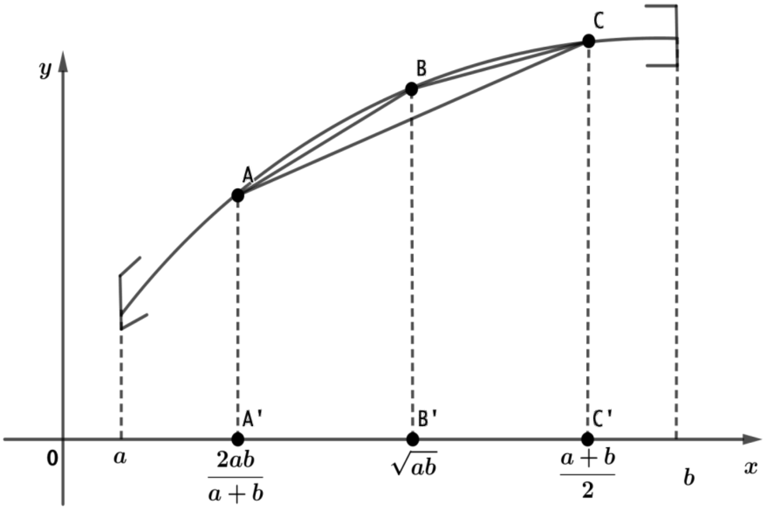
2.68. If $a, b > 1$ then:

$$\left(\frac{2ab}{a+b}\right)^{\sqrt{ab}-\frac{a+b}{2}} \cdot \left(\sqrt{ab}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}} \cdot \left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b}-\sqrt{ab}} \geq 1$$

Solution:

If $a = b$ the equality holds. WLOG $a < b$. Let be $f: [a, b] \to \mathbb{R}$.

$f(x) = \log x$; $f'(x) = \frac{1}{x}$; $f''(x) = \frac{-1}{x^2} < 0$; f *concave function*



$$A\left(\frac{2ab}{a+b}, \log\left(\frac{2ab}{a+b}\right)\right); B\left(\sqrt{ab}, \log\left(\sqrt{ab}\right)\right); C\left(\frac{a+b}{2}, \log\left(\frac{a+b}{2}\right)\right)$$

$$A'\left(\frac{2ab}{a+b}, 0\right); B\left(\sqrt{ab}, 0\right); C\left(\frac{a+b}{2}, 0\right)$$

$$S[ABB'A'] + S[BCC'B'] > S[ACC'A']$$

$$\left(\log\left(\frac{2ab}{a+b}\right) + \log\left(\sqrt{ab}\right)\right)\left(\sqrt{ab} - \frac{2ab}{a+b}\right) +$$

$$+\left(\log\sqrt{ab} + \log\left(\frac{a+b}{2}\right)\right)\left(\frac{a+b}{2} - \sqrt{ab}\right) >$$

$$> \left(\log\left(\frac{2ab}{a+b}\right) + \log\left(\frac{a+b}{2}\right)\right)\left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)$$

$$\left(\sqrt{ab} - \frac{a+b}{2}\right)\log\left(\frac{2ab}{a+b}\right) + \left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)\log\left(\sqrt{ab}\right) +$$

$$+\left(\frac{2ab}{a+b} - \sqrt{ab}\right)\log\left(\frac{a+b}{2}\right) > 0$$

$$\log\left(\left(\frac{2ab}{a+b}\right)^{\sqrt{ab}-\frac{a+b}{2}}\cdot\left(\sqrt{ab}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}}\cdot\left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b}-\sqrt{ab}}\right)>\log 1$$

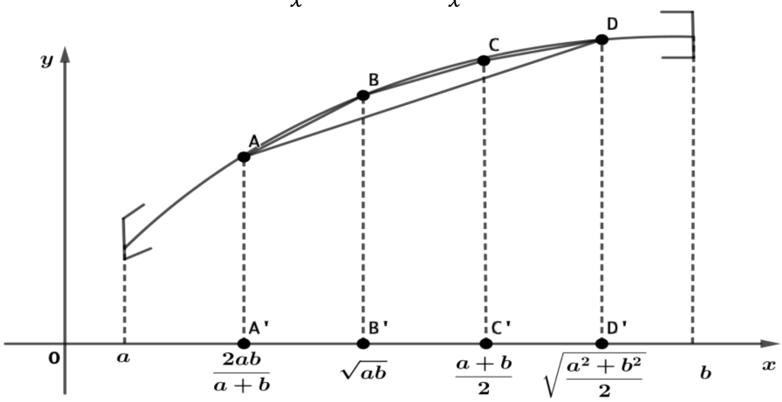
$$\left(\frac{2ab}{a+b}\right)^{\sqrt{ab}-\frac{a+b}{2}}\cdot\left(\sqrt{ab}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}}\cdot\left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b}-\sqrt{ab}}>1$$

2.69. If $a, b \geq 1$ then:

$$\left(\sqrt{\frac{2ab}{a^2+b^2}}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}}\cdot\left(\frac{(a+b)^2}{4ab}\right)^{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}\geq 1$$

Solution:

If $a=b$ the following inequality holds. WLOG $a<b$. Let be
$f:[a,b]\to\mathbb{R}$.
$f(x)=\log x\,;f'(x)=\frac{1}{x};f''(x)=\frac{-1}{x^2}<0.$ *f concave function.*



$$A\left(\frac{2ab}{a+b},\log\left(\frac{2ab}{a+b}\right)\right);B\left(\sqrt{ab},\log\left(\sqrt{ab}\right)\right);C\left(\frac{a+b}{2},\log\left(\frac{a+b}{2}\right)\right);$$

$$D\left(\sqrt{\frac{a^2+b^2}{2}},\log\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right);A'\left(\frac{2ab}{a+b},0\right);$$

$$B'\left(\sqrt{ab},0\right);C'\left(\frac{a+b}{2},0\right);D'\left(\sqrt{\frac{a^2+b^2}{2}},0\right)$$

$$S[ABB'A']+S[BCC'B']+S[CDD'C']>S[ADD'A']$$

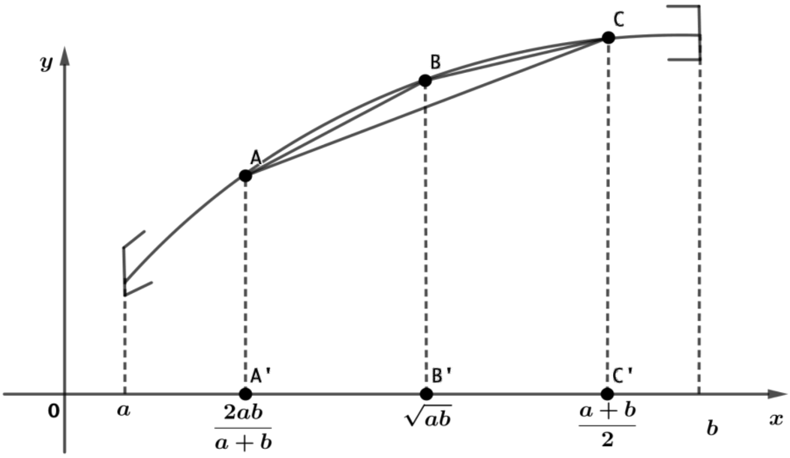
$$\left(\log\left(\frac{2ab}{a+b}\right)+\log\left(\sqrt{ab}\right)\right)\left(\sqrt{ab}-\frac{2ab}{a+b}\right)+$$
$$+\left(\log\left(\frac{a+b}{2}\right)+\log\left(\sqrt{ab}\right)\right)\left(\frac{a+b}{2}-\sqrt{ab}\right)+$$
$$+\left(\log\left(\sqrt{\frac{a^2+b^2}{2}}\right)+\log\left(\frac{a+b}{2}\right)\right)\left(\sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2}\right)>0$$
$$\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)\left(\log\left(\sqrt{ab}\right)-\log\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)+$$
$$+\left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)\left(\log\left(\frac{a+b}{2}\right)-\log\left(\frac{2ab}{a+b}\right)\right)>\log 1$$
$$\log\left(\left(\frac{\sqrt{ab}}{\sqrt{\frac{a^2+b^2}{2}}}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}}\cdot\left(\frac{\frac{a+b}{2}}{\frac{2ab}{a+b}}\right)^{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}\right)>\log 1$$
$$\left(\sqrt{\frac{2ab}{a^2+b^2}}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}}\cdot\left(\frac{(a+b)^2}{4ab}\right)^{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}>1$$

2.70. If $\boldsymbol{a,b\geq 1}$ then:

$$\left(\sqrt{ab}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}}\cdot\left(\frac{a+b}{2}\right)^{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}\cdot\left(\frac{2ab}{a+b}\right)^{\sqrt{ab}-\frac{a+b}{2}}\geq 1$$

Solution:

If $a=b$ the equality holds. WLOG $a<b$. Let be $f:[a,b]\to\mathbb{R}$
$f(x)=\log x$; $f'(x)=\frac{1}{x}$; $f''(x)=\frac{-1}{x^2}<0$. f concave function.



$$A\left(\sqrt{ab}, \log\left(\sqrt{ab}\right)\right); B\left(\frac{a+b}{2}, \log\left(\frac{a+b}{2}\right)\right); C\left(\sqrt{\frac{a^2+b^2}{2}}, \log\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)$$

$$A'\left(\sqrt{ab}, 0\right); B'\left(\frac{a+b}{2}, 0\right); C'\left(\sqrt{\frac{a^2+b^2}{2}}, 0\right)$$

$$S[ABB'A'] + S[BCC'B'] > S[ACC'A']$$

$$\left(\log\left(\sqrt{ab}\right) + \log\left(\frac{a+b}{2}\right)\right) \cdot \left(\frac{a+b}{2} - \sqrt{ab}\right) +$$

$$+\left(\log\left(\frac{a+b}{2}\right) + \log\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\left(\sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2}\right) +$$

$$+\left(\log\left(\sqrt{ab}\right) + \log\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\left(\sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab}\right) > 0$$

$$\left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)\log\left(\sqrt{ab}\right) + \left(\frac{2ab}{a+b} - \sqrt{ab}\right)\log\left(\frac{a+b}{2}\right) +$$

$$+\left(\sqrt{ab} - \frac{a+b}{2}\right)\log\left(\frac{2ab}{a+b}\right) > 0$$

$$\log\left(\left(\sqrt{ab}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}} \cdot \left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b}-\sqrt{ab}} \cdot \left(\frac{2ab}{a+b}\right)^{\sqrt{ab}-\frac{a+b}{2}}\right) > \log 1$$

$$\left(\sqrt{ab}\right)^{\frac{a+b}{2}-\frac{2ab}{a+b}}\cdot\left(\frac{a+b}{2}\right)^{\frac{2ab}{a+b}-\sqrt{ab}}\cdot\left(\frac{2ab}{a+b}\right)^{\sqrt{ab}-\frac{a+b}{2}}>1$$

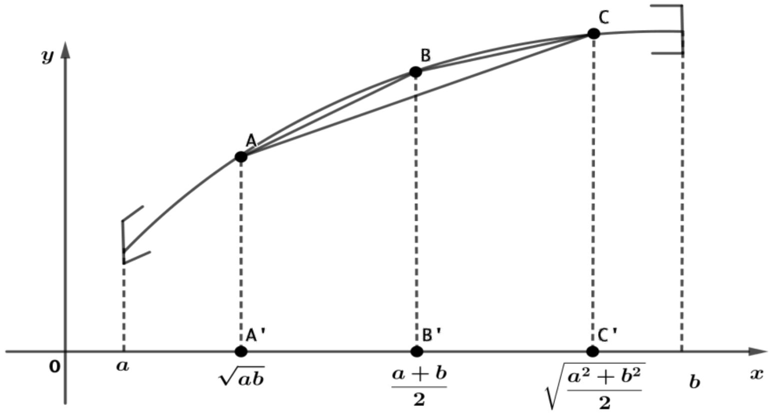
2.71. If $\boldsymbol{a},\boldsymbol{b}>0$ then:

$$\left(\frac{\boldsymbol{a}+\boldsymbol{b}}{\boldsymbol{2}}-\frac{\boldsymbol{2ab}}{\boldsymbol{a}+\boldsymbol{b}}\right)\mathbf{arctan}\left(\sqrt{\boldsymbol{ab}}\right)+\left(\frac{\boldsymbol{2ab}}{\boldsymbol{a}+\boldsymbol{b}}-\sqrt{\boldsymbol{ab}}\right)\mathbf{arctan}\left(\frac{\boldsymbol{a}+\boldsymbol{b}}{\boldsymbol{2}}\right)+$$
$$+\left(\sqrt{\boldsymbol{ab}}-\frac{\boldsymbol{a}+\boldsymbol{b}}{\boldsymbol{2}}\right)\mathbf{arctan}\left(\sqrt{\frac{\boldsymbol{a}^2+\boldsymbol{b}^2}{\boldsymbol{2}}}\right)>0$$

Solution:

If $a=b$ then the equality holds. WLOG $a<b$. Let be $f:[a,b]\to\mathbb{R}$;

$$f(x)=\arctan x\,;f'(x)=\frac{1}{1+x^2};f''(x)=\frac{-2x}{(1+x^2)^2}<0$$



$$A\left(\sqrt{ab},\arctan\left(\sqrt{ab}\right)\right);B\left(\frac{a+b}{2},\arctan\left(\frac{a+b}{2}\right)\right);C\left(\sqrt{\frac{a^2+b^2}{2}},\arctan\sqrt{\frac{a^2+b^2}{2}}\right)$$

$$A'\left(\sqrt{ab},0\right);B'\left(\frac{a+b}{2},0\right);C'\left(\sqrt{\frac{a^2+b^2}{2}},0\right)$$

$$S[ABB'A']+S[BCC'B']>S[ACC'A']$$

$$\left(\arctan\left(\sqrt{ab}\right)+\arctan\left(\frac{a+b}{2}\right)\right)\left(\frac{a+b}{2}-\sqrt{ab}\right)+$$

$$+\left(\arctan\left(\frac{a+b}{2}\right)+\arctan\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\left(\sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2}\right)>$$

$$>\left(\arctan(\sqrt{ab})+\arctan\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)$$

$$\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)\arctan(\sqrt{ab})+\left(\frac{2ab}{a+b}-\sqrt{ab}\right)\arctan\left(\frac{a+b}{2}\right)+$$

$$+\left(\sqrt{ab}-\frac{a+b}{2}\right)\arctan\left(\frac{\sqrt{a^2+b^2}}{2}\right)>0$$

2.72. If $\boldsymbol{a},\boldsymbol{b}>0$ then:

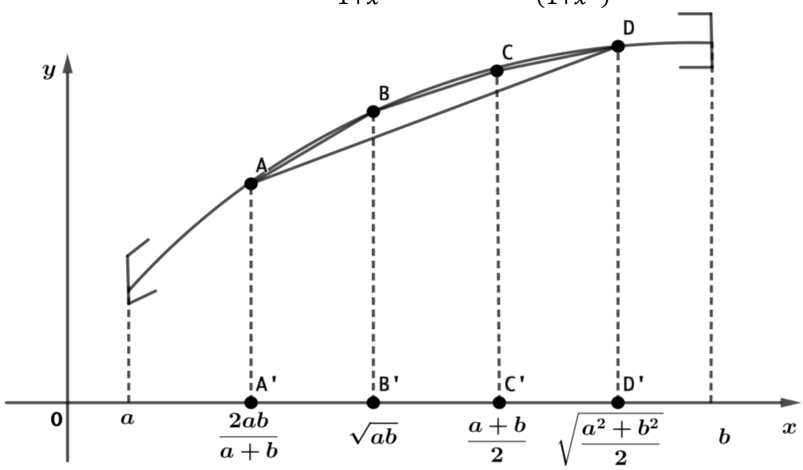
$$\left(\frac{\boldsymbol{a}+\boldsymbol{b}}{\mathbf{2}}-\frac{\mathbf{2}\boldsymbol{ab}}{\boldsymbol{a}+\boldsymbol{b}}\right)\mathbf{arctan}\left(\frac{\sqrt{\mathbf{2}\boldsymbol{ab}}-\sqrt{\boldsymbol{a}^2+\boldsymbol{b}^2}}{\sqrt{\mathbf{2}}+\sqrt{\boldsymbol{ab}(\boldsymbol{a}^2+\boldsymbol{b}^2)}}\right)+$$

$$+\left(\sqrt{\frac{\boldsymbol{a}^2+\boldsymbol{b}^2}{\mathbf{2}}}-\sqrt{\boldsymbol{ab}}\right)\mathbf{arctan}\left(\frac{(\boldsymbol{a}-\boldsymbol{b})^2}{\mathbf{2}+\mathbf{2}\boldsymbol{ab}}\right)>0$$

Solution:

If $a=b$ then the equality holds. WLOG $a<b$. Let be $f:[a,b]\to\mathbb{R}$

$f(x)=\arctan x$; $f'(x)=\frac{1}{1+x^2}$; $f''(x)=\frac{-2x}{(1+x^2)^2}<0$; *$f$ concave*



$$A\left(\frac{2ab}{a+b},\arctan\left(\frac{2ab}{a+b}\right)\right);B(\sqrt{ab},\arctan(\sqrt{ab}));$$

$$C\left(\frac{a+b}{2}, \arctan\left(\frac{a+b}{2}\right)\right); D\left(\sqrt{\frac{a^2+b^2}{2}}, \arctan\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)$$

$$A\left(\frac{2ab}{a+b}, 0\right); B\left(\sqrt{ab}, 0\right); C\left(\frac{a+b}{2}, 0\right); D\left(\sqrt{\frac{a^2+b^2}{2}}, 0\right)$$

$$S[ABB'A'] + S[BCC'B'] + S[CDD'C'] > S[ADD'A']$$

$$\left(\arctan\left(\frac{2ab}{a+b}\right) + \arctan\left(\sqrt{ab}\right)\right)\left(\sqrt{ab} - \frac{2ab}{a+b}\right) +$$

$$+\left(\arctan\left(\frac{a+b}{2}\right) + \arctan\sqrt{ab}\right)\left(\frac{a+b}{2} - \sqrt{ab}\right) +$$

$$+\left(\arctan\left(\sqrt{\frac{a^2+b^2}{2}}\right) + \arctan\left(\frac{a+b}{2}\right)\right)\left(\sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2}\right) >$$

$$> \left(\arctan\left(\frac{2ab}{a+b}\right) + \arctan\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\left(\sqrt{\frac{a^2+b^2}{2}} - \frac{2ab}{a+b}\right)$$

$$\left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)\left(\arctan\left(\sqrt{ab}\right) - \arctan\left(\frac{2ab}{a+b}\right)\right) +$$

$$+\left(\sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab}\right)\left(\arctan\left(\frac{a+b}{2}\right) - \arctan\left(\frac{2ab}{a+b}\right)\right) > 0$$

$$\left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)\arctan\left(\frac{\sqrt{ab} - \sqrt{\frac{a^2+b^2}{2}}}{1 + \sqrt{\frac{ab(a^2+b^2)}{2}}}\right) +$$

$$+\left(\sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab}\right)\arctan\left(\frac{\frac{a+b}{2} - \frac{2ab}{a+b}}{1 + \frac{a+b}{2}\cdot\frac{2ab}{a+b}}\right) > 0$$

$$\left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)\arctan\left(\frac{\sqrt{2ab} - \sqrt{a^2+b^2}}{\sqrt{2} + \sqrt{ab(a^2+b^2)}}\right) +$$

$$+\left(\sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab}\right)\arctan\left(\frac{(a-b)^2}{2+2ab}\right) > 0$$

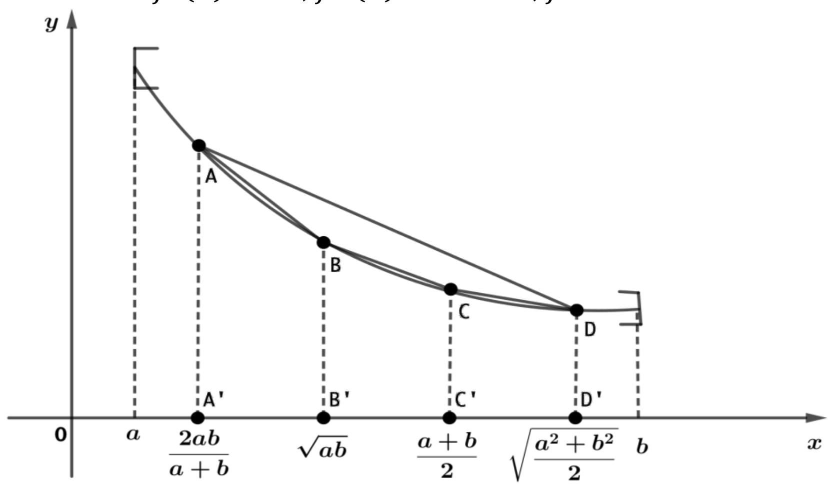
2.73. If $a, b > 0$ then:

$$\left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)\left(e^{\frac{a+b}{2}}-e^{\frac{2ab}{a+b}}\right)\geq\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)\left(e^{\sqrt{\frac{a^2+b^2}{2}}}-e^{\sqrt{ab}}\right)$$

Solution:

If $a = b$ then the equality holds. WLOG: $a < b$. Let be $f:[a,b]\to\mathbb{R}; f(x) = e^x$

$f'(x) = e^x; f''(x) = e^x > 0; f$ *convexe.*



$$S[ABB'A'] + S[BCC'B'] + S[CDD'C'] > S[ADD'A']$$

$$\left(e^{\frac{2ab}{a+b}}+e^{\sqrt{ab}}\right)\left(\sqrt{ab}-\frac{2ab}{a+b}\right)+\left(e^{\sqrt{ab}}+e^{\frac{a+b}{2}}\right)\left(\frac{a+b}{2}-\sqrt{ab}\right)+$$

$$+\left(e^{\sqrt{\frac{a^2+b^2}{2}}}+e^{\frac{a+b}{2}}\right)\left(\sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2}\right)>$$

$$>\left(e^{\frac{2ab}{a+b}}+e^{\sqrt{\frac{a^2+b^2}{2}}}\right)\left(\sqrt{\frac{a^2+b^2}{2}}-\frac{2ab}{a+b}\right)$$

$$e^{\frac{2ab}{a+b}}\left(\sqrt{ab}-\frac{2ab}{a+b}-\sqrt{\frac{a^2+b^2}{2}}+\frac{2ab}{a+b}\right)+$$

$$+e^{\sqrt{ab}}\left(\sqrt{ab}-\frac{2ab}{a+b}+\frac{a+b}{2}-\sqrt{ab}\right)+$$

$$+e^{\sqrt{\frac{a^2+b^2}{2}}}\left(\sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2}-\sqrt{\frac{a^2+b^2}{2}}+\frac{2ab}{a+b}\right)+$$

$$+e^{\frac{a+b}{2}}\left(\frac{a+b}{2}-\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2}\right)>0$$

$$e^{\frac{2ab}{a+b}}\left(\sqrt{ab}-\sqrt{\frac{a^2+b^2}{2}}\right)+e^{\sqrt{ab}}\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)+$$

$$+e^{\sqrt{\frac{a^2+b^2}{2}}}\left(\frac{2ab}{a+b}-\frac{a+b}{2}\right)+e^{\frac{a+b}{2}}\left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)>0$$

$$\left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)\left(e^{\frac{a+b}{2}}-e^{\frac{2ab}{a+b}}\right)+\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)$$

$$\cdot\left(e^{\sqrt{ab}}-e^{\sqrt{\frac{a^2+b^2}{2}}}\right)>0$$

$$\left(\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\right)\left(e^{\frac{a+b}{2}}-e^{\frac{2ab}{a+b}}\right)>$$

$$>\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right)\left(e^{\sqrt{\frac{a^2+b^2}{2}}}-e^{\sqrt{ab}}\right)$$

2.74. If $\boldsymbol{f}:\mathbb{R}\to(\mathbf{0},\infty)$, $\boldsymbol{f}$ continuous; $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}$; $\boldsymbol{a}\le\boldsymbol{b}$ then:

$$\int_a^b\int_a^b\left(\frac{\left(f^2(x)+f(y)\right)\left(f^2(y)+f(x)\right)}{\left(1+f(x)\right)\left(1+f(y)\right)}\right)dx\,dy\ge\left(\int_a^b f(x)\,dx\right)^2$$

Solution:

First, we prove: $(u^2+v)(v^2+u)\ge uv(1+u)(1+v)$

$$u^2v^2+u^3+v^3+vu\ge uv(1+u+v+uv)$$

$$u^2v^2+u^3+v^3+vu\ge uv+u^2v+uv^2+v^2u^2$$

$$u^3+v^3-u^2v-uv^2\ge 0$$

$$u^2(u-v)-v^2(u-v)\geq 0$$
$$(u-v)(u^2-v^2)\geq 0\Rightarrow (u-v)^2(u+v)\geq 0$$
$$\frac{(u^2+v)(v^2+u)}{(1+u)(1+v)}\geq uv \quad (1)$$
We take $u=f(x); v=f(y)$ *in (1):*
$$\frac{\big(f^2(x)+f(y)\big)\big(f^2(y)+f(x)\big)}{\big(1+f(x)\big)\big(1+f(y)\big)}\geq f(x)f(y)$$
$$\int_a^b\int_a^b\left(\frac{\big(f^2(x)+f(y)\big)\big(f^2(y)+f(x)\big)}{\big(1+f(x)\big)\big(1+f(y)\big)}\right)dx\,dy\geq$$
$$\geq\int_a^b\int_a^b\big(f(x)f(y)\big)\,dx\,dy=\left(\int_a^b f(x)\,dx\right)\left(\int_a^b f(y)\,dy\right)=\left(\int_a^b f(x)\,dx\right)^2$$

2.75. If $a,b\geq 1$ then:

$$2^{\frac{2}{a+b}}\cdot 3^{\frac{a+b}{2}}+2^{\sqrt{ab}}\cdot 3^{\frac{1}{\sqrt{ab}}}+2^{\frac{a+b}{2}}\cdot 3^{\frac{2}{a+b}}+2^{\frac{1}{\sqrt{ab}}}\cdot 3^{\sqrt{ab}}\geq 24$$

Solution:

Let be $f:[1,\infty)\to\mathbb{R}; f(x)=2^x\cdot 3^{\frac{1}{x}}+2^{\frac{1}{x}}\cdot 3^x$
$$f(x)=2^{x+\frac{1}{x}}\left(\left(\frac{3}{2}\right)^{\frac{1}{x}}+\left(\frac{3}{2}\right)^x\right)$$
Let be $g:[1,\infty)\to\mathbb{R}; g(x)=2^{x+\frac{1}{x}}$
$$h:[1,\infty)\to\mathbb{R}; h(x)=\left(\frac{3}{2}\right)^{\frac{1}{x}}+\left(\frac{3}{2}\right)^x$$
$$f(x)=g(x)\cdot h(x)$$
$$g'(x)=\left(x+\frac{1}{x}\right)'\cdot 2^{x+\frac{1}{x}}\cdot\ln 2=$$
$$=\left(1-\frac{1}{x^2}\right)\cdot 2^{x+\frac{1}{x}}\cdot\ln 2=\frac{x^2-1}{x^2}\cdot 2^{x+\frac{1}{x}}\cdot\ln 2\geq 0$$
because $x\geq 1$. *Hence* $g(x)\geq g(1)=4$
$$h'(x)=-\frac{1}{x^2}\ln\frac{3}{2}\cdot\left(\frac{3}{2}\right)^{\frac{1}{x}}+\left(\frac{3}{2}\right)^x\ln\frac{3}{2}=\ln\frac{3}{2}\left[-\frac{1}{x^2}\cdot\left(\frac{3}{2}\right)^{\frac{1}{x}}+\left(\frac{3}{2}\right)^x\right]\overset{(x\geq 1)}{\geq}$$

$$\geq \ln\frac{3}{2}\left[\left(\frac{3}{2}\right)^{x}-\frac{1}{x^2}\left(\frac{3}{2}\right)^{\frac{1}{x}}\right]\geq \ln\frac{3}{2}\left[\left(\frac{3}{2}\right)^{x}-\left(\frac{3}{2}\right)^{\frac{1}{x}}\right]\overset{\left(x\geq\frac{1}{x}\right)}{\geq} 0$$

$$\Rightarrow h(x)\geq h(1)=3$$

$$f(x)=g(x)h(x)\geq g(1)\cdot h(1)=3\cdot 4=12, (\forall)x\geq 1$$

$$2^x\cdot 3^{\frac{1}{x}}+2^{\frac{1}{x}}\cdot 3^x\geq 12; (\forall)x\geq 1$$

If $a,b\geq 1\Rightarrow 1\leq a\leq\sqrt{ab}\leq\frac{a+b}{2}$

$$\sqrt{ab}\geq 1\Rightarrow 2^{\sqrt{ab}}\cdot 3^{\frac{1}{\sqrt{ab}}}+2^{\frac{1}{\sqrt{ab}}}\cdot 3^{\sqrt{ab}}\geq 12 \quad (2)$$

$$\frac{a+b}{2}\geq 1\Rightarrow 2^{\frac{a+b}{2}}\cdot 3^{\frac{2}{a+b}}+2^{\frac{2}{a+b}}\cdot 3^{\frac{a+b}{2}}\geq 12 \quad (3)$$

By adding (2); (3):

$$2^{\frac{2}{a+b}}\cdot 3^{\frac{a+b}{2}}+2^{\sqrt{ab}}\cdot 3^{\frac{1}{\sqrt{ab}}}+2^{\frac{a+b}{2}}\cdot 3^{\frac{2}{a+b}}+2^{\frac{1}{\sqrt{ab}}}\cdot 3^{\sqrt{ab}}\geq 24$$

Equality holds for $a=b$.

2.76. Find:

$$\boldsymbol{\Omega=\int_{-1}^{1}\left(\frac{x^4}{(e^x+1)(x^2+1)}+\frac{(2x^5+4x^3)\log(e^x+1)}{(x^2+1)^2}\right)dx}$$

Solution:

$$\Omega=\int_{-1}^{1}\left(\frac{x^4}{(e^x+1)(x^2+1)}+\left(\frac{x^4}{x^2+1}\right)'\log(e^x+1)\right)dx=$$

$$=\int_{-1}^{1}\frac{x^4}{(e^x+1)(x^2+1)}dx+\int_{-1}^{1}\left(\frac{x^4}{x^2+1}\right)'\log(e^x+1)\,dx=$$

$$=\int_{-1}^{1}\frac{x^4}{(e^x+1)(x^2+1)}dx+\frac{x^4}{x^2+1}\log(e^x+1)\Big|_{-1}^{1}-$$

$$-\int_{-1}^{1}\frac{x^4}{x^2+1}\cdot\frac{e^x}{e^x+1}dx=$$

$$=\frac{1^4}{1^2+1}\log(e+1)-\frac{1}{(-1)^2+1}\log\left(\frac{1}{e}+1\right)+$$

$$+\int_{-1}^{1}\left(\frac{x^4}{x^2+1}\cdot\frac{1-e^x}{1+e^x}\right)dx=$$

$$=\frac{1}{2}\left(\log(e+1)-\log\left(\frac{e+1}{e}\right)\right)+I=\frac{1}{2}\log\left(\frac{e+1}{\frac{e+1}{e}}\right)+I=\frac{1}{2}+I$$

$$I=\int_{-1}^{1}\left(\frac{x^4}{x^2+1}\cdot\frac{1-e^x}{1+e^x}\right)dx$$

$$Let\ be\ f:[-1,1]\to\mathbb{R};f(x)=\frac{x^4}{x^2+1}\cdot\frac{1-e^x}{1+e^x}$$

$$f(-x)=\frac{(-x)^4}{(-x)^2+1}\cdot\frac{1-e^{-x}}{1+e^{-x}}=$$

$$=\frac{x^4}{x^2+1}\cdot\frac{1-\frac{1}{e^x}}{1+\frac{1}{e^x}}=\frac{x^4}{x^2+1}\cdot\frac{e^x-1}{e^x+1}=-f(x)\Rightarrow I=\int_{-1}^{1}f(x)\,dx=0$$

$$\Omega=\frac{1}{2}+I=\frac{1}{2}+0=\frac{1}{2}$$

2.77. Find:

$$\boldsymbol{\Omega=\int_{-1}^{1}\left(\frac{e^{x^2}}{2(e^x+1)}+xe^{x^2}\log(e^x+1)\right)dx}$$

Solution:

$$\Omega=\frac{1}{2}\int_{-1}^{1}\left(\frac{e^{x^2}}{e^x+1}+2xe^{x^2}\log(e^x+1)\right)dx=$$

$$=\frac{1}{2}\int_{-1}^{1}\frac{e^{x^2}}{e^x+1}dx+\frac{1}{2}\int_{-1}^{1}\left(\left(e^{x^2}\right)'\log(e^x+1)\right)dx=$$

$$=\frac{1}{2}\int_{-1}^{1}\frac{e^{x^2}}{e^x+1}dx+\frac{1}{2}e^{x^2}\log(e^x+1)\Big|_{-1}^{1}-\frac{1}{2}\int_{-1}^{1}e^{x^2}\cdot\frac{e^x}{e^x+1}dx=$$

$$=\frac{1}{2}\int_{-1}^{1}\frac{e^{x^2}(1-e^x)}{1+e^x}dx+\frac{1}{2}e\log(e+1)-\frac{1}{2}e\log\left(\frac{1}{e}+1\right)=$$

$$=\frac{1}{2}e\log\left(\frac{e+1}{\frac{1+e}{e}}\right)+\frac{1}{2}I=\frac{e}{2}\log e+\frac{1}{2}I$$

$$Let\ be\ f:[-1,1]\to\mathbb{R};f(x)=\frac{e^{x^2}(1-e^x)}{1+e^x}$$

$$f(-x)=\frac{e^{(-x)^2}(1-e^{-x})}{1+e^{-x}}=\frac{e^{x^2}\left(1-\frac{1}{e^x}\right)}{1+\frac{1}{e^x}}=\frac{e^{x^2}(e^x-1)}{e^x+1}=-f(x)$$

$$f(-x)=-f(x)\Rightarrow\int_{-1}^{1}f(x)\,dx=0\Rightarrow I=0$$

$$\Omega=\frac{e}{2}\log e+\frac{1}{2}I=\frac{e}{2}\cdot 1+\frac{1}{2}\cdot 0=\frac{e}{2}$$

2.78. If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}>0; a+b+c=3$ then:

$$\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{a}}\right)^{\boldsymbol{a^2+2ac}}\cdot\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{b}}\right)^{\boldsymbol{b^2+2ba}}\cdot\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{c}}\right)^{\boldsymbol{c^2+2cb}}\geq\mathbf{512}$$

Solution:

$$Let\ be\ f:(0,\infty)\to\mathbb{R}; f(x)=\log(x+1)$$

$$f'(x)=\frac{1}{x+1}; f''(x)=\frac{-1}{(x+1)^2}<0\Rightarrow f\ concave$$

By Jensen's inequality:

$$f(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)\geq\lambda_1f(x_1)+\lambda_2f(x_2)+\lambda_3f(x_3)$$

$$\lambda_1+\lambda_2+\lambda_3=1$$

$$\log(\lambda_1x_1+\lambda_2x_2+\lambda_2x_3+1)\geq$$

$$\geq\lambda_1\log(x_1+1)+\lambda_2\log(x_2+1)+\lambda_3\log(x_3+1)$$

$$\log(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3+1)\geq$$

$$\geq\log\left((x_1+1)\,)^{\lambda_1}\cdot(x_2+1)^{\lambda_2}\cdot(x_3+1)^{\lambda_3}\right)$$

$$\lambda_1x_1+\lambda_2x_2+\lambda_3x_3+1\geq(x_1+1)^{\lambda_1}\cdot(x_2+1)^{\lambda_2}\cdot(x_3+1)^{\lambda_3}$$

$$Let\ be\ \lambda_1=\frac{a^2+2ac}{(a+b+c)^2};\ \lambda_2=\frac{b^2+2ba}{(a+b+c)^2};\lambda_3=\frac{c^2+2cb}{(a+b+c)^2}$$

$$\lambda_1+\lambda_2+\lambda_3=\frac{a^2+2ac+b^2+2ba+c^2+2cb}{(a+b+c)^2}=\frac{(a+b+c)^2}{(a+b+c)^2}=1$$

$$Let\ be\ x_1=\frac{1}{a};x_2=\frac{1}{b};x_3=\frac{1}{c}$$

$$\frac{a^2+2ac}{(a+b+c)^2}\cdot\frac{1}{a}+\frac{b^2+2ba}{(a+b+c)^2}\cdot\frac{1}{b}+\frac{c^2+2cb}{(a+b+c)^2}\cdot\frac{1}{c}+1\geq$$

$$\geq\left(\frac{1}{a}+1\right)^{\frac{a^2+2bc}{(a+b+c)^2}}\cdot\left(\frac{1}{b}+1\right)^{\frac{b^2+2ba}{(a+b+c)^2}}\cdot\left(\frac{1}{c}+1\right)^{\frac{c^2+2cb}{(a+b+c)^2}}$$

$$\frac{(a+2c)+(b+2a)+(c+2b)}{(a+b+c)^2}+1 \geq \prod_{cyc}\left(1+\frac{1}{a}\right)^{\frac{a^2+2ac}{(a+b+c)^2}}$$

$$\left(1+\frac{3}{a+b+c}\right)^{(a+b+c)^2} \geq \prod_{cyc}\left(1+\frac{1}{a}\right)^{a^2+2ac}$$

$$\left(1+\frac{3}{3}\right)^{3^2} \geq \prod_{cyc}\left(1+\frac{1}{a}\right)^{a^2+2ac}$$

$$\prod_{cyc}\left(1+\frac{1}{a}\right)^{a^2+2ac} \leq 2^9 = 512$$

2.79. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\frac{1}{2}\int_a^b\int_a^b (1+\tan x)(1+\tan y)(1+\tan x\tan y)\,dx\,dy \geq (\tan b-\tan a)^2$$

Solution:

$$\frac{2}{\cos^2 x\cos^2 y} = 2\cdot\frac{1}{\cos^2 x}\cdot\frac{1}{\cos^2 y} =$$

$$= 2\cdot\frac{\sin^2 x+\cos^2 x}{\cos^2 x}\cdot\frac{\sin^2 y+\cos^2 y}{\cos^2 y} = 2(\tan^2 x+1)(\tan^2 y+1) =$$

$$= (\tan^2 x+1)(\tan^2 y+1)+(\tan^2 x+1)(\tan^2 y+1) \geq$$

$$\overset{QM-AM}{\geq} 2\left(\frac{\tan x+1}{2}\right)^2(\tan^2 y+1)+(\tan^2 x+1)\cdot 2\left(\frac{\tan y+1}{2}\right)^2 =$$

$$= \frac{1}{2}\left[(\tan x+1)^2(\tan^2 y+1)+(\tan^2 x+1)(\tan y+1)^2\right] =$$

$$= \frac{1}{2}\left[(\tan^2 x+2\tan x+1)(\tan^2 y+1)+(\tan^2 x+1)(\tan^2 y+2\tan y+1)\right] =$$

$$= \frac{1}{2}\begin{pmatrix}\tan^2 x\tan^2 y+\tan^2 x+2\tan x\tan^2 y+2\tan x+ \\ +\tan^2 y+1+\tan^2 x\tan^2 y+2\tan^2 x\tan y+\tan^2 x+\tan^2 y+2\tan y+1\end{pmatrix} =$$

$$= \tan^2 x\tan^2 y+\tan x\tan^2 y+\tan^2 x\tan y+\tan x+\tan y+1+$$

$$+\tan^2 x+\tan^2 y \overset{AM-GM}{\geq} \tan^2 x\tan^2 y+\tan x\tan^2 y+$$

$$+\tan^2 x\tan y+\tan x+\tan y+1+2\tan x\tan y =$$

$$= \tan^2 x\tan^2 y+\tan x\tan^2 y+\tan^2 x\tan y+\tan x\tan y+$$

$$+\tan x\tan y+\tan x+$$

$$+\tan y+1=\tan x\tan y\,(\tan x\tan y+\tan x+\tan y+1)+$$
$$+(\tan x\tan y+\tan x+\tan y+1)=$$
$$=(\tan x\tan y+\tan x+\tan y+1)(\tan x\tan y+1)=$$
$$=[\tan x\,(\tan y+1)+(\tan y+1)](\tan x\tan y+1)=$$
$$=(\tan x+1)(\tan y+1)(\tan x\tan y+1)$$
$$\int_a^b\int_a^b(1+\tan x)\,(1+\tan y)(1+\tan x\tan y)dxdy\geq$$
$$\geq\int_a^b\int_a^b\frac{2}{\cos^2 x\cos^2 y}dxdy=2\left(\int_a^b\frac{1}{\cos^2 x}dx\right)\left(\int_a^b\frac{1}{\cos^2 y}dy\right)$$
$$\frac{1}{2}\int_a^b\int_a^b(1+\tan x)\,(1+\tan y)(1+\tan x\tan y)dxdy\geq$$
$$\geq\left(\tan x\Big|_a^b\right)\cdot\left(\tan y\Big|_a^b\right)=(\tan b-\tan a)^2$$

Equality holds for $a=b$.

