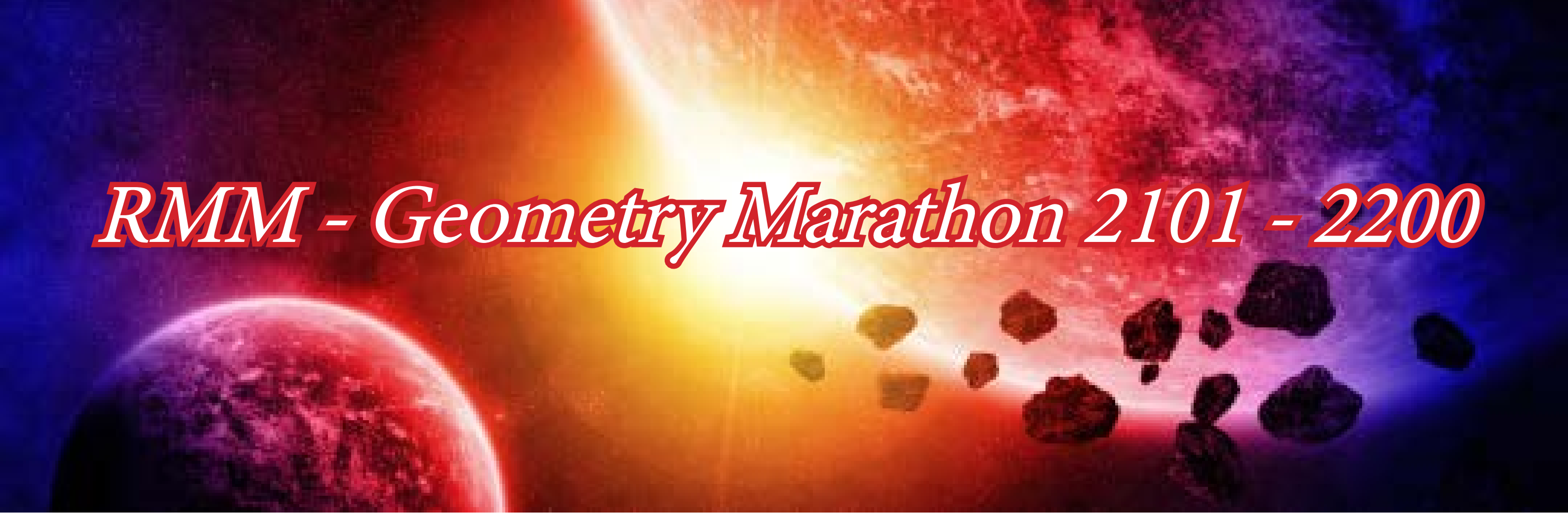

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2101. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a+r}{r_a-r}=\frac{8r_ar_br_c}{w_aw_bw_c}-2=3+\sum_{cyc}\frac{h_a}{r_a}=\sum_{cyc}\frac{s^2+r_br_c}{r_a(r_b+r_c)}=\sum_{cyc}\frac{b+c}{a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

1) $\sum_{cyc}\frac{r_a+r}{r_a-r}$

$$\sum_{cyc}\frac{r_a+r}{r_a-r}=\sum_{cyc}\frac{s\tan\frac{A}{2}+(s-a)\tan\frac{A}{2}}{s\tan\frac{A}{2}-(s-a)\tan\frac{A}{2}}=\sum_{cyc}\frac{(2s-a)\tan\frac{A}{2}}{a.\tan\frac{A}{2}}=\sum_{cyc}\frac{b+c}{a}\quad(1)$$

2) $\frac{8r_ar_br_c}{w_aw_bw_c}$

$$\frac{8r_ar_br_c}{w_aw_bw_c}-2=\frac{8sF}{\frac{2bc}{b+c}\cdot\frac{2ac}{a+c}\cdot\frac{2ba}{b+a}.\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}-2=$$

$$=\frac{8sF(a+b)(b+c)(a+c)}{8(abc)^2.\frac{s}{4R}}-2=$$

$$=\frac{s\frac{abc}{4R}.(a+b)(b+c)(a+c)}{(abc)^2.\frac{s}{4R}}-2=\frac{(a+b)(b+c)(a+c)}{abc}-2=$$

$$=\frac{(a^2b+b^2c+c^2a+a^2c+c^2b+b^2a)+2ab}{abc}-2=$$

$$=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}=\sum_{cyc}\frac{b+c}{a}\quad(2)$$

3) $3+\sum_{cyc}\frac{h_a}{r_a}$

$$3+\sum_{cyc}\frac{h_a}{r_a}=3+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}=3+\frac{h_ar_br_c+h_br_ar_c+h_cr_ar_b}{r_ar_cr_b}=$$

$$=3+\frac{\frac{2F}{a}s(s-a)+\frac{2F}{b}s(s-b)+\frac{2F}{c}s(s-c)}{sF}=3+\frac{2s-2a}{a}+\frac{2s-2b}{b}+\frac{2s-2c}{c}=$$

$$=\left(\frac{2s-2a}{a}+1\right)+\left(\frac{2s-2b}{b}+1\right)+\left(\frac{2s-2c}{c}+1\right)=$$

$$=\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}=\sum_{cyc}\frac{b+c}{a}\quad(3)$$