2.80. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\frac{\int_0^1 x^2(x+n)^n dx}{(n+1)^n}\right)}$$

Solution:

Let be: $I_n=\int_0^1 x^2(n+x)^n\,dx$

$$I_n=\int_0^1 x^2\left(\frac{(n+x)^{n+1}}{n+1}\right)'dx=\frac{x^2(n+x)^{n+1}}{n+1}\Bigg|_0^1-\int_0^1 2x\cdot\frac{(n+x)^{n+1}}{n+1}dx$$
$$=\frac{(n+1)^{n+1}}{n+1}-\frac{2}{n+1}\int_0^1 x\cdot\left(\frac{(n+x)^{n+2}}{n+2}\right)'dx=$$
$$=(n+1)^n-\frac{2}{n+1}\left(\frac{x(n+x)^{n+2}}{n+2}\Bigg|_0^1-\int_0^1\frac{(n+x)^{n+2}}{n+2}dx\right)=$$
$$=(n+1)^n-\frac{2(n+1)^{n+2}}{(n+1)(n+2)}+\frac{2(n+x)^{n+3}}{(n+1)(n+2)(n+3)}\Bigg|_0^1=$$

$$= (n+1)^n - \frac{2(n+1)^{n+1}}{n+2} + \frac{2(n+1)^{n+3}}{(n+1)(n+2)(n+3)} -$$

$$- \frac{2n^{n+3}}{(n+1)(n+2)(n+3)}$$

$$I_n = n^n\left[\left(\frac{n+1}{n}\right)^n - \frac{2(n+1)}{n+2}\left(\frac{n+1}{n}\right)^n + \frac{2(n+1)^2}{(n+2)(n+3)}\left(\frac{n+1}{n}\right)^n - \frac{2n^3}{(n+1)(n+2)(n+3)}\right]$$

$$\frac{I_n}{n^n} = \left(1+\frac{1}{n}\right)^n - \frac{2(n+1)}{n+2}\left(1+\frac{1}{n}\right)^n + \frac{2(n+1)^2}{(n+2)(n+3)}\left(1+\frac{1}{n}\right)^n -$$

$$- \frac{2n^3}{(n+1)(n+2)(n+3)}$$

$$\lim_{n\to\infty}\frac{I_n}{n^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n - \lim_{n\to\infty}\frac{2(n+1)}{n+2}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n +$$

$$+ \lim_{n\to\infty}\frac{2(n+1)^2}{(n+2)(n+3)}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n - \lim_{n\to\infty}\frac{2n^3}{(n+1)(n+2)(n+3)}$$

$$\lim_{n\to\infty}\frac{I_n}{n^n} = e - 2e + 2e - 2 = e - 2$$

$$\Omega = \lim_{n\to\infty}\left(\frac{\int_0^1 x^2(x+n)^n\,dx}{(n+1)^n}\right) = \lim_{n\to\infty}\left(\frac{I_n}{(n+1)^n}\right) =$$

$$= \lim_{n\to\infty}\frac{I_n}{n^n}\cdot\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\frac{I_n}{n^n}\cdot\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = (e-2)\cdot\frac{1}{e} = 1 - \frac{2}{e}$$

2.81. If $0 < a, b, c < \frac{\pi}{2}$ then:

$$\left(\sqrt[3]{\prod_{cyc}(\boldsymbol{\pi - 2a})}\right)\left(\sum_{cyc}\left(\frac{\tan \boldsymbol{a}}{\boldsymbol{a}}\right)\right) > \frac{\mathbf{12}}{\boldsymbol{\pi}}$$

Solution:

$$\textit{Because } 0 < a, b, c < \frac{\pi}{2} \to \tan a, \tan b, \tan c > 0$$

$$\textit{We have: } \sum\frac{\tan a}{a} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod\frac{\tan a}{a}}. \textit{ Then:} LHS \geq 3\sqrt[3]{\prod(\pi - 2a)\frac{\tan a}{a}}$$

$$\textit{We must show that: } (\pi - 2a)\frac{\tan a}{a} > \frac{4}{\pi}\ \textit{(etc)} \leftrightarrow$$

$$\leftrightarrow \pi\cdot\tan a - 2a\tan a - \frac{4a}{\pi} > 0$$

$$Let: f(a) = \pi \cdot \tan a - 2a \tan a - \frac{4a}{\pi}; 0 < a < \frac{\pi}{2}$$

$$f'(a) = -2\tan a + (\pi - 2x)\sec^2 a - \frac{4}{\pi}$$

$$f''(a) = 2\sec^2 a\,[(\pi - 2x)\tan x - 2]$$

$$f'''(a) = -12\tan a \sec^2 a - (2x + \pi)(2\sec^4 a + 4\tan^2 a \sec^2 a) < 0$$

$$\to f''(a) \downarrow on \left(0; \frac{\pi}{2}\right) \to f''(a) < f''(0) = -4 < 0 \to f(a)\ is\ concave$$

$$on\left(0; \frac{\pi}{2}\right). Hence,\ we\ just\ check: \lim_{a\to 0^+} f(a) = 0;\ and$$

$$\lim_{a\to\left(\frac{\pi}{2}\right)^-} f(a) = \lim_{a\to\left(\frac{\pi}{2}\right)^-} a \cdot \left[(\pi - 2a)\frac{\tan a}{a} - \frac{4}{\pi}\right] = \frac{\pi}{2}\left(\frac{4}{\pi} - \frac{4}{\pi}\right) = 0$$

Because:

$$\lim_{a\to\left(\frac{\pi}{2}\right)^-}(\pi - 2a)\frac{\tan a}{a} = \lim_{a\to\left(\frac{\pi}{2}\right)^-} \frac{(\pi-2a)}{\frac{a\cos a}{\sin a}} = \lim_{a\to 0^+} \frac{4x}{2(\pi - x)\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} =$$

$$= \frac{4}{\pi} \cdot 1 = \frac{4}{\pi}$$

$$So,\ LHS \geq 3\sqrt[3]{\prod(\pi - 2a)\frac{\tan a}{a}} > 3\sqrt[3]{\frac{4^3}{\pi^3}} = \frac{12}{\pi}$$

2.82. If $a, b, c > 0, ab + bc + ca = 3$ then:

$$\mathbf{4 \cdot \tan^{-1} 2 \cdot \tan^{-1}\left(\sqrt[3]{abc}\right) \leq \pi \cdot \tan^{-1}\left(1 + \sqrt[3]{abc}\right)}$$

Solution:

$$4\tan^{-1}(2)\tan^{-1}\left(\sqrt[3]{abc}\right) \leq \pi \tan^{-1}\left(1 + \sqrt[3]{abc}\right)$$

$$The\ inequality\ can\ be\ written\ as: \frac{\tan^{-1}\left(\sqrt[3]{abc}\right)}{\tan^{-1}\left(\sqrt[3]{abc}+1\right)} \leq \frac{\frac{\pi}{4}}{\tan^{-1}(2)} = \frac{\tan^{-1}(1)}{\tan^{-1}(2)}$$

$$ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} \Rightarrow abc \leq 1,\ let\ f: (0; 1] \to \mathbb{R}\ such\ that$$

$$f(x) = \frac{\tan^{-1}(x)}{\tan^{-1}(x+1)}$$

$$f'(x) = \frac{\frac{\tan^{-1}(x+1)}{x^2} - \frac{\tan^{-1}(x)}{x^2+1}}{(\tan^{-1}(x+1))^2} \quad (1)$$

$$\frac{1}{x^2} > \frac{1}{x^2+1}\ and\ \tan^{-1}(x + 1) > \tan^{-1}(x)\ (as\ increasing\ function) \overset{\text{“}\therefore\text{”}}{\Rightarrow}$$

$$\Rightarrow \frac{\tan^{-1}(x+1)}{x^2} > \frac{\tan^{-1}(x)}{x^2+1} (\tan^{-1}(x + 1) > 0\ and\ \tan^{-1}(x) > 0\ because$$

$$x > 0\ and\ -\frac{\pi}{2} < \tan^{-1}(x) < \tan^{-1}(x + 1) < \frac{\pi}{2}) \Rightarrow$$

$$\Rightarrow \frac{\tan^{-1}(x+1)}{x^2} - \frac{\tan^{-1}(x)}{x^2+1} > 0 \Rightarrow \frac{\frac{\tan^{-1}(x+1)}{x^2} - \frac{\tan^{-1}(x)}{x^2+1}}{(\tan^{-1}(x+1)^2)} > 0 \overset{(1)}{\Rightarrow} f'(x) > 0 \Rightarrow f$$

is increasing, $\sqrt[3]{abc} \le 1 \Rightarrow f(\sqrt[3]{abc}) \le f(1) \Leftrightarrow$

$$\Leftrightarrow \frac{\tan^{-1}(\sqrt[3]{abc})}{\tan^{-1}(\sqrt[3]{abc}+1)} \le \frac{\tan^{-1}(1)}{\tan^{-1}(2)} = \frac{\frac{\pi}{4}}{\tan^{-1}(2)} \Leftrightarrow$$

$$\Leftrightarrow 4\tan^{-1}(2)\tan^{-1}(\sqrt[3]{abc}) \le \pi \tan^{-1}(1+\sqrt[3]{abc}) \quad \textit{(Q.E.D.)}$$

2.83. If $a, b \in \mathbb{R}, a \le b$ then:

$$\int_a^b \int_a^b \sqrt[16]{e^{(3x+y)^2}}\, dxdy \ge (b-a)^2 \cdot \sqrt[4]{e^{(a+b)^2}}$$

Solution:

$$\varphi_y(x) = e^{\frac{(3x+y)^2}{16}}, \varphi_y''(x) = \frac{9}{8}e^{\frac{(3x+y)^2}{16}} + \frac{9}{64}(3x+y)^2 e^{\frac{(3x+y)^2}{16}} \ge 0$$

$$\varphi_y - convexe \rightarrow$$

$$\rightarrow \int_a^b \varphi_y(x)dx \ge (b-a)\varphi_y\left(\frac{a+b}{2}\right) = (b-a)e^{\frac{(3\left(\frac{a+b}{2}\right)+y)^2}{16}}$$

$$Let\ \Psi(y) = exp\left(\frac{3(a+b)}{8} + \frac{y}{4}\right)^2$$

$$\Psi''(y) = \frac{1}{8}exp\left(\frac{3(a+b)}{8} + \frac{y}{4}\right)^2 + \frac{1}{4}\left(\frac{3(a+b)}{8} + \frac{y}{4}\right)^2 exp\left(\frac{3(a+b)}{8} + \frac{y}{4}\right)^2$$

$$\Psi''(y) > 0, \Psi - convexe \rightarrow \int_a^b \Psi(y)dy \ge (b-a)\Psi\left(\frac{a+b}{2}\right)$$

$$\int_a^b \int_a^b \sqrt[16]{e^{(3x+y)^2}}dxdy \ge (b-a)^2 exp\left(\frac{3(a+b)}{8} + \frac{a+b}{8}\right)^2 =$$

$$= (b-a)^2 exp\left(\frac{(a+b)^2}{4}\right) = (b-a)^2 \cdot \sqrt[4]{e^{(a+b)^2}}$$

2.84. $\boldsymbol{\gamma_n = H_n - \log n\,, n \geq 1, x \in \mathbb{R}}$**. Find:**

$$\boldsymbol{\Omega(x) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\gamma_i^{\sin^2 x}\cdot\gamma_{n+1-i}^{\cos^2 x}\right)\right)}$$

Solution:

Let $(a_n)_n : (b_n)_n$ *be two sequence so that*

$\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to+\infty} b_n = b$

We will prove that $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k=n} a_k\, b_{n+1-k} = ab$*. We have*

$$\frac{1}{n}\sum_{k=1}^{k=n}(a_k b_{n+1-k} - ab) =$$

$$= \frac{1}{n}\sum_{k=1}^{k=n}(a_k - a)\, b_{n+1-k} + \frac{a}{n}\sum_{k=1}^{k=n}(b_n - b) + \frac{b_{n+1}\cdot a}{n}$$

$\exists M > 0$ *so that* $|b_n| \leq M, \forall n \in \mathbb{N}^*$*. By Cesaro theorem:*

$$0 \leq \frac{1}{n}\left|\sum_{k=1}^{k=n}(a_k - a)b_{n+1-k}\right| \leq \frac{M}{n}\sum_{k=1}^{k=1}|a_k - a|$$

We have $\lim_{n\to\infty}\frac{M}{n}\sum_{k=1}^{k=n}|a_k - a| = 0$

So, $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k=n}(a_k - a)\, b_{n+1-k} = 0$

We have also that: $\lim_{n\to\infty}\frac{a}{n}\sum_{k=1}^{k=n}(b_n - b) = 0$ *then:*

$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{k=n} a_k\, b_{n+1-k} = ab$ *(**)*

Let $a_n = \gamma_n^{\sin^2 x}$ *and* $b_n = \gamma_n^{\cos^2 x}$

We have $\lim_{n\to+\infty} a_n = \gamma^{\sin^2 x}$ *and* $\lim_{n+\infty} b_n = y^{\cos^2 x}$

So, by (**): $\Omega(x) = \lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{i=n} a_i\, b_{n+1-i} = \gamma^{\sin^2 x + cos^2 x} = \gamma$

2.85. If $\frac{\sqrt{3}}{3} < a \leq b < 1$ **then:**

$$\boldsymbol{\int_a^b\int_a^b\left(\frac{x+y}{\tan^{-1}\left(\frac{x+y}{2}\right)}\right)dx\,dy \geq \frac{(b^2-a^2)(b-a)}{2\tan^{-1}\left(\frac{a+b}{2}\right)}}$$

Solution:

Let $\varphi(t) = \frac{t}{\arctan(t)}$ *where* $t > 0$

$$\varphi''(t) = \frac{2(t - \arctan t)}{(\arctan t)^3(1 + t^2)^2} > 0$$

Since $t \geq \arctan t$ *for all* $t \in [0, +\infty[$ *we have* φ *is convexe on* $]0, +\infty[$

then $\int_a^b \varphi(t)dt \geq (v - u)\varphi\left(\frac{v+u}{2}\right)$ *for all* $0 < u < v$

Let $I = \int_a^b \int_a^b \frac{x+y}{\arctan\left(\frac{x+y}{2}\right)} dx\, dy = 2\int_a^b \int_a^b \varphi\left(\frac{x+y}{2}\right) dx\, dy$

$$\int_a^b \varphi\left(\frac{x+y}{2}\right) dx \geq (b - a)\varphi\left(\frac{b+a}{4} + \frac{y}{2}\right)$$

then

$$I \geq 2\int_a^b (b - a)\varphi\left(\frac{b+a}{4} + \frac{y}{2}\right) dy \geq$$

$$\geq 2(b - a)(b - a)\varphi\left(\frac{b+a}{4} + \frac{b+a}{4}\right)$$

$$\geq 2(b - a)(b - a) \cdot \varphi\left(\frac{b+a}{2}\right) \geq 2(b - a)(b - a) \times \frac{\frac{b+a}{2}}{\tan^{-1}\left(\frac{b+a}{2}\right)}$$

$$\geq \frac{(b - a)(b^2 - a^2)}{\tan^{-1}\left(\frac{a+b}{2}\right)} \geq \frac{(b - a)(b^2 - a^2)}{2\tan^{-1}\left(\frac{a+b}{2}\right)}$$

2.86. $\boldsymbol{m, n > 0, \Omega(0, \pi) = -\log 2}$. **Find:**

$$\boldsymbol{\Omega(m,n) = \int \left(\frac{(m - n)e^{mx}\sin(nx) - (m + n)e^{mx}\cos(nx) + n}{(e^{mx} - \sin(nx))(e^{mx} - \cos(nx))}\right) dx}$$

Solution:

$$\Omega(m,n) = \int \frac{(m - n)e^{mx}\sin nx - (m + n)e^{mx} \cdot \cos nx + n}{(e^{mx} - \sin nx)(e^{mx} - \cos nx)} dx$$

$$\frac{e^{mx} - \sin nx}{e^{mx} - \cos nx} = t \Rightarrow \frac{1}{t} = \frac{e^{mx} - \cos nx}{e^{mx} - \sin nx}$$

$$dt = \frac{(me^{mx} - n\cos nx)(e^{mx} - \cos nx) - (e^{mx} - \sin nx)(me^{mx} + n\sin nx)}{(e^{mx} - \cos nx)^2} dx$$

$$dt = \frac{me^{2mx} - me^{mx} \cdot \cos nx - ne^{mx} \cdot \cos nx + n\cos^2 nx + me^{2mx} - ne^{mx} \cdot \sin nx + me^{mx}\sin x + n\sin^2 nx}{(e^{mx} - \cos nx)^2} dx$$

$$dt = \frac{(m-n)e^{mx}\cdot \sin nx - (m+n)e^{mx}\cdot \cos nx + n(\cos^2 nx + \sin^2 nx)}{(e^{mx}-\cos nx)^2}dx$$

$$\Omega(m,n) = \int \underbrace{\frac{e^{mx}-\cos nx}{e^{mx}-\sin nx}}_{\frac{1}{t}} \cdot \underbrace{\frac{(m-n)e^{mx}\cdot \sin nx - (m+n)e^{mx}\cdot \cos nx + n}{(e^{mx}-\cos nx)^2}dx}_{dt}$$

$$\Omega(m,n) = \int \frac{dt}{t} = \ln t = \ln\left|\frac{e^{mx}-\sin nx}{e^{mx}-\cos nx}\right| + C$$

$$\Omega(0,\pi) = \ln\left|\frac{1-\sin \pi x}{1-\cos \pi x}\right| = -\ln 2$$

$$\ln\left|\frac{1-\sin \pi x}{1-\cos \pi x}\right| = \ln\frac{1}{2}$$

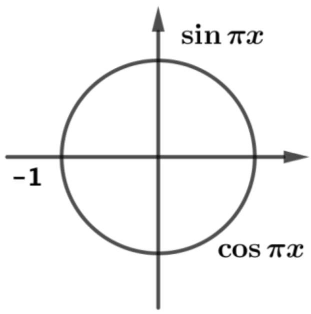
$$1-\cos \pi x \neq 0;\ \pi x \neq 2k\pi;\ x \neq 2k$$

$$2(1-\sin \pi x) = 1-\cos \pi x$$

$$2\sin \pi x - \cos \pi x = 1$$

$$(2\sin \pi x - \cos \pi x)^2 + (-\sin \pi x - 2\cos \pi x)^2 = 5$$

$$(-\sin \pi x - 2\cos \pi x)^2 = 4$$



$$-\sin \pi x - 2\cos \pi x = 2$$

$$\begin{cases} 2\sin \pi x - \cos \pi x = 1 \\ -\sin \pi x - 2\cos \pi x = 2 \end{cases} \Rightarrow \begin{matrix} \sin \pi x = 0 \\ \cos \pi x = -1 \end{matrix}$$

$$\tan \pi x = \frac{\sin \pi x}{\cos \pi x} = 0 \Rightarrow \pi x = (2k+1)\pi \Rightarrow x = 2k+1; k \in \mathbb{Z}$$

$$\Omega(m,n) = \ln\left|\frac{e^{(2k+1)m} - \sin(2k+1)n}{e^{(2k+1)m} - \cos(2k+1)n}\right|$$

2.87. If $\boldsymbol{a,b,c} > 0$ then:

$$\frac{\boldsymbol{a}}{\boldsymbol{a}+\sin \boldsymbol{a}} + \frac{\boldsymbol{b}}{\boldsymbol{b}+\sin \boldsymbol{b}} + \frac{\boldsymbol{c}}{\boldsymbol{c}+\sin \boldsymbol{c}} \leq \frac{3\sqrt[3]{\boldsymbol{abc}}}{\sqrt[3]{\boldsymbol{abc}} + \sqrt[3]{\sin \boldsymbol{a} \sin \boldsymbol{b} \sin \boldsymbol{c}}}$$

Solution:

$$Let\ be\ f:(-\infty,0)\to\mathbb{R}; f(x)=\frac{1}{1+e^x}; f'(x)=\frac{-e^x}{(1+e^x)^2}$$

$$f''(x)=\frac{e^x(e^x-1)}{(1+e^x)^3}<0; (\forall)x<0\Rightarrow f\ concave$$

$$By\ Jensen's\ inequality: f\left(\frac{y_1+y_2+y_3}{3}\right)\geq\frac{1}{3}\left(f(y_1)+f(y_2)+f(y_3)\right)$$

$$y_1,y_2,y_3<0\Rightarrow(\exists)x_1,x_2,x_3\in(0,1)$$

$$y_1=\ln x_1; y_2=\ln x_2; y_3=\ln x_3$$

$$f\left(\frac{\ln x_1+\ln x_2+\ln x_3}{3}\right)\geq\frac{1}{3}\left(f(\ln x_1)+f(\ln x_2)+f(\ln x_3)\right)$$

$$f\left(\ln\sqrt[3]{x_1x_2x_3}\right)\geq\frac{1}{3}\left(\frac{1}{1+e^{\ln x_1}}+\frac{1}{1+e^{\ln x_2}}+\frac{1}{1+e^{\ln x_3}}\right)$$

$$\frac{1}{1+e^{\ln\sqrt[3]{x_1x_2x_3}}}\geq\frac{1}{3}\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}\right)$$

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}\leq\frac{3}{1+\sqrt[3]{x_1x_2x_3}}$$

$$x_1=\frac{\sin a}{a}<1; x_2=\frac{\sin b}{b}<1; x_3=\frac{\sin c}{c}<1$$

$$\frac{1}{1+\frac{\sin a}{a}}+\frac{1}{1+\frac{\sin b}{b}}+\frac{1}{1+\frac{\sin c}{c}}\leq\frac{3}{1+\sqrt[3]{\frac{\sin a}{a}\cdot\frac{\sin b}{b}\cdot\frac{\sin c}{c}}}$$

$$\frac{a}{a+\sin a}+\frac{b}{b+\sin b}+\frac{c}{c+\sin c}\leq\frac{3\sqrt[3]{abc}}{\sqrt[3]{abc}+\sqrt[3]{\sin a\sin b\sin c}}$$

Equality holds for $a=b=c$.

2.88. If $1<a\leq b$ then:

$$\int_a^b\left(\int_a^b\left(\int_a^b\left(\frac{1}{1+\sqrt[3]{xyz}}\right)dx\right)dy\right)dz\leq(b-a)^2\log\left(\frac{1+b}{1+a}\right)$$

Solution:

$$Let\ be\ f:(0,\infty)\to\mathbb{R}; f(x)=\frac{1}{1+e^x}; f'(x)=\frac{-e^x}{(1+e^x)^2}$$

$$f''(x)=\frac{-e^x(1+e^x)^2+e^x\cdot 2\cdot e^x(1+e^x)}{(1+e^x)^4}=\frac{-e^x(1+e^x)+2e^{2x}}{(1+e^x)^3}$$

$$f''(x)=\frac{-e^x+e^{2x}}{(1+e^x)^3}=\frac{e^x(e^x-1)}{(1+e^x)^3}>0;(\forall)x>0$$

By Jensen's inequality:

$$f\left(\frac{y_1+y_2+y_3}{3}\right)\le\frac{1}{3}\big(f(y_1)+f(y_2)+f(y_3)\big);y_1,y_2,y_3>0$$

$$y_1,y_2,y_3>0\Rightarrow(\exists)x_1,x_2,x_3>1;\ y_1=\ln x_1;y_2=\ln x_2;y_3=\ln x_3$$

$$f\left(\frac{\ln x_1+\ln x_2+\ln x_3}{3}\right)\le\frac{1}{3}\big(f(\ln x_1)+f(\ln x_2)+f(\ln x_3)\big)$$

$$f\left(\ln\sqrt[3]{x_1x_2x_3}\right)\le\frac{1}{3}\left(\frac{1}{1+e^{\ln x_1}}+\frac{1}{1+e^{\ln x_2}}+\frac{1}{1+e^{\ln x_3}}\right)$$

$$\frac{1}{1+e^{\ln\sqrt[3]{x_1x_2x_3}}}\le\frac{1}{3}\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}\right)$$

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}\ge\frac{3}{1+\sqrt[3]{x_1x_2x_3}}$$

$$\frac{3}{1+\sqrt[3]{x_1x_2x_3}}\le\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}$$

$$x_1=x,x_2=y,x_3=z$$

$$\frac{3}{1+\sqrt[3]{xyz}}\le\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}$$

$$3\int_a^b\left(\int_a^b\left(\int_a^b\left(\frac{1}{1+\sqrt[3]{xyz}}\right)dx\right)dy\right)dz\le$$

$$\le\int_a^b\left(\int_a^b\left(\int_a^b\left(\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}\right)dx\right)dy\right)dz=$$

$$=\int_a^b\left(\int_a^b\left(\int_a^b\left(\frac{1}{1+x}\right)dx\right)dy\right)dz+\int_a^b\left(\int_a^b\left(\int_a^b\left(\frac{1}{1+y}\right)dx\right)dy\right)dz+$$

$$+\int_a^b\left(\int_a^b\left(\int_a^b\left(\frac{1}{1+z}\right)dx\right)dy\right)dz=$$

$$=(b-a)^2(\log(1+b)-\log(1+a)+$$

$$+(b-a)^2(\log(1+b)-\log(1+a)+$$

$$+(b-a)^2(\log(1+b)-\log(1+a)=$$

$$= 3(b-a)^2(\log(1+b) - \log(1+a) = 3(b-a)^2 \log\left(\frac{1+b}{1+a}\right)$$

$$3\int_a^b \left(\int_a^b \left(\int_a^b \left(\frac{1}{1+\sqrt[3]{xyz}}\right) dx\right) dy\right) dz \le 3(b-a)^2 \log\left(\frac{1+b}{1+a}\right)$$

$$\int_a^b \left(\int_a^b \left(\int_a^b \left(\frac{1}{1+\sqrt[3]{xyz}}\right) dx\right) dy\right) dz \le (b-a)^2 \log\left(\frac{1+b}{1+a}\right)$$

2.89. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\left(\int_a^{\sqrt{ab}} e^{-x^2} \sin x\, dx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2} \cos x\, dx\right) \le \left(\int_0^{\sqrt{ab}} e^{-x^2} \cos x\, dx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2} \sin x\, dx\right)$$

Solution:

By Cauchy's theorem $(\exists)c \in \left[a, \sqrt{ab}\right]; d \in \left[\frac{a+b}{2}, b\right]$ *such that:*

$$\frac{\int_a^{\sqrt{ab}} e^{-x^2} \sin x\, dx}{\int_a^{\sqrt{ab}} e^{-x^2} \cos x\, dx} = \frac{\sin c}{\cos c} = \tan c$$

$$\frac{\int_{\frac{a+b}{2}}^{b} e^{-x^2} \sin x\, dx}{\int_{\frac{a+b}{2}}^{b} e^{-x^2} \cos x\, dx} = \frac{\sin d}{\cos d} = \tan d$$

$$0 < a \le c \le \sqrt{ab} \overset{AM-GM}{\le} \frac{a+b}{2} \le d \le b < \frac{\pi}{2}$$

$\tan x$ – *increasing on* $\left(0, \frac{\pi}{2}\right) \Rightarrow \tan c \le \tan d$

$$\Rightarrow \frac{\int_a^{\sqrt{ab}} e^{-x^2} \sin x\, dx}{\int_a^{\sqrt{ab}} e^{-x^2} \cos x\, dx} \le \frac{\int_{\frac{a+b}{2}}^{b} e^{-x^2} \sin x\, dx}{\int_{\frac{a+b}{2}}^{b} e^{-x^2} \cos x\, dx}$$

$$\left(\int_a^{\sqrt{ab}} e^{-x^2} \sin x\, dx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2} \cos x\, dx\right) \le$$

$$\leq \left(\int_{a}^{\sqrt{ab}} e^{-x^2}\cos x\,dx\right)\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2}\sin x\,dx\right)$$

2.90. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\left(\int_0^{\sqrt{ab}}\left(\frac{\sin t}{1+e^t}\right)dt\right)\left(\int_0^{\frac{a+b}{2}}\left(\frac{\cos t}{1+e^t}\right)dt\right) \leq \left(\int_0^{\sqrt{ab}}\left(\frac{\cos t}{1+e^t}\right)dt\right)\left(\int_0^{\frac{a+b}{2}}\left(\frac{\sin t}{1+e^t}\right)dt\right)$$

Solution:

Let be $f:[a,b]\to\mathbb{R}$;

$$f(x) = \frac{\int_0^x\left(\frac{\sin t}{1+e^t}\right)dt}{\int_0^x\left(\frac{\cos t}{1+e^t}\right)dt}$$

By Cauchy's theorem exists $c(x)\in[a,b]$ *such that:*

$$f(x) = \frac{\frac{\sin c(x)}{1+e^{c(x)}}}{\frac{\cos c(x)}{1+e^{c(x)}}} = \tan c\,(x)$$

$$f'(x) = \frac{c'(x)}{\cos^2 c(x)} > 0 \Rightarrow f \text{ increasing}$$

$$\sqrt{ab} \leq \frac{a+b}{2} \Rightarrow f\left(\sqrt{ab}\right) \leq f\left(\frac{a+b}{2}\right)$$

$$\frac{\int_0^{\sqrt{ab}}\left(\frac{\sin t}{1+e^t}\right)dt}{\int_0^{\sqrt{ab}}\left(\frac{\cos t}{1+e^t}\right)dt} \leq \frac{\int_0^{\frac{a+b}{2}}\left(\frac{\sin t}{1+e^t}\right)dt}{\int_0^{\frac{a+b}{2}}\left(\frac{\cos t}{1+e^t}\right)dt}$$

$$\left(\int_0^{\sqrt{ab}}\left(\frac{\sin t}{1+e^t}\right)dt\right)\left(\int_0^{\frac{a+b}{2}}\left(\frac{\cos t}{1+e^t}\right)dt\right) \leq \left(\int_0^{\sqrt{ab}}\left(\frac{\cos t}{1+e^t}\right)dt\right)\left(\int_0^{\frac{a+b}{2}}\left(\frac{\sin t}{1+e^t}\right)dt\right)$$

2.91. If $0 < a \le b; n \in \mathbb{N}; n \ge 1$ then:

$$\left(\int_0^{\sqrt{ab}} x^n e^{x^2} dx\right)\left(\int_0^{\frac{a+b}{2}} x^{n-1} e^{x^2}\, dx\right) \le \left(\int_0^{\sqrt{ab}} x^{n-1} e^{x^2} dx\right)\left(\int_0^{\frac{a+b}{2}} x^n e^{x^2}\, dx\right)$$

Solution:

$$Let\ be\ g:(0,\infty)\to\mathbb{R};\ g(x)=\frac{\int_0^x t^n e^{t^2} dt}{\int_0^x t^{n-1} e^{t^2} dt}$$

$$g'(x)=\frac{x^n e^{x^2}\int_0^x t^{n-1}e^{t^2}\,dt - x^{n-1}e^{x^2}\int_0^x t^n\, e^{t^2}dt}{\left(\int_0^x t^{n-1}e^{t^2}dt\right)^2}$$

$$g'(x)=\frac{x^{n-1}e^{x^2}\left(x\int_0^x t^{n-1}\, e^{t^2}dt - \int_0^x t^n e^{t^2}\, dt\right)}{\left(\int_0^x t^{n-1}e^{t^2}\, dt\right)^2}$$

$$Let\ be\ h:(0,\infty)\to\mathbb{R},\ h(x)=x\int_0^x t^{n-1}\, e^{t^2}dt - \int_0^x t^n e^{t^2}dt$$

$$h'(x)=\int_0^x t^{n-1}\, e^{t^2}dt + x\cdot x^{n-1}e^{x^2} - x^n e^{x^2} =$$

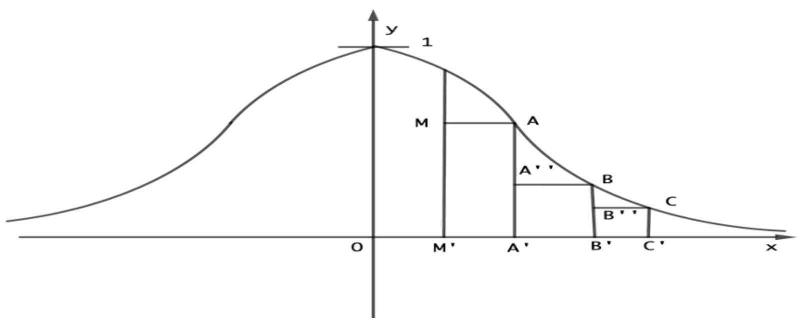
$$=\int_0^x t^{n-1}\, e^{t^2}dt > 0 \Rightarrow h\ increasing \Rightarrow h\left(\sqrt{ab}\right)\le h\left(\frac{a+b}{2}\right)\ because$$

$$\sqrt{ab} \overset{AM-GM}{\le} \frac{a+b}{2}$$

2.92. If $0 < a \le b$ then:

$$\int_a^b e^{-x^2}\, dx \ge \frac{\sqrt{a}\left(\sqrt{b}-\sqrt{a}\right)}{e^{ab}} + \frac{\left(\sqrt{b}-\sqrt{a}\right)^2}{2\sqrt[4]{e^{(a+b)^2}}} + \frac{(b-a)^2}{2e^{b^2}}$$

Solution:



$$\int_a^b e^{-x^2}\,dx \geq A[AMM'A'] + A[BA''A'B'] + A[CB''B'C']$$

$$M'(a,0);\, A'\left(\sqrt{ab},0\right);\, B'\left(\frac{a+b}{2},0\right);\, C'(b,0)$$

$$M\big(a,f(a)\big);\, A''\left(\sqrt{ab},f\left(\sqrt{ab}\right)\right);\, B''\left(\frac{a+b}{2},f\left(\frac{a+b}{2}\right)\right);$$

$$\int_a^b e^{-x^2}dx \geq \left(\sqrt{ab}-a\right)e^{-ab} + \left(\frac{a+b}{2}-\sqrt{ab}\right)e^{-\frac{(a+b)^2}{4}} + \left(b-\frac{a+b}{2}\right)e^{-b^2} =$$

$$= \frac{\sqrt{a}\left(\sqrt{b}-\sqrt{a}\right)}{e^{ab}} + \frac{\left(\sqrt{b}-\sqrt{a}\right)^2}{2\sqrt[4]{e^{(a+b)^2}}} + \frac{b-a}{2e^{b^2}}$$

2.93. If $a, b, c > 0$ then:

$$\sum_{cyc}\left(1+\frac{1}{a}\right)^b > e^{\frac{b}{2a+1}+\frac{c}{2b+1}} + e^{\frac{c}{2b+1}+\frac{a}{2c+1}} + e^{\frac{a}{2c+1}+\frac{b}{2a+1}}$$

Solution:

$$Let\ be\ f:(0,\infty)\to\mathbb{R};\, f(x) = \ln\left(1+\frac{1}{x}\right) - \frac{2}{2x+1}$$

$$f'(x) = \frac{-\frac{1}{x^2}}{\frac{x+1}{x}} + \frac{4}{(2x+1)^2} = \frac{-4(x+1)}{x(x+1)(2x+1)^2} = \frac{-4}{x(2x+1)^2} < 0$$

$$f\ decreasing \Rightarrow \inf f(x) = \lim_{x\to\infty} f(x) = 0$$

$$\Rightarrow f(x) > 0 \Rightarrow \ln\left(1+\frac{1}{x}\right) - \frac{2}{2x+1} > 0$$

$$\ln\left(1+\frac{1}{a}\right) - \frac{2}{2a+1} > 0 \Rightarrow \ln\left(1+\frac{1}{a}\right) > \frac{2}{2a+1}$$

$$b\ln\left(1+\frac{1}{a}\right) > \frac{2b}{2a+1} \Rightarrow \ln\left(1+\frac{1}{a}\right)^b > \frac{2b}{2a+1}$$

$$\left(1+\frac{1}{a}\right)^b > e^{\frac{2b}{2a+1}}$$

$$\sum_{cyc}\left(1+\frac{1}{a}\right)^b > \sum_{cyc} e^{\frac{2b}{2a+1}} \geq \sum_{cyc}\left(e^{\frac{b}{2a+1}}\cdot e^{\frac{c}{2b+1}}\right) = \sum_{cyc}\left(e^{\frac{b}{2a+1}+\frac{c}{2b+1}}\right)$$

2.94. **If $a, b > 1$ then:**

$$\frac{2}{(a-1)(b-1)}\int_1^a\int_1^a\left(\frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y}\right)dx\,dy < 1$$

Solution:

For $x \neq y, 1 < x < a, 1 < y < b$.

By Cauchy's mean value theorem: $\frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y} = \frac{\frac{1}{(1+c^2)}}{\frac{1}{c}}$

For the same c lying between x and y.

$$\Rightarrow \frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y} \leq \frac{c}{1+c^2} < \frac{1}{2}\ \ [\because c > 1] \Rightarrow$$

$$\Rightarrow \int_1^a\int_1^b \frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y}dx\,dy$$

$$< \frac{1}{2}\int_1^a\int_1^b dx\,dy = \frac{1}{2}(a-1)(b-1) \Rightarrow$$

$$\Rightarrow \frac{2}{(a-1)(b-1)}\int_1^a\int_1^b \frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y}dx\,dy < 1$$

2.95. **If $f \in C^2((0,\infty)), f(e) = e, f(xy) = xf(y) + yf(x), \forall x, y > 0$ then:**

$$f\left(\frac{x+3y}{4}\right) + f\left(\frac{3x+y}{4}\right) \geq f(x) + f(y), \forall x, y > 0$$

Solution:

$$f \in C^2((0,\infty)) \Rightarrow f \text{ is a continuous function}$$

$$\left.\begin{array}{r}\frac{f(xy)}{xy} = \frac{f(x)}{x} + \frac{f(y)}{y}, \forall x, y > 0 \\ \text{Let } g: (0,+\infty) \to \mathbb{R}; g(x) = \frac{f(x)}{x}\end{array}\right\} \Rightarrow \text{we have}$$

$$g(xy) = g(x) + g(y), g \text{ continuous}$$

Because $x > 0, y > 0 \Rightarrow \exists t_1, t_2 \in \mathbb{R}$ such that $x = e^{t_1}, y = e^{t_2}$

$\Rightarrow$ we have: $g(e^{t_1+t_2}) = g(e^{t_1}) + g(e^{t_2})$

Let $h(x) = g(e^x) \Rightarrow h(t_1 + t_2) = h(f_1) + h(f_2)$

$$g \text{ continuous and from Cauchy equation we have:}$$
$$h(x) = kx \Rightarrow g(e^x) = kx \Rightarrow g(t) = k\ln t$$
$$\left.\begin{array}{r}\Rightarrow \dfrac{f(x)}{x} = k\ln x \Rightarrow f(x) = kx\ln x \\ \text{But } f(e) = e \Rightarrow ke = e \Rightarrow k = 1\end{array}\right\} \Rightarrow f(x) = x\ln x$$
$$f'(x) = \ln x + 1, f''(x) = \frac{1}{x} > 0 \Rightarrow f \text{ convexe} \Rightarrow$$
$$f\left(\frac{x+3y}{4}\right) + f\left(\frac{3x+y}{4}\right) \geq f(x) + f(y), \forall x, y > 0 \text{ (M.O. DRÎMBE)}$$

2.96. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{1\leq i<j\leq n}\left((-1)^{i+j}\cdot i\cdot j\right)\right)}$$

Solution:

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\leq i<j\leq n}(-1)^i i(-1)^j j = \lim_{n\to\infty}\frac{\sum_{1\leq i<j\leq n}(-1)^i i(-1)^j j}{n^3} \overset{CS}{=}$$
$$\lim_{n\to\infty}\frac{\sum_{1\leq i<j\leq n}(-1)^i i\,(-1)^j j - \sum_{1\leq i<j\leq n}(-1)^i i(-1)^j j}{(n+1)^3 - n^3} =$$
$$= \lim_{n\to\infty}\frac{\sum_{i=1}^{n}(-1)^i i\,(-1)^{n+1}(n+1)}{3n^2+3n+1} = \lim_{n\to\infty}\underbrace{\frac{(-1)^{n+1}(n+1)\sum_{i=1}^{n}(-1)^i j}{3n^2+3n+1}}_{x_n}$$
$$\lim_{n\to\infty}x_{2n} = \lim_{n\to\infty}\frac{(-1)^{2n+1}(2n+1)\sum_{i=1}^{2n}(-1)^i i}{12n^2+6n+1} =$$
$$= \lim_{n\to\infty}\frac{-(2n+1)\left(\overbrace{-1+2}^{1}\overbrace{-3+4}^{1}+\cdots+\overbrace{2n}^{1}\right)}{12n^2+6n+1}$$
$$= \lim_{n\to\infty}\frac{-(2n+1)n}{12n^2+6n+1} = -\frac{1}{6}$$
$$\lim_{n\to\infty}x_{2n+1} = \lim_{n\to\infty}\frac{(-1)^{2n+2}(2n+2)\sum_{i=1}^{2n+1}(-1)^i i}{3(2n+1)^2+3(2n+1)+1}$$
$$= \lim_{n\to\infty}\frac{(2n+2)\big(-1+2-3+4+\cdots+2n-(2n+1)\big)}{3(2n+1)^2+3(2n+1)+1} =$$
$$= \lim_{n\to\infty}\frac{(2n+2)(-n-1)}{3(2n+1)^2+3(2n+1)+1} = -\frac{1}{6}$$

$$\Omega = -\frac{1}{6}$$

2.97. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\frac{\sin n}{n^3}\left(\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}} - \sum_{1\le i<j\le n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)\right)}$$

Solution:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}} = -\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sqrt[3]{i}-\sqrt[3]{j}\right)\left(\frac{1}{\sqrt[3]{i}}-\frac{1}{\sqrt[3]{j}}\right) =$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+1-\sqrt[3]{\frac{i}{j}}-\sqrt[3]{\frac{j}{i}}\right) =$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n}2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt[3]{\frac{i}{j}} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt[3]{\frac{j}{i}} =$$

$$= -2n^2 + 2\left(\sum_{k=1}^{n}\sqrt[3]{k}\right)\left(\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right) =$$

$$= -2n^2 + 2n^2\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}} = -2n^2 + 2n^2\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right)$$

$$2\sum_{1\le i<j\le n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}} = -2n^2 + 2n^2\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right)$$

$$\frac{1}{n^2}\left(\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}} - \sum_{1\le i<j\le n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right) = 1 \quad \textit{(1)}$$

$$\Omega = \lim_{n\to\infty}\left(\frac{\sin n}{n}\cdot\frac{1}{n^2}\left(\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}} - \sum_{1\le i<j\le n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)\right) =$$

$$\overset{(1)}{=} \lim_{n\to\infty}\left(\frac{\sin n}{n}\cdot 1\right) = \lim_{n\to\infty}\frac{\sin n}{n} = 0$$

2.98. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\frac{1}{n^2}\left(\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{i\cdot j}}\right)\right)}$$

Solution:

$$\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}}=$$

$$=\sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}\cdot\sqrt{k}\right)+\frac{1}{\sqrt{1}}\left(\sqrt{2}+\cdots+\sqrt{n}\right)+\frac{1}{\sqrt{2}}\left(\sqrt{3}+\cdots+\sqrt{n}\right)+\cdots+$$

$$+\frac{1}{\sqrt{n-1}}\cdot\sqrt{n}+\sqrt{1}\left(\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\right)+\sqrt{2}\left(\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\right)+\cdots+$$

$$+\sqrt{n-1}\cdot\frac{1}{\sqrt{n}}=n+\sum_{1\le i<\le n}\sqrt{\frac{j}{i}}+\sum_{1\le i<j\le n}\sqrt{\frac{i}{j}}=$$

$$=n+\sum_{1\le i<j\le n}\left(\sqrt{\frac{j}{i}}+\sqrt{\frac{i}{j}}-2+2\right)=n+\sum_{1\le i<j\le n}\left(\frac{i+j-2\sqrt{ij}}{\sqrt{ij}}+2\right)$$

$$=n+\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}+2\cdot\frac{n(n-1)}{2}=$$

$$=n+n^2-n+\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}=$$

$$=n^2+\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}\Rightarrow\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}}=$$

$$=n^2+\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}\Rightarrow\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}=$$

$$=n^2+\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}-\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}=n^2\Rightarrow$$

$$\Rightarrow \frac{1}{n^2}\left(\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}\right)=\frac{n^2}{n^2}=1\Rightarrow\Omega=1$$

2.99. If $\mathbf{0}<a\le b$ then:

$$\int_a^b\left(\int_a^b\left(\int_a^b\left(\frac{\sqrt[3]{(\boldsymbol{x^3+y^3})(\boldsymbol{y^3+z^3})(\boldsymbol{z^3+x^3})}}{(\boldsymbol{x^2+y^2})(\boldsymbol{y^2+z^2})(\boldsymbol{z^2+x^2})}\right)\boldsymbol{dx}\right)\boldsymbol{dy}\right)\boldsymbol{dz}\le\frac{\mathbf{1}}{\mathbf{4}}\left(\mathbf{log}\frac{\boldsymbol{b}}{\boldsymbol{a}}\right)^{\mathbf{3}}$$

Solution:

$$\frac{\sqrt[3]{\prod(x^3+y^3)}}{\prod(x^2+y^2)}\overset{Holder}{\le}\frac{xyz+xyz}{\prod(x^2+y^2)}\overset{x^2+y^2\ge 2xy}{\le}\frac{2xyz}{8(xyz)^2}=\frac{1}{4xyz}$$

$$so,\ \int_a^b\int_a^b\int_a^b\frac{\sqrt[3]{\prod(x^3+y^3)}}{\prod(x^2+y^2)}dx\,dy\,dz\le\int_a^b\int_a^b\int_a^b\frac{dx\,dy\,dz}{4xyz}$$

$$=\int_a^b\int_a^b\frac{dx\,dy}{4xy}\cdot(\ln z)_a^b=\frac{1}{4}(\ln x)_a^b\cdot(\ln y)_a^b\cdot(\ln z)_a^b=$$

$$=\frac{1}{4}(\ln b-\ln a)^3=\frac{\left(\ln\left(\frac{b}{a}\right)\right)^3}{4}$$

2.100. Find:

$$\boldsymbol{\Omega}=\lim_{\boldsymbol{n}\to\infty}\left(\mathbf{1}-\frac{\mathbf{1}}{\boldsymbol{e}}\right)\left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{e}}\right)\cdot\ldots\cdot\left(\mathbf{1}-\frac{\mathbf{1}}{\boldsymbol{ne}}\right)$$

Solution:

Let be $f:[0,1]\to\mathbb{R}, f(x)=e^{-x}-1+x\ , f'(x)=1-e^{-x}\ge 0$

$$minf(x)=f(0)=0\Rightarrow f(x)\ge 0\Rightarrow 1-x\le e^{-x}\,,\qquad \frac{1}{e}\in[0,1]$$

$$1-\frac{1}{e}<e^{-\frac{1}{e}}$$

$$1-\frac{1}{2e}<e^{-\frac{1}{2e}}$$

$$1-\frac{1}{3e}<e^{-\frac{1}{2e}}$$

....

$$1-\frac{1}{ne}<e^{-\frac{1}{ne}}$$

$$0<\left(1-\frac{1}{e}\right)\left(1-\frac{1}{2e}\right)\cdot\ldots\cdot\left(1-\frac{1}{ne}\right)<e^{-\frac{1}{e}}\cdot e^{-\frac{1}{2e}}\cdot\ldots\cdot e^{-\frac{1}{ne}}=$$
$$=e^{-\frac{1}{e}-\frac{1}{2e}-\cdots-\frac{1}{ne}}=e^{-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}$$
$$\lim_{n\to\infty}0\le\lim_{n\to\infty}\left(1-\frac{1}{e}\right)\left(1-\frac{1}{2e}\right)\cdot\ldots\cdot\left(1-\frac{1}{ne}\right)\le\lim_{n\to\infty}e^{-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}$$
$$0\le\Omega\le e^{-\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}=e^{-\infty}=0$$
$$\Omega=\lim_{n\to\infty}\left(1-\frac{1}{e}\right)\left(1-\frac{1}{2e}\right)\cdot\ldots\cdot\left(1-\frac{1}{ne}\right)=0$$

2.101. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\sum_{k=0}^{2n}\frac{(-1)^k}{\binom{2n}{k}}\right)}$$

Solution:

$$\frac{1}{\binom{2n}{k}}=\frac{1}{\frac{(2n)!}{k!\,(2n-k)!}}=\frac{k!\,(2n-k)!}{(2n)!}=\frac{\Gamma(k+1)\cdot\Gamma(2n-k+1)}{\Gamma(2n+1)}=$$
$$=\frac{\Gamma(k+1)\cdot\Gamma(2n-k+1)}{(2n+1)\cdot\Gamma(2n+1)}\cdot(2n+1)=$$
$$=\frac{\Gamma(k+1)\cdot\Gamma(2n-k+1)}{\Gamma(2n+2)}\cdot(2n+1)=$$
$$=(2n+1)\cdot B(k+1,2n-k+1)=(2n+1)\int_0^1 x^k(1-x)^{2n-k}dx$$
$$\sum_{k=0}^{2n}\frac{(-1)^k}{\binom{2n}{k}}=\sum_{k=0}^{2n}(-1)^k\,(2n+1)\int_0^1 x^k(1-x)^{2n-k}dx=$$
$$=(2n+1)\int_0^1\left(\sum_{k=1}^{2n}(-1)^k x^k(1-x)^{2n-k}\right)dx=$$
$$=(2n+1)\int_0^1\left(\sum_{k=0}^{2n}\frac{(-x)^k(1-x)^{2n}}{(1-x)^k}\right)dx=$$
$$=(2n+1)\int_0^1\left((x-1)^{2n}\cdot\frac{\frac{x^{2n+1}}{(x-1)^{2n+1}}-1}{\frac{x}{x-1}-1}\right)dx=$$

$$= (2n+1)\int_0^1 \left(\frac{\frac{x^{2n+1}}{x-1} - (x-1)^{2n}}{\frac{1}{x-1}}\right)dx =$$

$$= (2n+1)\int_0^1 (x^{2n+1} - (x-1)^{2n+1})dx =$$

$$= (2n+1)\cdot\left(\frac{1}{2n+2} + \frac{1}{2n+2}\right) = \frac{2n+1}{n+1}$$

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=0}^{2n} \frac{(-1)^k}{\binom{2n}{k}}\right) = \lim_{n\to\infty} \frac{2n+1}{n+1} = 2$$

2.102. If $a, b, c > 0$ then:

$$a^{\frac{a}{a+b}+\frac{a}{a+c}} \cdot b^{\frac{b}{b+a}+\frac{b}{b+c}} \cdot c^{\frac{c}{c+a}+\frac{a}{a+c}} \geq abc$$

Solution:

$$\textit{Let be } f_1, f_2: (0,\infty) \to \mathbb{R};\ f_1(x) = \tfrac{1}{2}x^2 - a\log x$$

$$f_1'(x) = x - \frac{a}{x} = \frac{x^2 - a}{x} = 0 \Rightarrow x^2 = a \Rightarrow x = \sqrt{a}$$

$$\min f_1(x) = f_1(\sqrt{a}) = \frac{1}{2}(\sqrt{a})^2 - a\log\sqrt{a} =$$

$$= \tfrac{1}{2}a - \tfrac{a}{2}\log a = \tfrac{a}{2}(1 - \log a).\ \textit{Analogous: } \min f_2(x) = \tfrac{b}{2}(1 - \log b)$$

$$\min(f_1 + f_2)(x) = (f_1 + f_2)\left(\frac{a+b}{2}\right) = \frac{a+b}{2}\left(1 - \log\left(\frac{a+b}{2}\right)\right)$$

$$\min f_1(x) + \min f_2(x) \leq \min(f_1 + f_2)(x)$$

$$\frac{a}{2}(1 - \log a) + \frac{b}{2}(1 - \log b) \leq \frac{a+b}{2}\left(1 - \log\left(\frac{a+b}{2}\right)\right)$$

$$-\frac{a}{2}\log a - \frac{b}{2}\log b \leq -\frac{a+b}{2}\log\left(\frac{a+b}{2}\right)$$

$$a\log a + b\log b \geq (a+b)\log\left(\frac{a+b}{2}\right)$$

$$\log(a^a \cdot b^b) \geq \log\left(\frac{a+b}{2}\right)^{a+b}$$

$$a^a \cdot b^b \geq \left(\frac{a+b}{2}\right)^{a+b}$$

$$a^{\frac{a}{a+b}} \cdot b^{\frac{b}{a+b}} \ge \frac{a+b}{2} \quad (1)$$

Analogous: $b^{\frac{b}{b+c}} \cdot c^{\frac{c}{b+c}} \ge \frac{b+c}{2}$ *(2);* $c^{\frac{c}{c+a}} \cdot a^{\frac{a}{c+a}} \ge \frac{c+a}{2}$ *(3)*

By multiplying (1); (2); (3):

$$a^{\frac{a}{a+b}+\frac{a}{a+c}} \cdot b^{\frac{b}{b+a}+\frac{b}{b+c}} \cdot c^{\frac{c}{c+a}+\frac{a}{a+c}} \ge \frac{(a+b)(b+c)(c+a)}{8} \overset{AM-GM}{\ge}$$

$$\ge \frac{2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}}{8} = abc. \text{ Equality holds for } a = b = c.$$

2.103. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \left(\int_a^b \left(\int_a^b \left(\cos\left(\frac{\pi}{4} - x\right) \cos\left(\frac{\pi}{4} - y\right) \cos\left(\frac{\pi}{4} - z\right) \right) dx \right) dy \right) dz \ge$$

$$\ge \sin^3(b + a) \cdot \sin^3(b - a)$$

Solution:

$$\frac{\sin x + \cos x}{2} \overset{W-AM-GM}{\ge} \left(\frac{\sin^{-2} x + \cos^{-2} x}{2} \right)^{-\frac{1}{2}}$$

$$\frac{\sqrt{2}}{2} \cos x + \frac{\sqrt{2}}{2} \sin x \ge \frac{\sqrt{2}}{2} \cdot \frac{2}{2^{-\frac{1}{2}}} \left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \right)^{-\frac{1}{2}}$$

$$\cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4} \ge \frac{\sqrt{2}}{2} \cdot \frac{1}{\frac{1}{\sqrt{2}}} \cdot \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \right)^{-\frac{1}{2}}$$

$$\cos\left(\frac{\pi}{4} - x\right) \ge 2 \cdot \frac{1}{\frac{1}{\sin x \cos x}} = 2 \sin x \cos x = \sin 2x$$

Analogous: $\cos\left(\frac{\pi}{4} - y\right) \ge \sin 2y$; $\cos\left(\frac{\pi}{4} - z\right) \ge \sin 2z$

By multiplying:

$$\cos\left(\frac{\pi}{4} - x\right) \cos\left(\frac{\pi}{4} - y\right) \cos\left(\frac{\pi}{4} - z\right) \ge \sin 2x \sin 2y \sin 2z$$

$$\int_a^b \left(\int_a^b \left(\int_a^b \left(\cos\left(\frac{\pi}{4} - x\right) \cos\left(\frac{\pi}{4} - y\right) \cos\left(\frac{\pi}{4} - z\right) \right) dx \right) dy \right) dz \ge$$

$$\geq \int_a^b \sin 2x\,dx \cdot \int_a^b \sin 2y\,dy \cdot \int_a^b \sin 2z\,dz =$$

$$= \left(-\frac{1}{2}(\cos 2b - \cos 2a)\right)^3 = \left(\frac{1}{2}(\cos 2a - \cos 2b)\right)^3 =$$

$$= \left(\frac{1}{2} \cdot 2\sin\frac{2a+2b}{2}\sin\frac{2b-2a}{2}\right)^3 = \sin^3(b+a) \cdot \sin^3(b-a)$$

2.104. If $f: (0,\infty) \to \mathbb{R}, f$ −twice derivable,

$$\boldsymbol{f(e) = e, f(xy) = yf(x) + xf(y), \forall x, y > 0} \textbf{ then:}$$

$$\boldsymbol{f\left(\frac{3x+2y}{5}\right) + f\left(\frac{2x+3y}{5}\right) \leq f(x) + f(y), \forall x, y > 0}$$

Solution:

$$f(xy) = yf(x) + xf(y) \Rightarrow \frac{f(xy)}{xy} = \frac{f(x)}{x} + \frac{f(y)}{y}, \forall x, y > 0$$

Let be $g: (0,\infty) \to \mathbb{R}, g(x) = \frac{f(x)}{x} \Rightarrow g(xy) = g(x) + g(y), \forall x, y > 0$

$x, y > 0$ *then* $\exists a, b > 0, x = e^a, y = e^b \Rightarrow g(e^{a+b}) = g(e^a) + g(e^b)$

Let be $h: (0,\infty) \to \mathbb{R}, h(x) = g(e^x) \Rightarrow h(x+y) = h(x) + h(y)$

h *−continuous. By Cauchy:* $\exists c \in \mathbb{R}, h(x) = cx.$

$$g(e^x) = cx \Rightarrow g(x) = clnx \Rightarrow \frac{f(x)}{x} = clnx \Rightarrow f(x) = cxlnx$$

$$f(e) = ce = e \Rightarrow c = 1 \Rightarrow f(x) = xlnx$$

$f'(x) = lnx + 1, f''(x) = \frac{1}{x} > 0, f$ *−convexe. By Jensen's inequality:*

$$f\left(\frac{3x+2y}{5}\right) \leq \frac{3}{5}f(x) + \frac{2}{5}f(y), \forall x, y > 0$$

$$f\left(\frac{2x+3y}{5}\right) \leq \frac{2}{5}f(x) + \frac{3}{5}f(y), \forall x, y > 0$$

By adding: $f\left(\frac{3x+2y}{5}\right) + f\left(\frac{2x+3y}{5}\right) \leq f(x) + f(y), \forall x, y > 0$

2.105. Find:

$$\boldsymbol{\Omega(a,b) = \lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}\left(a^{\frac{1}{n+k}} \cdot b^{\frac{2}{n+k}}\right) - n - log2 \cdot log\left(ab^2\right)\right)\right), a, b > 1}$$

Solution:

$$\Omega(a,b)=\lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}\left(a^{\frac{1}{n+k}}\cdot b^{\frac{2}{n+k}}\right)-n-log2\cdot log(ab^2)\right)\right)=$$

$$=\lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}(ab^2)^{\frac{1}{n+k}}-n-log2\cdot log(ab^2)\right)\right)\ (Not\ c=ab^2)=$$

$$=\lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}(c)^{\frac{1}{n+k}}-n-log2\cdot log(c)\right)\right)=$$

$$=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}(c)^{\frac{1}{n+k}}-n-log2\cdot log(c)}{\frac{1}{n}}=$$

$$\overset{TCS}{\cong}\lim_{n\to\infty}\frac{c^{\frac{1}{2n+1}}+c^{\frac{1}{2n+2}}-1-c^{\frac{1}{n+1}}}{\frac{-1}{n^2+n}}=\lim_{n\to\infty}\frac{c^{\frac{1}{2n+1}}+c^{\frac{1}{2n+2}}-1-c^{\frac{1}{n+1}}}{\frac{-1}{n^2+n}\cdot\frac{n^2}{n^2}}=$$

$$=\lim_{n\to\infty}\frac{-c^{\frac{1}{2n+1}}-c^{\frac{1}{2n+2}}+1+c^{\frac{1}{n+1}}}{\frac{1}{n^2}}=\lim_{\substack{x\to0\\x>0}}\frac{c^{\frac{x}{x+1}}+1-c^{\frac{x}{x+2}}-c^{\frac{x}{2x+2}}}{x^2}=$$

$$\overset{L'H}{\cong}\lim_{\substack{x\to0\\x>0}}\frac{\frac{1}{(x+1)^2}e^{\frac{x}{x+1}}c^{\frac{x}{x+1}}logc+1-\frac{2}{(x+2)^2}e^{\frac{x}{x+2}}c^{\frac{x}{x+2}}logc-\frac{2}{(2x+2)^2}e^{\frac{x}{2x+2}}c^{\frac{x}{2x+2}}logc}{2x}$$

$$\overset{L'H}{\cong}\frac{logc\left(logc-2-logc-\frac{4}{16}logc+\frac{4}{8}-logc\cdot\frac{4}{24}+\frac{8}{8}\right)}{2}=$$

$$=logc\left(\frac{1}{2}logc-\frac{1}{2}\right)^2=\frac{logc(logc-1)}{4}=\frac{log(ab^2)(log(ab^2)-1)}{4}$$

2.106. If $0<a\le b<\frac{\pi}{2}$ then:

$$\int_a^b\int_a^b\left(\frac{\sin x\sin y\sin(x+y)}{xy(\pi-x-y)}\right)dx\,dy\le\frac{81\sqrt{3}(b-a)^2}{8\pi^3}$$

Solution:

$$Let\ be\ f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R};\ f(x)=\log(\sin x)-\log x$$

$$f'(x)=\cot x-\frac{1}{x}$$
$$f''(x)=-\frac{1}{\sin^2 x}+\frac{1}{x^2}=\frac{\sin^2 x-x^2}{x^2\sin^2 x}\le 0\Rightarrow f\ concave$$
$$f\left(\frac{x+y+\pi-(x+y)}{3}\right)\ge\frac{1}{3}\Big(f(x)+f(y)+f\big(\pi-(x+y)\big)\Big)$$
$$f\left(\frac{\pi}{3}\right)\ge\frac{1}{3}\left(\log\left(\frac{\sin x}{x}\right)+\log\left(\frac{\sin y}{y}\right)+\log\left(\frac{\sin\big(\pi-(x+y)\big)}{\pi-(x+y)}\right)\right)$$
$$3\log\left(\frac{\sin\frac{\pi}{3}}{\frac{\pi}{3}}\right)\ge\log\left(\frac{\sin x\sin y\sin(x+y)}{xy(\pi-x-y)}\right)$$
$$\log\left(\frac{\sin x\sin y\sin(x+y)}{xy(\pi-x-y)}\right)\le\log\left(\frac{\frac{\sqrt{3}}{2}}{\frac{\pi}{3}}\right)^3$$
$$\frac{\sin x\sin y\sin(x+y)}{xy(\pi-x-y)}\le\frac{81\sqrt{3}}{8\pi^3}$$
$$\int_a^b\int_a^b\left(\frac{\sin x\sin y\sin(x+y)}{xy(\pi-x-y)}\right)dx\,dy\le\int_a^b\int_a^b\frac{81\sqrt{3}}{8\pi^3}dx\,dy=\frac{81\sqrt{3}}{8\pi^3}(b-a)^2$$

2.107. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\frac{6}{7}\cdot\frac{13}{14}\cdot\ldots\cdot\frac{7n-1}{7n}\right)}$$

Solution:

$$Let\ be\ f:[0,1]\to\mathbb{R};f(x)=e^{-x}-1+x$$
$$f'(x)=-e^{-x}+1=1-\frac{1}{e^x}=\frac{e^x-1}{e^x}\le 0;(\forall)x\in[0,1]$$
$$M=\max_{x\in[0,1]}f(x)=f(0)=0\Rightarrow f(x)\ge 0,(\forall)x\in[0,1]$$
$$e^{-x}\ge 1-x;x\in[0,1]$$
$$e^{-\frac{1}{7}}\ge 1-\frac{1}{7}$$
$$e^{-\frac{1}{14}}\ge 1-\frac{1}{14}$$
$$-------\,-$$
$$e^{-\frac{1}{7n}}\ge 1-\frac{1}{7n}$$

By multiplying:

$$0 < \left(1-\frac{1}{7}\right)\left(1-\frac{1}{14}\right)\cdot \ldots \cdot \left(1-\frac{1}{7n}\right) < e^{-\frac{1}{7}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}$$
$$0 < \frac{6}{7}\cdot\frac{13}{14}\cdot \ldots \cdot \frac{7n-1}{7n} < e^{-\frac{1}{7}H_n}$$
$$0 \le \Omega \le \lim_{n\to\infty} e^{-\frac{1}{7}H_n} = e^{-\infty} = 0$$
$$\Omega = 0$$

2.108. If $a,b,c > 0$ then:

$$\left(\boldsymbol{a}+\frac{\boldsymbol{b}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})}{\boldsymbol{a}}\right)(\boldsymbol{2a}+\boldsymbol{b}+\boldsymbol{c})\left(\boldsymbol{a}+\frac{\boldsymbol{b}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})}{\boldsymbol{c}}\right) \ge (\boldsymbol{a}+\boldsymbol{3b})^{3}$$

Solution:

$$\ln\left(a+\frac{b(a+b+c)}{a}\right)+\ln\left(a+\frac{b(a+b+c)}{b}\right)+\ln\left(a+\frac{b(a+b+c)}{c}\right) \overset{?}{\ge} 3\ln(a+3b)$$

Suppose: $f(x) = \ln\left(a+\frac{b(a+b+c)}{x}\right), x \in]0,+\infty[$

$$f'(x) = \frac{-\frac{b(a+b+c)}{x^2}}{a+\frac{b(a+b+c)}{x}} = \frac{-b(a+b+c)}{ax^2+b(a+b+c)x} =$$
$$= -b(a+b+c)\left[\frac{1}{ax^2+b(a+b+c)x}\right]$$
$$f''(x) = \frac{-b(a+b+c)}{1}\cdot\frac{-2\big(ax+b(a+b+c)\big)}{(ax^2+b(a+b+c)x)^2}$$

$f''(x) = \frac{b(a+b+c)\left(2ax+b(a+b+c)\right)}{(ax^2+b(a+b+c)x)^2} > 0$*, so, f is a convexe function.*

So: $f\left(\frac{x+y+z}{3}\right) \le \frac{f(x)+f(y)+f(z)}{3}$

$$3f\left(\frac{a+b+c}{3}\right) \le f(a)+f(b)+f(c)$$
$$\ln\left(a+\frac{b(a+b+c)}{a}\right)+\ln\left(a+\frac{b(a+b+c)}{a}\right)+\ln\left(a+\frac{b(a+b+c)}{c}\right)$$
$$\ge 3\ln\left(a+\frac{b(a+b+c)}{\frac{(a+b+c)}{3}}\right)$$
$$\ln\left(a+\frac{b(a+b+c)}{a}\right)+\ln\left(a+\frac{b(a+b+c)}{a}\right)+\ln\left(a+\frac{b(a+b+c)}{c}\right)$$
$$\ge 3\ln(a+3b)$$

So, the inequality is true.

2.109. Find:

$$\Omega = \lim_{n\to\infty}\left(\int_0^{\frac{1}{\sqrt{n}}e^{\sum_{k=1}^{n}\left(\frac{1}{2k-1}\right)}}\left(\frac{\sin^2 x+\sin x}{\sin x+\cos x+1}\right)dx+\int_{\frac{1}{\sqrt{n+1}}e^{\sum_{k=1}^{n+1}\left(\frac{1}{2k-1}\right)}}^{\frac{\pi}{2}}\left(\frac{\cos^2 x+\cos x}{\cos x+\sin x+1}\right)dx\right)$$

Solution:

$$\text{Let's compute } I=\int\frac{\sin^2 x+\sin x}{\sin x+\cos x+1}dx \text{ and } J=\int\frac{\cos^2 x+\cos x}{\cos x+\sin x+1}dx \Rightarrow$$

$$\Rightarrow I+J=\int\frac{\sin x+\cos x+1}{\cos x+\sin x+1}dx=\int dx=x$$

$$J-I=\int\frac{(\cos x-\sin x)(\cos x+\sin x)+\cos x-\sin x}{\cos x+\sin x+1}dx=$$

$$=\int\frac{(\cos x-\sin x)(\cos x+\sin x+1)}{\cos x+\sin x+1}dx=$$

$$=\int(\cos x-\sin x)\,dx=\sin x+\cos x$$

$$\text{So, } I+J=x$$

$$J-I=\sin x+\cos x$$

$$-----------\text{“}+\text{”}$$

$$2J=\sin x+\cos x+x\Rightarrow J=\frac{\sin x+\cos x+x}{2}\Rightarrow$$

$$\Rightarrow I=x-\frac{\sin x+\cos x+x}{2}=\frac{x-\sin x-\cos x}{2}\Rightarrow$$

$$\Rightarrow 2\Omega=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\cdot e^{\sum_{k=1}^{n}\frac{1}{2k-1}}-\sin\left(\frac{1}{\sqrt{n}}e^{\sum_{k=1}^{n}\frac{1}{2k-1}}\right)-$$

$$-\cos\left(\frac{1}{\sqrt{n}}e^{\sum_{k=1}^{n}\frac{1}{2k-1}}\right)+1+\frac{\pi}{2}+1-\frac{1}{\sqrt{n+1}}e^{\sum_{k=1}^{n+1}e^{\frac{1}{2k-1}}}-$$

$$-\sin\left(\frac{1}{\sqrt{n+1}}e^{\sum_{k=1}^{n+1}\frac{1}{2k-1}}\right)-\cos\left(\frac{1}{\sqrt{n+1}}e^{\sum_{k=1}^{n+1}\frac{1}{2k-1}}\right)\quad (1)$$

Let's compute

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}e^{\sum_{k=1}^{n}\frac{1}{2k-1}}=\lim_{n\to\infty}\frac{e^{\sum_{k=1}^{n}\frac{1}{2k-1}}}{e^{\ln(\sqrt{n})}}=\lim_{n\to\infty}e^{\sum_{k=1}^{n}\frac{1}{2k-1}-\ln(\sqrt{n})}=$$

$$=\lim_{n\to\infty}e^{H_{2n-1}-\frac{H_{n-1}}{2}-\ln(\sqrt{n})}=$$

$$=\lim_{n\to\infty}e^{H_{2n-1}-\ln(2n-1)+\ln(2n-1)-\frac{H_{n-1}-\ln(n-1)}{2}-\frac{\ln(n-1)}{2}-\ln(\sqrt{n})}=$$

$$= e^{\frac{\gamma}{2}} \cdot \lim_{n\to\infty} e^{\ln\left(\frac{2n-1}{\sqrt{n^2-n}}\right)} = e^{\frac{\gamma}{2}} \cdot \lim_{n\to\infty} e^{\ln\left(\frac{n\left(2-\frac{1}{n}\right)}{n\sqrt{1-\frac{1}{n}}}\right)} = e^{\frac{\gamma}{2}} \cdot 2 \overset{(1)}{\Rightarrow}$$

$$\overset{(1)}{\Rightarrow} \Omega = 1 + \frac{\pi}{4} - \sin\left(2e^{\frac{\gamma}{2}}\right) - \cos\left(2e^{\frac{\gamma}{2}}\right)$$

2.110. If $\boldsymbol{a, b, c \in \left(0, \frac{\pi}{2}\right)}$ then:

$$\frac{\sin a}{\sin a + a\cos a} + \frac{\sin b}{\sin b + b\cos b} + \frac{\sin c}{\sin c + c\cos c} \le$$

$$\le \frac{3\sqrt[3]{\sin a \sin b \sin c}}{\sqrt[3]{\sin a \sin b \sin c} + \sqrt[3]{abc\cos a \cos b \cos c}}$$

Solution:

$$\textit{Considering } f: (-\infty; 0) \to \mathbb{R}, f(x) = \frac{1}{1+e^x}$$

$$f'(x) = \frac{-e^x}{(1+e^x)^2}; f''(x) = \frac{-e^x(1+e^x)^2 + e^x \cdot 2(1+e^x)\cdot e^x}{(1+e^x)^4}$$

$$= \frac{-e^x(1+e^x) + 2e^{2x}}{(1+e^x)^3} = \frac{-e^x - e^{2x} + 2e^{2x}}{(1+e^x)^3} = \frac{e^{2x} - e^x}{(1+e^x)^3}$$

$$= \frac{e^x(e^x-1)}{(1+e^x)^3} < 0, \textit{for } x < 0 \Rightarrow f \textit{ concave}$$

$$\Rightarrow f\left(\frac{x+y+z}{2}\right) \ge \frac{f(x)+f(y)+f(z)}{3}; \forall x, y, z \in (-\infty, 0)$$

$$\textit{For } a, b, c \in \left(0, \frac{\pi}{2}\right); a < \tan a; b < \tan b; c < \tan c$$

$$\Rightarrow \frac{a}{\tan a} < 1; \frac{b}{\tan b} < 1; \frac{c}{\tan c} < 1 \Rightarrow$$

$$x = \ln\frac{a}{\tan a} < a; y = \ln\frac{b}{\tan b} < 0; z = \frac{c}{\tan c} < 0$$

$$\textit{Replacing} \Rightarrow f\left(\frac{\ln\frac{a}{\tan a} + \ln\frac{b}{\tan b} + \ln\frac{c}{\tan c}}{3}\right) \ge \frac{f\left(\ln\frac{a}{\tan a}\right) + f\left(\ln\frac{b}{\tan b}\right) + f\left(\ln\frac{c}{\ln c}\right)}{3}$$

$$3f\left(\ln\sqrt[3]{\frac{abc}{\tan a \tan b \tan c}}\right) \ge f\left(\ln\frac{a}{\tan a}\right) + f\left(\ln\frac{b}{\tan b}\right) + f\left(\ln\frac{c}{\tan c}\right)$$

$$3\frac{1}{1 + e^{\ln\sqrt[3]{\frac{abc}{\tan a \tan b \tan c}}}} \ge \frac{1}{1 + e^{\ln\frac{a}{\tan a}}} + \frac{1}{1 + e^{\ln\frac{b}{\tan b}}} + \frac{1}{1 + e^{\ln\frac{c}{\tan c}}}$$

$$\frac{3}{1+\sqrt[3]{\frac{abc}{\tan a\tan b\tan c}}}\geq\frac{1}{1+\frac{a}{\tan a}}+\frac{1}{1+\frac{b}{\tan b}}+\frac{1}{1+\frac{c}{\tan c}}$$

$$\frac{3\sqrt[3]{\sin a\sin b\sin c}}{\sqrt[3]{\sin a\sin b\sin c}+\sqrt[3]{abc\cdot\cos a\cdot\cos b\cdot\cos c}}\geq$$

$$\geq\frac{\sin a}{\sin a+a\cos a}+\frac{\sin b}{\sin b+b\cos b}+\frac{\sin c}{\sin c+c\cos c}$$

2.111. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\frac{1}{n^8}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{l=1}^{k}(ijkl)\right)}$$

Solution:

$$Let\ be\ x_k=k;k\geq 1$$

$$\sum_{k=1}^{n}x_k=\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k\right)^4=\lim_{n\to\infty}\frac{n^4(n+1)^4}{16n^8}=\frac{1}{16}$$

$$\sum_{k=1}^{n}x_k^2=\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k\right)^2\cdot\left(\sum_{k=1}^{n}x_k^2\right)=$$

$$=\lim_{n\to\infty}\frac{1}{n^8}\cdot\left(\frac{n(n+1)}{2}\right)^2\cdot\frac{n(n+1)(2n+1)}{6}=\frac{1}{12}$$

$$\sum_{k=1}^{n}x_k^3=\sum_{k=1}^{n}k^3=\left(\frac{n(n+1)}{2}\right)^2=\frac{n^2(n+1)^2}{4}$$

$$\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}x_k^3\right)=\lim_{n\to\infty}\frac{1}{n^8}\cdot\frac{n^3(n+1)^3}{8}=0$$

$$\sum_{k=1}^{n}x_k^4=\sum_{k=1}^{n}k^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k^2\right)^2=\lim_{n\to\infty}\frac{1}{n^8}\cdot\frac{n^4(n+1)^4}{16}=\frac{1}{16}$$

$$\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k^4\right)=\lim_{n\to\infty}\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30n^8}=0$$

We will use the identity:

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}x_ix_jx_kx_l=$$

$$=\frac{1}{24}\left(\left(\sum_{k=1}^{n}x_k\right)^4+6\left(\sum_{k=1}^{n}x_k\right)^2+8\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}x_k^3\right)+3\left(\sum_{k=1}^{n}x_k^2\right)^2+6\sum_{k=1}^{n}x_k^4\right)$$

$$\Omega=\frac{1}{24}\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k\right)^4+\frac{1}{4}\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k\right)^2\cdot\left(\sum_{k=1}^{n}x_k^2\right)+$$

$$+\frac{1}{3}\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}x_k^3\right)+\frac{1}{8}\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k^2\right)^2+$$

$$+\frac{1}{4}\lim_{n\to\infty}\frac{1}{n^8}\left(\sum_{k=1}^{n}x_k^4\right)$$

$$\Omega=\frac{1}{24}\cdot\frac{1}{16}+\frac{1}{4}\cdot\frac{1}{12}+\frac{1}{3}\cdot 0+\frac{1}{8}\cdot\frac{1}{16}+\frac{1}{4}\cdot 0$$

$$\Omega=\frac{1}{24\cdot 16}+\frac{1}{4\cdot 12}+\frac{1}{8\cdot 16}=\frac{1}{16}\left(\frac{1}{24}+\frac{1}{3}+\frac{1}{8}\right)$$

$$\Omega=\frac{1}{16}\cdot\frac{1+8+3}{24}=\frac{1}{16}\cdot\frac{1}{2}=\frac{1}{32}$$

2.112. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}\left(\frac{1}{2^{i+j+k+l}}\right)\right)}$$