4) $\sum_{cyc}\frac{s^2+r_br_c}{r_a(r_b+r_c)}$

$$\sum_{cyc}\frac{s^2+r_br_c}{r_a(r_b+r_c)}=\sum_{cyc}\frac{s^2+r_br_c}{r_ar_b+r_ar_c}=\sum_{cyc}\frac{s^2+s(s-a)}{s(s-c)+s(s-b)}=$$
$$=\sum_{cyc}\frac{2s-a}{2s-(b+c)}=\sum_{cyc}\frac{b+c}{a}\quad(4)$$

2102. In ΔABC the following relationship holds:

$$tan\frac{A}{2}tan\frac{B}{2}=1-\frac{2r}{h_c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$tan\frac{A}{2}tan\frac{B}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=$$
$$=\frac{s-c}{s}=1-\frac{c}{s}=1-\frac{\frac{F}{s}}{\frac{2F}{c}}\cdot 2=1-\frac{2r}{h_c}$$

2103. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{n_a}{\sin B\sin C}\ge\sum_{cyc}\left(\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right).\sqrt{4r^2+(b-c)^2}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$\Rightarrow a^2n_a^2=a^2s^2-sa(a^2-(b-c)^2)\Rightarrow a^2n_a^2=a^2s^2-sa^3+sa(b-c)^2$$
$$\stackrel{?}{=}4r^2s^2+s^2(b-c)^2\Leftrightarrow a^2s^2-sa^3+sa(b-c)^2$$
$$\stackrel{?}{=}s(s-a)(a^2-(b-c)^2)+s^2(b-c)^2$$
$$=a^2s^2-sa^3-s(s-a)(b-c)^2+s^2(b-c)^2$$
$$=a^2s^2-sa^3-s^2(b-c)^2+sa(b-c)^2+s^2(b-c)^2$$

$$\Leftrightarrow a^2s^2 - sa^3 + sa(b-c)^2 \stackrel{?}{=} a^2s^2 - sa^3 + sa(b-c)^2 \to \text{true}$$

$$\therefore \frac{n_a^2}{(b-c)^2+4r^2} = \frac{s^2}{a^2} \Rightarrow \frac{n_a}{\sqrt{4r^2+(b-c)^2}}.\frac{1}{\sin B \sin C} = \frac{s}{a}.\frac{4R^2}{bc}$$

$$= \frac{4R^2 s}{4Rrs} = \frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b}$$

$$\left(\because \frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \ldots \text{reference : article titled ''New Triangle Inequalities With Brocard's Angle'' by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 − 7, published at : www.ssmrmh.ro}\right)$$

$$\Rightarrow \frac{n_a}{\sin B \sin C} \geq \left(\frac{m_b}{h_c} + \frac{m_c}{h_b}\right).\sqrt{4r^2+(b-c)^2} \text{ and analogs}$$

$$\therefore \sum_{cyc} \frac{n_a}{\sin B \sin C} \geq \sum_{cyc}\left(\left(\frac{m_b}{h_c} + \frac{m_c}{h_b}\right).\sqrt{4r^2+(b-c)^2}\right) \forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

2104. **In any Δ ABC, the following relationship holds :**

$$R + r - \sqrt{R^2 - 4r^2} \leq h_a \leq w_a \leq R + r + \sqrt{R(R-2r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$w_a = \frac{2bc}{b+c}.\cos\frac{A}{2} = \frac{4R^2(\cos(B-C)+\cos A).\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}$$

$$= \frac{R\left(2\cos^2\frac{B-C}{2} - 1 + 1 - 2\sin^2\frac{A}{2}\right)}{\cos\frac{B-C}{2}} = \frac{2R(c^2-s^2)}{c}\left(c = \cos\frac{B-C}{2} \text{ and } s = \sin\frac{A}{2}\right)$$

$$\leq R + r + \sqrt{R(R-2r)} = R + 2R\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right) + R.\sqrt{1-4sc+4s^2}$$

$$\Leftrightarrow 2c - \frac{2s^2}{c} \leq 1 + 2sc - 2s^2 + \sqrt{1-4sc+4s^2}$$

$$\Leftrightarrow 1 + 2sc - 2c + \frac{2s^2}{c} - 2s^2 + \sqrt{1-4sc+4s^2} \stackrel{①}{\geq} 0$$

$$\text{Now, } \frac{2s^2}{c} - 2s^2 = \frac{2s^2\left(1-\cos\frac{B-C}{2}\right)}{\cos\frac{B-C}{2}} \geq 0 \text{ and } 1 - 4sc + 4s^2 \stackrel{0<c\leq 1}{\geq} 1 - 4s + 4s^2$$