Solution:

Let be $x_k=\frac{1}{2^k};k\geq 1$

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{\frac{1}{2}\left(\frac{1}{2^n} - 1\right)}{\frac{1}{2} - 1} = 1 - \frac{1}{2^n}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$$

$$\sum_{k=1}^{n} x_k^2 = \sum_{k=1}^{n} \frac{1}{2^{2k}} = \frac{\frac{1}{4}\left(\frac{1}{2^{2n}} - 1\right)}{\frac{1}{4} - 1} = \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k^2 = \lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right) = \frac{1}{3}$$

$$\sum_{k=1}^{n} x_k^3 = \sum_{k=1}^{n} \frac{1}{2^{3k}} = \frac{\frac{1}{8}\left(\frac{1}{2^{3n}} - 1\right)}{\frac{1}{8} - 1} = \frac{1}{7}\left(1 - \frac{1}{2^{3n}}\right)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k^3 = \lim_{n\to\infty} \frac{1}{7}\left(1 - \frac{1}{2^{3n}}\right) = \frac{1}{7}$$

$$\sum_{k=1}^{n} x_k^4 = \sum_{k=1}^{n} \frac{1}{2^{4k}} = \frac{\frac{1}{16}\left(\frac{1}{2^{4n}} - 1\right)}{\frac{1}{16} - 1} = \frac{1}{15}\left(1 - \frac{1}{2^{4n}}\right)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k^4 = \lim_{n\to\infty} \frac{1}{15}\left(1 - \frac{1}{2^{4n}}\right) = \frac{1}{15}$$

We will use the identity:

$$\sum_{i=1}^{n} \sum_{j=1}^{j} \sum_{k=1}^{j} \sum_{l=1}^{k} x_i x_j x_k x_l =$$

$$= \frac{1}{24}\left(\begin{array}{c} \left(\sum_{k=1}^{n} x_k\right)^4 + 6\left(\sum_{k=1}^{n} x_k\right)^2 \cdot \left(\sum_{k=1}^{n} x_k^2\right) + \\ +8\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^3\right) + 3\left(\sum_{k=1}^{n} x_k^2\right)^2 + 6\sum_{k=1}^{n} x_k^4 \end{array}\right)$$

$$\Omega = \frac{1}{24} \lim_{n\to\infty} \left(\sum_{k=1}^{n} x_k\right)^4 + \frac{1}{4} \lim_{n\to\infty} \left(\sum_{k=1}^{n} x_k\right)^2 \cdot \left(\sum_{k=1}^{n} x_k^2\right) +$$

$$+\frac{1}{3}\lim_{n\to\infty}\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}x_k^3\right)+\frac{1}{8}\lim_{n\to\infty}\left(\sum_{k=1}^{n}x_k^2\right)^2+\frac{1}{4}\lim_{n\to\infty}\left(\sum_{k=1}^{n}x_k^4\right)$$

$$\Omega=\frac{1}{24}\cdot 1^4+\frac{1}{4}\cdot 1^2\cdot\frac{1}{3}+\frac{1}{3}\cdot 1\cdot\frac{1}{7}+\frac{1}{8}\cdot\frac{1}{3^2}+\frac{1}{4}\cdot\frac{1}{15}$$

$$\Omega=\frac{1}{24}+\frac{1}{12}+\frac{1}{21}+\frac{1}{71}+\frac{1}{60}$$

$$\Omega=\frac{105+420+120+35+42}{2520}=\frac{717}{2520}=\frac{239}{840}$$

2.113. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\binom{n}{3}\cdot\lim_{m\to\infty}\left(\frac{1}{m}\sum_{k=0}^{m}\left(\frac{\binom{m}{k}}{\binom{m+n}{k+3}}\right)\right)\right)}$$

Solution:

$$Let\ \omega=\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{m+n}{k+3}}=\sum_{k=0}^{m}\binom{m}{n}\frac{(k+3)!(m+n-k-3)!}{(m+n)!}=$$

$$=(n+m+1)\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(k+4)\Gamma(m+n-k-2)}{\Gamma(m+n+2)}$$

$$=(n+m+1)\sum_{k=0}^{m}\binom{m}{k}\int_0^1 x^{k+3}\,(1-x)^{m+n-k-3}dx=$$

$$=(m+n+1)\int_0^1 x^3(1-x)^{n-3}\sum_{k=0}^{m}\binom{m}{k}x^k(1-x)^{m-k}dx=$$

$$=(n+m+1)\int_0^1 x^3(1-x)^{n-3}\,dx$$

$$=\frac{(n+m+1)\Gamma(4)\Gamma(n-2)}{\Gamma(n+2)}=\frac{6(m+n+1)}{(n-1)n(n+1)(n-2)}\Rightarrow$$

$$\Rightarrow\lim_{m\to\infty}\frac{\omega}{m}=\lim_{m\to\infty}\frac{6(m+n+1)}{m(n-1)n(n+1)(n-2)}=$$

$$=\frac{6}{(n-1)n(n+1)(n-2)}$$

Now:

$$\Omega = \lim_{n\to\infty}\left(\binom{n}{3}\lim_{m\to\infty}\frac{\omega}{m}\right) = \lim_{n\to\infty}\frac{n!}{6(n-3)!}\cdot\frac{6}{(n-1)n(n+1)(n-2)} = 0$$

2.114. If $0 \le a \le b < \frac{\pi}{2}$ then:

$$\frac{1}{\cos a}+\frac{1}{\cos b}\ge\frac{1}{\cos\left(a-\sqrt{ab}+b\right)}+\frac{1}{\cos\left(\sqrt{ab}\right)}$$

Solution:

The result is obvious if $a = 0$ or if $a = b$, so, we'll assume that

$$0 < a < b < \frac{\pi}{2}.$$

The function $f(x) = \frac{1}{\cos x}$ is convex on $\left[0, \frac{\pi}{2}\right)$: $f''(x) = \frac{(2-\cos^2 x)}{\cos^3 x} > 0$

Let $x_1, x_2, x_3, x_4 \in \left[0, \frac{\pi}{2}\right)$ be such that $x_1 < x_2 < x_3 < x_4$. By the definition of 'convex function' (one whose derivative is increasing), we have: $\frac{f(x_4)-f(x_3)}{x_4-x_3} \ge \frac{f(x_2)-f(x_1)}{x_2-x_1}$

If we choose our points so that $x_4 - x_3 = x_2 - x_1 > 0$, we can just write $f(x_4) - f(x_3) \ge f(x_2) - f(x_1)$ (1)

Choosing $x_1 = a, x_2 = \sqrt{ab}, x_3 = a + \left(b - \sqrt{ab}\right)$, and $x_4 = b$, we have $x_4 - x_3 = \sqrt{ab} - a = x_2 - x_1 > 0$, and (1) becomes

$$\frac{1}{\cos b}-\frac{1}{\cos\left(a-\sqrt{ab}+b\right)}\ge\frac{1}{\cos\left(\sqrt{ab}\right)}-\frac{1}{\cos a}$$

or $\frac{1}{\cos a}+\frac{1}{\cos b}\ge\frac{1}{\cos\left(a-\sqrt{ab}+b\right)}+\frac{1}{\cos\sqrt{ab}}$

Note: $a < \sqrt{ab} < b \Rightarrow b - \sqrt{ab} > 0$

2.115. Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k}\frac{1}{ijkl(i+1)(j+1)(k+1)(l+1)}\right)$$

Solution:

Let $M = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{ijkl(i+1)(j+1)(k+1)(l+1)}$

$$N = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{ij(i+1)(j+1)(k+1)^2k^2}$$

$$P=\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{i^2(i+1)^2(k+1)^2k^2};S=\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{i(i+1)(k+1)^3k^3};$$

$$T=\sum_{i=1}^{\infty}\frac{1}{i^4(i+1)^4}$$

We will prove that: $\Omega=\frac{M+6N+3P+8S+12P}{24}$

In first note that there are $4!$ *of permutations of* $\frac{1}{i(1+i)}\cdot\frac{1}{j(1+j)}\cdot\frac{1}{k(1+k)}\cdot\frac{1}{l(1+l)}$ *let's discuss the 04 cases:*

Case 01: if $l<k<j<i$ *all 4! combinations are then contained with in* M

Case 02: if $l=k$ *and* i,j *are distinct we need 6 of* N *because some possibilities are contained in* M

Case 03: if $i=j\neq l=k$ *we need 3 of* P *because some possibilities are contained in* M *and* N

Case 03: if $ii\neq j=k=l$ *we need 8 of* S *because some possibilities are contained in* M,N *and* P

Case 03: $i=j=k=l$ *this appears exactly once in every sum, which thus us 12 copies of* T

Since: $A=\sum_{i=1}^{\infty}\frac{1}{i(1+i)}=1\Rightarrow M=1$

$$\therefore\sum_{i=1}^{\infty}\frac{1}{i^2(1+i)^2}=\sum_{i=1}^{\infty}\frac{1}{i^2}+\sum_{i=1}^{\infty}\frac{1}{(i+1)^2}-2A=2\zeta(2)-3\Rightarrow$$

$$\Rightarrow\begin{matrix}N=2\zeta(2)-3\\P=(2\zeta(2)-3)^2\end{matrix}$$

$$\therefore\sum_{i=1}^{\infty}\frac{1}{i^3(1+i)^3}=\sum_{i=1}^{\infty}\left(\frac{1}{i^2}-\frac{1}{(1+i)^3}\right)-3\sum_{i=1}^{\infty}\left(\frac{1}{i^2}+\frac{1}{(1+i)^2}\right)+6A=$$

$$=10-\pi^3\Rightarrow S=10-\pi^2\therefore T=\sum_{i=1}^{\infty}\frac{1}{i^4(1+i)^4}=$$

$$=\sum_{i=1}^{\infty}\left(\frac{1}{i^4}+\frac{1}{(1+i)^4}\right)-4\sum_{i=1}^{\infty}\left(\frac{1}{i^2}-\frac{1}{(1+i)^3}\right)+$$

$$+10\sum_{i=1}^{\infty}\left(\frac{1}{i^2}+\frac{1}{(1+i)^2}\right)-20A=$$

$$=2\zeta(A)+20\zeta(2)-35\therefore\Omega=\frac{9\zeta(4)+28\zeta(2)-55}{4}$$

GEOMETRY

3.1. In ΔABC the following relationship holds:

$$\frac{(a^2-bc\cos A)^4}{\left(\sin^{-1}\left(\frac{4}{5}\right)\right)^3}+\frac{(b^2-ac\cos B)^2}{\left(\sin^{-1}\left(\frac{5}{13}\right)\right)^3}+\frac{(c^2-ab\cos C)^4}{\left(\sin^{-1}\left(\frac{16}{15}\right)\right)^3}>\frac{3888r^8}{\pi^3}$$

Solution:

$$\textit{Let } \sin^{-1}\left(\frac{4}{5}\right)=\theta, \sin^{-1}\left(\frac{5}{13}\right)=\phi, \sin^{-1}\left(\frac{16}{65}\right)=\omega, \textit{ of course}$$

$$0<\theta,\phi,\omega<\frac{\pi}{2}$$

$$\sin\theta=\frac{4}{5}\Rightarrow \csc^2\theta-1=\frac{25}{16}-1=\frac{9}{16}\Rightarrow \cot\theta=\frac{3}{4}\Rightarrow \tan\theta\overset{(1)}{=}\frac{4}{3}$$

$$\textit{Again, } \sin\phi=\frac{5}{13}\Rightarrow \csc^2\phi-1=\frac{169}{25}-1=\frac{144}{25}\Rightarrow$$

$$\Rightarrow \cot\phi=\frac{12}{5}\Rightarrow \tan\phi\overset{(2)}{=}\frac{5}{12}$$

$$\textit{Also, } \sin\omega=\frac{16}{65}\Rightarrow \csc^2\omega-1=\frac{65^2-16^2}{16^2}=\left(\frac{63}{16}\right)^2\Rightarrow$$

$$\Rightarrow \cot\omega=\frac{63}{16}\Rightarrow \tan\omega\overset{(3)}{=}\frac{16}{63}$$

$$\textit{(1), (2), (3)}\Rightarrow 1-\tan\theta\tan\phi-\tan\phi\tan\omega-\tan\omega\cdot\tan\theta$$

$$=1-\left(\frac{4}{3}\right)\left(\frac{5}{12}\right)-\left(\frac{5}{12}\right)\left(\frac{16}{63}\right)-\left(\frac{16}{63}\right)\left(\frac{4}{3}\right)=1-\frac{5}{9}-\frac{20}{64\times 3}-\frac{64}{63\times 3}$$

$$=\frac{189-105-20-64}{63\times 3}=0 \textit{ and } \because 0<\theta+\phi+\omega<\frac{3\pi}{2}\therefore \theta+\phi+\omega\overset{(4)}{=}\frac{\pi}{2}$$

$$\textit{Now, LHS}=\frac{|c^2-ab\cos A|^4}{\theta^3}+\frac{|b^2-ca\cos B|^4}{\phi^3}+\frac{|c^2-ab\cos C|^4}{\omega^3}$$

$$\overset{Radon}{>}\frac{(|a^2-bc\cos A|+|b^2-ca\cos B|+|c^2-abc\cos C|)^4}{(\theta+\phi+\omega)^3}$$

$$\geq\frac{(1\sum a^2-\sum bc\cos A)^4}{\left(\frac{\pi}{2}\right)^3}=\frac{\left(\left|\sum a^2-\sum\frac{b^2+c^2-a^2}{2}\right|\right)^4}{\left(\frac{\pi}{2}\right)^3}$$

$$=\frac{(\sum a^2)^4}{16}\cdot\frac{8}{\pi^3}\overset{Ionescu-Weitzenbock}{\geq}\frac{\left(4\sqrt{3}rs\right)^4}{2\pi^3}$$

$$\overset{Mitrinovic}{\geq} \frac{\left(4\sqrt{3}r\cdot 3\sqrt{3}r\right)^4}{2\pi^3} = \frac{(36)^4 r^8}{2\pi^3} = \frac{839808 r^8}{\pi^3} = 216\left(\frac{3888r^8}{\pi^3}\right) > \frac{3888r^8}{\pi^3}$$

3.2. If $0 < x, y < \frac{\pi}{2}$ then:

$$\left(\cos^2 x \cdot \cos^2 y \cdot (\tan x + \tan y)\right)^{\tan x + \tan y} \leq \left(\cos^2 x\right)^{\tan y} \cdot \left(\cos^2 y\right)^{\tan x}$$

Solution:

$$\sqrt[(\tan y + \tan x)]{(\cos^2 x)^{\tan y} \cdot (\cos^2 y)^{\tan x}}$$

$$\overset{\substack{weighted\ GM \\ \geq \\ weighted\ HM}}{\geq} \frac{\tan y + \tan x}{\frac{\tan y}{\cos^2 x} + \frac{\tan x}{\cos^2 y}} = \frac{(\tan x + \tan y)(\cos^2 x + \cos^2 y)}{\tan y \cos^2 y + \tan x \cos^2 x}$$

$$= \frac{2\cos^2 x \cos^2 y (\tan x + \tan y)}{\sin 2x + \sin 2y} \geq \frac{2\cos^2 x \cos^2 y\, (\tan x + \tan y)}{2}$$

$$(\because 0 < \sin 2x, \sin 2y \leq 1 \ as\ 0 < 2x, 2y < \pi)$$

$$\Rightarrow \left(\cos^2 x \cos^2 y\, (\tan x + \tan y)\right)^{\tan x + \tan y} \leq (\cos^2 x)^{\tan y} (\cos^2 y)^{\tan x}$$

(Proved)

3.3. In ΔABC; AA', BB', CC' - internal bisectors, $A' \in (BC)$; $B' \in (CA)$; $C' \in (AB)$. Prove that:

$A'B' + B'C' + C'A' \geq 2F \sum_{cyc} \frac{\sec\frac{A}{2}}{\sqrt{(a+b)(a+c)}}$; F – area.

Solution:

$$B'C'^2 = AC'^2 + AB'^2 - 2AC' \cdot AB' \cdot \cos A =$$

$$= AC'^2 + AB'^2 - 2AC'AB' + 2AC'AB' - 2AC'AB' \cos A =$$

$$= (AC' - AB')^2 + 2AC'AB'(1 - \cos A) \geq$$

$$\geq 2AC'AB'\left(1 - 1 + 2\sin^2\frac{A}{2}\right) = 4AC'AB' \sin^2\frac{A}{2}$$

$$B'C' \geq 2\sin\frac{A}{2}\sqrt{AC'AB'} = 2\sin\frac{A}{2}\sqrt{\frac{bc}{a+b} \cdot \frac{bc}{a+c}}$$

$$B'C' \geq 2bc\sin\frac{A}{2} \cdot \frac{1}{\sqrt{(a+b)(a+c)}} =$$

$$= \frac{2bc\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}\sqrt{(a+b)(a+c)}} = \frac{bc\sin A}{\cos\frac{A}{2}\sqrt{(a+b)(a+c)}} =$$

$$= \frac{2F}{\cos\frac{A}{2} \cdot \sqrt{(a+b)(a+c)}} = \frac{2F\sec\frac{A}{2}}{\sqrt{(a+b)(a+c)}}$$

$$A'B' + B'C' + C'A' \geq 2F\sum_{cyc}\frac{\sec\frac{A}{2}}{\sqrt{(a+b)(a+c)}}$$

3.4. In ΔABC the following relationship holds:

$$\boldsymbol{\frac{a^a \cdot b^b \cdot c^c}{r_a^a \cdot r_b^b \cdot r_c^c} \geq \left(\frac{4s}{9R}\right)^{2s}}$$

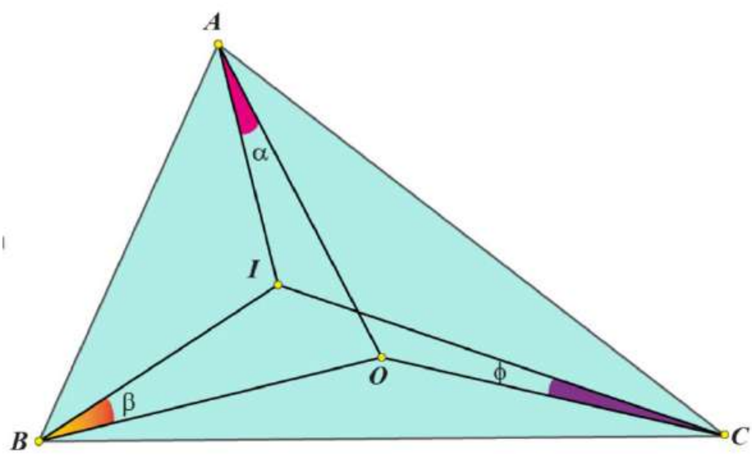
Solution:

$$\prod\left(\frac{a}{r_a}\right)^a \geq \left(\frac{4s}{9R}\right)^{2s}. \textit{Weighted GM} \geq \textit{weighted HM} \Rightarrow$$

$$\Rightarrow \sqrt[a+b+c]{\prod\left(\frac{a}{r_a}\right)^a} \geq \frac{2s}{\sum\left(\frac{a}{\left(\frac{a}{r_a}\right)}\right)} = \frac{2s}{4R+r} \overset{Euler}{\geq} \frac{2s}{4R+\frac{R}{2}} = \frac{4s}{9R}$$

$$\Rightarrow \ln\left(\sqrt[a+b+c]{\prod\left(\frac{a}{r_a}\right)^a}\right) \geq \ln\left(\frac{4s}{9R}\right) \Rightarrow 2s\ln\left(\sqrt[a+b+c]{\prod\left(\frac{a}{r_a}\right)^a}\right) \geq 2s\ln\left(\frac{4s}{9R}\right)$$

$$\Rightarrow \ln\left(\left(\sqrt[a+b+c]{\prod\left(\frac{a}{r_a}\right)^a}\right)^{a+b+c}\right) \geq \ln\left(\frac{4s}{9R}\right)^{2s} \Rightarrow \prod\left(\frac{a}{r_a}\right)^a \geq \left(\frac{4s}{9R}\right)^{2s}$$



3.5.

I – incenter of ABC, O – circumcenter, R, r circumradius and inradius

Prove: $\cos^2\alpha+\cos^2\beta+\cos^2\phi\ge\frac{4r}{R}+1$

Abdilkadir Altintas

Solution:

$$\sum_{cyc}\cos^2\alpha=\sum_{cyc}\left(\frac{AI^2+OA^2-OI^2}{2\cdot AI\cdot OA}\right)^2=$$

$$=\frac{1}{4R^2}\sum_{cyc}\left(\frac{\frac{r^2}{\sin^2\frac{A}{2}}+R^2-R^2+2Rr}{\frac{r}{\sin\frac{A}{2}}}\right)^2=$$

$$=\frac{1}{4R^2}\sum_{cyc}\left(\sin\frac{A}{2}\left(\frac{r}{\sin^2\frac{A}{2}}+2R\right)\right)^2=\frac{1}{4R^2}\sum_{cyc}\left(\frac{r}{\sin\frac{A}{2}}+2R\sin\frac{A}{2}\right)^2=$$

$$=\frac{1}{4R^2}\left(r^2\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}}+12Rr+4R^2\sum_{cyc}\sin^2\frac{A}{2}\right)=$$

$$=\frac{1}{4R^2}\left(r^2\cdot\frac{s^2+r^2-8Rr}{r^2}+12Rr+4R^2\left(1-\frac{r}{2R}\right)\right)=$$

$$=\frac{1}{4R^2}(s^2+r^2-8Rr+12Rr+4R^2-2Rr)=$$

$$=\frac{1}{4R^2}(s^2+2Rr+r^2+4R^2)\ge$$

$$\overset{GERRETSEN}{\ge}\frac{1}{4R^2}(16Rr-5R^2+2Rr+r^2+4R^2)=$$

$$=1+\frac{18Rr-4r^2}{4R^2}=1+\frac{9Rr-2r^2}{2R^2}\ge$$

$$\overset{EULER}{\ge}1+\frac{9Rr-2r\cdot\frac{R}{2}}{2R^2}=1+\frac{9Rr-Rr}{2R^2}=1+\frac{4r}{R}$$

3.6. In ΔABC, n_a, n_b, n_c are Nagel's cevians. Prove that:

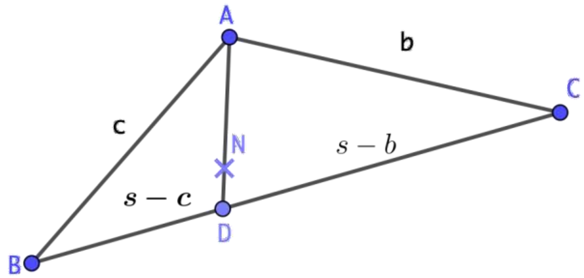
$$n_an_bn_c\ge r_ar_br_c$$

Solution:

Lemma 1

$$In\ \Delta ABC: n_a\ge m_a\quad (1)$$

Proof:



$$\text{Let } AD \text{ be the Nagel's cevian of } A; AD = n_a$$

By Stewart's theorem in ΔABC:

$$a \cdot n_a^2 = c^2(s-b) + b^2(s-c) - a(s-b)(s-c)$$

$$n_a^2 = \frac{c^2(s-b) + b^2(s-c)}{a} - (s-b)(s-c)$$

$$n_a \geq m_a \Leftrightarrow n_a^2 \geq m_a^2$$

$$\frac{c^2(s-b) + b^2(s-c)}{a} - (s-b)(s-c) \geq \frac{2(b^2+c^2) - a^2}{4}$$

$$\frac{c^2(a+c-b) + b^2(a+b-c)}{2a} \geq$$

$$\geq \frac{(a+b-c)(a+c-b) + 2(b^2+c^2) - a^2}{4}$$

$$\frac{c^2(a+c-b) + b^2(a+b-c)}{2a} \geq \frac{b^2+c^2+2bc}{4}$$

$$2(c^2a + c^3 - bc^2 + b^2a + b^3 - b^2c) \geq a(b^2+c^2+2bc)$$

$$2c^2a + 2c^3 - 2bc^2 + 2b^2a + 2b^3 - 2b^2c - ab^2 - ac^2 - 2abc \geq 0$$

$$ab^2 + ac^2 + 2c^3 + 2b^3 - 2bc^2 - 2b^2c - 2abc \geq 0$$

$$2c^2(c-b) - 2b^2(c-b) + ac(c-b) - ab(c-b) \geq 0$$

$$(c-b)[2c^2 - 2b^2 + a(c-b)] \geq 0$$

$$(c-b)^2(2c+2b+a) \geq 0 \text{ which is true.}$$

Lemma 2.

$$\text{In } \Delta ABC: m_a \geq \sqrt{s(s-a)}$$

Proof:

$$m_a \geq \sqrt{s(s-a)} \Leftrightarrow m_a^2 \geq s(s-a)$$

$$\frac{2(b^2+c^2) - a^2}{4} \geq \frac{(a+b+c)(b+c-a)}{4},$$

$$2b^2 + 2c^2 - a^2 \geq (b+c)^2 - a^2, b^2 + c^2 - 2bc \geq 0 \Leftrightarrow (b-c)^2 \geq 0$$

Back to the problem:

$$n_a n_b n_c \overset{Lemma\ 1}{\geq} m_a m_b m_c \overset{Lemma\ 2}{\geq}$$

$$\geq \sqrt{s(s-a)} \cdot \sqrt{s(s-b)} \cdot \sqrt{s(s-c)} =$$

$$= s\sqrt{s(s-b)(s-c)(s-a)} = sS =$$

$$= \frac{sS^3}{s^2} = \frac{s}{s(s-a)(s-b)(s-c)} \cdot S^3 =$$
$$= \frac{S^3}{(s-a)(s-b)(s-c)} = \frac{S}{s-a} \cdot \frac{S}{s-b} \cdot \frac{S}{s-c} = r_a r_b r_c$$

3.7. In ΔABC the following relationship holds:

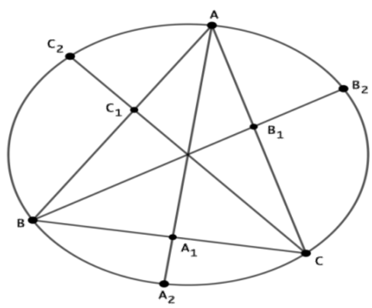
$$\boldsymbol{b\left(\frac{a}{b}\right)^{\frac{2r}{R}} + c\left(\frac{b}{c}\right)^{\frac{2r}{R}} + a\left(\frac{c}{a}\right)^{\frac{2r}{R}} \leq a + b + c}$$

Solution:

$$Let\ f(x) = x^{\alpha}, (x > 0, 0 < \alpha \leq 1) \rightarrow$$
$$\rightarrow f''(x) = \alpha(\alpha - 1)x^{\alpha-2} \leq 0, (x > 0, 0 < \alpha \leq 1)$$

We have: $0 < \frac{2r}{R} \leq 1$. *Using Jensen's inequality:*

$$\frac{b}{2s}\left(\frac{a}{b}\right)^{\frac{2r}{R}} + \frac{c}{2s}\left(\frac{b}{c}\right)^{\frac{2r}{R}} + \frac{a}{2s}\left(\frac{c}{a}\right)^{\frac{2r}{R}} \leq$$
$$\leq \left(\frac{b}{2s} \cdot \frac{a}{b} + \frac{c}{2s} \cdot \frac{b}{c} + \frac{a}{2s} \cdot \frac{c}{a}\right)^{\frac{2r}{R}} = \left(\frac{a+b+c}{2s}\right)^{\frac{2r}{R}} = 1$$
$$\rightarrow b\left(\frac{a}{b}\right)^{\frac{2r}{R}} + c\left(\frac{b}{c}\right)^{\frac{2r}{R}} + a\left(\frac{c}{a}\right)^{\frac{2r}{R}} \leq 2s = a + b + c \ \ (proved)$$



3.8.

$$\boldsymbol{AA_1 = m_a, BB_1 = m_b, CC_1 = m_c \Rightarrow}$$
$$\boldsymbol{AA_2 \cdot BB_2 \cdot CC_2 \cdot (AA_2 + BB_2 + CC_2) \geq 3abc\sqrt[3]{abc}}$$

Solution:

$$A_1A_2 \cdot A_1A = A_1B \cdot A_1C \ (the\ power\ of\ the\ point\ A\ towards\ the\ circle)$$

$$\left.\begin{array}{l} A_1A_2 \cdot A_1A = \frac{a}{2}\cdot\frac{a}{2} \Rightarrow AA_1 \cdot A_1A_2 = \frac{a^2}{4} \\ A_1A \cdot A_1A_2 \overset{MG<MA}{\leq} \left(\frac{A_1A + A_1A_2}{2}\right)^2 = \frac{AA_2^2}{4} \end{array}\right\} \Rightarrow \frac{a^2}{4} \leq \frac{AA_2^2}{4} \Rightarrow AA_2 \geq a$$

$$\text{Similarly: } BB_2 \geq b; CC_2 \geq c$$

$$\text{So: } AA_2 \cdot BB_2 \cdot CC_2(AA_2 + BB_2 + CC_2) \geq abc(a+b+c) \overset{MA\geq MG}{\geq}$$

$$\geq abc \cdot 3\sqrt[3]{abc} = 3abc\sqrt[3]{abc}$$

3.9. If in $\Delta ABC, \Delta A'B'C', s \leq s'$ then:

$$\left(\frac{a}{a'}\right)^{a'} \cdot \left(\frac{b}{b'}\right)^{b'} \cdot \left(\frac{c}{c'}\right)^{c'} \leq 1$$

Solution:

$$\Delta ABC \to s, \Delta A'B'C' \to s', s < s' \Rightarrow \frac{s}{s'} < 1$$

$$\left(\frac{a}{a'}\right)^{a'} \cdot \left(\frac{b}{b'}\right)^{b'} \cdot \left(\frac{c}{c'}\right)^{c'} \leq 1$$

$$\left(\frac{a}{a'}\right)^{a'} \cdot \left(\frac{b}{b'}\right)^{b'} \cdot \left(\frac{c}{c'}\right)^{c'} \overset{MG<MA}{\leq} \left(\frac{a' \cdot \frac{a}{a'} + b' \cdot \frac{b}{b'} + c' \cdot \frac{c}{c'}}{a' + b' + c'}\right)^{a'+b'+c'} =$$

$$= \left(\frac{a+b+c}{a'+b'+c'}\right)^{a'+b'+c'} = \left(\frac{2s}{2s'}\right)^{2s'} = \left(\frac{s}{s'}\right)^{2s'} < 1 \text{ because } \frac{s}{s'} < 1$$

3.10. If a, b, c, d – sides in a bicentric quadrilateral, s – semiperimeter, e, f – diagonals, R – circumradii then:

$$\frac{2(ab + ac + ad + bc + bd + cd)}{s^3(s^2 + ef)} \leq \frac{efR^2}{abcd(abc + abd + bcd + cda)}$$

Solution:

$$LHS = \frac{2[(ab + ad) + (bc + cd) + (ac + bd)]}{s^3(s^2 + ef)} \overset{Ptolemy}{=} \frac{2\{a(b+d) + c(b+d) + ef\}}{s^3(s^2 + ef)}$$

$$= \frac{2\{(b+d)(a+c) + ef\}}{s^3(s^2 + ef)} = \frac{2(s \cdot s + ef)}{s^3(s^2 + ef)} \overset{(1)}{=} \frac{2}{s^3}$$

$$\text{Also, } RHS = \frac{afR^2}{abcd\{ac(b+d) + bd(a+c)\}} =$$

$$= \frac{afR^2}{abcd(ac \cdot s + bds)} \overset{Ptolemy}{=} \frac{efR^2}{sabcdef} \overset{(2)}{=} \frac{R^2}{sabcd}$$

(1), (2) $\Rightarrow$ *given inequality* $\Leftrightarrow \frac{2}{s^3} \le \frac{R^2}{sabcd} \Leftrightarrow s^2R^2 \ge 2abcd \Leftrightarrow$

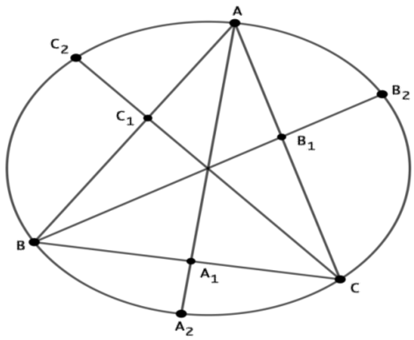
$$\Leftrightarrow 2sR \ge 2\sqrt{2abcd}$$

Now, $2s = \sum a \overset{A-G}{\ge} 4\sqrt[4]{abcd} \Rightarrow 2sR \ge 4R\sqrt[4]{abcd} \overset{?}{\ge} 2\sqrt{2abcd}$

$$\Leftrightarrow 4R^2\sqrt{abcd} \overset{?}{\ge} 2abcd \Leftrightarrow 2R^2 \underset{(3)}{\overset{?}{\ge}} \sqrt{abcd}$$

Now, Parameshvara $\Rightarrow 2R^2 = \frac{2(ab+cd)(ac+bd)(ad+bc)}{16abcd}$

$\overset{A-G}{\ge} \frac{2\sqrt{abcd}\cdot 2\sqrt{acbd}\cdot 2\sqrt{adbc}}{8abcd} = \sqrt{abcd} \Rightarrow$ *(3) is true (Proved)*



3.11.

If $AA_1 \cap BB_1 \cap CC_1 = \{0\}$ – circumcentre then:

$$AA_2 + BB_2 + CC_2 \ge 8\sqrt[3]{\prod_{cyc}\left(\frac{a\sin 2A}{\sin 2B + \sin 2C}\right)}$$

Solution:

Because: $AA_1 \cap BB_1 \cap CC_1 = \{0\} \rightarrow$

$$\rightarrow AA_2 = BB_2 = CC_2 = 2OA = 2OB = 2OC = 2R$$

We must show that: $2R + 2R + 2R \ge 8\sqrt[3]{\prod_{cyc}\left(\frac{a\sin 2A}{\sin 2B+\sin 2C}\right)}$

$$\leftrightarrow 6R \ge 8\sqrt[3]{\prod_{cyc}\left(\frac{a\sin 2A}{\sin 2B + \sin 2C}\right)}$$

$$\Omega = \prod_{cyc}\left(\frac{a\sin 2A}{\sin 2B + \sin 2C}\right) =$$

$$= s\frac{a\sin 2A \cdot b\sin 2B \cdot c\sin 2C}{(\sin 2B + \sin 2C)(\sin 2C + \sin 2A)(\sin 2A + \sin 2B)} =$$

$$abc\left(\frac{\sin 2A \cdot \sin 2B \cdot \sin 2C}{(\sin 2B + \sin 2C)(\sin 2C + \sin 2A)(\sin 2A + \sin 2B)}\right) \le$$

$$abc \cdot \frac{(\sin 2B + \sin 2C)(\sin 2C + \sin 2A)(\sin 2A + \sin 2B)}{8} \cdot$$

$$\cdot \frac{1}{(\sin 2B + \sin 2C)(\sin 2C + \sin 2A)(\sin 2A + 2B)} =$$

$$\frac{8abc}{8} = \frac{4Rrs}{8} = \frac{Rrs}{2}$$

(Because: $(x + y(y + z)(z + x) \geq 8xyz$*; with* $x = \sin A\,; y = \sin 2B\,; z = \sin 2C$,

$x, y, x > 0, A, B, C$ *– acute). Now, we need to prove:* $8\sqrt[3]{\frac{Rrs}{2}} \leq 6R$*. It is true because:*

$$\begin{cases} R \geq 2r \\ s \leq \frac{3\sqrt{3}R}{2} \end{cases} \rightarrow \begin{cases} r \leq \frac{R}{2} \\ s \leq \frac{3\sqrt{3}R}{2} \end{cases} \rightarrow 8\sqrt[3]{\frac{Rrs}{2}} \leq 8\sqrt[3]{\frac{R \cdot \frac{R}{2} \cdot \frac{3\sqrt{3}R}{2}}{2}} = R \cdot \sqrt{3} \leq 6R \text{ (true)}$$

3.12. In ΔABC the following relationship holds:

$$\sum_{cyc} \left(c\sqrt{((b-c)^2 + 4r^2)((a-c)^2 + 4r^2)} \right) \geq 24\sqrt{3}r^3$$

Solution:

$$\textit{We have: } ((b-c)^2 + 4r^2)((a-c)^2 + 4r^2) \geq$$

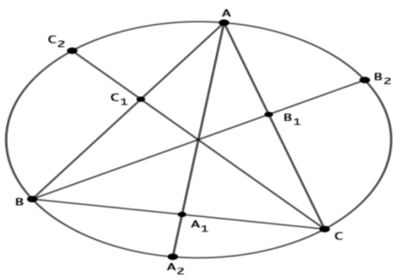
$$\geq (0 + 4r^2)(0 + 4r^2) = 4^2 r^4$$

$$\rightarrow c\sqrt{((b-c)^2 + 4r^2)((a-c)^2 + 4r^2)} \geq 4r^2 c; \textit{ etc}$$

$$\rightarrow LHS = 4r^2(a + b + c) = 8r^2 s$$

$$\textit{But: } s \geq 3\sqrt{3}r \rightarrow LHS = 8r^2 s \geq 8r^2 \cdot 3\sqrt{3}r = 24\sqrt{3}r^3$$

Proved. Equality $\leftrightarrow a = b = c$.



3.13.

$$AA_1 = m_a, BB_1 = m_b, CC_1 = m_c \Rightarrow [A_2B_2C_2] \geq \frac{3\sqrt{3}F^4}{m_a^2 m_b^2 m_c^2},$$

$$F = [ABC]$$

Solution:

$$\frac{S_{A_2B_2C_2}}{S_{ABC}} = \frac{A_2B_2 \cdot B_2C_2 \cdot C_2A_2}{4R} \cdot \frac{4R}{AB \cdot BC \cdot AC} \Rightarrow$$

$$\Rightarrow \frac{S_{A_2B_2C_2}}{S_{ABC}} = \frac{A_2B_2 \cdot B_2C_2 \cdot C_2A_2}{AB \cdot BC \cdot AC} \quad (1)$$

$$\Delta GAB \sim \Delta GB_2A_2 \Rightarrow \frac{GA}{GB_2} = \frac{AB}{A_2B_2} \Rightarrow$$

$$\Rightarrow A_2B_2 = c \cdot \frac{GB_2}{GA} = c \cdot \frac{GB \cdot GB_2}{GA \cdot GB} = \frac{c \cdot \rho(G)}{GAGB} \text{ and similarly } (2)$$

$$\text{From (1)+(2)} \Rightarrow \frac{S_{A_2B_2C_2}}{S_{ABC}} = \frac{\left(\rho(G)\right)^3}{(GAGBGC)^2} \quad (3)$$

$$\text{But } \rho(G) = R^2 - OG^2 = \frac{1}{9}(a^2 + b^2 + c^2) = \frac{1}{3}(GA^2 + GB^2 + GC^2) \quad (4)$$

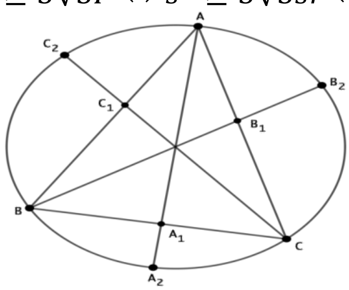
$$\text{From (3)+(4)} \Rightarrow \frac{S_{A_2B_2C_2}}{S_{ABC}} = \frac{\left(GA^2 + GB^2 + GC^2\right)^3}{27(GAGBGC)^2} \geq \frac{27GA^2GB^2GC^2}{27GA^2GB^2GC^2} \Rightarrow$$

$$\Rightarrow \frac{S_{A_2B_2C_2}}{S_{ABC}} \geq 1 \Rightarrow \text{we must show:} 1 \geq \frac{3\sqrt{3}F^3}{m_a^2m_b^2m_c^2} \Leftrightarrow m_a^2m_b^2m_c^2 \geq 3\sqrt{3}F^3 \quad (5)$$

$$\text{But in any } \Delta ABC \text{ we have: } m_a \geq \sqrt{s(s-a)}, s = \frac{a+b+c}{2} \Rightarrow$$

$$\Rightarrow m_a^2m_b^2m_c^2 \geq s^2F^2 \text{ (6). From (5)+(6) we must show:}$$

$$s^2F^3 \geq 3\sqrt{3}F^3 \Leftrightarrow s^2 \geq 3\sqrt{3}F \Leftrightarrow s^2 \geq 3\sqrt{3}sr \Leftrightarrow s \geq 3\sqrt{3}r \text{ true}$$



3.14.