$$= (1-2s)^2 \Rightarrow \sqrt{1-4sc+4s^2} \geq |1-2s| \text{ and so, in order to prove ①,}$$

$$\text{it suffices to prove : } \boxed{1 + 2sc - 2c + |1 - 2s| \overset{②}{\geq} 0}$$

$$\boxed{Case\ 1}\ 1 - 2s \geq 0 \text{ and then : LHS of ② } = 1 + 2sc - 2c + 1 - 2s$$

$$= 1 - c - c(1 - 2s) + 1 - 2s = (1 - 2s)(1 - c) + (1 - c) = 2(1 - c)(1 - s) \geq 0$$

$$\because c = \cos\frac{B - C}{2} \leq 1 \text{ and } s = \sin\frac{A}{2} < 1 \Rightarrow ② \text{ is true}$$

$$\boxed{Case\ 2}\ 1 - 2s < 0 \text{ and then : LHS of ② } = 1 + 2sc - 2c + 2s - 1$$

$$= 1 - c + c(2s - 1) + (2s - 1) \overset{c \leq 1}{\geq} (2s - 1)(1 + c) > 0 \because 1 - 2s < 0$$

$$\Rightarrow ② \text{ is true (strict inequality)}$$

$$\therefore \text{combining both cases, ② is true } \forall \Delta ABC \therefore w_a \leq R + r + \sqrt{R(R - 2r)}$$

$$\text{Now, } \sqrt{R^2 - 4r^2} = \sqrt{(R - 2r)(R + 2r)} = \sqrt{R(1 - 4sc + 4s^2)\big(R + 4Rs(c - s)\big)}$$

$$= R.\sqrt{(1 - 4sc + 4s^2)\big(1 + 4s(c - s)\big)} \geq R.\sqrt{1 - 4sc + 4s^2}$$

$$\left(\because c = \cos\frac{B - C}{2} = \frac{b + c}{a}.\sin\frac{A}{2} > \sin\frac{A}{2} = s\right) \therefore h_a + \sqrt{R^2 - 4r^2} \geq$$

$$2R(c^2 - s^2) + R.\sqrt{1 - 4sc + 4s^2} \overset{?}{\geq} R + r = R + 2Rs(c - s)$$

$$\Leftrightarrow \boxed{\sqrt{1 - 4sc + 4s^2} \underset{③}{\overset{?}{\geq}} 1 + 2sc - 2c^2} \text{ and its's trivially true when}$$

$$1 + 2sc - 2c^2 < 0 \text{ and so we now focus on the scenario when :}$$

$$1 + 2sc - 2c^2 \geq 0 \text{ and then : ③ } \Leftrightarrow 1 - 4sc + 4s^2 \geq (1 + 2sc - 2c^2)^2$$

$$\Leftrightarrow -c^4 + 2c^3s - c^2s^2 + c^2 - 2cs + s^2 \geq 0 \Leftrightarrow -c^2(c - s)^2 + (c - s)^2 \geq 0$$

$$\Leftrightarrow (c - s)^2(1 - c^2) \geq 0 \to \text{true} \because 1 \geq \cos^2\frac{B - C}{2} \Rightarrow ③ \text{ is true}$$

$$\therefore h_a \geq R + r - \sqrt{R^2 - 4r^2} \text{ and so, } R + r - \sqrt{R^2 - 4r^2} \leq h_a \leq w_a \leq$$

$$R + r + \sqrt{R(R - 2r)}\ \forall \Delta ABC, " = " \text{ iff } b = c \text{ (QED)}$$

2105. **In ΔABC the following relationship holds:**

$$\frac{288}{w_a w_b w_c} \leq \frac{36}{r_a r_b r_c} + \frac{243}{h_a h_b h_c} + \frac{8\sqrt{3}}{abc}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$w_a = \frac{2\sqrt{bc}}{b + c}\sqrt{r_b r_c}, w_b = \frac{2\sqrt{ac}}{a + c}\sqrt{r_a r_c}, w_c = \frac{2\sqrt{ab}}{a + b}\sqrt{r_a r_b}$$

$$\frac{w_a w_b w_c}{r_a r_b r_c} = \frac{8abc}{(a + b)(b + c)(c + a)} = \frac{8.4Rrs}{2s(s^2 + r^2 + 4Rr) - 4Rrs} =$$

$$=\frac{16Rr}{s^2+2Rr+r^2}\overset{Gerretsen}{\geq}\frac{16Rr}{4R^2+4Rr+3r^2+2Rr+r^2}=\frac{16Rr}{4R^2+6Rr+4r^2}=$$

$$=\frac{16\left(\frac{R}{r}\right)}{4\left(\frac{R}{r}\right)^2+6\left(\frac{R}{r}\right)+4}\overset{\frac{R}{r}=x\geq 2}{=}\frac{16x}{4x^2+6x+4}\ (1)$$