If $AA_1 = w_a, BB_1 = w_b, CC_1 = w_c$ then:

$$AA_2 \cdot BB_2 \cdot CC_2 \geq \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

Solution:

$$\rho(A_1) = AA_1 \cdot A_1A_2 = BA_1 \cdot A_1C \quad (1)$$

$$AA_2 = AA_1 + A_1A_2 \geq 2\sqrt{AA_1 \cdot A_1A_2} \quad (2)$$

$$\text{From (1)+(2)} \Rightarrow AA_2 \geq 2\sqrt{BA_1 \cdot A_1C} \quad (3)$$

$$\text{But } \frac{BA_1}{A_1C} = \frac{c}{b} \Rightarrow \frac{BA_1}{a} = \frac{c}{b+c} \Rightarrow BA_1 = \frac{ac}{b+c} \text{ and } CA_1 = \frac{ab}{b+c} \quad (4)$$

$$\text{From (3)+(4)} \Rightarrow AA_2 \geq \frac{2a\sqrt{bc}}{(b+c)} \text{ and similarly} \Rightarrow$$

$$AA_2BB_2CC_2 \geq \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

3.15. In ΔABC the following relationship holds:

$$\mathbf{3\sum_{cyc}\left(\frac{h_b+h_c}{h_b+4m_a+h_c}\right) \leq 2\sum_{cyc}\left(\frac{m_a}{h_b+h_c}\right)}$$

Solution:

$$2\sum\frac{m_a}{h_b+h_c} \geq 3\sum\frac{h_b+h_c}{h_b+h_c+4m_a}$$

$$h_b+h_c+4m_a \overset{Tereshin}{\geq} \frac{ca+ab}{2R}+\frac{b^2+c^2}{R} \geq \frac{a(b+c)}{2R}+\frac{(b+c)^2}{2R}$$

$$= \frac{(b+c)(a+b+c)}{2R} = \frac{(b+c)s}{R} \Rightarrow \frac{h_b+h_c}{h_b+h_c+4m_a} \overset{(1)}{\leq} \frac{\left(\frac{ca+ab}{2R}\right)}{\frac{(b+c)s}{R}}$$

$$= \frac{a(b+c)}{2R}\cdot\frac{R}{s(b+c)} = \frac{a}{2s}$$

Similarly, $\frac{h_c+h_a}{h_c+h_a+4m_b} \overset{(2)}{\leq} \frac{b}{2s}$ *and* $\frac{h_a+h_b}{h_a+h_b+4m_c} \overset{(3)}{\leq} \frac{c}{2s}$

(1)+(2)+(3)$\Rightarrow 3\sum\frac{h_b+h_c}{h_b+h_c+4m_a} \leq 3\sum\frac{a}{2s} = \frac{3(2s)}{2s} = 3 \Rightarrow RHS \overset{(i)}{\leq} 3$

Again, $2\sum\frac{m_a}{h_b+h_c} \overset{Tereshin}{\geq} 2\sum\frac{\left(\frac{b^2+c^2}{4R}\right)}{\left(\frac{ca+ab}{2R}\right)} \geq \sum\frac{(b+c)^2}{2a(b+c)} = \frac{1}{2}\sum\frac{b+c}{a}$

$$= \frac{1}{2}\sum\left(\frac{b}{a}+\frac{a}{b}\right) \overset{A-G}{\geq} \frac{1}{2}\sum(2) = \frac{6}{2} = 3 \overset{by\ (i)}{\geq} RHS\ (Proved)$$

3.16. In ΔABC the following relationship holds:

$$\left(\sum_{cyc}(\boldsymbol{a}+5\boldsymbol{h_a})^2\right)\left(\sum_{cyc}\frac{\mathbf{1}}{(\boldsymbol{a}+5\boldsymbol{h_a})^2}\right) \geq \mathbf{6}\sum_{cyc}\frac{\boldsymbol{a}+5\boldsymbol{h_a}}{\boldsymbol{b}+\boldsymbol{c}+5(\boldsymbol{h_b}+\boldsymbol{h_c})}$$

Solution:

Let $x = 5+h_a; y = b+5h_b; z = c+5h_c \quad (x,y,z>0)$

Inequality $\leftrightarrow (x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \geq 6\left(\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}\right)$

$$\leftrightarrow \left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2+\left(\frac{x}{z}\right)^2+\left(\frac{z}{x}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{y}\right)^2+3 \geq 6\left(\frac{1}{\frac{y}{x}+\frac{z}{x}}+\frac{1}{\frac{x}{y}+\frac{z}{y}}+\frac{1}{\frac{x}{z}+\frac{y}{z}}\right) \quad (*)$$

$$\frac{1}{\frac{y}{x}+\frac{z}{x}}+\frac{1}{\frac{x}{y}+\frac{z}{y}}+\frac{1}{\frac{x}{z}+\frac{y}{z}} \overset{Schwarz}{\leq} \frac{1}{4}\left(\frac{1}{\frac{y}{x}}+\frac{1}{\frac{z}{x}}+\frac{1}{\frac{x}{y}}+\frac{1}{\frac{z}{y}}+\frac{1}{\frac{x}{z}}+\frac{1}{\frac{y}{z}}\right)=$$

$$=\frac{1}{4}\left(\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+\frac{y}{z}+\frac{z}{x}+\frac{z}{y}\right)$$

$$\to 6\left(\frac{1}{\frac{y}{x}+\frac{z}{x}}+\frac{1}{\frac{x}{y}+\frac{z}{y}}+\frac{1}{\frac{x}{z}+\frac{y}{z}}\right) \leq \frac{3}{2}\left(\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+\frac{y}{z}+\frac{z}{x}+\frac{z}{y}\right)$$

We must show that:

$$\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2+\left(\frac{x}{z}\right)^2+\left(\frac{z}{x}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{y}\right)^2+3$$

$$\geq \frac{3}{2}\left(\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+\frac{y}{z}+\frac{z}{x}+\frac{z}{y}\right)$$

$$\leftrightarrow 2\left[\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2+\left(\frac{x}{z}\right)^2+\left(\frac{z}{x}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{y}\right)^2+3\right] \geq$$

$$\geq 3\left(\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+\frac{y}{z}+\frac{z}{x}+\frac{z}{y}\right) \quad (**)$$

$$\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2+\left(\frac{x}{z}\right)^2+\left(\frac{z}{x}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{y}\right)^2 \geq$$

$$\geq \frac{\left(\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+\frac{y}{z}+\frac{z}{x}+\frac{z}{y}\right)^2}{6}=\frac{t^2}{6}$$

$$\left(t=\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+\frac{y}{z}+\frac{z}{x}+\frac{z}{y} \overset{AM-GM}{\geq} 6\right)$$

$$(**) \leftrightarrow 2\left(\frac{t^2}{6}+3\right) \geq 3t$$

$\leftrightarrow t^2-9t+18 \geq 0 \leftrightarrow (t-6)(t-3) \geq 0$ *(true because* $t \geq 6$*)* $\to$ *(*) true.*

Proved. Equality if and only if $x=y=z$.

3.17. In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$\sqrt[4]{aa'^3}+\sqrt[4]{bb'^3}+\sqrt[4]{cc'^3} \leq 2\sqrt[4]{ss'^3}$$

Solution:

$$Let\ f(x)=\sqrt[4]{x^3}; x>0$$

$$\to f'(x)=\frac{3}{4}\cdot x^{-\frac{1}{4}} \to f''(x)=-\frac{3}{16}x^{-\frac{5}{4}}<0\ (x>0)$$

$$Let\ f(y)=\sqrt[4]{y}; y>0 \to f'(y)=\frac{1}{4}\cdot x^{-\frac{3}{4}} \to f''(y)=-\frac{3}{16}\cdot x^{-\frac{7}{4}}<0\ (x>0)$$

Using Jensen's inequality: $\sqrt[4]{a}f(a')+\sqrt[4]{b}f(b')+\sqrt[4]{c}f(c') \leq$

$$\leq \left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)\cdot f\left(\frac{\sqrt[4]{a}\cdot a'+\sqrt[4]{b}\cdot b'+\sqrt[4]{c}\cdot c'}{\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)}\right)$$

$$=\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)\sqrt[4]{\left(\frac{\sqrt[4]{a}\cdot a'+\sqrt[4]{b}\cdot b'+\sqrt[4]{c}\cdot c'}{\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)}\right)^3}$$

$$=\sqrt[4]{\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)}\sqrt[4]{\left(\sqrt[4]{a}\cdot a'+\sqrt[4]{b}\cdot b'+\sqrt[4]{c}\cdot c'\right)^3}$$

$$\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\overset{Jensen}{\leq}3\sqrt[4]{\frac{a+b+c}{3}}=3\sqrt[4]{\frac{2s}{3}}$$

$$\sqrt[4]{a}\cdot a'+\sqrt[4]{b}\cdot b'+\sqrt[4]{c}\cdot c'\overset{Jensen}{\leq}2s'\sqrt[4]{\frac{aa'+bb'+cc'}{2s'}}$$

We must show that: $3\sqrt[4]{\frac{2s}{3}}\left(2s'\sqrt[4]{\frac{aa'+bb'+cc'}{2s'}}\right)^3\leq 16ss'^3$

$$\leftrightarrow (aa'+bb'+cc')^3\leq 3^3\cdot s^3\cdot s'^3=\frac{(a+b+c)^3(a'+b'+c')^3}{3^3}$$

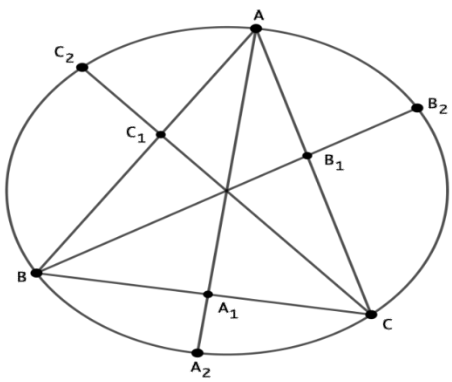
$$\leftrightarrow (a+b+c)^3(a'+b'+c')^3\geq 27(aa'+bb'+cc')^3$$

It is true because: Suppose: $a\leq b\leq c, a'\geq b'\geq c'$

$$(a+b+c)(a'+b'+c')\overset{Chebyshev}{\geq}27(aa'+bb'+cc')$$

$$\leftrightarrow (a+b+c)^3(a'+b'+c')^3\geq 27(aa'+bb'+cc')^3$$

Proved. Equality if and only if $\begin{cases}a=b=c\\a'=b'=c'\end{cases}$



3.18.

$$\boldsymbol{AA_1=s_a, BB_1=s_b, CC_1=s_c\Rightarrow [A_2B_2C_2]\geq\frac{81\sqrt{3}a^2b^2c^2r^2}{64m_a^2m_b^2m_c^2}}$$

Solution:

$$\frac{S_{A_2B_2C_2}}{S_{ABC}} = \frac{\rho(k)^3}{(KA\cdot KB\cdot KC)^2} \quad (1)$$

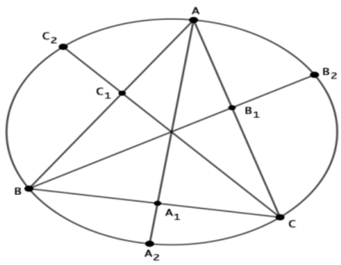
$$But\ \rho(k) = R^2 - OK^2 = \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2} \quad (2)$$

$$KA^2 = \left(\frac{b^2+c^2}{a^2+b^2+c^2}s_a\right)^2 = \left(\frac{(b^2+c^2)}{a^2+b^2+c^2}\cdot\frac{2bc}{b^2+c^2}\cdot m_a\right)^2 = \frac{4b^2c^2}{(a^2+b^2+c^2)^2}m_a^3 \quad (3)$$

$$From\ (1)+(2)+(3) \Rightarrow S_{A_2B_2C_2} = \frac{27}{64}\cdot\frac{a^2b^2c^2}{m_a^2m_b^2m_c^2}S\ (4)$$

$$From\ (4)\ we\ must\ show: \frac{27}{64}\cdot\frac{a^2b^2c^2}{m_a^2\cdot m_b^2\cdot m_c^2}\cdot S \geq \frac{81\sqrt{3}a^2b^2c^2r^2}{64m_a^2m_b^2m_c^2} \Leftrightarrow$$

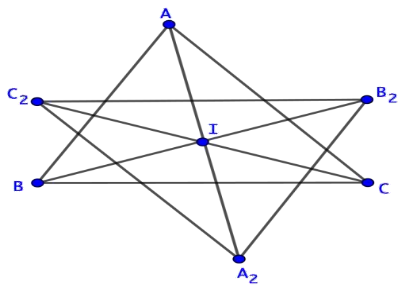
$S \geq 3\sqrt{3}r^2 \Leftrightarrow sr \geq 3\sqrt{3}r^2 \Leftrightarrow s \geq 3\sqrt{3}r$, true because it is Mitrinovic's inequality.



3.19.

$$\boldsymbol{AA_1 = w_a, BB_1 = w_b, CC_1 = w_c \Rightarrow [A_2B_2C_2] \geq 3\sqrt{3}r^2}$$

Solution:



$$From\ inscribed\ angle\ theorem: \angle CC_2A_2 = \angle OAA_2 = \frac{A}{2}\ and$$

$$\angle B_2C_2C = \angle B_2B_C = \frac{B}{2} \therefore \angle B_2C_2A_2 = \frac{A+B}{2} = \frac{180-C}{2} = 90 - \frac{C}{2}$$

similarly, we can get other formulas

$$\therefore \frac{B_2A_2}{\sin\left(90-\frac{C}{2}\right)} \overset{law\ of\ sines}{=} 2R \therefore B_2A_2 = 2R\cos\frac{C}{2}, similarly,\ we\ can\ get\ the\ other$$

$$formulas \therefore \Delta_{A_2B_2C_2} = \frac{a'b'c'}{4R} = \frac{8R^3\prod\cos\frac{A}{2}}{4R} = 2R^2\cdot\frac{s}{4R}$$

$= \frac{Rs}{2} \geq \frac{(2r)\cdot(3\sqrt{3}r)}{2} = 3\sqrt{3}r^2$ *[where are used Euler and Mitrinovic's inequalities]*

Equality $\Leftrightarrow \Delta A_2B_2C_2$ *is equilateral* $\Leftrightarrow \Delta ABC$ *is equilateral. (Proved).*

3.20. If a, b, c, d are sides of a convexe quadrilater then:

$$\frac{(a+b+c-d)(b+c+d-a)(c+d+a-b)(d+a+b-c)}{(a+b)(b+c)(c+d)(d+a)} < 1$$

Solution:

$$\overset{a,b,c,d>0}{\leftrightarrow} (a+b+c-d)(b+c+d-a)(c+d+a-b)(d+a+b-c) \leq$$
$$\leq (a+b)(b+c)(c+d)(d+a) \quad (*)$$

Let: $x = a+b+c-d; y = b+c+d-a; z = c+d+a-b$

$$t = d+a+b-c \quad (x,y,z,t > 0)$$
$$\to x+y+z+t = 2(a+b+c+d) \to a+b+c+d$$
$$= \frac{x+y+z+t}{2}; a+b = \frac{x+t}{2}; b+c = \frac{x+y}{2};$$
$$c+d = \frac{y+z}{2}; d+a = \frac{z+t}{2}$$
$$(*) \leftrightarrow xyzt \leq \left(\frac{x+t}{2}\right)\left(\frac{x+y}{2}\right)\left(\frac{y+z}{2}\right)\left(\frac{z+t}{2}\right)$$
$$\leftrightarrow (x+y)(y+z)(z+t)(x+t) \geq 16xyzt$$
$$x+y \geq 2\sqrt{xy}, y+z \geq 2\sqrt{yz}, z+t \geq 2\sqrt{zt}, x+t \geq 2\sqrt{xt}$$
$$\to (x+y)(y+z)(z+t)(x+t) \geq 2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{zt}\cdot 2\sqrt{xt} = 16xyzt$$

Proved. Equality $\leftrightarrow a = b = c = d$*.*

3.21. In ΔABC the following relationship holds:

$$a\left(\left(\frac{b}{a}\right)^{\frac{h_c}{m_c}} + \left(\frac{c}{a}\right)^{\frac{h_b}{m_b}}\right) + b\left(\left(\frac{c}{b}\right)^{\frac{h_a}{m_a}} + \left(\frac{a}{b}\right)^{\frac{h_c}{m_c}}\right) + c\left(\left(\frac{a}{c}\right)^{\frac{h_b}{m_b}} + \left(\frac{b}{c}\right)^{\frac{h_a}{m_a}}\right) \leq 4s$$

Solution:

$$\left(\frac{b}{a}\right)^{\frac{h_c}{m_c}} = \left(1 + \frac{b-a}{a}\right)^{\frac{h_c}{m_c}} \overset{Bernoulli}{\leq}$$
$$\leq 1 + \frac{h_c}{m_c}\left(\frac{b-a}{a}\right) \left[\because \frac{b-a}{a} = \frac{b}{a} - 1 > -1 \,\&\, \frac{h_c}{m_c} \leq 1\right]$$

$$\Rightarrow a\left(\frac{b}{a}\right)^{\frac{h_c}{m_c}} \overset{(1)}{\le} a+\frac{h_c}{m_c}(b-a)$$

Similarly, $a\left(\frac{c}{a}\right)^{\frac{h_b}{m_b}} \overset{(2)}{\le} a+\frac{h_b}{m_b}(c-a), b\left(\frac{c}{b}\right)^{\frac{h_a}{m_a}} \overset{(3)}{\le} b+\frac{h_a}{m_a}(c-b)$

$$b\left(\frac{a}{b}\right)^{\frac{h_c}{m_c}} \overset{(4)}{\le} b+\frac{h_c}{m_c}(a-b), c\left(\frac{a}{c}\right)^{\frac{h_b}{m_b}} \overset{(5)}{\le} c+\frac{h_b}{m_b}(a-c), c\left(\frac{b}{c}\right)^{\frac{h_a}{m_a}} \overset{(6)}{\le}$$

$$\le c+\frac{h_a}{m_a}(b-c)$$

(1)+(2)+(3)+(4)+(5)+(6)⇒ $LHS \le 4s+\sum\frac{h_a}{m_a}(b-c+c-b)=4s$

3.22. In ΔABC the following relationship holds:

$$\boldsymbol{a^2|\cos A|+b^2|\cos B|+c^2|\cos C| \ge 4S\sqrt{6|\cos A\cos B\cos C|}}$$

Solution:

By Weizenbock's inequality: $xa^2+yb^2+zc^2 \ge 4\sqrt{xy+yz+zx}\cdot S$

(For $x,y,z>0, a,b,c$*: lengths side* ΔABC*)*

Let $x=|\cos A|; y=|\cos B|; z=|\cos C|\ \ (x,y,z>0)$

We must show that: $xy+yz+zx \ge 6xyz$

$$\because (xy+yz+zx)(x+y+z) \ge 9xyz$$

$$\Leftrightarrow xy+yz+zx \ge \frac{9xyz}{x+y+z} \overset{(1)}{\ge} 6xyz$$

$$(1) \Leftrightarrow \frac{3}{2} \ge x+y+z$$

If $0<A,B,C\le\frac{\pi}{2} \Rightarrow x+y+z=\cos A+\cos B+\cos C=$

$$=1+\frac{r}{R} \overset{Euler}{\le} \frac{3}{2}\ (true)$$

Let $A=\max\{A,B,C\}$ *and* $\pi>A>\frac{\pi}{2}>B,C>0$

$$\Rightarrow x+y+z=\cos B+\cos C-\cos A$$

$$=2\cos\frac{B+C}{2}\cos\frac{B-C}{2}-\cos A=2\sin\frac{A}{2}\cos\frac{B-C}{2}-\cos A$$

$$\overset{\left(\sin\frac{A}{2}>0;\cos\frac{B-C}{2}\le 1\right)}{\le} 2\sin\frac{A}{2}-\cos A=2\sin\frac{A}{2}-\left(1-2\sin^2\frac{A}{2}\right)$$

$$=2\sin^2\frac{A}{2}+2\sin\frac{A}{2}-1 \overset{(*)}{\le} \frac{3}{2} \Leftrightarrow 2t^2+2t-\frac{5}{2}\le 0$$

$$\left(\because t = \sin\frac{A}{2}; 0 < t < 1\right)$$

$$\Leftrightarrow 0 < t \le \tfrac{1}{2}\left[\sqrt{6} - 1\right] < 1 \text{ (true) Proved.}$$

3.23. Solve for real numbers:

$$(\cos 2x)^{15} \cdot (\cos 4x)^{6} \cdot \cos 6x = \cos^{192} x$$

Solution:

$$\text{If } \cos 2x = 0, \text{ then } LHS = 0 \Rightarrow RHS = 0 \Rightarrow \cos x = 0 \Rightarrow$$

$$\Rightarrow \cos 2x = 2\cos^2 x - 1 = -1, \text{ a contradiction} \Rightarrow \cos 2x \overset{(1)}{\neq} 0$$

$$\text{If } \cos 4x = 0, \text{ then } LHS = RHS = 0 \Rightarrow \cos x = 0 \Rightarrow \cos 2x = -1 \Rightarrow$$

$$\Rightarrow \cos 4x = 2\cos^2 2x - 1 = 1, \text{ a contradiction} \Rightarrow \cos 4x \overset{(2)}{\neq} 0$$

$$\text{If } \cos 6x = 0, \text{ then } LHS = RHS = 0 \Rightarrow \cos x = 0 \Rightarrow \cos 2x = -1 \Rightarrow$$

$$\Rightarrow \cos 6x = 4\cos^3 2x - 3\cos 2x = -4 + 3 = -1, \text{ a contradiction}$$

$$\Rightarrow \cos 6x \overset{(3)}{\neq} 0$$

$$\text{If } \cos x = 0, \text{ then } RHS = 0 \Rightarrow LHS = 0. \text{ But (1), (2), (3)} \Rightarrow LHS \neq 0 \Rightarrow$$

$$\cos x \overset{(4)}{\neq} 0$$

$$(4) \Rightarrow RHS > 0 \Rightarrow LHS > 0 \Rightarrow (\cos 2x)^{14}(\cos 4x)^{6}(\cos 2x \cos 6x) > 0$$

$$\Rightarrow \cos 2x \cos 6x \overset{(5)}{>} 0. \text{ Now, given equation} \Leftrightarrow$$

$$\sqrt[24]{(\cos 2x)^{15}(\cos 4x)^{6}\cos 6x} \overset{(i)}{=} \cos^8 x \;\; (\because LHS > 0)$$

$$\text{Now, } \because (\cos 2x)^{14} \overset{by\ (1)}{>} 0, (\cos 4x)^{6} \overset{by\ (2)}{>} 0, \cos 2x \cos 6x \overset{by\ (5)}{>} 0$$

$$\therefore \sqrt[24]{(\cos 2x)^{15}(\cos 4x)^{6}\cos 6x} =$$

$$= \sqrt[24]{(\cos 2x)^{14}(\cos 4x)^{6}(\cos 2x \cos 6x) \cdot 1 \cdot 1 \cdot 1}$$

$$\overset{weighted\ GM \le weighted\ AM}{\le} \frac{14\cos 2x + 6\cos 4x + \cos 2x \cos 6x + 3}{24}$$

$$= \frac{14t + 6(2t^2 - 1) + (4t^3 - 3t)t + 3}{24} \quad (t = \cos 2x) \Rightarrow$$

$$\Rightarrow LHS \text{ of (i)} \le \frac{14t + 6(2t^2 - 1) + t(4t^3 - 3t) + 3}{24}$$

$$\Rightarrow \cos^8 x \le \frac{14t + 6(2t^2 - 1) + t(4t^3 - 3t) + 3}{24}$$

$$\Rightarrow \frac{(1 + t)^4}{16} \le \frac{14t + 6(2t^2 - 1) + t(4t^3 - 3t) + 3}{24}$$

$$\Rightarrow \frac{3(1+t)^4 - 28t - 12(2t^2-1) - 2t(4t^3-3t) - 6}{48} \le 0$$

$$\Rightarrow -(t-1)^2(5t^2-2t-9) \le 0 \Rightarrow (t-1)^2(5t^2-2t-9) \overset{(ii)}{\ge} 0$$

$$\because -1 \le t \le 1, \therefore 5t^2 - 2t - 9 \le 5 + 2 - 9 < 0$$

$$\therefore (ii) \Rightarrow (t-1)^2 \le 0. \text{ But } (t-1)^2 \ge 0$$

$$\therefore (t-1)^2 = 0 \Rightarrow t = 1 \Rightarrow \cos 2x = 1 \Rightarrow 2x = 2n\pi (n \in \mathbb{Z}) \Rightarrow$$

$$\Rightarrow x = n\pi \ (n \in \mathbb{Z}) \text{ (Answer)}$$

3.24. In acute or right ΔABC the following relationship holds:

$$\frac{\sin A}{\mu(A)} + \frac{\sin B}{\mu(B)} + \frac{\sin C}{\mu(C)} > \frac{9}{\pi} - \frac{4}{\pi^3}\left(\mu^2(A) + \mu^2(B) + \mu^2(C)\right)$$

Solution:

Using inequality: $\sin x \ge \frac{2}{\pi}x \left(0 < x \le \frac{\pi}{2}\right)$

$$\Rightarrow \frac{\sin A}{\mu(A)} + \frac{4}{\pi^3}\cdot\mu^2(A) = \frac{\sin A}{\mu(A)} + \frac{2}{\pi^3}\cdot\mu^2(A) + \frac{2}{\pi^3}\cdot\mu^2(A) \ge$$

$$\ge \frac{2}{\pi} + \frac{2}{\pi^3}\cdot\mu^2(A) + \frac{2}{\pi^3}\cdot\mu^2(A) = \frac{1}{\pi} + \frac{1}{\pi} + \frac{2}{\pi^3}\cdot\mu^2(A) + \frac{2}{\pi^3}\mu^2(A)$$

$$\overset{AM-GM}{\ge} 4\cdot\sqrt[3]{\frac{4\mu^4(A)}{\pi^8}} = \frac{4\sqrt[3]{4}}{\sqrt[3]{\pi^8}}\sqrt[3]{\mu^4(A)}$$

$$\Rightarrow \sum \frac{\sin A}{\mu(A)} + \frac{4}{\pi^3}\cdot\mu^2(A) \ge \frac{4\sqrt[3]{4}}{\sqrt[3]{\pi^8}}\sum\sqrt{\mu^4(A)} \quad (*)$$

Let $f(x) = \sqrt[3]{x^4} \quad (x > 0) \Rightarrow f'(x) = \frac{4}{3}\cdot x^{\frac{1}{3}} \Rightarrow f''(x) = \frac{4}{9}x^{-\frac{2}{3}} > 0$ $(x > 0)$

$$\Rightarrow RHS_{(*)} \overset{Jensen}{\ge} \frac{4\sqrt[3]{4}}{\sqrt[3]{\pi^8}}\cdot 4\sqrt[3]{\left[\frac{\mu(A)+\mu(B)+\mu(C)}{3}\right]^4} = \frac{12\sqrt[3]{4}}{\sqrt[3]{\pi^8}}\cdot\sqrt[3]{\frac{\pi^4}{3^4}} =$$

$$= 36\cdot\sqrt[3]{\frac{4}{3}}\cdot\frac{1}{\pi\sqrt[3]{\pi}} \overset{(1)}{\ge} \frac{9}{\pi}$$

$$(1) \Leftrightarrow 4\sqrt[3]{\frac{4}{3}}\cdot\frac{1}{\sqrt[3]{\pi}} > 1 \Leftrightarrow 4\sqrt[3]{\frac{4}{3}} > \sqrt[3]{\pi} \Leftrightarrow 64\cdot\frac{4}{3} > \pi \quad (true)$$

3.25. In ΔABC the following relationship holds:

$$\sqrt[3]{\left(\frac{r_a r_b}{s^2}+\frac{s^2}{r_a r_b}\right)\left(\frac{r_b r_c}{s^2}+\frac{s^2}{r_b r_c}\right)\left(\frac{r_c r_a}{s^2}+\frac{s^2}{r_c r_a}\right)} \geq \frac{10}{3}$$

Solution:

$$Let\ x=\frac{r_a r_b}{s^2};\ y=\frac{r_b r_c}{s^2};\ z=\frac{r_c r_a}{s^2}\ (x,y,z>0) \Rightarrow$$

$$\Rightarrow x+y+z=\frac{r_a r_b+r_b r_c+r_c r_a}{s^2}=\frac{s^2}{s^2}=1$$

$$Now,\ inequality \Leftrightarrow \sqrt[3]{\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)} \geq \frac{10}{3}$$

$$Using\ AM\text{-}GM,\ we\ have: \sqrt[3]{\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)} \geq \frac{3}{\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}}$$

$$Let\ f(t)=\frac{t}{1+t^2}(0<t<1) \Rightarrow f''(t)=\frac{2t(t^2-3)}{(t^2+1)^3} \Rightarrow f''(t)<0,$$
$$\forall t \in (0,1)$$

$$\Rightarrow \sum_{cyc}\frac{x}{1+x^2} \overset{Jensen}{\leq} 3\cdot f\left(\frac{x+y+z}{3}\right)=3f\left(\frac{1}{3}\right)=\frac{9}{10} \Rightarrow \frac{3}{\sum\frac{x}{1+x^2}} \geq \frac{3\cdot 10}{9}=\frac{10}{3}$$

(proved) Equality $\Leftrightarrow x=y=z=\frac{1}{3}$.

3.26. In ΔABC the following relationship holds:

$$(\boldsymbol{R}+\boldsymbol{r})\cos\left(\frac{\boldsymbol{\pi r}}{\boldsymbol{R}+\boldsymbol{r}}\right)>r+R\cos\left(\frac{\boldsymbol{\pi r}}{\boldsymbol{R}}\right)$$

Solution:

$$(R+r)\cos\left(\frac{\pi r}{R+r}\right)>r+R\cos\left(\frac{\pi r}{R}\right)$$

$$\Leftrightarrow \left(1+\frac{r}{R}\right)\cos\left[\frac{\pi\cdot\frac{r}{R}}{1+\frac{r}{R}}\right]>\frac{r}{R}+\cos\left(\frac{\pi r}{R}\right) \quad (*)$$

$$\Leftrightarrow (1+t)\cos\left(\frac{\pi t}{1+t}\right)-t-\cos(\pi t)>0;\left(0<t=\frac{r}{R}\leq\frac{1}{2}\right)$$

$$Let{:}\ f(t)=(1+t)\cos\left(\frac{\pi t}{1+t}\right)-t-\cos(\pi t);\ with\ 0<t\leq\frac{1}{2}$$

$$\Rightarrow f'(t) = \frac{-t + \pi(1+t)\sin(\pi t) - \pi \sin\left(\frac{\pi t}{1+t}\right) + (1+t)\cos\left(\frac{\pi t}{1+t}\right) - 1}{1+x}$$

$$\therefore f'(t) = 0 \Leftrightarrow \pi(1+t)\sin(\pi t) + (1+t)\cos\left(\frac{\pi t}{1+t}\right) =$$

$$= 1 + t + \pi \sin\left(\frac{\pi t}{1+t}\right)$$

$$\Leftrightarrow t = 0 \notin \left(0, \frac{1}{2}\right] \Rightarrow f'(t) > 0, \forall t \in \left(0, \frac{1}{2}\right] \Rightarrow f(t) > f(0) = 0 \Rightarrow (*)$$

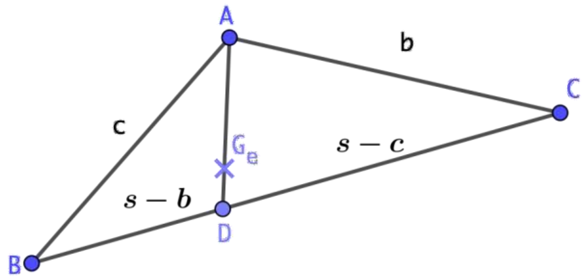
true. Proved.

3.27. In ΔABC, g_a, g_b, g_c are lengths of Gergonne's cevians. Prove that:

$$\sqrt[3]{g_a g_b g_c} \le \frac{3R}{2}$$

Solution:

First we prove that $g_a \le m_a$ (1)



Let AD be the Gergonne's cevian of A.

$AD = g_a$. By Stewart's theorem in ΔABC:

$$a \cdot g_a^2 = c^2(s-c) + b^2(s-b) - a(s-b)(s-c)$$

$$g_a^2 = \frac{s(c^2+b^2) - b^3 - c^3}{a} - (s-b)(s-c)$$

$$g_a \le m_a \Rightarrow g_a^2 \le m_a^2 \Leftrightarrow$$

$$\frac{s(c^2+b^2) - b^3 - c^3}{a} - (s-b)(s-c) \le \frac{2(b^2+c^2) - a^2}{4}$$

$$4s(c^2+b^2) - 4(b^3+c^3) - 4a(s-b)(s-c) \le a(2(b^2+c^2) - a^2)$$

$$4s(c^2+b^2) - 4(b^3+c^3) - a(a+c-b)(a+b-c) \le$$

$$\le 2a(b^2+c^2) - a^3$$

$$4s(c^2+b^2) - 4(b^3+c^3) - a(a^2-(b-c)^2) \le 2a(b^2+c^2) - a^3$$

$$4s(b^2+c^2) - 4(b^3+c^3) - a^3 + a(b-c)^2 - 2a(b^2+c^2) + a^3 \le 0$$

$$(b^2+c^2)(4s-2a) - 4(b^3+c^3) + a(b-c)^2 \le 0$$

$$(b^2+c^2)(2b+2c)-4(b^3+c^3)+a(b-c)^2 \le 0$$
$$2(b+c)(b^2+c^2)-4(b+c)(b^2-bc+c^2)+a(b-c)^2 \le 0$$
$$(b+c)(2b^2+2c^2-4b^2+4bc-4c^2)+a(b-c)^2 \le 0$$
$$2(b+c)(-b^2+2bc-c^2)+a(b-c)^2 \le 0$$
$$2(b+c)(b-c)^2-a(b-c)^2 \ge 0$$
$$(b-c)^2(2b+2c-a) \ge 0$$
which is true because $b+c>a \Rightarrow 2b+2c>a \Rightarrow 2b+2c-a>0$

$$\text{By (1): } \sqrt[3]{g_a g_b g_c} \le \sqrt[3]{m_a m_b m_c} \overset{AM-GM}{\le} \frac{m_a+m_b+m_c}{3} =$$
$$= \frac{1}{3}(1\cdot m_a + 1\cdot m_b + 1\cdot m_c) \overset{CAUCHY-SCHWARZ}{\le}$$
$$\le \frac{1}{3}\cdot\sqrt{(1^2+1^2+1^2)(m_a^2+m_b^2+m_c^2)} =$$
$$= \frac{\sqrt{3}}{3}\sqrt{\sum m_a^2} = \frac{\sqrt{3}}{3}\sqrt{\frac{3}{4}\sum a^2} =$$
$$= \frac{3}{6}\sqrt{a^2+b^2+c^2} \overset{LEIBNIZ}{\le} \frac{1}{2}\sqrt{9R^2} = \frac{9R}{2}$$
Equality holds if $a=b=c$.