$$\frac{r_ar_br_c}{h_ah_bh_c}=s^2r.\frac{4Rrs}{8r^3s^3}=\frac{R}{2r}\overset{\frac{R}{r}=x\geq 2}{=}\frac{x}{2}\ (2)$$

$$\frac{8\sqrt{3}r_ar_br_c}{abc}\geq\frac{8\sqrt{3}s^2r}{4Rrs}\overset{Mitrinovic}{\geq}\frac{8\sqrt{3}}{4}\frac{3\sqrt{3}r}{R}=\frac{18r}{R}\overset{\frac{R}{r}=x\geq 2}{=}\frac{18}{x}\ (3)$$

$$We\ need\ to\ show:$$

$$\frac{288}{w_aw_bw_c}\leq\frac{36}{r_ar_br_c}+\frac{243}{h_ah_bh_c}+\frac{8\sqrt{3}}{abc}$$

$$288\frac{r_ar_br_c}{w_aw_bw_c}\leq 36+\frac{r_ar_br_c}{h_ah_bh_c}243+\frac{8\sqrt{3}r_ar_br_c}{abc}$$

$$Using\ (1),(2)\&(3),\frac{4x^2+6x+4}{16x}288\leq 36+243.\frac{x}{2}+\frac{18}{x}$$

$$\frac{2}{x}(4x^2+6x+4)\leq 4+\frac{27x}{2}+\frac{2}{x}$$

$$16x^2+24x+16\leq 8x+27x^2+4 or, 11x^2-16x-12\geq 0$$

$$(x-2)(11x+6)\geq 0\ true\ as\ x\geq 2\ Euler$$

$$Equality\ for\ a=b=c.$$

2106. In any ΔABC the following relationship holds :

$$2R-r-2\sqrt{R(R-2r)}\leq m_a\leq 2R-r+2\sqrt{R(R-2r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$4\leq(m_b+m_c)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)=[(m_a+m_b+m_c)-m_a]\left[\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)-\frac{1}{m_a}\right]$$

$$\overset{\substack{\text{Leuenberger} \\ m_a \geq h_a \text{ (and analogs)}}}{\lesssim} (4R + r - m_a)\left(\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) - \frac{1}{m_a}\right)$$

$$= (4R + r - m_a)\left(\frac{1}{r} - \frac{1}{m_a}\right)$$

$$\Leftrightarrow 4rm_a \leq (4R + r - m_a)(m_a - r) \Leftrightarrow {m_a}^2 - 2(2R - r)m_a + r(4R + r) \leq 0$$

$$\Leftrightarrow \left(m_a - 2R + r + 2\sqrt{R(R - 2r)}\right)\left(m_a - 2R + r - 2\sqrt{R(R - 2r)}\right) \leq 0$$

$$\Leftrightarrow 2R - r - 2\sqrt{R(R - 2r)} \leq m_a \leq 2R - r + 2\sqrt{R(R - 2r)}.$$

Equality holds iff ΔABC is equilateral.

2107. In the non – obtuse ΔABC, prove that

$$r_a + r \leq \sqrt{7b^2 + 7c^2 - 2bc - 4a^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$r_a + r = r\left(\frac{s}{s - a} + 1\right) = (b + c).\frac{r}{s - a}$$

$$= (b + c)\tan\frac{A}{2} \overset{A \leq \frac{\pi}{2}}{\lesssim} b + c \overset{?}{\lesssim} \sqrt{7b^2 + 7c^2 - 2bc - 4a^2}$$

$$0 \overset{?}{\lesssim} 4(b^2 + c^2 - a^2) + 2(b - c)^2,$$

which is true for non – obtuse ΔABC.

Equality holds iff ABC is isosceles right triangle at A.

2108. In any Δ ABC, the following relationship holds :

$$\frac{(w_a + w_b + w_c)^2}{h_a h_b + h_b h_c + h_c h_a} \leq \frac{1}{12} + \frac{35R}{24r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} w_a = \sum_{cyc}\left(2\sqrt{bc}.\frac{\sqrt{s(s-a)}}{b+c}\right) \overset{CBS}{\leq} 2\sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}} \leq$$

$$\overset{A-G}{\leq}\sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{bc}} = \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}} = \sqrt{\sum_{cyc} bc}.\sqrt{\frac{4R+r}{2R}}$$

$$\Rightarrow \frac{(w_a+w_b+w_c)^2}{h_ah_b+h_bh_c+h_ch_a} \leq \frac{(s^2+4Rr+r^2)(4R+r)}{2R.\sum_{cyc}\frac{bc.ca}{4R^2}} = \frac{(s^2+4Rr+r^2)(4R+r)}{2R.\frac{4Rrs}{4R^2}.2s}$$

$$\overset{CBS}{\leq}\frac{35R+2r}{24r} \Leftrightarrow (11R-4r)s^2 \overset{?}{\underset{(1)}{\geq}} r(96R^2+48Rr+6r^2)$$

$$\text{Now, } (11R-4r)s^2 \overset{Gerretsen}{\geq} (11R-4r)(16Rr-5r^2) \overset{?}{\geq} r(96R^2+48Rr+6r^2)$$

$$\Leftrightarrow 80R^2-167Rr+14r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(80R-7r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{Euler}{\geq} 2r$$

$$\Rightarrow (1) \text{ is true} \therefore \frac{(w_a+w_b+w_c)^2}{h_ah_b+h_bh_c+h_ch_a} \leq \frac{1}{12}+\frac{35R}{24r}\ \forall\Delta ABC,$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