3.28. If in $\Delta ABC, s=\sqrt{3}$ then:

$$\boldsymbol{r_a\sqrt{\frac{r_b}{r_c}} + r_b\sqrt{\frac{r_c}{r_a}} + r_c\sqrt{\frac{r_a}{r_b}} \ge 3}$$

Solution:

$$\text{Let be } f:(0,\infty)\to\mathbb{R};\ f(x)=x^{-\frac{1}{2}}$$
$$f'(x) = -\tfrac{1}{2}x^{-\frac{3}{2}};\ f''(x)=\tfrac{3}{4}x^{-\frac{5}{2}}>0, f \text{ convexe}$$
$$s=\sqrt{3} \Rightarrow s^2=3 \Rightarrow r_ar_b+r_br_c+r_cr_a = 3 \quad (1)$$
$$\tfrac{r_ar_b}{3}+\tfrac{r_br_c}{3}+\tfrac{r_cr_a}{3} = 1. \text{ Denote:}$$
$$p_1=\frac{r_ar_b}{3};\ p_2=\frac{r_br_c}{3};\ p_3=\frac{r_cr_a}{3} \Rightarrow p_1+p_2+p_3=1$$
The inequality that we have to prove can be written:
$$\frac{r_ar_b}{3}\cdot\frac{1}{\sqrt{r_br_c}}+\frac{r_br_c}{3}\cdot\frac{1}{\sqrt{r_cr_a}}+\frac{r_cr_a}{3}\cdot\frac{1}{\sqrt{r_ar_b}} \ge 1$$
$$p_1f(r_br_c)+p_2f(r_cr_a)+p_3f(r_ar_b) \ge 1$$

By Jensen's inequality: $p_1f(r_br_c)+p_2f(r_cr_a)+p_3f(r_ar_b)\ge$
$$\ge f(p_1r_br_c+p_2r_cr_a+p_3r_ar_b)=\frac{1}{\sqrt{p_1r_br_c+p_2r_cr_a+p_3r_ar_b}}\ge 1 \text{ (to prove)}$$
$$\sqrt{p_1r_br_c+p_2r_cr_a+p_3r_ar_b}\le 1, p_1r_br_c+p_2r_cr_a+p_3r_ar_b\le 1$$
$$\frac{r_ar_b}{3}\cdot r_br_c+\frac{r_br_c}{3}r_cr_a+\frac{r_cr_a}{3}r_ar_b\le 1, r_ar_br_c(r_a+r_b+r_c)\le 3$$ *(to prove). Denote* $r_ar_b=x; r_br_c=y; r_cr_a=z$

We must prove that: $xy+yz+zx\le 3$

By (1): $x+y+z=3$
$$9=(x+y+z)^2\ge 3(xy+yz+zx)$$
$$xy+yz+zx\le 3$$

3.29. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{y+z}{x}\left(bc\csc\frac{B}{2}\csc\frac{C}{2}\right)^8\right)\ge 6\left(\frac{abc(a+b+c)}{\sqrt{3}S}\right)^8$$

Solution:

By Cesaro's inequality: $(x+y)(y+z)(z+x)\ge 8xyz$ *(1)*
$$\sum_{cyc}\left(\frac{y+z}{x}\left(bc\csc\frac{B}{2}\csc\frac{C}{2}\right)^8\right)\overset{AM-GM}{\ge}$$
$$\ge 3\sqrt[3]{\frac{y+z}{x}\cdot\frac{z+x}{y}\cdot\frac{x+y}{z}\cdot\prod_{cyc}\left(\frac{bc}{\sin\frac{B}{2}\sin\frac{C}{2}}\right)^8}\ge$$
$$\overset{(1)}{\ge} 3\sqrt[3]{\frac{8xyz}{xyz}\cdot\left(\frac{a^2b^2c^2}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}\right)^8}=$$
$$=6\sqrt[3]{\left(\frac{4Rrs}{\frac{r}{4R}}\right)^{16}}=6\cdot\left(\frac{4Rrs\cdot 4R}{r}\right)^{\frac{16}{3}}=6\cdot(16R^2s)^{\frac{16}{3}}\quad (2)$$
$$6\left(\frac{abc(a+b+c)}{\sqrt{3}S}\right)^8=6\left(\frac{4RS\cdot 2s}{\sqrt{3}S}\right)^8=6\left(\frac{8Rs}{\sqrt{3}}\right)^8\quad (3)$$
By (2); (3) we must prove that: $6\cdot(16R^2s)^{\frac{16}{3}}\ge 6\left(\frac{8Rs}{\sqrt{3}}\right)^8$
$$(16R^2s)^{\frac{2}{3}}\ge\frac{8Rs}{\sqrt{3}},\qquad (16R^2s)^2\ge\left(\frac{8Rs}{\sqrt{3}}\right)^3$$

$2^8 \cdot R^4 s^2 \geq \frac{2^9 \cdot R^3 \cdot s^3}{3\sqrt{3}}, Rs^2 \geq \frac{2s^3}{3\sqrt{3}}, R \geq \frac{2s}{3\sqrt{3}}, s \leq \frac{3\sqrt{3}R}{2}$ *which is Mitrinovic's inequality.*

3.30. In $\Delta ABC, g_a, g_b, g_c$ – Gergonne's cevians, n_a, n_b, n_c – Nagel's cevians. Prove that:

$$\frac{1}{g_a} + \frac{1}{g_b} + \frac{1}{g_c} \geq \frac{9}{4R + r}, \qquad \frac{1}{n_a} + \frac{1}{n_b} + \frac{1}{n_c} \leq \frac{1}{r}$$

Solution:

It's known that: $n_a \geq h_a; n_b \geq h_b; n_c \geq h_c \Rightarrow$

$$\frac{1}{n_a} \leq \frac{1}{h_a}; \frac{1}{n_b} \leq \frac{1}{h_b}; \frac{1}{n_c} \leq \frac{1}{h_c}$$

$$\frac{1}{n_a} + \frac{1}{n_b} + \frac{1}{n_c} \leq \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a}{2S} + \frac{b}{2S} + \frac{c}{2S} = \frac{a+b+c}{2S} =$$

$= \frac{2s}{2rs} = \frac{1}{r}$ *It's known that:* $g_a \leq m_a; g_b \leq m_b; g_c \leq m_c \Rightarrow$

$$\Rightarrow g_a + g_b + g_c \leq m_a + m_b + m_c \leq 4R + r$$

$$\Rightarrow \frac{1}{g_a + g_b + g_c} \geq \frac{1}{m_a + m_b + m_c} \geq \frac{1}{4R + r} \quad (1)$$

$$\frac{1}{g_a} + \frac{1}{g_b} + \frac{1}{g_c} \overset{CBS}{\geq} \frac{(1+1+1)^2}{g_a + g_b + g_c} \overset{(1)}{\geq} \frac{9}{4R + r}$$

3.31. In $\Delta ABC, n_a, n_b, n_c$ are Nagel's cevians. Prove that:

$$\frac{a^5}{m_c} + \frac{b^5}{m_b} + \frac{c^5}{m_a} \geq \frac{2592\sqrt{3}r^5}{n_a + n_b + n_c}$$

Solution:

$$n_a \geq m_a, n_b \geq m_b, n_c \geq m_c \Rightarrow n_a + n_b + n_c \Rightarrow m_a + m_b + m_c$$

$$\Rightarrow \frac{1}{m_a + m_b + m_c} \geq \frac{1}{n_a + n_b + n_c} \quad (1)$$

$$\frac{a^5}{m_c} + \frac{b^5}{m_b} + \frac{c^5}{m_a} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{(abc)^5}{m_a m_b m_c}} =$$

$$= \frac{3\sqrt[3]{(4Rrs)^5}}{\sqrt[3]{m_a m_b m_c}} \overset{AM-GM}{\geq} \frac{3\sqrt[3]{(4Rrs)^5}}{\frac{m_a + m_b + m_c}{3}} \geq$$

$$\overset{EULER}{\geq} \frac{9\sqrt[3]{(4 \cdot 2r \cdot r \cdot s)^5}}{m_a + m_b + m_c} \overset{(1)}{\geq} \frac{9\sqrt[3]{(8r^2 s)^5}}{n_a + n_b + n_c} \geq$$

$$\overset{MITRINOVIC}{\geq} \frac{9\left(8r^2\cdot 3\sqrt{3}r^3\right)^{\frac{5}{3}}}{n_a+n_b+n_c} = \frac{9\left(2\cdot\sqrt{3}\right)^5\cdot r^5}{n_a+n_b+n_c} =$$
$$= \frac{9\cdot 32\cdot 9\sqrt{3}r^5}{n_a+n_b+n_c} = \frac{2592\sqrt{3}r^5}{n_a+n_b+n_c}$$

3.32. If in acute ΔABC, I – incentre, G – centroid, H – orthocentre, N – Nagel's point then:

$$\boldsymbol{NI\ \cdot HI + 3HI\cdot GI + 3NI\cdot GI \leq s^2 + 4R^2 - 28Rr + 13r^2}$$

Solution:

$$IG^2 = \frac{s^2 - 16Rr + 5r^2}{9} \Rightarrow IG = \frac{\sqrt{s^2 - 16Rr + 5r^2}}{3}$$
$$NI = 3IG = 3\cdot\frac{\sqrt{s^2 - 16Rr + 5r^2}}{3} = \sqrt{s^2 - 16Rr + 5r^2}$$
$$HI = \sqrt{4R^2 + 2r^2 - \frac{1}{2}(a^2+b^2+c^2)} = \sqrt{4R^2 + 3r^2 - s^2 + 4Rr}$$
$$NI\cdot HI + 3HI\cdot GI + 3NI\cdot GI \leq NI^2 + HI^2 + 9GI^2 =$$
$$= s^2 - 16Rr + 5r^2 + 4R^2 + 3r^2 - s^2 + 4Rr + s^2 - 16Rr + 5r^2 =$$
$$= s^2 + 4R^2 - 28Rr + 13r^2$$

3.33. In ΔABC the following relationship holds:

$$\boldsymbol{\sum_{cyc}\left(\frac{b\csc\frac{B}{2} + c\csc\frac{C}{2}}{a\csc\frac{A}{2}}\right) + \prod_{cyc}\left(\frac{b\csc\frac{B}{2} + c\csc\frac{C}{2} - a\csc\frac{A}{2}}{a\csc\frac{A}{2}}\right) \geq 7}$$

Solution:

Let $a' = a\csc\frac{A}{2}$; $b' = b\csc\frac{B}{2}$; $c' = c\csc\frac{C}{2}$ be sides of anticevian Hexyl triangle of ΔABC.

Let $s' = \frac{a'+b'+c'}{2}$; R', r' - circumradii and inradii of this triangle.

$$\sum\frac{b'+c'}{a'} + \prod\left(\frac{b'+c'-a'}{a'}\right) = \sum\frac{2s'-a'}{a'} + \prod\left(\frac{2s'-2a'}{a'}\right) =$$

$$= 2s\sum\frac{1}{a'} - 3 + \frac{8}{a'b'c'}\prod(s'-a') =$$
$$= 2s'\cdot\frac{a'b'+b'c'+c'a'}{a'b'c'} - 3 + \frac{8S'^2}{s'a'b'c'} =$$
$$= 2s'\cdot\frac{s'^2+4R'r'+r'^2}{4R'r's'} + \frac{8S'r's'}{s'a'b'c'} - 3 =$$
$$= \frac{2s'(s'^2+4R'r'+r'^2)+8r'^2s'}{4R'r's'} - 3 =$$
$$= \frac{s'^2+4R'r'+r'^2+4r'^2}{2R'r'} - 3 \geq$$
$$\overset{GERRETSEN}{\geq} \frac{16R'r'-5r'^2+4R'r'+5r'^2}{2R'r'} - 3 =$$
$$= \frac{20R'r'}{2R'r'} - 3 = 10 - 3 = 7$$

3.34. In $\Delta ABC, R_A, R_B, R_C$ are radii of Lucas circles. Prove that:

$$(xR_A+yR_B+zR_C)^2 \geq abc\sum_{cyc}\frac{yza^2}{(ac+2bR)(ab+2cR)};\ x,y,z\geq 0$$
$$aR_A+bR_B+cR_C < 3R\sqrt{\frac{S}{2}}$$

Solution:

It's known that if $u,v,w\in\mathbb{R}$ then:
$$(u+v+w)^2 \geq 3(uv+vw+wu) \quad (1)$$
For $u = xR_A; v = yR_B; w = zR_C$ in (1):
$$(xR_A+yR_B+zR_C)^2 \geq 3(xyR_AR_B+yzR_BR_C+zxR_CR_A)$$
$$= 3\sum\frac{yza^2bc}{(ac+2bR)(ab+2cR)} =$$
$$= 3abc\sum\frac{ayz}{(ac+2bR)(ab+2cR)}$$
$$aR_A+bR_B+cR_C = abcR\left(\frac{1}{bc+2aR}+\frac{1}{ac+2bR}+\frac{1}{ab+2cR}\right) =$$
$$= 4R^2S\left(\frac{1}{bc+2aR}+\frac{1}{ac+2bR}+\frac{1}{ab+2cR}\right)$$
Inequality that we have to prove can be written:
$$4R^2S\left(\frac{1}{bc+2aR}+\frac{1}{ac+2bR}+\frac{1}{ab+2cR}\right) < 3R\sqrt{\frac{S}{2}}$$

$$4R^2S\left(\frac{1}{bc+2aR}+\frac{1}{ac+2bR}+\frac{1}{ab+2cR}\right)<$$

$$\overset{AM-GM}{<}4R^2S\cdot\sum_{cyc}\frac{1}{2\sqrt{2abcR}}=$$

$$=\frac{4R^2S\cdot 3}{2\sqrt{2\cdot 4RSR}}=\frac{3R^2S}{\sqrt{2SR^2}}=3R\sqrt{\frac{S^2R^2}{2SR^2}}=3R\sqrt{\frac{S}{2}}$$

3.35. If in $\Delta ABC; r_a=2; r_b=3, r_c=4$ then the following relationship holds:

$$\mathbf{2r^2s<\frac{4a}{3}+\frac{8b}{9}+\frac{2c}{3}}<rsR$$

Solution:

$$a=\frac{r_a(r_b+r_c)}{r_ar_b+r_br_c+r_cr_a}=\frac{2(3+4)}{2\cdot 3+2\cdot 4+3\cdot 4}=\frac{14}{28}=\frac{1}{2}$$

$$b=\frac{r_b(r_c+r_a)}{r_ar_b+r_br_c+r_cr_a}=\frac{3(2+4)}{28}=\frac{18}{28}=\frac{9}{14}$$

$$c=\frac{r_c(r_a+r_b)}{r_ar_b+r_br_c+r_cr_a}=\frac{4(2+3)}{28}=\frac{20}{28}=\frac{5}{7}$$

$$s=\frac{1}{2}(a+b+c)=\frac{1}{2}\left(\frac{1}{2}+\frac{9}{14}+\frac{5}{7}\right)=\frac{1}{2}\cdot\frac{7+9+10}{14}=\frac{26}{2\cdot 14}=\frac{13}{14}$$

$$\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}$$

$$r=\frac{12}{13}$$

$$R=\frac{4R+r-r}{4}=\frac{r_a+r_b+r_c-r}{4}=$$

$$=\frac{2+3+4-\frac{12}{13}}{4}=\frac{9-\frac{12}{13}}{4}=\frac{117-12}{4\cdot 13}=\frac{105}{4\cdot 13}=\frac{105}{52}$$

$$2r^2s=2\cdot\frac{12^2}{13^2}\cdot\frac{13}{14}=\frac{2\cdot 12\cdot 12}{13\cdot 14}=\frac{12\cdot 12}{13\cdot 14}=\frac{71}{7\cdot 13}=\frac{72}{91}$$

$$rsR=\frac{12}{13}\cdot\frac{13}{14}\cdot\frac{105}{52}=\frac{12\cdot 105}{14\cdot 52}=\frac{3\cdot 105}{7\cdot 26}=\frac{315}{182}$$

$$\frac{4a}{3}+\frac{8b}{9}+\frac{2c}{3}=\frac{4}{3}\cdot\frac{1}{2}+\frac{8}{9}\cdot\frac{9}{14}+\frac{2}{3}\cdot\frac{5}{7}=$$

$$= \frac{4}{6} + \frac{8}{14} + \frac{10}{21} = \frac{2}{3} + \frac{4}{7} + \frac{10}{21} = \frac{14 + 12 + 10}{21} = \frac{26 + 10}{21} = \frac{36}{21} = \frac{12}{7}$$

$$2r^2 s < \frac{4a}{3} + \frac{8b}{9} + \frac{2c}{3} < Rsr$$

$$\frac{72}{91} < \frac{12}{7} < \frac{315}{182} \quad \textit{(to prove)}$$

$$\begin{cases} \frac{72}{91} < \frac{12}{7} \Leftrightarrow 72 < 156 \\ \frac{12}{7} < \frac{315}{182} \Leftrightarrow 312 < 315 \end{cases}$$

3.36. If in $\Delta ABC; r_a = 2; r_b = 3; r_c = 4$ then find:
$\Omega = R_i + R_e + R_i R_e$ where R_i, R_e are radii of Soddy's circles

Solution:

$$a = \frac{r_a(r_b + r_c)}{r_a r_b + r_b r_c + r_c r_a} = \frac{2(3 + 4)}{2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4} = \frac{14}{28} = \frac{1}{2}$$

$$b = \frac{r_b(r_c + r_a)}{r_a r_b + r_b r_c + r_c r_a} = \frac{3(2 + 4)}{28} = \frac{18}{28} = \frac{9}{14}$$

$$c = \frac{r_c(r_a + r_b)}{r_a r_b + r_b r_c + r_c r_a} = \frac{4(2 + 3)}{28} = \frac{20}{28} = \frac{5}{7}$$

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}\left(\frac{1}{2} + \frac{9}{14} + \frac{5}{7}\right) = \frac{1}{2} \cdot \frac{7 + 9 + 10}{14} = \frac{1}{2} \cdot \frac{26}{14} = \frac{13}{14}$$

$$r_a + r_b + r_c = 2 + 3 + 4 = 9$$

$$R_i = \frac{S}{r_a + r_b + r_c + 2s}; R_e = \frac{S}{|r_a + r_b + r_c - 2s|}$$

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

$$r = \frac{12}{13}$$

$$R_i = \frac{rs}{9 + 2 \cdot \frac{13}{14}} = \frac{\frac{12}{13} \cdot \frac{13}{14}}{9 + \frac{13}{7}} = \frac{\frac{6}{7}}{\frac{63 + 13}{7}} = \frac{6}{76} = \frac{3}{38}$$

$$R_e = \frac{rs}{|r_a + r_b + r_c - 2s|} = \frac{\frac{12}{13} \cdot \frac{13}{14}}{\left|9 - 2 \cdot \frac{13}{14}\right|} = \frac{\frac{12}{14}}{\left|9 - \frac{13}{7}\right|} = \frac{\frac{6}{7}}{\frac{50}{7}} = \frac{3}{25}$$

$$\Omega = R_i + R_e + R_iR_e = \frac{3}{28} + \frac{3}{25} + \frac{9}{38\cdot 25} =$$
$$= 3\left(\frac{1}{38} + \frac{1}{25} + \frac{3}{38\cdot 25}\right) = 3\cdot\frac{38+25+3}{38\cdot 25} =$$
$$= 3\cdot\frac{38+28}{38\cdot 25} = 3\cdot\frac{66}{38\cdot 25} = 3\cdot\frac{33}{19\cdot 25} = \frac{99}{475}$$

3.37. In ΔABC; n_a, n_b, n_c – Nagel's cevians; g_a, g_b, g_c – Gergonne's cevians. Find $\min \Omega$:

$$\boldsymbol{\Omega = \frac{n_a^2 + n_b^2 + n_c^2}{ag_a + bg_b + cg_c}}$$

Solution:

It's known that:

$n_a \geq m_a \geq g_a$ *and analogs.*

$$n_a^2 + n_b^2 + n_c^2 \geq m_a^2 + m_b^2 + m_c^2 = \tfrac{3}{4}(a^2+b^2+c^2) \quad (1)$$

$$ag_a + bg_b + cg_c \leq am_a + bm_b + cm_c \overset{CBS}{\leq}$$
$$\leq \sqrt{(a^2+b^2+c^2)(m_a^2+m_b^2+m_c^2)} =$$
$$= \sqrt{(a^2+b^2+c^2)\cdot\frac{3}{4}(a^2+b^2+c^2)} = \frac{\sqrt{3}}{2}(a^2+b^2+c^2)$$

$$\frac{1}{ag_a+bg_b+cg_c} \geq \frac{1}{\frac{\sqrt{3}}{2}(a^2+b^2+c^2)} \quad (2)$$

By multiplying (1); (2): $\Omega = \frac{n_a^2+n_b^2+n_c^2}{ag_a+bg_b+cg_c} \geq \frac{\frac{3}{4}(a^2+b^2+c^2)}{\frac{\sqrt{3}}{2}(a^2+b^2+c^2)} = \frac{\sqrt{3}}{2}$

$$\min\Omega = \frac{\sqrt{3}}{2}$$

Equality holds for $a = b = c$.

3.38. If in $\Delta ABC, r_1, r_2, r_3$ are radii of Malfatii's circles then:

$$\boldsymbol{\frac{r_1^4}{r_2^3} + \frac{r_2^4}{r_3^3} + \frac{r_3^4}{r_1^3} \geq \frac{3r}{2}\sqrt[3]{\left(1+\tan\frac{A}{4}\right)\left(1+\tan\frac{B}{4}\right)\left(1+\tan\frac{C}{4}\right)}}$$

Solution:

$$\frac{r_1^4}{r_2^3} + \frac{r_2^4}{r_3^3} + \frac{r_3^4}{r_1^3} \overset{RADON}{\geq} \frac{(r_1+r_2+r_3)^4}{(r_1+r_2+r_3)^3} =$$

$$= r_1 + r_2 + r_3 = \sum_{cyc} \frac{r}{2} \cdot \frac{\left(1 + \tan\frac{B}{4}\right)\left(1 + \tan\frac{C}{4}\right)}{1 + \tan\frac{A}{4}} \geq$$

$$\overset{AM-GM}{\geq} \frac{r}{2} \cdot 3 \sqrt[3]{\prod_{cyc} \frac{\left(1 + \tan\frac{B}{4}\right)\left(1 + \tan\frac{C}{4}\right)}{\left(1 + \tan\frac{A}{4}\right)}} =$$

$$= \frac{3r}{2} \sqrt[3]{\left(1 + \tan\frac{A}{4}\right)\left(1 + \tan\frac{B}{4}\right)\left(1 + \tan\frac{C}{4}\right)}$$

3.39. In any triangle ABC the following relationship holds:

$$\sum_{cyc} \frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}} \leq \frac{3}{2} + \frac{R}{4\sqrt{2F}} \sum_{cyc} \frac{(\sin A - \sin B)^2}{\sqrt{\sin C}}$$

Solution:

$$\sum_{cyc} \frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}} = \sum_{cyc} \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos\frac{A}{2}\cos\frac{B-C}{2}} = \sum_{cyc} \frac{\sin A}{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\frac{B-C}{2}} =$$

$$= \sum_{cyc} \frac{\sin A}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}} = \sum_{cyc} \frac{\sin A}{\sin B + \sin C} =$$

$$= \sum_{cyc} \frac{2R\sin A}{2R\sin B + 2R\sin C} =$$

$$= \sum_{cyc} \frac{a}{b+c} = \sum_{cyc} \left(\frac{a}{b+c} - \frac{1}{2}\right) + \frac{3}{2} = \frac{3}{2} + \sum_{cyc} \frac{2a - b - c}{2(b+c)} =$$

$$= \frac{3}{2} + \sum_{cyc} \frac{a-b}{2(b+c)} + \sum_{cyc} \frac{a-c}{2(b+c)}$$

$$= \frac{3}{2} + \frac{a-b}{2(b+c)} + \frac{b-c}{2(c+a)} + \frac{c-a}{2(a+b)} + \frac{a-c}{2(b+c)} + \frac{b-a}{2(c+a)} + \frac{c-b}{2(a+b)} =$$

$$= \frac{3}{2} + \frac{a-b}{2}\left(\frac{1}{b+c} - \frac{1}{c+a}\right) + \frac{b-c}{2}\left(\frac{1}{c+a} - \frac{1}{a+b}\right) + \frac{c-a}{2}\left(\frac{1}{a+b} - \frac{1}{b+c}\right)$$

$$= \frac{3}{2} + \frac{(a-b)(a-b)}{2(c+b)(c+a)} + \frac{(b-c)(b-c)}{2(c+a)(a+b)} + \frac{(c-a)(c-a)}{2(b+c)(b+c)} =$$

$$= \frac{3}{2} + \frac{1}{2}\sum_{cyc} \frac{(a-b)^2}{(c+b)(c+a)} \overset{AM-GM}{\leq} \frac{3}{2} + \frac{1}{2}\sum \frac{(a-b)^2}{2\sqrt{bc} \cdot 2\sqrt{ca}} =$$

$$= \frac{3}{2} + \frac{1}{8}\sum_{cyc} \frac{(a-b)^2}{c\sqrt{ab}} = \frac{3}{2} + \frac{1}{8\sqrt{abc}}\sum_{cyc} \frac{(a-b)^2}{\sqrt{c}} =$$

$$= \frac{3}{2} + \frac{1}{8\sqrt{4RF}}\sum_{cyc} \frac{(2R\sin A - 2R\sin B)^2}{\sqrt{2R\sin C}} =$$

$$= \frac{3}{2} + \frac{1}{16\sqrt{RF}} \cdot \frac{4R^2}{\sqrt{2R}} \cdot \sum_{cyc} \frac{(\sin A - \sin B)^2}{\sqrt{\sin C}} = \frac{3}{2} + \frac{R}{4\sqrt{2F}}\sum_{cyc} \frac{(\sin A - \sin B)^2}{\sqrt{\sin C}}$$

3.40. If $a, b, c \in (0, \pi)$ then:

$$\cos^2 a \cos^2 b \cos^2 c + (\sin a + \sin b + \sin c)^2 > 1$$

Solution:

Denote $x = \sin a\,; y = \sin b\,; z = \sin c\,; t = \sin d$

$$\cos^2 a = 1 - x^2; \cos^2 b = 1 - y^2; \cos^2 c = 1 - z^2; \cos^2 d = 1 - t^2$$

Inequality to prove becomes:

$$(1-x^2)(1-y^2)(1-z^2) + (x+y+z)^2 > 1$$

$$1 - (x^2+y^2+z^2) + (x^2y^2 + y^2z^2 + z^2x^2) - x^2y^2z^2 +$$

$$+(x^2+y^2+z^2) + 2(xy + xz + yz) > 1$$

$$x^2y^2 + y^2z^2 + z^2x^2 + 2(xy + xz + yz) > x^2y^2z^2$$

$$x^2y^2 + y^2z^2 + z^2x^2 + 2xy + 2yz + 2zx \overset{AM-GM}{\geq}$$

$$\geq 6\sqrt[6]{x^2y^2 \cdot y^2z^2 \cdot z^2x^2 \cdot 2xy \cdot 2yz \cdot 2zx} =$$

$$= 6\sqrt[6]{2^3 \cdot x^6y^6z^6} = 6\sqrt{2}xyz > xyz > (xyz)^2 \textit{ because}$$

$$x, y, z \in (0,1) \Rightarrow (xyz) > (xyz)^2$$

3.41. In any triangle ABC the following relationship holds:

$$(3a+b)\sqrt{\frac{b}{a+b}} + (3b+c)\sqrt{\frac{c}{b+c}} + (3c+a)\sqrt{\frac{a}{c+a}} \leq 6\sqrt{6}R$$

Solution:

We first prove that: $\frac{(3a+b)\sqrt{b}}{\sqrt{a+b}} \leq \sqrt{2}(a+b); (\forall)a, b > 0$ *(1)*

By squaring: $\frac{(3a+b)^2 \cdot b}{a+b} \leq 2(a+b)^2$

$$b(9a^2 + 6ab + b^2) \leq 2(a+b)^3$$

$$9a^2b + 6ab^2 + b^3 \leq 2a^3 + 6a^2b + 6ab^2 + 2b^3$$
$$2a^3 - 3a^2b + b^3 \geq 0, \qquad 2a^3 - 2a^2b - a^2b + b^3 \geq 0$$
$$2a^2(a-b) - b(a^2 - b^2) \geq 0, 2a^2(a-b) - b(a-b)(a+b) \geq 0$$
$$(a-b)(2a^2 - ab - b^2) \geq 0, (a-b)(2a^2 - 2ab + ab - b^2) \geq 0$$
$$(a-b)[2a(a-b) + b(a-b)] \geq 0, (a-b)^2(2a+b) \geq 0$$

Equality holds for $a = b$.

$$\text{By (1): } (3a+b)\sqrt{\frac{b}{a+b}} + (3b+c)\sqrt{\frac{c}{b+c}} + (3c+a)\sqrt{\frac{a}{c+a}} \leq$$
$$\leq \sqrt{2}(a+b) + \sqrt{2}(b+c) + \sqrt{2}(c+a) =$$
$$= 2\sqrt{2}(a+b+c) = 2\sqrt{2} \cdot 2s = 4\sqrt{2}s \overset{MITRINOVIC}{\leq} 4\sqrt{2} \cdot \frac{3R\sqrt{3}}{2} = 6R\sqrt{6}$$

Equality holds for an equilateral triangle: $a = b = c$.

3.42. If in $\Delta ABC, \frac{\pi}{2} > \mu(A), \mu(B), \mu(C) \geq \frac{\pi}{4}$ then:

$$\tan^2 A + \tan^2 B + \tan^2 C + \tan^2 A \cdot \tan^2 B \cdot \tan^2 C \geq 36$$

Solution:

$$LHS \overset{(a)}{\geq} \frac{1}{3}\left(\sum \tan A\right)^2 + \pi \tan^2 A = \frac{(\pi \tan A)^2}{3} + (\pi \tan A)^2$$
$$= \frac{4x^2}{3} (x = \pi \tan A) \overset{?}{\geq} 36 \Leftrightarrow x^2 \overset{?}{\geq} 27 \Leftrightarrow x \overset{?}{\geq} 3\sqrt{3}$$
$$\left(\because \tan A \tan B \tan C > 0 \text{ as } A, B, C < \frac{\pi}{2}\right)$$
$$\Leftrightarrow \sum \tan A \underset{(1)}{\overset{?}{\geq}} 3\sqrt{3}$$
$$\text{Let } f(x) = \tan x \ \forall x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$$
$$f''(x) = 2\tan x \sec^2 x > 0 \Rightarrow f(x) \text{ is convex}$$
$$\therefore \sum \tan A \overset{Jensen}{\geq} 3\tan\left(\frac{A+B+C}{3}\right) = 3\sqrt{3}$$
$$\Rightarrow \text{(1) is true} \Rightarrow \frac{4x^2}{3} \geq 36 \Rightarrow LHS \geq 36 \text{ (by (a)) (Proved)}$$

3.43. In ΔABC the following relationship holds:

$$\frac{(3a+b)(3b+c)(3c+a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} \leq \frac{1}{r\sqrt{s}}$$

Solution:

$$\text{We prove that: } \frac{3a+b}{a+b} \le \sqrt{2}\cdot\sqrt{\frac{a+b}{b}}$$

$$\text{By squaring: } \frac{(3a+b)^2}{(a+b)^2} \le \frac{2(a+b)}{b}$$

$$b(3a+b)^2 \le 2(a+b)^3$$

$$b(9a^2+6ab+b^2) \le 2(a^3+3a^2b+3ab^2+b^3)$$

$$9a^2b+6ab^2+b^3 \le 2a^3+6a^2b+6ab^2+2b^3$$

$$2a^3-3a^2b+b^3 \ge 0$$

$$2a^3-2a^2b-a^2b+b^3 \ge 0$$

$$2a^2(a-b)-b(a^2-b^2) \ge 0$$

$$2a^2(a-b)-b(a-b)(a+b) \ge 0$$

$$(a-b)(2a^2-ba-b^2) \ge 0$$

$$(a-b)(2a^2-2ab+ab-b^2) \ge 0$$

$$(a-b)\big(2a(a-b)+b(a-b)\big) \ge 0$$

$$(a-b)^2(2a+b) \ge 0. \text{ Which is true.}$$

$$\frac{3a+b}{a+b} \le \sqrt{2}\cdot\sqrt{\frac{a+b}{b}} \quad (1)$$

$$\text{Analogous: } \frac{3b+c}{b+c} \le \sqrt{2}\cdot\sqrt{\frac{b+c}{c}} \quad (2);\ \frac{3c+a}{c+a} \le \sqrt{2}\cdot\sqrt{\frac{c+a}{a}} \quad (3)$$

$$\text{By multiplying (1); (2); (3): } \frac{(3a+b)(3b+c)(3c+a)}{(a+b)(b+c)(c+a)} \le 2\sqrt{2}\cdot\sqrt{\frac{(a+b)(b+c)(c+a)}{abc}}$$

$$\frac{(3a+b)(3b+c)(3c+a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} \le \sqrt{\frac{8}{abc}} = \sqrt{\frac{8}{4RF}} =$$

$$= \sqrt{\frac{2}{Rrs}} \overset{EULER}{\le} \sqrt{\frac{2}{2r\cdot r\cdot s}} = \frac{1}{r\sqrt{s}}$$

Equality holds for an equilateral triangle: $a=b=c$.

3.44. In ΔABC the following relationship holds:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} = \frac{5s^2+r^2+4rR}{2s(s^2+2rR+r^2)}$$

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \ge \frac{2\sqrt{3}r^2}{R^3}$$

Solution:

Let be $P(x)=(x-a)(x-b)(x-c);\ a,b,c$ *sides in* ΔABC.

$$P'(x)=(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)$$

$$\frac{P'(x)}{P(x)}=\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c} \quad (1)$$

$$P(x)=x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$$

$$P(x)=x^3-2sx^2+(s^2+r^2+4rR)x-4srR$$

$$P'(x)=3x^2-4sx+s^2+r^2+4sR$$

Replacing in (1):

$$\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=\frac{3x^2-4sx+s^2+r^2+4rR}{x^3-2sx^2+(s^2+r^2+4rR)x-4srR} \quad (2)$$

Replacing $x=a+b+c=2s$ in (2):

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}=\frac{12s^2-4s\cdot 2s+s^2+r^2+4rR}{8s^3-8s^3+(s^2+r^2+4rR)\cdot 2s-4srR}=$$

$$=\frac{5s^2+r^2+4rR}{2s(s^2+2rR+r^2)}\overset{MITRINOVIC}{\geq}\frac{5\cdot 27r^2+r^2+4rR}{2\cdot\frac{3\sqrt{3}R}{2}\left(\frac{27R^2}{4}+2rR+r^2\right)}\overset{EULER}{\geq}$$

$$\geq\frac{135r^2+r^2+4r\cdot 2r}{3\sqrt{3}R\left(\frac{27R^2}{4}+2\cdot\frac{R}{2}\cdot R+\frac{R^2}{4}\right)}=$$

$$=\frac{144r^2}{3\sqrt{3}R\cdot 8R^2}=\frac{144r^2}{24\sqrt{3}R^3}=\frac{6\sqrt{3}r^2}{3R^3}=\frac{2\sqrt{3}r^2}{R^3}$$

3.45. In ΔABC the following relationship holds (I – incenter):

$$\frac{AI\cdot BI}{ab}+\frac{BI\cdot CI}{bc}+\frac{CI\cdot AI}{ca}\geq 3\sqrt[3]{\prod_{cyc}\left(1-\frac{2r}{h_a}\right)}$$

Solution:

$$AI^2=\frac{r^2}{\sin^2\frac{A}{2}}=\frac{r^2}{\frac{(s-b)(s-c)}{bc}}=\frac{bc\cdot\left(\frac{S}{s}\right)^2}{(s-b)(s-c)}=$$

$$=\frac{bc\cdot s(s-a)(s-b)(s-c)}{s^2(s-b)(s-c)}=$$

$$=\frac{bc(s-a)}{s}=bc-\frac{abc}{s}=bc-\frac{4Rrs}{s}=bc-4Rr=$$

$$=\frac{2S}{\sin A}-4Rr=\frac{ah_a}{\sin A}-4Rr=\frac{2R\sin A\cdot h_a}{\sin A}-4Rr=$$

$$= 2R(h_a - 2r) = 2Rh_a\left(1 - \frac{2r}{h_a}\right) = 2R \cdot \frac{2S}{a}\left(1 - \frac{2r}{h_a}\right) =$$

$$= \frac{4RS}{a}\left(1 - \frac{2r}{h_a}\right) = \frac{abc}{a}\left(1 - \frac{2r}{h_a}\right) = bc\left(1 - \frac{2r}{h_a}\right)$$

$$\frac{AI^2}{bc} = 1 - \frac{2r}{h_a} \quad (1)$$

$$\frac{AI \cdot BI}{ab} + \frac{BI \cdot CI}{bc} + \frac{CI \cdot AI}{ca} \geq 3\sqrt[3]{\frac{AI^2}{bc} \cdot \frac{BI^2}{ca} \cdot \frac{CI^2}{ab}} =$$

$$= 3\sqrt[3]{\left(1 - \frac{2r}{h_a}\right)\left(1 - \frac{2r}{h_b}\right)\left(1 - \frac{2r}{h_c}\right)} = 3\sqrt[3]{\prod_{cyc}\left(1 - \frac{2r}{h_a}\right)}$$

3.46. If a, b, c, d sides in a cyclic quadrilateral with Δ – area, s – semiperimeter then:

$$\left(s - a + \tan\frac{\pi}{20}\right)\left(s - b + \tan\frac{9\pi}{20}\right)\left(s - c + \tan\frac{13\pi}{20}\right)\left(s - d + \tan\frac{17\pi}{20}\right) > \left(\sqrt{\Delta} + 1\right)^4$$

Solution:

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)} \to \textit{Archimedes' formula}$$

$$\left(s - a + \tan\frac{\pi}{20}\right)\left(s - b + \tan\frac{9\pi}{20}\right)\left(s - c + \tan\frac{13\pi}{20}\right)\left(s - d + \tan\frac{17\pi}{20}\right) > \left(\sqrt{\Delta} + 1\right)^4$$

$$\left(s - a + \tan\frac{\pi}{20}\right)\left(s - b + \tan\frac{9\pi}{20}\right)\left(s - c + \tan\frac{13\pi}{20}\right)\left(s - d + \tan\frac{17\pi}{20}\right) \overset{J.Holder}{\geq}$$

$$\geq \left(\sqrt[4]{(s-a)(s-b)(s-c)(s-d)} + \sqrt[4]{\underbrace{\tan\frac{\pi}{20} \cdot \tan\frac{9\pi}{20} \cdot \tan\frac{13\pi}{20} \cdot \tan\frac{17\pi}{20}}_{s}}\right)^4 =$$

$$= \left(\sqrt[4]{\Delta^2} + \sqrt[4]{s}\right)^4 = \left(\sqrt{\Delta} + \sqrt[4]{s}\right)^4$$

$$\left.\begin{array}{l} \tan\frac{\pi}{20} = 1 + \sqrt{5} - \sqrt{5 + 2\sqrt{5}} \\ \tan\frac{9\pi}{20} = 1 + \sqrt{5} + \sqrt{5 + 2\sqrt{5}} \end{array}\right\} \Rightarrow \tan\frac{\pi}{20} \cdot \tan\frac{9\pi}{20} = \left(1 + \sqrt{5}\right)^2 - \left(5 + 2\sqrt{5}\right) =$$

$$= 1 + 2\sqrt{5} + 5 - 5 - 2\sqrt{5} = 1 \quad (1)$$

$$\left.\begin{array}{l}\tan\frac{13\pi}{20} = -\tan\frac{2\pi}{20} = -1+\sqrt{5}+\sqrt{5-2\sqrt{5}} \\ \tan\frac{17\pi}{20} = -\tan\frac{3\pi}{20} = -1+\sqrt{5}-\sqrt{5-2\sqrt{5}}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \tan\frac{13\pi}{20}\cdot\tan\frac{17\pi}{20} = \left(-1+\sqrt{5}\right)^2 - \left(5-2\sqrt{5}\right) =$$

$$= 1-2\sqrt{5}+5-5+2\sqrt{5} = 1 \quad (2)$$

From (1) and (2) $\Rightarrow s = 1$

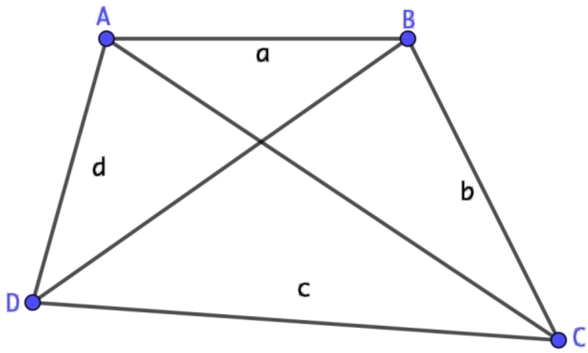
So, $\left[(s-a)+\tan\frac{\pi}{20}\right]\left[(s-b)+\tan\frac{9\pi}{20}\right]\left[(B-C)+\tan\frac{13\pi}{20}\right]\left[(s-d)+\tan\frac{17\pi}{20}\right] \geq$

$$\geq \left(\sqrt{\Delta}+1\right)^4$$

3.47. In $ABCD$ convexe quadrilateral the following relationship holds:

$$\mathbf{16\cdot[ABC]\cdot[BCD]\cdot[CDA]\cdot[DAB] = [ABCD]^4}$$

Solution:



$$Let\ AB = a, BC = b, CD = c, DA = d$$

$$LHS = 16\left(\frac{1}{2}ab\sin B\right)\left(\frac{1}{2}bc\sin C\right)\left(\frac{1}{2}cd\sin D\right)\left(\frac{1}{2}da\sin A\right)$$

$$= [(ab\sin B)(cd\sin D)][(bc\sin C)(da\sin A)]$$

$$\overset{G-A}{\leq} \frac{(ab\sin B + cd\sin D)^2}{4}\cdot\frac{(bc\sin C + da\sin A)^2}{4}$$

$$= \left(\frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D\right)\left(\frac{1}{2}bc\sin C + \frac{1}{2}da\sin A\right)^2$$

$$= ([ABC]+[CDA])^2([BCD]+[ABD])^2$$

$$= [ABCD]^2[ABCD]^2 = [ABCD]^4 \quad (Proved)$$

3.48. $\boldsymbol{\Omega_1 = (\tan(ax) + \tan(by) + \tan(cz))^2; a, b, c, x, y, z \in (0,1)}$

$$\boldsymbol{\Omega_2 = \left(\tan\sqrt{ax^3} + \tan\sqrt{by^3} + \tan\sqrt{cz^3}\right)\left(\tan\sqrt{xa^3} + \tan\sqrt{yb^3} + \tan\sqrt{zc^3}\right)}$$