2109. In any Δ ABC, the following relationship holds :

$$(R-s)(R-r)+(s-r)(s-R)+(r-R)(r-s) \geq \frac{10\sqrt{3}-9}{3}\cdot F$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(R-s)(R-r)+(s-r)(s-R)+(r-R)(r-s) \geq \frac{10\sqrt{3}-9}{3}.F$$

$$\Leftrightarrow s^2+R^2+r^2-Rr-(R+r)s+3rs \geq \frac{10}{\sqrt{3}}.rs$$

$$\Leftrightarrow s^2+R^2+r^2-Rr \geq (R-2r)s+\frac{10}{\sqrt{3}}.rs$$

$$\Leftrightarrow (s^2+R^2+r^2-Rr)^2 \geq (R-2r)^2s^2+\frac{100r^2s^2}{3}+\frac{20}{\sqrt{3}}.r(R-2r)s^2 \text{ and}$$

$$\because \frac{20}{\sqrt{3}} < \frac{35}{3} \therefore \text{ it suffices to prove :}$$

$$(s^2+R^2+r^2-Rr)^2 \ge (R-2r)^2s^2 + \frac{100r^2s^2}{3} + \frac{35}{3}r(R-2r)s^2$$

$$\Leftrightarrow 3s^4 + (3R^2-29Rr-36r^2)s^2 + 3R^4 - 6R^3r + 9R^2r^2 - 6Rr^3 + 3r^4 \overset{①}{\ge} 0$$

$$\text{Now, } \xi = 3(s^2-16Rr+5r^2)^2 + (3R^2+67Rr-66r^2)(s^2-16Rr+5r^2) \ge$$

$$\overset{Gerretsen}{\ge} 0 \therefore \text{ in order to prove ①, it suffices to prove : LHS of ①} \ge \xi$$

$$\Leftrightarrow 3t^4+42t^3+298t^2-917t+258 \ge 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(3t^3+48t^2+394t-129) \ge 0 \to \text{true} \because t \overset{Euler}{\ge} 2 \Rightarrow ① \text{ is true}$$

$$\therefore (R-s)(R-r)+(s-r)(s-R)+(r-R)(r-s) \ge \frac{10\sqrt{3}-9}{3}.F\ \forall\ \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

2110. In any acute Δ ABC, the following relationship holds :

$$(s-R)(R-r)(s-r) \ge (29-9\sqrt{3}) \cdot \sqrt{\left(\frac{F}{3\sqrt{3}}\right)^3}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(s-R)(R-r)(s-r) \overset{Walker}{\ge}$$

$$\ge \left(\sqrt{2R^2+8Rr+3r^2}-R\right)(R-r)\left(\sqrt{2R^2+8Rr+3r^2}-r\right) \to (1) \text{ and}$$

$$\frac{rs}{3\sqrt{3}} \overset{Mitrinovic}{\le} \frac{Rr}{2} \Rightarrow (29-9\sqrt{3}).\sqrt{\left(\frac{F}{3\sqrt{3}}\right)^3} \le (29-9\sqrt{3}).\left(\frac{Rr}{2}\right)^{\frac{3}{2}} \to (2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \text{ it suffices to prove :}$$

$$\left(\sqrt{2R^2+8Rr+3r^2}-R\right)(R-r)\left(\sqrt{2R^2+8Rr+3r^2}-r\right) \ge (29-9\sqrt{3}).\left(\frac{Rr}{2}\right)^{\frac{3}{2}}$$

$$\Leftrightarrow \left(\sqrt{2t^2+8t+3}-t\right)(t-1)\left(\sqrt{2t^2+8t+3}-1\right) \overset{①}{\ge} (29-9\sqrt{3}).\left(\frac{t}{2}\right)^{\frac{3}{2}} \left(t = \frac{R}{r}\right)$$

$$\text{Let } f(t) = \left(\sqrt{2t^2+8t+3}-t\right)(t-1)\left(\sqrt{2t^2+8t+3}-1\right) - (29-9\sqrt{3}).\left(\frac{t}{2}\right)^{\frac{3}{2}}$$

$\forall\ t \in [2,\infty)$ *and* then :

$$f'(t) = 6t^2 + 8t - 6 - \frac{6t^3 + 20t^2 + 4t - 4}{\sqrt{2t^2 + 8t + 3}} + 6t + \frac{\left(3^{\frac{7}{2}} - 87\right).\sqrt{t}}{2^{\frac{5}{2}}}$$

$$\overset{\text{Euler}}{\geq} 6t^2 + 8t - 6 - \frac{6t^3 + 20t^2 + 4t - 4}{\sqrt{2t^2 + 8t + 3}} + \left(6\sqrt{2} + \frac{3^{\frac{7}{2}} - 87}{4\sqrt{2}}\right).\sqrt{t}$$

$$> \frac{\sqrt{2t^2 + 8t + 3}.(6t^2 + 8t - 6) - (6t^3 + 20t^2 + 4t - 4)}{\sqrt{2t^2 + 8t + 3}}$$

$$= \frac{2(9t^6 + 60t^5 + 103t^4 - 20t^3 - 144t^2 + 8t + 23)}{\sqrt{2t^2 + 8t + 3}.\left(\sqrt{2t^2 + 8t + 3}.(6t^2 + 8t - 6) + (6t^3 + 20t^2 + 4t - 4)\right)}$$

$$= \frac{2(9t^6 + 60t^5 + 57t^4 + 10t^3(t - 2) + 36t^2(t^2 - 4) + 8t + 23)}{\sqrt{2t^2 + 8t + 3}.\left(\sqrt{2t^2 + 8t + 3}.(6t^2 + 8t - 6) + (6t^3 + 20t^2 + 4t - 4)\right)} \overset{\text{Euler}}{>} 0$$

$$\Rightarrow f(t) \text{ is } \uparrow \text{ on } [2,\infty) \Rightarrow f(t) \geq f(2) = 0 \Rightarrow ① \text{ is true}$$

$$\therefore (s - R)(R - r)(s - r) \geq (29 - 9\sqrt{3}).\sqrt{\left(\frac{F}{3\sqrt{3}}\right)^3}\ \forall\ \textit{acute}\ \Delta\ ABC,$$

$$'' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

2111. In *any* Δ ABC, the following relationship holds :

$$3\sqrt{3r^3} \leq h_a\sqrt{r_a} \leq \frac{2s^2}{3\sqrt{6R}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$h_a.\sqrt{r_a} \leq \frac{2s^2}{3.\sqrt{6R}} \Leftrightarrow \left(\frac{2rs}{a}.\sqrt{\frac{rs}{s-a}}\right)^2 \leq \left(\frac{2s^2}{3.\sqrt{6R}}\right)^2 \Leftrightarrow \frac{4r^2s^2}{a^2}.\frac{rs}{s-a} \leq \frac{4s^4}{54R}$$

$$\Leftrightarrow a^2bc.\frac{s(s-a)}{bc} \geq 54Rr^3 \Leftrightarrow 4Rrs.a.\cos^2\frac{A}{2} \geq 54Rr^3 \Leftrightarrow 2s^2a\cos^2\frac{A}{2} \overset{(*)}{\boxed{\geq}} 27r^2s$$

$$\text{Now, } 2s^2.a\cos^2\frac{A}{2} \overset{\text{Gerretsen + Euler}}{\geq} 27Rr.a\cos^2\frac{A}{2} \overset{?}{\geq} 27r^2s \Leftrightarrow \frac{R}{r}.a \overset{?}{\geq} s\sec^2\frac{A}{2}$$

$$\Leftrightarrow \frac{R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.4R\sin\frac{A}{2}\cos\frac{A}{2} \overset{?}{\geq} \frac{4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos^2\frac{A}{2}}$$

$$\Leftrightarrow \cos^2\frac{A}{2} \overset{?}{\geq} \left(2\cos\frac{B}{2}\cos\frac{C}{2}\right)\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$\Leftrightarrow \cos^2\frac{A}{2} \overset{?}{\underset{(**)}{\geq}} \left(\sin\frac{A}{2}+\cos\frac{B-C}{2}\right)\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)$$

$$\text{Now}, \because \cos\frac{B-C}{2} \leq 1 \therefore \left(\sin\frac{A}{2}+\cos\frac{B-C}{2}\right)\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right) \leq$$

$$\left(1+\sin\frac{A}{2}\right)\left(1-\sin\frac{A}{2}\right) = 1-\sin^2\frac{A}{2} = \cos^2\frac{A}{2} \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore h_a.\sqrt{r_a} \leq \frac{2s^2}{3.\sqrt{6R}}$$

$$\text{Again}, h_a.\sqrt{r_a} \geq 3.\sqrt{3r^3} \Leftrightarrow \left(\frac{2rs}{a}.\sqrt{\frac{rs}{s-a}}\right)^2 \geq 3.\sqrt{3r^3} \Leftrightarrow \frac{4s^3}{a^2(s-a)} \geq 27$$

$$\Leftrightarrow 8s^3 \geq 27a^2(b+c-a) \to \text{true} \because \sqrt[3]{a^2(b+c-a)} \overset{A-G}{\leq} \frac{a+a+b+c-a}{3} = \frac{2s}{3}$$

$$\Rightarrow a^2(b+c-a) \leq \frac{8s^3}{27} \therefore h_a.\sqrt{r_a} \geq 3.\sqrt{3r^3} \text{ and so}, \forall \Delta ABC, h_a.\sqrt{r_a} \leq \frac{2s^2}{3.\sqrt{6R}},$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral and } h_a.\sqrt{r_a} \geq 3.\sqrt{3r^3}, ''='' \text{ iff } b+c=2a \text{ (QED)}$$