Prove that: $\boldsymbol{\Omega_1 \le \Omega_2}$

Solution:

We consider $f(x) = \ln\tan e^x\,; f: (-\infty, 0) \to \mathbb{R}$

$$f'(x) = \frac{1}{\tan e^x}(1 + \tan^2 e^x)e^x = \frac{e^x}{\tan e^x} + e^x \tan e^x$$

$$f''(x) = \frac{e^x \tan e^x - e^x \cdot e^x(1 + \tan^2 e^x)}{\tan^2 e^x} + e^x \tan e^x + e^x(1 + \tan^2 e^x)e^x$$

$$= \frac{e^x(\tan e^x - e^x - e^x \tan^2 e^x)}{\tan^2 e^x} + e^x \tan e^x + e^{2x} + e^{2x}\tan^2 e^x$$

$$= \frac{e^x(\tan e^x - e^x - e^x\tan^2 e^x + \tan^3 e^x + e^x \tan^2 e^x + e^x \tan^4 e^x)}{\tan^2 e^x}$$

$$= \frac{e^x(e^x \tan^4 e^x + \tan^3 e^x + \tan e^x - e^x)}{\tan^2 e^x} =$$

$$= \frac{e^x\left(e^x(\tan^4 e^x - 1) + \tan e^x(\tan^2 e^x + 1)\right)}{\tan^2 e^x} =$$

$$= \frac{e^x\left(e^x(\tan^2 e^x - 1)(\tan^2 e^x + 1) + \tan e^x(\tan^2 e^x + 1)\right)}{\tan^2 e^x}$$

$$= \frac{e^x(\tan^2 e^x + 1)(e^x(\tan^2 e^x - 1) + \tan e^x)}{\tan^2 e^x}$$

Obviously, $\frac{e^x(\tan^2 e^x + 1)}{\tan^2 e^x} > 0$*. We prove that*

$$e^x(\tan^2 e^x - 1) + \tan e^x > 0$$

We denote $c^x = t$ *and as* $x < 0 \Rightarrow t \in (0,1)$

So, we prove that $t(\tan^2 t - 1) + \tan t > 0 \Rightarrow$

$\tan t > t(1 - \tan^2 t)$*. Inequality is obvious for* $t \in \left[\frac{\pi}{4}; 1\right)$

For $t \in \left(0; \frac{\pi}{4}\right)$ *the inequality becomes* $\frac{\tan t}{1 - \tan^2 t} > t$ *or* $\frac{\tan t}{1 - \tan^2 t} - t > 0$

Let be $g(t) = \frac{\tan t}{1 - \tan^2 t} - t; g: \left(0; \frac{\pi}{4}\right) \to \mathbb{R}$

$$g'(t) = \frac{(1 + \tan^2 t)(1 - \tan^2 t) + 2\tan t\,(1 + \tan^2 t)\tan t}{(1 - \tan^2 t)^2} - 1 =$$

$$= \frac{1 - \tan^4 t + 2\tan^2 t + 2\tan^4 t - (1 - \tan^2 t)^2}{(1 - \tan^2 t)^2} =$$

$$= \frac{1 - \tan^4 t + 2\tan^2 t + 2\tan^4 t - 1 + 2\tan^2 t - \tan^4 t}{(1 - \tan^2 t)^2} = \frac{4\tan^2 t}{(1 - \tan^2 t)^2} > 0$$

$$\Rightarrow g \uparrow on \left(0, \frac{\pi}{4}\right) \Rightarrow g(t) > \lim_{t\to 0} g(t) = 0$$

So, $f''(x) > 0 \Rightarrow f$ *convexe* $\Rightarrow f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$ *or*

$$\ln \tan e^{\frac{x+y}{2}} \le \frac{\ln \tan e^x + \ln \tan e^y}{2}$$

$$\ln \tan e^{\frac{x+y}{2}} \le \ln \sqrt{\tan e^x \tan e^y}$$

$$\tan e^{\frac{x+y}{2}} \le \sqrt{\tan e^x \tan e^y} \quad (1)$$

Let be $m, n \in (0,1) \Rightarrow \ln m, \ln n \in (-\infty, 0)$

In inequality (1) we consider $x = \ln m, y = \ln n$

$$\tan e^{\frac{\ln m + \ln n}{2}} \le \sqrt{\tan e^{mn} \tan e^{\ln n}}$$

$$\tan e^{\ln\sqrt{mn}} \le \sqrt{\tan m \tan n}$$

$$\tan\sqrt{mn} \le \sqrt{\tan m \tan n}; m, n \in (0,1)$$

For $m = \sqrt{ax^3}$ *and* $n = \sqrt{xa^3}$ *we obtain*

$$\left.\begin{array}{l} \tan ax \le \sqrt{\tan\sqrt{ax^3}\tan\sqrt{xa^3}} \\ \tan by \le \sqrt{\tan\sqrt{by^3}\tan\sqrt{yb^3}} \\ \tan cz \le \sqrt{\tan\sqrt{cz^3}\cdot\tan\sqrt{zc^3}} \end{array}\right\} \quad (2)$$

According to CBS

$$\left(\sqrt{\tan\sqrt{ax^3}}\cdot\sqrt{\tan\sqrt{xa^3}} + \sqrt{\tan\sqrt{by^3}}\sqrt{\tan\sqrt{yb^3}} + \sqrt{\tan\sqrt{cz^3}\tan\sqrt{zc^3}}\right)^2$$

$$\le \left(\tan\sqrt{ax^3} + \tan\sqrt{by^3} + \tan\sqrt{cz^3}\right)\left(\tan\sqrt{xa^3} + \tan\sqrt{yb^3} + \tan\sqrt{zc^3}\right)$$

Using inequality (2) $\Rightarrow (\tan ax + \tan by + \tan cz)^2 \le$

$$\le \left(\tan\sqrt{ax^3} + \tan\sqrt{by^3} + \tan\sqrt{cz^3}\right)\left(\tan\sqrt{xa^2} + \tan\sqrt{yb^3} + \tan\sqrt{zc^3}\right)$$

3.49. In any triangle ABC the following relationship holds:

$$s^2 \ge \frac{\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^2\left(\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2}\right)}{4\left(\frac{a}{(b+c)^4} + \frac{b}{(c+a)^4} + \frac{c}{(a+b)^4}\right)}$$

Solution:

First, we prove that if $a, b, c, x_1, x_2, x_3 > 0$ *then:*

$$(a+b+c)^2(ax_1^4 + bx_2^4 + cx_3^4) \ge (ax_1 + bx_2 + cx_3)^2(ax_1^2 + bx_2^2 + cx_3^2) \quad (1)$$

$$ax_1^4+bx_2^4+cx_3^4=\frac{a(x_1^2)^3}{x_1^2}+\frac{b(x_2^2)^3}{x_2^2}+\frac{c(x_3^2)^3}{x_3^2}=$$

$$=\frac{(ax_1^2)^3}{(ax_1)^2}+\frac{(bx_2^2)^3}{(bx_2)^2}+\frac{(cx_3^2)^3}{(cx_3)^2}\overset{RADON}{\geq}\frac{(ax_1^2+bx_2^2+cx_3^2)^3}{(ax_1+bx_2+cx_3)^2}\quad (2)$$

$$ax_1^2+bx_2^2+cx_3^2=\frac{(ax_1)^2}{a}+\frac{(bx_2)^2}{b}+\frac{(cx_3)^2}{c}\overset{RADON}{\geq}\frac{(ax_1+bx_2+cx_3)^2}{a+b+c}$$

$$a+b+c\geq\frac{(ax_1+bx_2+cx_3)^2}{ax_1^2+bx_2^2+cx_3^2}$$

$$(a+b+c)^2\geq\frac{(ax_1+bx_2+cx_3)^4}{(ax_1^2+bx_2^2+cx_3^2)^2}\quad (3)$$

By multiplying (2); (3) we obtain (1):

$$(a+b+c)^2(ax_1^4+bx_2^4+cx_3^4)\geq\frac{(ax_1^2+bx_2^2+cx_3^2)^3}{(ax_1+bx_2+cx_3)^2}\cdot\frac{(ax_1+bx_2+cx_3)^4}{(ax_1^2+bx_2^2+cx_3^2)^2}=$$

$$=(ax_1+bx_2+cx_3)^2(ax_1^2+bx_2^2+cx_3^2)$$

We take in (1): $x_1=\frac{1}{b+c}; x_2=\frac{1}{c+a}; x_3=\frac{1}{a+b}$

$$a+b+c=2s$$

$$(2s)^2\left(a\cdot\frac{1}{(b+c)^4}+b\cdot\frac{b}{(c+a)^4}+c\cdot\frac{1}{(a+b)^4}\right)\geq$$

$$\geq\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\left(a\cdot\frac{1}{(b+c)^2}+b\cdot\frac{1}{(c+a)^2}+c\cdot\frac{1}{(a+b)^2}\right)$$

$$s^2\geq\frac{\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\left(\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right)}{4\left(\frac{a}{(b+c)^4}+\frac{b}{(c+a)^4}+\frac{c}{(a+b)^4}\right)}$$

Equality holds for an equilateral triangle: $a=b=c$.

3.50. In any triangle ABC the following relationship holds:

$$\frac{\tan^2 1^\circ\cdot\tan^2 2^\circ\cdot\tan^2 3^\circ}{h_a}+\frac{\tan^2 2^\circ}{h_b}+\frac{\tan^2 1^\circ}{h_c}>\frac{\left(2\sqrt{2}+1\right)\tan^2 3^\circ}{7s}$$

Solution:

Lemma 1

$$\tan 3^\circ-\tan 2^\circ-\tan 1^\circ=\tan 3^\circ\cdot\tan 2^\circ\cdot\tan 1^\circ$$

Proof:

Denote $\tan 1^\circ=x$. *We must prove that:*

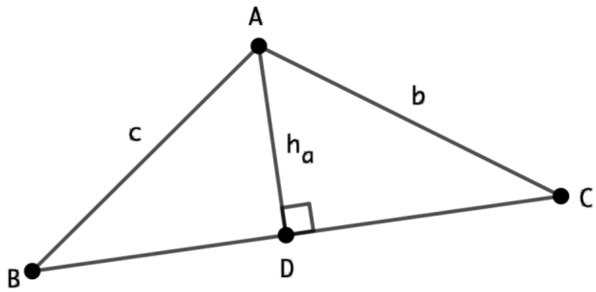
$$\frac{3x - x^3}{1 - 3x^2} - \frac{2x}{1 - x^2} - x = \frac{3x - x^3}{1 - 3x^2} \cdot \frac{2x}{1 - x^2} \cdot x$$
$$(3x - x^3)(1 - x^2) - 2x(1 - 3x^2) - x(1 - x^2)(1 - 3x^2) =$$
$$= 2x^2(3x - x^3)$$
$$(3 - x^2)(1 - x^2) - 2(1 - 3x^2) - (1 - x^2)(1 - 3x^2) =$$
$$= 2x(3x - x^3)$$
$$3 - 3x^2 - x^2 + x^4 - 2 + 6x^2 - 1 + 3x^2 + x^2 - 3x^4 =$$
$$= 6x^2 - 2x^4$$
$$3 - 3 - 2x^4 + 6x^2 = 6x^2 - 2x^4$$
$$0 = 0$$

Lemma 2

In any triangle ABC the following relationship holds:

$$h_a + h_b + h_c < \frac{7s}{2\sqrt{2} + 1}$$

Proof:



$$\cos B = \frac{BD}{AB} = \frac{BD}{c} \Rightarrow BD = c \cos B$$
$$\cos C = \frac{DC}{AC} = \frac{DC}{b} \Rightarrow DC = b \cos C$$
$$a = BD + DC = c \cos B + b \cos C \quad (1)$$
$$\sin B = \frac{AD}{AB} = \frac{h_a}{c} \Rightarrow h_a = c \sin B$$
$$\sin C = \frac{AD}{AC} = \frac{h_a}{b} \Rightarrow h_a = b \sin C$$

By adding: $2h_a = h_a + h_a = c \sin B + b \sin C \quad (2)$

By (1); (2): $a + 2h_a = c(\sin B + \cos B) + b(\sin C + \cos C) \quad (3)$

$$\sin B + \cos B = \sin B + 1 \cdot \cos B = \sin B + \tan\frac{\pi}{4} \cdot \cos B =$$

$$= \sin B + \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} \cdot \cos B = \frac{1}{\cos\frac{\pi}{4}}\left(\sin B \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos B\right) =$$
$$= \frac{1}{\frac{\sqrt{2}}{2}}\sin\left(B + \frac{\pi}{4}\right) = \sqrt{2}\sin\left(B + \frac{\pi}{4}\right) \le \sqrt{2}$$

Analogous: $\sin C + \cos C \le \sqrt{2}$

By (3): $a + 2h_a = c(\sin B + \cos B) + b(\sin C + \cos C) \le$
$$\le c \cdot \sqrt{2} + b \cdot \sqrt{2} = (b + c)\sqrt{2}$$
$$a + 2h_a \le (b + c)\sqrt{2}; a + 2h_b \le (c + a)\sqrt{2};$$
$$c + 2h_c \le (a + b)\sqrt{2}$$

Equality case can't be true in all three inequalities. By adding:
$$a + b + c + 2(h_a + h_b + h_c) < 2(a + b + c)\sqrt{2}$$
$$2s + 2(h_a + h_b + h_c) < 4s\sqrt{2}$$
$$2(h_a + h_b + h_c) < 2s(2\sqrt{2} - 1)$$
$$h_a + h_b + h_c < s(2\sqrt{2} - 1) = \frac{s(8 - 1)}{2\sqrt{2} + 1} = \frac{7s}{2\sqrt{2} + 1}$$

Back to the problem: $\frac{\tan^2 1°\cdot\tan^2 2°\cdot\tan^2 3°}{h_a} + \frac{\tan^2 2°}{h_b} + \frac{\tan^2 1°}{h_c} >$
$$\overset{BERGSTROM}{>} \frac{(\tan 1° \cdot \tan 2° \cdot \tan 3° + \tan 2° + \tan 1°)^2}{h_a + h_b + h_c} >$$
$$\overset{LEMMA\ 1}{>} \frac{\tan^2 3°}{h_b + h_c + h_a} \overset{LEMMA\ 2}{>} \frac{7s}{2\sqrt{2} + 1}$$

3.51. In ΔABC the following relationship holds:

$$\boldsymbol{a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}} + b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}} + b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}} + c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}} + c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}} + a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}} \le 4s}$$

Solution:

$$\text{Let be } f: [0,1] \to \mathbb{R}; f(x) = a\left(\frac{b}{a}\right)^x + b\left(\frac{a}{b}\right)^x$$
$$f'(x) = a\left(\frac{b}{a}\right)^x \log\left(\frac{b}{a}\right) + b\left(\frac{a}{b}\right)^x \log\left(\frac{a}{b}\right) = \log\left(\frac{b}{a}\right)\left[a\left(\frac{b}{a}\right)^x - b\left(\frac{a}{b}\right)^x\right] =$$
$$= \log\left(\frac{b}{a}\right)\left(a \cdot \frac{b^x}{a^x} - b \cdot \frac{a^x}{b^x}\right) = \log\left(\frac{b}{a}\right) \cdot \frac{a \cdot b^{2x} - b \cdot a^{2x}}{a^x b^x} = 0$$

$$a\cdot b^{2x}-b\cdot a^{2x}=0\Rightarrow\frac{a}{b}=\left(\frac{a}{b}\right)^{2x}\Rightarrow 2x=1\Rightarrow x=\frac{1}{2}$$

x	0		$\frac{1}{2}$		1
$f'(x)$	$-----------$		0	$+++++++++$	
$f(x)$	$a+b$	↘	$2\sqrt{ab}$	↗	$a+b$

$$\min f(x)=f\left(\frac{1}{2}\right)=2\sqrt{ab}$$

$$\max f(x)=f(0)=f(1)=a+b$$

$$\frac{h_a}{w_a}\le 1\Rightarrow 0<h_a\le w_a\Rightarrow 0<\frac{h_a}{w_a}\le 1$$

$$f(x)\le a+b;(\forall)x\in[0,1]$$

$$a\left(\frac{b}{a}\right)^x+b\left(\frac{a}{b}\right)^x\le a+b,(\forall)x\in[0,1]$$

$$For\ x=\frac{h_c}{w_c}$$

$$a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}\le a+b$$

$$\sum_{cyc}\left(a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}\right)\le\sum_{cyc}(a+b)=4s$$

$$a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}+b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}}+c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}}+c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}}+a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}\le 4s$$

3.52. In ΔABC the following relationship holds:

$$\left(a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}\right)\left(b\left(\frac{c}{b}\right)^{\frac{h_a}{w_a}}+c\left(\frac{b}{c}\right)^{\frac{h_a}{w_a}}\right)\left(c\left(\frac{a}{c}\right)^{\frac{h_b}{w_b}}+a\left(\frac{c}{a}\right)^{\frac{h_b}{w_b}}\right)\ge 8abc$$

Solution:

$$Let\ be\ f:[0,1]\to\mathbb{R};f(x)=a\left(\frac{b}{a}\right)^x+b\left(\frac{a}{b}\right)^x$$

$$f'(x)=a\left(\frac{b}{a}\right)^x\log\left(\frac{b}{a}\right)+b\left(\frac{a}{b}\right)^x\log\left(\frac{a}{b}\right)=\log\left(\frac{b}{a}\right)\left[a\left(\frac{b}{a}\right)^x-b\left(\frac{a}{b}\right)^x\right]=$$

$$=\log\left(\frac{b}{a}\right)\left(a\cdot\frac{b^x}{a^x}-b\cdot\frac{a^x}{b^x}\right)=\log\left(\frac{b}{a}\right)\cdot\frac{a\cdot b^{2x}-b\cdot a^{2x}}{a^xb^x}=0$$

$$a\cdot b^{2x}-b\cdot a^{2x}=0\Rightarrow\frac{a}{b}=\left(\frac{a}{b}\right)^{2x}\Rightarrow 2x=1\Rightarrow x=\frac{1}{2}$$

x	0 $\qquad\qquad\qquad$ $\frac{1}{2}$ $\qquad\qquad\qquad$ 1
$f'(x)$	$- - - - - - - - - - -0 + + + + + + + + + +$
$f(x)$	$a+b$ ↘ $2\sqrt{ab}$ ↗ $a+b$

$$\min f(x)=f\left(\frac{1}{2}\right)=2\sqrt{ab}$$

$$\frac{h_a}{w_a}\le 1\Rightarrow 0<h_a\le w_a\Rightarrow 0<\frac{h_a}{w_a}\le 1$$

$$f(x)=a\left(\frac{b}{a}\right)^x+b\left(\frac{a}{b}\right)^x\ge 2\sqrt{ab};(\forall)x\in[0,1]$$

$$For\ x=\frac{h_c}{w_c}$$

$$a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}\ge 2\sqrt{ab}$$

$$\prod_{cyc}\left(a\left(\frac{b}{a}\right)^{\frac{h_c}{w_c}}+b\left(\frac{a}{b}\right)^{\frac{h_c}{w_c}}\right)\ge 8abc$$

3.53. In ΔABC the following relationship holds:

$$\frac{(s^2+r_ar_b)(s^2+r_br_c)(s^2+r_cr_a)}{(s^2-r_ar_b)(s^2-r_br_c)(s^2-r_cr_a)}\ge \mathbf{8}$$

Solution:

$$\frac{r_ar_b}{s^2}+\frac{r_br_c}{s^2}+\frac{r_cr_a}{s^2}=\frac{\frac{S}{s-a}\cdot\frac{S}{s-b}}{s^2}+\frac{\frac{S}{s-b}\cdot\frac{S}{s-c}}{s^2}+\frac{\frac{S}{s-c}\cdot\frac{S}{s-a}}{s^2}=$$

$$=\frac{S^2}{s^2}\left(\frac{1}{(s-a)(s-b)}+\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}\right)=$$

$$=\frac{S^2}{s^2}\cdot\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)}=$$

$$=\frac{S^2}{s^2}\cdot\frac{s}{(s-a)(s-b)(s-c)}=\frac{s(s-a)(s-b)(s-c)\cdot s}{s^2(s-a)(s-b)(s-c)}=1$$

$$Let\ be\ f:(0,1)\to\mathbb{R};f(x)=\log\left(\frac{1+x}{1-x}\right)$$

$$f(x)=\log(1+x)-\log(1-x),f'(x)=\frac{1}{1+x}+\frac{1}{1-x}$$

$$f''(x) = \frac{-1}{(1+x)^2} + \frac{1}{(1-x)^2} = \frac{4x}{(1-x^2)^2} > 0$$

f convexe. By Jensen's inequality:

$$f\left(\frac{\frac{r_ar_b}{s^2} + \frac{r_br_c}{s^2} + \frac{r_cr_a}{s^2}}{3}\right) \le \frac{1}{3}\left(f\left(\frac{r_ar_b}{s^2}\right) + f\left(\frac{r_br_c}{s^2}\right) + f\left(\frac{r_cr_a}{s^2}\right)\right)$$

$$f\left(\frac{1}{3}\right) \le \frac{1}{3}\log\left(\prod_{cyc}\left(\frac{1+\frac{r_ar_b}{s^2}}{1-\frac{r_ar_b}{s^2}}\right)\right), \log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) \le \frac{1}{3}\log\left(\prod_{cyc}\left(\frac{1+\frac{r_ar_b}{s^2}}{1-\frac{r_ar_b}{s^2}}\right)\right)$$

$$\log\left(\frac{3+1}{3-1}\right) \le \frac{1}{3}\log\left(\prod_{cyc}\left(\frac{s^2+r_ar_b}{s^2-r_ar_b}\right)\right), 3\log 2 \le \log\left(\prod_{cyc}\left(\frac{s^2+r_ar_b}{s^2-r_ar_b}\right)\right)$$

$$\prod_{cyc}\left(\frac{s^2+r_ar_b}{s^2-r_ar_b}\right) \ge 8, \frac{(s^2+r_ar_b)(s^2+r_br_c)(s^2+r_cr_a)}{(s^2-r_ar_b)(s^2+r_br_c)(s^2-r_cr_a)} \ge 8$$

3.54. If $x, y \in \mathbb{R}$ then:

$$|\cos x \cos y \sin(x+y)| \le \frac{3\sqrt{3}}{8}$$

Solution:

First, we prove that for $x, y \in \mathbb{R}$

$$\cos^2 x + \cos^2 y + \sin^2(x+y) \le \tfrac{9}{4} \quad (1)$$

$$\frac{1+\cos 2x}{2} + \frac{1+\cos 2y}{2} + 1 - \cos^2(x+y) \le \frac{9}{4}$$

$$2 + 2\cos 2x + 2 + 2\cos 2y + 4 - 4\cos^2(x+y) \le 9$$

$$2(\cos 2x + \cos 2y) - 4\cos^2(x+y) \le 1$$

$$2 \cdot 2\cos\frac{2x+2y}{2}\cos\frac{2x-2y}{2} - 4\cos^2(x+y) \le 1$$

$$4\cos(x+y)\cos(x-y) - 4\cos^2(x+y) \le 1$$

$$4\cos(x+y)\left[\cos(x-y) - \cos(x+y)\right] \le 1$$

Denote $x + y = u; x - y = v$

$$4\cos u\,(\cos v - \cos u) \le 1$$

$$4\cos u \cos v - 4\cos^2 u \le 1$$

$$4\cos^2 u - 4\cos u\cos v + \cos^2 v + \sin^2 v \ge 0$$

$$(2\cos u - \cos v)^2 + \sin^2 v \ge 0$$

By AM-GM:

$$\sqrt[3]{\cos^2 x \cos^2 y \sin^2(x+y)} \le \frac{\cos^2 x + \cos^2 y + \sin^2(x+y)}{3} \overset{(1)}{\le} \frac{\frac{9}{4}}{3} = \frac{3}{4}$$

$$\cos^2 x \cos^2 y \sin^2(x+y) \le \frac{27}{64}$$

$$|\cos x \cos y \sin(x+y)| \le \frac{3\sqrt{3}}{8}$$

Equality holds for $x = y = \frac{\pi}{6}$.

3.55. If $x, y, z > 0; x + y + z = 2\pi$ then:

$$\frac{\cos^4 x}{y+z} + \frac{\cos^4 y}{z+x} + \frac{\cos^4(x+y)}{x+y} \ge \frac{9}{64\pi}$$

Solution:

First, we prove that for $x, y \in \mathbb{R}$

$$\cos^2 x + \cos^2 y + \cos^2(x+y) \ge \frac{3}{4} \quad (1)$$

$$\frac{1+\cos 2x}{2} + \frac{1+\cos 2y}{2} + 1 - \sin^2(x+y) \ge \frac{3}{4}$$

$$2(2 + \cos 2x + \cos 2y) + 4 - 4\sin^2(x+y) \ge 3$$

$$4 + 2(\cos 2x + \cos 2y) + 1 - 4\sin^2(x+y) \ge 0$$

$$2 \cdot 2\cos\frac{2x+2y}{2}\cos\frac{2x-2y}{2} - 4\sin^2(x+y) + 5 \ge 0$$

$$4\cos(x+y)\cos(x-y) - 4(1-\cos^2(x+y)) + 5 \ge 0$$

Denote $x + y = y; x - y = v$

$$4\cos u \cos v - 4 + 4\cos^2 u + 5 \ge 0$$

$$4\cos^2 u + 4\cos u \cos v + 1 \ge 0$$

$$4\cos^2 u + 4\cos u \cos v + \cos^2 v + \sin^2 v \ge 0$$

$$(2\cos u + \cos v)^2 + \sin^2 v \ge 0$$

$$\frac{\cos^4 x}{y+z} + \frac{\cos^4 y}{z+x} + \frac{\cos^4(x+y)}{x+y} \overset{BERGSTROM}{\ge}$$

$$\ge \frac{(\cos^2 x + \cos^2 y + \cos^2(x+y))^2}{2(x+y+z)} \overset{(1)}{\ge} \frac{\left(\frac{3}{4}\right)^2}{2 \cdot 2\pi} = \frac{9}{64\pi}$$

Equality holds for $x = y = z = \frac{2\pi}{3}$

3.56. In any triangle ABC the following relationship holds:

$$\sqrt[3]{r_a r_b} + \sqrt[3]{r_b r_c} + \sqrt[3]{r_c r_a} \le \sqrt[3]{3(4R+r)^2}$$

Solution:

$$Denote\ x = \frac{3r_a}{4R+r};\ y = \frac{3r_b}{4R+r};\ z = \frac{3r_c}{4R+r}$$

$$It's\ known\ that:\ r_a + r_b + r_c = 4R + r$$

$$\frac{x}{3} + \frac{y}{3} + \frac{z}{3} = \frac{r_a + r_b + r_c}{4R + r} = \frac{4R + r}{4R + r} = 1$$

$$x + y + z = 3$$

$$\sqrt[3]{xy} = \sqrt[3]{xy \cdot 1} \overset{AM-GM}{\leq} \frac{x + y + 1}{3}$$

$$\sqrt[3]{yz} = \sqrt[3]{yz \cdot 1} \overset{AM-GM}{\leq} \frac{y + z + 1}{3}$$

$$\sqrt[3]{zx} = \sqrt[3]{zx \cdot 1} \overset{AM-GM}{\leq} \frac{z + x + 1}{3}$$

$$By\ adding:$$

$$\sqrt[3]{xy} + \sqrt[3]{yz} + \sqrt[3]{zx} \leq \frac{2(x + y + z) + 1}{3} = \frac{2 \cdot 3 + 3}{3} = 3$$

$$\sqrt[3]{xy} + \sqrt[3]{yz} + \sqrt[3]{zx} \leq 3$$

$$\sqrt[3]{\frac{3r_a}{4R + r} \cdot \frac{3r_b}{4R + r}} + \sqrt[3]{\frac{3r_b}{4R + r} \cdot \frac{3r_c}{4R + r}} + \sqrt[3]{\frac{3r_c}{4R + r} \cdot \frac{3r_a}{4R + r}} \leq 3$$

$$\sqrt[3]{\frac{9}{(4R + r)^2}}\left(\sqrt[3]{r_a r_b} + \sqrt[3]{r_b r_c} + \sqrt[3]{r_c r_a}\right) \leq 3$$

$$\sum_{cyc} \sqrt[3]{\frac{9r_a r_b}{(4R + r)^2}} \leq 3, \qquad \sqrt[3]{9} \cdot \sum_{cyc} \sqrt[3]{r_a r_b} \leq 3\sqrt[3]{(4R + r)^2}$$

$$\sum_{cyc} \sqrt[3]{r_a r_b} \leq \sqrt[3]{3(4R + r)^2}$$

3.57. In ΔABC the following relationship holds:

$$\left(\mathbf{1} + \frac{\mathbf{2}\boldsymbol{r_a}}{\boldsymbol{r_b} + \boldsymbol{r_c}}\right)\left(\mathbf{1} + \frac{\mathbf{2}\boldsymbol{r_b}}{\boldsymbol{r_c} + \boldsymbol{r_a}}\right)\left(\mathbf{1} + \frac{\mathbf{2}\boldsymbol{r_c}}{\boldsymbol{r_a} + \boldsymbol{r_b}}\right) \geq \mathbf{8}$$

Solution:

$$Let\ be\ f: (0,1) \to \mathbb{R};\ f(x) = \log\left(\frac{1+x}{1-x}\right)$$

$$f(x) = \log(1 + x) - \log(1 - x)$$

$$f'(x) = \frac{1}{1 + x} + \frac{1}{1 - x}$$

$$f''(x) = \frac{-1}{(1+x)^2} + \frac{1}{(1-x^2)} = \frac{4x}{(1-x^2)^2} > 0; (\forall) x \in (0,1)$$

f convexe. By Jensen's inequality:

$$f\left(\frac{x_1+x_2+x_3}{3}\right) \le \frac{1}{3}\left(f(x_1) + f(x_2) + f(x_3)\right) \quad (1)$$

It is known that: $r_a + r_b + r_c = 4R + r$

$$\frac{r_a}{4R+r} + \frac{r_b}{4R+r} + \frac{r_c}{4R+r} = 1$$

In (1) we take: $x_1 = \frac{r_a}{4R+r}; x_2 = \frac{r_b}{4R+r}; x_3 = \frac{r_c}{4R+r}$

$$f\left(\frac{\frac{r_a}{4R+r} + \frac{r_b}{4R+r} + \frac{r_c}{4R+r}}{3}\right) \le$$

$$\le \frac{1}{3}\left(f\left(\frac{r_a}{4R+r}\right) + f\left(\frac{r_b}{4R+r}\right) + f\left(\frac{r_c}{4R+r}\right)\right)$$

$$f\left(\frac{1}{3}\right) \le \frac{1}{3}\sum_{cyc} \log\left(\frac{1+\frac{r_a}{4R+r}}{1-\frac{r_a}{4R+r}}\right)$$

$$\log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) \le \frac{1}{3}\log\left(\prod_{cyc}\left(\frac{4R+r+r_a}{4R+r-r_a}\right)\right)$$

$$3\log\left(\frac{\frac{4}{3}}{\frac{2}{3}}\right) \le \log\left(\prod_{cyc}\left(\frac{r_a+r_b+r_c+r_a}{r_a+r_b+r_c-r_a}\right)\right)$$

$$\log(2^3) \le \log\left(\prod_{cyc}\left(\frac{2r_a+r_b+r_c}{r_b+r_c}\right)\right)$$

$$8 \le \prod_{cyc}\left(1+\frac{2r_a}{r_b+r_c}\right)$$

Equality holds for an equilateral triangle $(a = b = c)$.

3.58. In acute ΔABC the following relationship holds:

$$\sin A \sin B \sin C > ABC\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)^2$$

Solution:

First, we prove Cusa-Huygens reversed inequality:

$$\frac{\sin x}{x} > \frac{1+\cos x}{2}; x \in \left(0, \frac{\pi}{2}\right) \quad (1)$$

$$Let\ be\ f: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}; f(x) = 2\sin x - x - x\cos x$$

$$f'(x) = 2\cos x - 1 - \cos x + x\sin x = \cos x - 1 + x\sin x$$

$$f''(x) = (\cos x - 1 + x\sin x)' = -\sin x + \sin x + x\cos x$$

$$f''(x) = x\cos x > 0; (\forall)x \in \left(0, \frac{\pi}{2}\right)$$

$$\inf f'(x) = \lim_{\substack{x\to 0\\ x>0}} f'(x) = \cos 0 - 1 + 0 \cdot \sin 0 = 0$$

$$\Rightarrow f'(x) > 0; (\forall)x \in \left(0, \frac{\pi}{2}\right)$$

$$\inf f(x) = \lim_{\substack{x\to 0\\ x>0}} f(x) = 2\sin 0 - 0 - 0 \cdot \cos 0 = 0$$

$$\Rightarrow f(x) > 0, (\forall)x \in \left(0, \frac{\pi}{2}\right)$$

$$2\sin x - x - x\cos x > 0 \Rightarrow \sin x > \frac{x(1+\cos x)}{2}$$

$$\sin x > \frac{x\left(1 + 2\cos^2\frac{x}{2} - 1\right)}{2} = x\cos^2\frac{x}{2} \quad (2)$$

We apply (2) for $x = A, x = B, x = C$ and multiplying:

$$\sin A > A\cos^2\frac{A}{2} \Rightarrow \prod_{cyc} \sin A > \prod_{cyc} \left(A\cos^2\frac{A}{2}\right)$$

$$\sin A \sin B \sin C > ABC\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)^2$$

3.59. In ΔABC the following relationship holds:

$$\mathbf{6(\sin A + \sin B + \sin C) + 3(\sin 2A + \sin 2B + \sin 2C) +}$$
$$\mathbf{+2(\sin 3A + \sin 3B + \sin 3C)} > 0$$

Solution:

$$6\sin A + 3\sin 2A + 2\sin 3A =$$
$$= 6\sin A + 6\sin A\cos A + 2\sin A\,(3 - 4\sin^2 A) =$$
$$= \sin A\,(6 + 6\cos A + 6 - 8\sin^2 A) =$$
$$= \sin A\left(12 + 6\cos A - 8(1 - \cos^2 A)\right) =$$
$$= \sin A\,(12 - 8 + 6\cos A + 8\cos^2 A) =$$
$$= \sin A\,(8\cos^2 A + 6\cos A + 4) = 2\sin A\,(4\cos^2 A + 3\cos A + 2) =$$
$$= 2\sin A\left[4\left(\cos A + \frac{3}{8}\right)^2 - \frac{9-32}{16}\right] = 2\sin A\left(4\left(\cos A + \frac{3}{8}\right)^2 + \frac{23}{16}\right) > 0$$
$$6\sin A + 3\sin 2A + 2\sin 3A > 0$$

$$6\sum_{cyc}\sin A+3\sum_{cyc}\sin 2A+2\sum_{cyc}\sin 3A>0$$

3.60. GENERALIZATION OF TAREK'S INEQUALITY

In $\Delta ABC, x, y, z$ are distances from $M\in Int(\Delta ABC)$ to sides of triangle. Prove that:

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\le 3\sqrt{\frac{R}{2}}$$

Solution:

$$S=\frac{xa}{2}+\frac{yb}{2}+\frac{zc}{2}=\frac{x}{\frac{2}{a}}+\frac{y}{\frac{2}{b}}+\frac{z}{\frac{2}{c}}\overset{BERGSTROM}{\geq}\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

$$\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2\le 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)S=2\cdot\frac{ab+bc+ca}{abc}\cdot\frac{abc}{4R}=$$

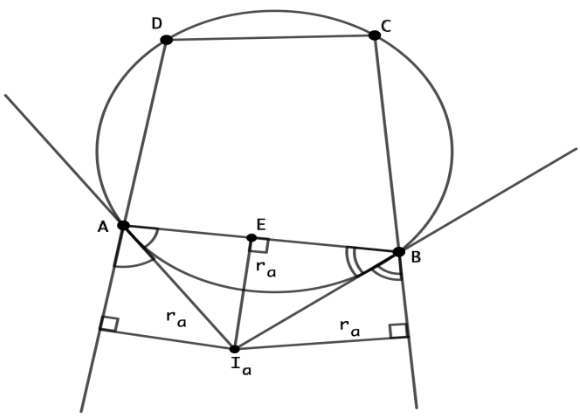
$$=2\cdot\frac{ab+bc+ca}{4R}\le\frac{a^2+b^2+c^2}{2R}\overset{LEIBNIZ}{\leq}\frac{9R^2}{2R}=9\cdot\frac{R}{2}$$

$$\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2\le 9\cdot\frac{R}{2},\qquad \sqrt{x}+\sqrt{y}+\sqrt{z}\le 3\sqrt{\frac{R}{2}}$$

3.61. If a, b, c, d are sides in a cyclic quadrilateral, r_a, r_b, r_c, r_d – exradii, s – semiperimeter then:

$$\frac{a}{r_a^2}+\frac{b}{r_b^2}+\frac{c}{r_c^2}+\frac{d}{r_d^2}\ge\frac{32}{s}$$

Solution:



$$\tan\left(\widehat{I_aAE}\right) = \frac{I_aE}{AE} = \frac{r_a}{AE} \quad (1)$$

$$\tan\left(\widehat{I_aAE}\right) = \tan\left(\frac{\pi - A}{2}\right) = \cot\frac{A}{2} \quad (2)$$

By (1); (2): $\frac{r_a}{AE} = \cot\frac{A}{2} \Rightarrow AE = r_a \tan\frac{A}{2} \quad (3)$

$$\tan\left(\widehat{I_aBE}\right) = \frac{I_aE}{BE} = \frac{r_a}{BE} \quad (4)$$

$$\tan\left(\widehat{I_aBE}\right) = \tan\left(\frac{\pi - B}{2}\right) = \cot\frac{B}{2} \quad (5)$$

By (4); (5): $\frac{r_a}{BE} = \cot\frac{B}{2} \Rightarrow BE = r_a \tan\frac{B}{2} \quad (6)$

$$a = AB = AE + EB \overset{(3);(6)}{=} r_a \tan\frac{A}{2} + r_a \tan\frac{B}{2} =$$

$$= r_a\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) \Rightarrow r_a = \frac{a}{\tan\frac{A}{2} + \tan\frac{B}{2}}$$

$$\frac{a}{r_a} = \tan\frac{A}{2} + \tan\frac{B}{2} \quad (7)$$

$$\frac{b}{8} = \tan\frac{B}{2} + \tan\frac{C}{2} \quad (8)$$

$$\frac{c}{r_c} = \tan\frac{C}{2} + \tan\frac{D}{2} \quad (9)$$

$$\frac{d}{r_d} = \tan\frac{D}{2} + \tan\frac{A}{2} \quad (10)$$

By adding (7); (8); (9); (10):

$$\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} + \frac{d}{r_d} = 2\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \tan\frac{D}{2}\right) \quad (11)$$

Let be $f: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}; f(x) = \tan\frac{x}{2}$

$$f'(x) = \frac{1}{2\cos^2\frac{x}{2}}; f''(x) = \frac{-\left(2\cos^2\frac{x}{2}\right)'}{\cos^4\frac{x}{2}} = \frac{4\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^4\frac{x}{2}} = \frac{2\sin x}{\cos^4\frac{x}{2}} > 0; f \text{ convexe}$$

By Jensen's inequality: $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \tan\frac{D}{2} \geq$

$$\geq 4\tan\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}+\frac{D}{2}}{2}\right) = 4\tan\left(\frac{A+B+C+D}{8}\right) = 4\tan\left(\frac{2\pi}{8}\right) = 4\tan\frac{\pi}{4} - 4 \quad (12)$$

By (11); (12): $\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} + \frac{d}{r_d} \geq 8 \quad (13)$

$$\frac{a}{r_a^2} + \frac{b}{r_b^2} + \frac{c}{r_c^2} + \frac{d}{r_d^2} = \frac{\left(\frac{a}{r_a}\right)^2}{a} + \frac{\left(\frac{b}{r_b}\right)^2}{b} + \frac{\left(\frac{c}{r_c}\right)^2}{c} + \frac{\left(\frac{d}{r_d}\right)^2}{d} \geq$$

$$\overset{BERGSTROM}{\geq} \frac{\left(\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} + \frac{d}{r_d}\right)^2}{a+b+c+d} \overset{(13)}{\geq} \frac{8^2}{2s} = \frac{64}{2s} = \frac{32}{s}$$

Equality holds for $a = b = c = d$.