2112. In ΔABC the following relationship holds:

$$\left(3+\sum\frac{a^2+b^2}{c^2}\right)\left(\sum\sec^2\frac{A}{2}\right) \leq \frac{9}{4}\left(\frac{R}{r}\right)^4$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$\left(\sum\sec^2\frac{A}{2}\right) = 3+\sum\tan^2\frac{A}{2} = 3+\left(\frac{4R+r}{s}\right)^2 - 2 =$$

$$= 1+\left(\frac{4R+r}{s}\right)^2 \overset{Euler\ \&\ Mitrinovic}{\leq} \left(\frac{R}{R}\right)^2 + \frac{\left(\frac{9R}{2}\right)^2}{27r^2} \overset{Euler}{\leq} \frac{1}{4}\left(\frac{R}{r}\right)^2 + \frac{3}{4}\left(\frac{R}{r}\right)^2 = \left(\frac{R}{r}\right)^2$$

$$\left(3+\sum\frac{a^2+b^2}{c^2}\right) = \left(\sum 1+\frac{a^2+b^2}{c^2}\right) =$$

$$= (a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \overset{Leibniz\ \&\ Steining}{\leq} 9R^2\cdot\frac{1}{4r^2} = \frac{9}{4}\left(\frac{R}{r}\right)^2$$

$$\left(3+\sum\frac{a^2+b^2}{c^2}\right)\left(\sum\sec^2\frac{A}{2}\right) \leq \frac{9}{4}\left(\frac{R}{r}\right)^2.\left(\frac{R}{r}\right)^2 = \frac{9}{4}\left(\frac{R}{r}\right)^4$$

$$\textit{Equality holds for } A=B=C.$$

2113. **In ΔABC the following relationship holds:**

$$\sum \cot A + \lambda \prod \cot\frac{A}{2} \geq \frac{1}{2}(3\lambda+1)\sum \csc A, \lambda \geq 0$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

It is known: In ΔABC :

$$\sum \tan A \tan B = \frac{s^2-4Rr-r^2}{s^2-(2R+r)^2}, \prod \tan A = \frac{2sr}{s^2-(2R+r)^2},$$

$$\prod \sin A = \frac{sr}{2R^2}, \sum \sin A \sin B = \frac{s^2+4Rr+r^2}{4R^2}$$

Using above result we get:

$$\sum \cot A = \frac{s^2-4Rr-r^2}{2sr}, \sum \csc A = \frac{s^2+4Rr+r^2}{2sr},$$

$$\prod \cot\frac{A}{2} = \frac{s}{r}$$

We need to show:

$$\sum \cot A + \lambda \prod \cot\frac{A}{2} \geq \frac{1}{2}(3\lambda+1)\sum \csc A$$

$$\frac{s^2-4Rr-r^2}{2sr} + \lambda\frac{s}{r} \geq \frac{1}{2}(3\lambda+1)\frac{s^2+4Rr+r^2}{2sr}$$

$$2s^2-8Rr-2r^2+4\lambda s^2 \geq (3\lambda+1)(s^2+r^2+4Rr)$$

$$2s^2-8Rr-2r^2+4\lambda s^2 \geq 3\lambda s^2+3\lambda r(4R+r)+s^2+r^2+4Rr$$

$$s^2-12Rr-3r^2+4\lambda s^2 \geq 3\lambda s^2+3\lambda r(4R+r)$$

$$s^2-12Rr-3r^2+4\lambda s^2 \overset{s^2\geq 3r(4R+r)}{\geq} 3\lambda s^2+\lambda s^2$$

$$s^2-12Rr-3r^2 \geq 0$$

$$16Rr-5r^2-12Rr-3r^2 \geq 0\ (Gerretsen)$$

$$or\ 4Rr-8r^2 \geq 0\ or\ R \geq 2r\ true\ Euler$$

$$Equality\ holds\ for\ A=B=C$$

2114. In ΔABC the following relationship holds:

$$\prod\left(\frac{8}{r_a^2}+\frac{1}{r_ar_b}\right)\le\frac{1}{r^6}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ \frac{r}{r_a}=x,\frac{r}{r_b}=y,\frac{r}{r_c}=z\ then:$$

$$x+y+z=r\sum\frac{1}{r_a}=1\ (1)\ and\ xyz\overset{AM-GM}{\le}\left(\frac{x+y+z}{3}\right)^3=\frac{1}{27}\ (2)$$