3.62. If a, b, c, d are sides in a bicentric quadrilateral; r_a, r_b, r_c, r_d – exradii; s – semiperimeter then:

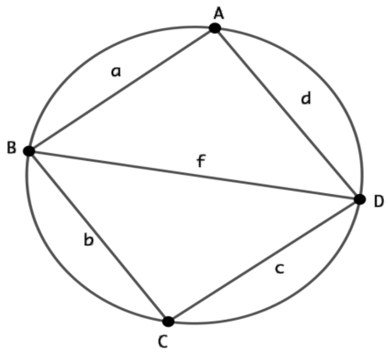
$$\frac{r_a^2}{a^3} + \frac{r_b^2}{b^3} + \frac{r_c^2}{c^3} + \frac{r_d^2}{d^3} \ge \frac{1}{s}$$

Solution:

Let be $AC = e; BD = f; AB = a; BC = b; CD = c; DA = d$.

By the law of cosines in $\Delta ABD; \Delta BCD$:

$$BD^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$$

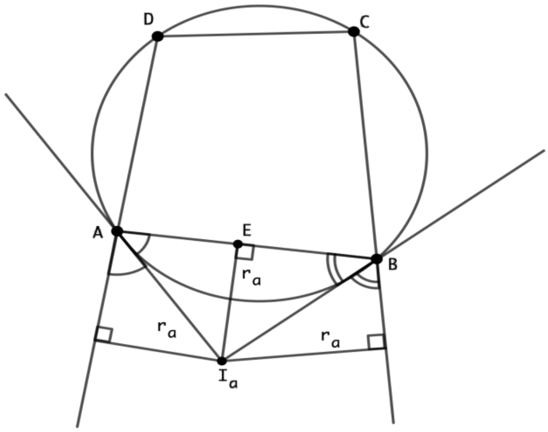


$$a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$$
$$a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos(\pi - A)$$
$$a^2 + d^2 - 2ad\cos A = b^2 + c^2 + 2bc\cos A$$
$$(2ad + 2bc)\cos A = a^2 + d^2 - b^2 - c^2$$
$$\cos A = \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)} \quad (1)$$

$$\tan^2\frac{A}{2} = \frac{1 - \cos A}{1 + \cos A} \overset{(1)}{=} \frac{1 - \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}}{1 + \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}} =$$

$$= \frac{2ad + 2bc - a^2 + b^2 + c^2 - d^2}{2ad + 2bc + a^2 - b^2 - c^2 + d^2} = \frac{(b + c)^2 - (a - d)^2}{(a + d)^2 - (b - c)^2} =$$

$$= \frac{(b+c-a+d)(b+c+a-d)}{(a+d+b-c)(a+d-b+c)} = \frac{(2s-2a)(2s-2d)}{(2s-2c)(2s-2b)} = \frac{(s-a)(s-d)}{(s-b)(s-c)} \quad (2)$$



$$\tan\left(\widehat{I_aAE}\right) = \frac{I_aE}{AE} = \frac{r_a}{AE} \quad (3)$$

$$\tan\left(\widehat{I_aAE}\right) = \tan\left(\frac{\pi - A}{2}\right) = \cot\frac{A}{2} \quad (4)$$

$$\text{By (3); (4): } \frac{r_a}{AE} = \cot\frac{A}{2} \Rightarrow AE = r_a \tan\frac{A}{2} \quad (5)$$

$$\tan\left(\widehat{I_aBE}\right) = \frac{I_aE}{BE} = \frac{r_a}{BE} \quad (6)$$

$$\tan\left(\widehat{I_aBE}\right) = \tan\left(\frac{\pi - B}{2}\right) = \cot\frac{B}{2} \quad (7)$$

$$\text{By (6); (7): } \frac{r_a}{BE} = \cot\frac{B}{2} \Rightarrow BE = r_a \tan\frac{B}{2} \quad (8)$$

$$a = AB = AE + EB \overset{(5);(8)}{=} r_a \tan\frac{A}{2} + r_a \tan\frac{B}{2} =$$

$$= r_a\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) \Rightarrow r_a = \frac{a}{\tan\frac{A}{2} + \tan\frac{B}{2}}$$

$$\text{Denote } \Delta = \text{Area } [ABCD]\text{: } r_a = \frac{a}{\tan\frac{A}{2} + \tan\frac{B}{2}} = \frac{a\Delta}{\Delta\tan\frac{A}{2} + \Delta\tan\frac{B}{2}} =$$

$$= \frac{a\Delta}{\sqrt{(s-a)(s-b)(s-c)(s-d)} \cdot \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}} + \sqrt{(s-a)(s-b)(s-c)(s-d)} \cdot \sqrt{\frac{(s-b)(s-a)}{(s-c)(s-d)}}}$$

$$= \frac{a\Delta}{(s-a)(s-d) + (s-b)(s-a)} = \frac{a\Delta}{(s-a)(2s-b-d)} = \frac{a\Delta}{(s-a)(a+c)}$$

$$\frac{r_a}{a} = \frac{\Delta}{(s-a)(a+c)} \quad (9)$$

$$\text{Analogous: } \frac{r_c}{c} = \frac{\Delta}{(s-c)(a+c)} \quad (10)$$

$$\frac{r_a}{a} + \frac{r_c}{c} = \frac{\Delta}{a+c}\left(\frac{1}{s-a} + \frac{1}{s-c}\right) =$$

$$= \frac{\Delta}{a+c} \cdot \frac{s-c+s-a}{(s-a)(s-c)} = \frac{\Delta}{a+c} \cdot \frac{b+d}{(s-a)(s-c)} \quad (11)$$

$$\text{Analogous: } \frac{r_b}{b}+\frac{r_d}{d}=\frac{\Delta}{b+c}\cdot\frac{a+c}{(s-a)(s-c)} \quad (12)$$

By adding (11); (12):

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}+\frac{r_d}{d}=\frac{\Delta}{a+c}\cdot\frac{b+d}{(s-a)(s-c)}+\frac{\Delta}{b+d}\cdot\frac{a+c}{(s-b)(s-c)}\overset{AM-GM}{\geq}$$

$$\geq \Delta\cdot 2\sqrt{\frac{(b+d)(a+c)}{(a+c)(b+d)(s-a)(s-b)(s-c)(s-d)}}=$$

$$=\Delta\cdot\frac{2}{\Delta}=2 \quad (13)$$

$$\frac{r_a}{a^2}+\frac{r_b}{b^2}+\frac{r_c}{c^2}+\frac{r_d}{d^2}=\frac{\left(\frac{r_a}{a}\right)^2}{a}+\frac{\left(\frac{r_b}{b}\right)^2}{b}+\frac{\left(\frac{r_c}{c}\right)^2}{c}+\frac{\left(\frac{r_d}{d}\right)^2}{d}\geq$$

$$\overset{BERGSTROM}{\geq}\frac{\left(\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}+\frac{r_d}{d}\right)^2}{a+b+c+d}\geq\frac{2^2}{2s}=\frac{2}{s}$$

3.63. In ΔABC the following relationship holds:

$$\prod_{cyc}\left(\cot\frac{A}{2}\cot\frac{B}{2}+\tan\frac{A}{2}\tan\frac{B}{2}\right)+\prod_{cyc}\left(\cot\frac{A}{2}\cot\frac{B}{2}-\tan\frac{A}{2}\tan\frac{B}{2}\right)\geq\frac{1064}{27}$$

Solution:

$$\text{Let } x=\tan\frac{A}{2}\tan\frac{B}{2};\, y=\tan\frac{B}{2}\tan\frac{C}{2};\, z=\tan\frac{C}{2}\tan\frac{A}{2}$$

$$\Rightarrow x+y+z=1;(0<x,y,z<1)\Rightarrow xyz\leq\frac{(x+y+z)^3}{27}=\frac{1}{27}$$

$$\text{Inequality}\Leftrightarrow\prod\left(\frac{1}{x}+x\right)+\prod\left(\frac{1}{x}-x\right)\geq\frac{1064}{27}$$

$$\Leftrightarrow(1+x^2)(1+y^2)(1+z^2)+(1-x^2)(1-y^2)(1-z^2)\geq\frac{1064}{27}xyz$$

$$\Leftrightarrow 2+2(x^2y^2+y^2z^2+z^2x^2)\geq\frac{1064}{27}xyz$$

$$\text{We have: }\Leftrightarrow 2[1+x^2y^2+y^2z^2+z^2x^2]\geq 2[1+xyz(x+y+z)]$$

$$=2[1+xyz].\text{ We must show that:}$$

$$2[1+xyz]\geq\frac{1064}{27}xyz\Leftrightarrow 1\geq\frac{505}{27}xyz\Leftrightarrow xyz\leq\frac{27}{505}$$

It is true because: $xyz\leq\frac{1}{27}\leq\frac{27}{505}$. Proved.

3.64. If in $\Delta ABC; s=\frac{1}{2}$ then:

$$\boldsymbol{a\cdot e^{\frac{m_a}{a}}+b\cdot e^{\frac{m_b}{b}}+c\cdot e^{\frac{m_c}{c}}\ge e^{m_a+m_b+m_c}}$$

Solution:

$$\textit{Let be } f_1,f_2,f_3:(0,\infty)\to\mathbb{R}$$
$$f_1(x)=ax\ln x-(a+m_a)x;\ f_2(x)=bx\ln x-(b+m_b)x$$
$$f_3(x)=cx\ln x-(c+m_c)x$$
$$f_1'(x)=a(\ln x+1)-(a+m_a)=a\ln x-m_a$$
$$f_1'(x)=0\Rightarrow a\ln x=m_a\Rightarrow \ln x=\frac{m_a}{a}$$
$$\ln x=\ln e^{\frac{m_a}{a}}\Rightarrow x=e^{\frac{m_a}{a}}$$
$$\min f_1(x)=f_1\left(e^{\frac{m_a}{a}}\right)=a\cdot e^{\frac{m_a}{a}}\cdot\ln e^{\frac{m_a}{a}}-(a+m_a)\cdot e^{\frac{m_a}{a}}=$$
$$=m_a\cdot e^{\frac{m_a}{a}}-ae^{\frac{m_a}{a}}-m_a\cdot e^{\frac{m_a}{a}}=-ae^{\frac{m_a}{a}}$$
$$\textit{Analogous: } \min f_2(x)=-be^{\frac{m_b}{b}};\ \min f_3(x)=-ce^{\frac{m_c}{c}}$$
$$f_1+f_2+f_3:(0,\infty)\to\mathbb{R}$$
$$(f_1+f_2+f_3)(x)=f_1(x)+f_2(x)+f_3(x)$$
$$\min(f_1+f_2+f_3)(x)=-(a+b+c)e^{\frac{m_a+m_b+m_c}{a+b+c}}$$
$$\min f_1(x)+\min f_2(x)+\min f_3(x)\le\min(f_1+f_2+f_3)(x)$$
$$-ae^{\frac{m_a}{a}}-be^{\frac{m_b}{b}}-ce^{\frac{m_c}{c}}\le-(a+b+c)e^{\frac{m_a+m_b+m_c}{a+b+c}}$$
$$ae^{\frac{m_a}{a}}+be^{\frac{m_b}{b}}+ce^{\frac{m_c}{c}}\ge(a+b+c)e^{\frac{m_a+m_b+m_c}{a+b+c}}=$$
$$=(2s)\cdot e^{\frac{m_a+m_b+m_c}{2s}}=\left(2\cdot\frac{1}{2}\right)\cdot e^{\frac{m_a+m_b+m_c}{2\cdot\frac{1}{2}}}=e^{m_a+m_b+m_c}$$
$$\textit{Equality holds for } a=b=c=\tfrac{1}{6}.$$

3.65. If $0<x,y,z<\frac{\pi}{6}$ then:

$$\boldsymbol{(\sin^2 x)^{\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right)}+(\sin^2 y)^{\sin\left(\frac{z+x}{2}\right)\cos\left(\frac{z-x}{2}\right)}+}$$
$$\boldsymbol{+(\sin^2 z)^{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}}>1$$

Solution:

$$\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right)=\frac{1}{2}[\sin y+\sin z]$$

$$\sin\left(\frac{x+z}{2}\right)\cos\left(\frac{z-x}{2}\right)=\frac{1}{2}[\sin z+\sin x]$$
$$\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)=\frac{1}{2}[\sin x+\sin y]$$
$$\Rightarrow LHS\geq(\sin x)^{(\sin y+\sin z)}+(\sin y)^{[\sin z+\sin x]}+(\sin z)^{[\sin x+\sin y]}$$
$$=(\sin x)^{\sin y}(\sin x)^{\sin z}+(\sin y)^{\sin z}\cdot(\sin y)^{\sin x}+(\sin z)^{\sin x}\cdot(\sin z)^{\sin y}$$
$$Let\ X=\sin x;Y=\sin y;Z=\sin z$$
$$\left(X,Y,Z\in\left(0;\frac{1}{2}\right)\right)$$
$$We\ prove\ that:\ X^Y\cdot X^Z+Y^Z\cdot Y^X+Z^X\cdot Z^Y>1$$
Using AM-GM:
$$X^Y\cdot X^Z+Y^Z\cdot Y^X+Z^X\cdot Z^Y\geq 3\sqrt[3]{(X^Y\cdot Y^Z\cdot Z^X)(Y^X\cdot X^Z\cdot Z^X)}$$
$$But\ X^Y\cdot Y^Z\cdot Z^X>\frac{1}{3\sqrt{3}};Y^X\cdot X^Z\cdot Z^X>\frac{1}{3\sqrt{3}}$$
Now, we want to prove: $X^Y\cdot Y^Z\cdot Z^X>\frac{1}{3\sqrt{3}}$ *(Similarly:* $Y^X\cdot X^Z\cdot Z^X>\frac{1}{3\sqrt{3}}$*)*
$$\Leftrightarrow Y\ln X+Z\ln Y+X\ln Z>-\ln(3\sqrt{3})$$
Using Jensen's inequality with $f(t)=\ln(\sin t),t\in\left(0,\frac{\pi}{2}\right)$
$$Y\ln X+Z\ln Y+X\ln Z\geq(X+Y+Z)\ln\left(\frac{XY+YZ+ZX}{X+Y+Z}\right)$$
$$>\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)\ln\left(\frac{\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}\right)=\frac{3}{2}\ln\frac{1}{2}=-\frac{3}{2}\ln 2>-\ln(3\sqrt{3})$$
(Because $g(n)=n\ln\frac{\alpha}{n}\searrow\left(0;\frac{1}{2}\right)$*)*

Hence, $X^Y\cdot X^Z+Y^Z\cdot Y^X+Z^X+Z^X\cdot Z^Y>1$ *(Proved)*

3.66. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{1}{\tan\frac{A}{2}+\tan\frac{B}{2}}-\frac{1}{\cot\frac{A}{2}+\cot\frac{B}{2}}\right)\geq\sqrt{3}$$

Solution:

Denote $\tan\frac{A}{2}=x;\tan\frac{B}{2}=y;\tan\frac{C}{2}=z\Rightarrow xy+yz+zx=1$
$$\sum_{cyc}\left(\frac{1}{\tan\frac{A}{2}+\tan\frac{B}{2}}-\frac{1}{\cot\frac{A}{2}+\cot\frac{B}{2}}\right)=$$

$$= \sum_{cyc} \left(\frac{1}{x+y} - \frac{1}{\frac{1}{x} + \frac{1}{y}} \right) = \sum_{cyc} \left(\frac{1}{x+y} - \frac{xy}{x+y} \right) =$$

$$= \sum_{cyc} \frac{xy + yz + zx - xy}{x+y} = \sum_{cyc} \frac{z(x+y)}{x+y} =$$

$$= \sum_{cyc} z = x + y + z = \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq$$

$$\overset{JENSEN}{\geq} 3\tan\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3} \right) = 3\tan\left(\frac{A+B+C}{6} \right) =$$

$$= 3\tan\left(\frac{\pi}{6} \right) = 3 \cdot \frac{\sqrt{3}}{3} = \sqrt{3}$$

$$f: (0, \pi) \to \mathbb{R}; f(x) = \tan\frac{x}{2}; f'(x) = \frac{1}{2\cos^2\frac{x}{2}}$$

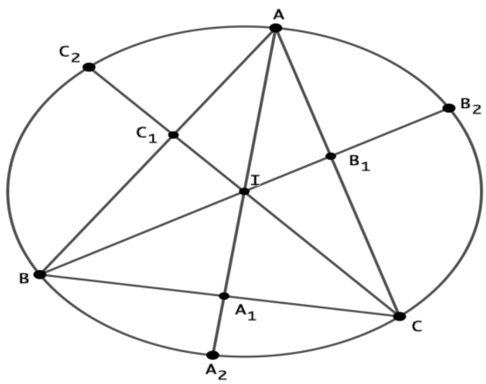
$$f''(x) = \frac{1}{2} \cdot \frac{-\left(\cos^2\frac{x}{2}\right)'}{\left(\cos^2\frac{x}{2}\right)^2} = \frac{1}{2} \cdot \frac{2 \cdot \frac{1}{2}\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^4\frac{x}{2}} = \frac{1}{4} \cdot \frac{\sin x}{\cos^4\frac{x}{2}} > 0;\ f\ convexe$$

Equality holds for an equilateral triangle.

3.67. Let w'_a, w'_b, w'_c be the circumpedal extensions of incentre in ΔABC. Prove that:

$$w'_a w'_b w'_c \geq \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

Solution:



$$AA_1 = w_a; BB_1 = w_b; CC_1 = w_c; AA_1 \cap BB_1 \cap CC_1 = \{I\}$$

$$\frac{A_1B}{A_1C} = \frac{c}{b} \Rightarrow \frac{A_1B}{A_1B + A_1C} = \frac{c}{b+c}$$

$$\frac{A_1B}{a} = \frac{c}{b+c} \Rightarrow A_1B = \frac{ac}{b+c}; A_1C = \frac{ab}{b+c}$$

$$\rho(A_1) = BA_1 \cdot A_1C = AA_1 \cdot A_1A_2$$

$$\frac{ac}{b+c} \cdot \frac{ab}{b+c} = w_a \cdot A_1A_2$$

$$A_1A_2 = \frac{a^2bc}{w_a(b+c)^2}$$

$$w_a' = AA_2 = AA_1 + A_1A_2 = w_a + \frac{a^2bc}{w_a(b+c)^2} \overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{w_a \cdot \frac{a^2bc}{w_a(b+c)^2}} = \frac{2a}{b+c}\sqrt{bc}$$

Analogous: $w_b' = \frac{2b}{c+a}\sqrt{ca}; w_c' = \frac{2c}{a+b}\sqrt{ab}$

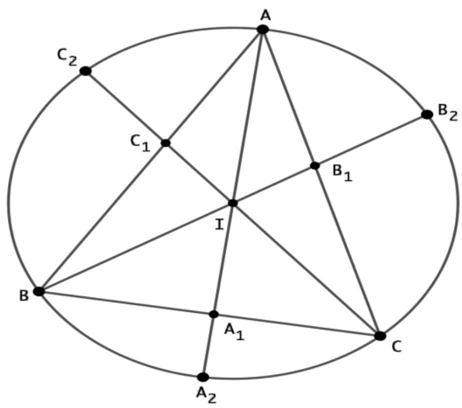
By multiplying: $w_a'w_b'w_c' \geq \frac{2a}{b+c}\sqrt{bc} \cdot \frac{2b}{c+a}\sqrt{ca} \cdot \frac{2c}{a+b}\sqrt{ab} =$

$$= \frac{8abc \cdot abc}{(a+b)(b+c)(c+a)} \geq \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

3.68. Let w_a', w_b', w_c' be the circumpedal extensions of incentre in ΔABC. Prove that:

$$\boldsymbol{w_a' + w_b' + w_c' \geq 6\sqrt[3]{\frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}}}$$

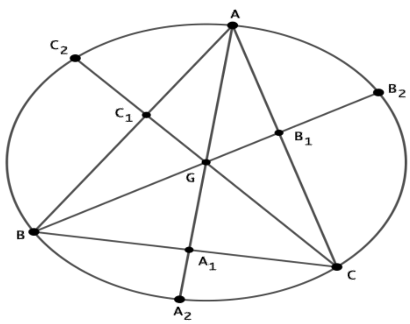
Solution:



$$AA_1 = w_a, BB_1 = w_b, CC_1 = w_c, AA_1 \cap BB_1 \cap CC_1 = \{I\}$$
$$\frac{A_1B}{A_1C} = \frac{c}{b} \Rightarrow \frac{A_1B}{A_1B + A_1C} = \frac{c}{b+c}$$
$$\frac{A_1B}{a} = \frac{c}{b+c} \Rightarrow A_1B = \frac{ac}{b+c}; A_1C = \frac{ab}{b+c}$$
$$\rho(A_1) = BA_1 \cdot A_1C = AA_1 \cdot A_1A_2$$
$$\frac{ac}{b+c} \cdot \frac{ab}{b+c} = w_a \cdot A_1A_2$$
$$A_1A_2 = \frac{a^2bc}{w_a(b+c)^2}$$
$$w_a' = AA_2 = AA_1 + A_1A_2 = w_a + \frac{a^2bc}{w_a(b+c)^2}$$
Analogous: $w_b' = w_b + \frac{ab^2c}{w_b(a+c)^2}; w_c' = w_c + \frac{abc^2}{w_c(b+a)^2}$

By summing:
$$w_a' + w_b' + w_c' =$$
$$= w_a + \frac{a^2bc}{w_a(b+c)^2} + w_b + \frac{ab^2c}{w_b(a+c)^2} + w_c + \frac{abc^2}{w_c(b+a)^2} \geq$$
$$\overset{AM-GM}{\geq} 6\sqrt[6]{w_aw_bw_c \cdot \frac{a^2bc \cdot ab^2c \cdot abc^2}{w_aw_bw_c(b+c)^2(a+c)^2(b+a)^2}} =$$
$$= 6\sqrt[6]{\frac{a^4b^4c^4}{(a+b)^2(b+c)^2(c+a)^2}} = 6\sqrt[3]{\frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}}$$

3.69. Let m_a', m_b', m_c' be the circumpedal extensions cevians of centroid in ΔABC. Prove that:
$$\boldsymbol{m_a' + m_b' + m_c' \geq 3\sqrt[3]{abc}}$$
Solution:



$$AA_1 = m_a; BB_1 = m_b; CC_1 = m_c; AA_1 \cap BB_1 \cap CC_1 = \{G\}$$

$$A_1B = A_1C = \frac{a}{2}$$

$$\rho(A_1) = BA_1 \cdot A_1C = AA_1 \cdot A_1A_2 \ \textit{(the power } A_1 \textit{ in circumcircle)}$$

$$\frac{a}{2} \cdot \frac{a}{2} = m_a \cdot A_1A_2 \Rightarrow A_1A_2 = \frac{a^2}{4m_a}$$

$$m'_a = AA_2 = AA_1 + A_1A_2 = m_a + \frac{a^2}{4m_a}$$

$$\textit{Analogous: } m'_b = m_b + \frac{b^2}{4m_b}; m'_c = m_c + \frac{c^2}{4m_c}$$

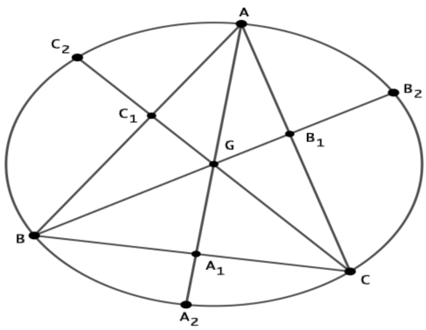
$$m'_a + m'_b + m'_c = m_a + m_b + m_c + \frac{a^2}{4m_a} + \frac{b^2}{4m_b} + \frac{c^2}{4m_c} \geq$$

$$\overset{AM-GM}{\geq} 6\sqrt[6]{m_am_bm_c \cdot \frac{a^2}{4m_a} \cdot \frac{b^2}{4m_b} \cdot \frac{c^2}{4m_c}} = 3\sqrt[3]{abc}$$

3.70. Let m'_a, m'_b, m'_c be the circumpedal extensions cevians of centroid in ΔABC. Prove that:

$$m'_am'_bm'_c \geq abc$$

Solution:



$$AA_1 = m_a; BB_1 = m_b; CC_1 = m_c; AA_1 \cap BB_1 \cap CC_1 = \{G\}$$

$$A_1B = A_1C = \frac{a}{2}$$

$$\rho(A_1) = BA_1 \cdot A_1C = AA_1 \cdot A_1A_2 \ \textit{(the power of } A_1 \textit{ in circumcircle)}$$

$$\frac{a}{2} \cdot \frac{a}{2} = m_a \cdot A_1A_2 \Rightarrow A_1A_2 = \frac{a^2}{4m_a}$$

$$m'_a = AA_2 = AA_1 + A_1A_2 = m_a + \frac{a^2}{4m_a} \overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{m_a \cdot \frac{a^2}{4m_a}} = 2 \cdot \frac{1}{2} \cdot a = a$$

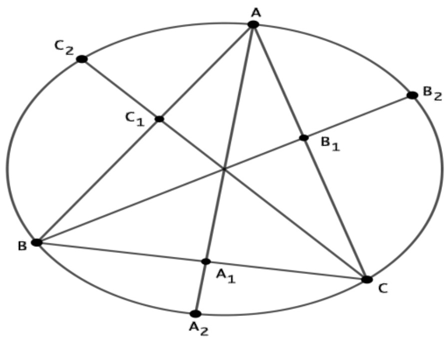
Analogous: $m'_b \geq b; m'_c \geq c$. *By multiplying:* $m'_a m'_b m'_c \geq abc$

3.71. Let k_a, k_b, k_c be the circumpedal extensions of three concurrent cevians in ΔABC. Denote $AA_1 = c_a; BB_1 = c_b; CC_1 = c_c$ and A_2, B_2, C_2 intersection points of cevians with circumcircle. If $BA_1 = x$; $CB_1 = y; AC_1 = z; k_a = AA_2; k_b = BB_2; k_c = CC_2$ then:

$$k_a + k_b + k_c \geq 6\sqrt[6]{xyz(a-x)(b-y)(c-z)}$$

$$k_a \cdot k_b \cdot k_c \geq 8\sqrt{xyz(a-x)(b-y)(c-z)}$$

Solution:



$\rho(A_1) = A_1B \cdot A_1C = x(a-x)$ *(power of the point* A_1 *to circumcircle)*

$$\rho(A_1) = AA_1 \cdot A_1A_2 = c_a \cdot A_1A_2$$

$$c_a \cdot A_1A_2 = x(a-x) \Rightarrow A_1A_2 = \frac{x(a-x)}{c_a}$$

$$k_a = AA_1 + A_1A_2 = c_a + \frac{x(a-x)}{c_a} \overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{c_a \cdot \frac{x(a-x)}{c_a}} = 2\sqrt{x(a-x)}$$

$$k_a \geq 2\sqrt{x(a-x)} \quad (1)$$

$$k_b \geq 2\sqrt{y(a-y)} \quad (2)$$

$$k_c \geq 2\sqrt{z(a-z)} \quad (3)$$

$$k_a + k_b + k_c = c_a + \frac{x(a-x)}{c_a} + c_b + \frac{y(b-y)}{c_b} + c_c + \frac{z(c-z)}{c_c} \geq$$

$$\overset{AM-GM}{\geq} 6\sqrt[6]{c_a \cdot c_b \cdot c_c \cdot \frac{x(a-x)}{c_a} \cdot \frac{y(b-y)}{c_b} \cdot \frac{z(c-z)}{c_c}} =$$

$$= 6\sqrt[6]{xyz(a-x)(b-y)(c-z)}$$

$$k_a \cdot k_b \cdot k_c \overset{(1);(2);(3)}{\geq} 8\sqrt{xyz(a-x)(b-y)(c-z)}$$

3.72. If $ABCD$ bicentric quadrilateral; I – incircle then:

$$(IA^2+IC^2)(IB^2+ID^2) \geq AB \cdot BC \cdot CD \cdot DA$$

Solution:

Let a,b,c,d be sides in quadrilateral; $s=\frac{a+b+c+d}{2}$ – semiperimeter;
$ABCD$ – tangential quadrilateral $\Rightarrow s-a=c; s-b=d;$
$s-c=a; s-d=b$. By Brahmagupta's formula ($ABCD$ – cyclic):

$$[ABCD]=\sqrt{(s-a)(s-b)(s-c)(s-d)}=\sqrt{abcd} \quad (1)$$

$$\sqrt{AB\cdot BC\cdot CD\cdot DA}=\sqrt{abcd}\overset{(1)}{=}[ABCD]=$$
$$=[AIB]+[BIC]+[CID]+[DIA]=$$
$$=\frac{AI\cdot BI\cdot\sin\left(\widehat{AIB}\right)}{2}+\frac{BI\cdot CI\cdot\sin\left(\widehat{BIC}\right)}{2}+\frac{CI\cdot DI\cdot\sin\left(\widehat{CID}\right)}{2}+$$
$$+\frac{DI\cdot AI\cdot\sin\left(\widehat{DIA}\right)}{2}\leq\frac{IA\cdot IB}{2}+\frac{IB\cdot IC}{2}+\frac{IC\cdot ID}{2}+\frac{ID\cdot IA}{2}=$$
$$=\frac{IA(IB+ID)+IC(IB+ID)}{2}=$$
$$=\frac{(IB+ID)(IA+IC)}{2}=2\cdot\frac{IA+IC}{2}\cdot\frac{IB+ID}{2}\leq$$
$$\overset{AM-QM}{\leq}2\cdot\sqrt{\frac{IA^2+IC^2}{2}}\cdot\sqrt{\frac{IB^2+ID^2}{2}}=\sqrt{(IA^2+IC^2)(IB^2+ID^2)}$$
$$\sqrt{(IA^2+IC^2)(IB^2+ID^2)}\geq\sqrt{AB\cdot BC\cdot CD\cdot DA}$$
$$(IA^2+IC^2)(IB^2+ID^2)\geq AB\cdot BC\cdot CD\cdot DA$$

3.73. In ΔABC the following relationship holds:

$$\sin^4 A+\sin^4 B+\sin^4 C+\sin^4\left(\frac{\pi}{3}+A\right)+\sin^4\left(\frac{\pi}{3}+B\right)+\sin^4\left(\frac{\pi}{3}+C\right)\leq\frac{27}{8}$$

Solution:

$$\left(\sin A-\sqrt{3}\cos A\right)^4\geq 0$$
$$\sin^4 A+9\cos^4 A+18\sin^2 A\cos^2 A-12\sqrt{3}\sin A\cos^3 A$$
$$-4\sqrt{3}\sin^3 A\cos A\geq 0$$
$$16\sin^4 A+9\cos^4 A+\sin^4 A+6\sin^2 A\cos^2 A+6\sin^2 A\cos^2 A+$$

$$+12\sqrt{3}\sin A\cos^3 A + 4\sqrt{3}\sin^3 A\cos A \le$$
$$\le 18\sin^4 A + 18\cos^4 A + 36\sin^2 A\cos^4 A$$
$$16\sin^4 A + \left(3\cos^2 A + \sin^2 A + \sqrt{3}\sin 2A\right)^2 \le$$
$$\le 18(\sin^4 A + 2\sin^2 A\cos^2 A + \cos^4 A)$$
$$16\sin^4 A + \left(\left(\sqrt{3}\cos A + \sin A\right)^2\right)^2 \le 18(\sin^2 A + \cos^2 A)^2$$
$$16\sin^4 A + \left(2\sin\left(\frac{\pi}{3} + A\right)\right)^4 \le 18 \cdot 1^2$$
$$16\sin^4 A + 16\sin^4\left(\frac{\pi}{3} + A\right) \le 18$$
$$\sin^4 A + \sin^4\left(\frac{\pi}{3} + A\right) \le \frac{18}{16} = \frac{9}{8} \quad (1)$$

Analogous: $\sin^4 B + \sin^4\left(\frac{\pi}{4} + B\right) \le \frac{9}{8}$ *(2);*

$$\sin^4 C + \sin^4\left(\frac{\pi}{3} + C\right) \le \frac{9}{8} \quad (3)$$

By adding (1); (2); (3):

$$\sin^4 A + \sin^4 B + \sin^4 C + \sin^4\left(\frac{\pi}{3} + A\right) + \sin^4\left(\frac{\pi}{3} + B\right) + \sin^4\left(\frac{\pi}{3} + C\right) \le \frac{27}{8}$$

Equality holds for an equilateral triangle: $A = B = C = \frac{\pi}{3}$

3.74. In ΔABC the following relationship holds:

$$\left(a^a \cdot b^b\right)^{\frac{1}{a+b}} + \left(b^b \cdot c^c\right)^{\frac{1}{b+c}} + \left(c^c \cdot a^a\right)^{\frac{1}{c+a}} \ge 6\sqrt{3}r$$

Solution:

Let be $f_1, f_2 : (0, \infty) \to \mathbb{R};\ f_1(x) = \frac{1}{2}x^2 - a\log x$

$$f_1'(x) = x - \frac{a}{x} = \frac{x^2 - a}{x} = 0 \Rightarrow x^2 = a \Rightarrow x = \sqrt{a}$$
$$\min f(x) = f_1\left(\sqrt{a}\right) = \frac{1}{2}\left(\sqrt{a}\right)^2 - a\log\sqrt{a} =$$
$$= \frac{1}{2}a - \frac{a}{2}\log a = \frac{a}{2}(1 - \log a)$$

Analogous: $\min f_2\,(x) = \frac{b}{2}(1 - \log b)$

$$\min(f_1 + f_2)\,(x) = (f_1 + f_2)\left(\frac{a+b}{2}\right) = \frac{a+b}{2}\left(1 - \log\left(\frac{a+b}{2}\right)\right)$$
$$\min f_1(x) + \min f_2(x) \le \min(f_1 + f_2)\,(x)$$

$$\frac{a}{2}(1-\log a)+\frac{b}{2}(1-\log b)\le\frac{a+b}{2}\left(1-\log\left(\frac{a+b}{2}\right)\right)$$
$$-\frac{a}{2}\log a-\frac{b}{2}\log b\le-\frac{a+b}{2}\log\left(\frac{a+b}{2}\right)$$
$$a\log a+b\log b\ge(a+b)\log\left(\frac{a+b}{2}\right)$$
$$\log(a^a\cdot b^b)\ge\log\left(\frac{a+b}{2}\right)^{a+b}$$
$$a^a\cdot b^b\ge\left(\frac{a+b}{2}\right)^{a+b}$$
$$a^{\frac{a}{a+b}}\cdot b^{\frac{b}{a+b}}\ge\frac{a+b}{2}\quad (1)$$

Analogous: $b^{\frac{b}{b+c}}\cdot c^{\frac{c}{b+c}}\ge\frac{b+c}{2}\quad (2)$

$$c^{\frac{c}{c+a}}\cdot a^{\frac{a}{c+a}}\ge\frac{c+a}{2}\quad (3)$$

By adding (1); (2); (3): $a^{\frac{a}{a+b}+\frac{a}{a+c}}+b^{\frac{b}{b+a}+\frac{b}{b+c}}+c^{\frac{c}{c+a}+\frac{a}{a+c}}\ge a+b+c=$
$=2s\overset{MITRINOVIC}{\ge}2\cdot 3\sqrt{3}r=6\sqrt{3}$. *Equality holds for* $a=b=c$.

3.75. If $x,y\in\mathbb{R}$ then:

$$\sin^4 x+\cos^4 x\sin^4 y+\cos^4 x\cos^4 y\ge\frac{1}{3}$$

When does the equality holds?

Solution:

$$\left(\frac{\sin^4 x+\cos^4 x\sin^4 y+\cos^4 x\cos^4 y}{3}\right)^{\frac{1}{4}}\overset{WAM-WGM}{\ge}$$
$$\ge\left(\frac{\sin^2 x+\cos^2 x\sin^2 y+\cos^2 x\cos^2 y}{3}\right)^{\frac{1}{2}}=$$
$$=\left(\frac{\sin^2 x+\cos^2 x}{3}\right)^{\frac{1}{2}}=\left(\frac{1}{3}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{3}}$$
$$\frac{\sin^4 x+\cos^4 x\sin^4 y+\cos^4 x\cos^4 y}{3}\ge\frac{1}{9}$$
$$\sin^4 x+\cos^4 x\sin^4 y+\cos^4 x\cos^4 y\ge\frac{1}{3}$$

Equality holds for:

$$\sin^4 x=\cos^4 x\sin^4 y=\cos^4 x\cos^4 y=\frac{1}{3}$$

$$x = \arcsin\left(\frac{1}{\sqrt[4]{3}}\right); y = \arcsin\left(\frac{1}{\sqrt{\sqrt{3}-1}}\right)$$

3.76. If $0 \le a \le b < \frac{\pi}{2}$ then:

$$\frac{1}{\cos a} + \frac{1}{\cos b} \ge \frac{1}{\cos(a - \sqrt{ab} + b)} + \frac{1}{\cos(\sqrt{ab})}$$

Solution:

$$\frac{1}{\cos a} + \frac{1}{\cos b} \ge \frac{1}{\cos(a-\sqrt{ab}+b)} + \frac{1}{\cos(\sqrt{ab})} \quad (1)$$

The function $\frac{1}{\cos t}$ on $\left[0, \frac{\pi}{2}\right)$ is increasing as $f'(t) > 0$. Also $f''(t)$ is increasing as $f''(t) > 0$. This means that for u, v, u', v' such that $u > u'$ and $u - v = u' - v' \ge 0$ we have $f(u) - f(v) \ge f(u') - f(v')$ due to $f'(t)$ is increasing. Consider that (1)→

$$\frac{1}{\cos b} - \frac{1}{\cos(\sqrt{ab})} \ge \frac{1}{\cos(a-\sqrt{ab}+b)} - \frac{1}{\cos a} \text{ and let } x = a + b - \sqrt{ab} \to$$

$\to a = x - \left[b - \sqrt{ab}\right]$. With $u = b, v = \sqrt{ab}, u' = x,$

$v' = x - \left(b - \sqrt{ab}\right)$ we have $u > u'$

(as $b > a + b - \sqrt{ab} \Rightarrow \sqrt{ab} > a$ true for $b > a$) and

$u - v = b - \sqrt{ab} \ge 0$ $(b \ge a), u' - v' = b - \sqrt{ab}$ therefore

(T)→ (1). Because of the symmetry of (1) it is valid for $a \ge b$. To show this we can replace b with a in the above proof. So (1) is true for any $a, b \in \left[0, \frac{\pi}{2}\right)$.

Equality for $a = b$.

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