$$(8x+y)(8y+z)(8z+x)\overset{AM-GM}{\le}\left(\frac{8x+y+8y+z+8z+x}{3}\right)^3=$$
$$=27(x+y+z)=27\big(using\ (1)\big)\ (3)$$

$$\prod\left(\frac{8}{r_a^2}+\frac{1}{r_ar_b}\right)=\prod\left(\frac{8x^2}{r^2}+\frac{xy}{r^2}\right)=\prod\frac{x}{r^2}(8x+y)=$$
$$=\frac{xyz}{r^6}(8x+y)(8y+z)(8z+x)\overset{(3)\&(2)}{\le}\frac{1}{r^6}\cdot\frac{1}{27}\cdot27=\frac{1}{r^6}$$

$$Equality\ holds\ for\ an\ equilateral\ triangle.$$

2115. In ΔABC the following relationship holds:

$$\prod\left(\frac{8}{h_a^2}+\frac{1}{h_ah_b}\right)\le\frac{1}{r^6}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ \frac{r}{h_a}=x,\frac{r}{h_b}=y,\frac{r}{h_c}=z\ then\ x+y+z=r\sum\frac{1}{h_a}=1\ (1)$$

$$and\ xyz\overset{AM-GM}{\le}\left(\frac{x+y+z}{3}\right)^3=\frac{1}{27}\ (2)$$

$$(8x+y)(8y+z)(8z+x)\overset{AM-GM}{\le}\left(\frac{8x+y+8y+z+8z+x}{3}\right)^3=$$

$$= 27(x + y + z) = 27(using\ (1))\ (3)$$

$$\prod\left(\frac{8}{h_a^2} + \frac{1}{h_a h_b}\right) = \prod\left(\frac{8x^2}{r^2} + \frac{xy}{r^2}\right) = \frac{xyz}{r^6}(8x + y)(8y + z)(8z + x) \overset{(2)\&(3)}{\leq}$$

$$\leq \frac{1}{r^6} \cdot \frac{27.1}{27} = \frac{1}{r^6}$$

Equality holds for $a = b = c$.

2116. **In ΔABC the following relationship holds:**

$$\sum bc\cos^2\frac{A}{2} \geq 3\sum bc\sin^2\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum bc\cos^2\frac{A}{2} = \sum bc\frac{s(s-a)}{bc} = s\sum(s-a) = s^2\ (1)$$

$$3\sum bc\sin^2\frac{A}{2} = 3\sum bc\frac{(s-b)(s-c)}{bc} =$$

$$= 3\sum(s-b)(s-c) \overset{\forall x,y,z>0\ 3\sum xy \leq (\sum x)^2}{\leq} \left(\sum(s-a)\right)^2 = s^2\ (2)$$

$$From\ (1)\&(2)\ we\ \ get \sum bc\cos^2\frac{A}{2} \geq 3\sum bc\sin^2\frac{A}{2}$$

Equality holds for $A = B = C$.

2117. **In ΔABC the following relationship holds:**

$$\frac{h_a}{a} + \frac{h_b}{b} + \frac{h_c}{c} \geq \frac{3\sqrt{3}r}{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{h_a}{a} = \sum_{cyc}\frac{2F}{a^2} = 2F\sum_{cyc}\frac{1}{a^2} = 2F\sum_{cyc}\frac{1^3}{a^2} \overset{RADON}{\geq} 2F \cdot \frac{(1+1+1)^3}{(a+b+c)^2} =$$

$$= 2F \cdot \frac{27}{4s^2} = \frac{27rs}{2s^2} = \frac{27r}{2s} \overset{MITRINOVIC}{\geq} \frac{27r}{2 \cdot \frac{3\sqrt{3}}{2}R} = \frac{9r}{\sqrt{3} \cdot R} = \frac{3\sqrt{3}r}{R}$$

Equality holds for $a = b = c$.

2118. In ΔABC the following relationship holds:

$$\frac{\cos\frac{A}{2}}{1 + \cos A} + \frac{\cos\frac{B}{2}}{1 + \cos B} + \frac{\cos\frac{C}{2}}{1 + \cos C} \geq \sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc} \frac{\cos\frac{A}{2}}{1 + \cos A} = \sum_{cyc} \frac{\cos\frac{A}{2}}{1 + 2\cos^2\frac{A}{2} - 1} = \frac{1}{2}\sum_{cyc} \frac{1}{\cos\frac{A}{2}} =$$

$$= \frac{1}{2}\sum_{cyc} \frac{1^2}{\cos\frac{A}{2}} \overset{BERGSTROM}{\geq} \frac{1}{2} \cdot \frac{(1+1+1)^2}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}} \overset{JENSEN}{\geq}$$

$$\geq \frac{9}{2} \cdot \frac{1}{3\cos\left(\frac{A+B+C}{6}\right)} = \frac{3}{2\cos\frac{\pi}{6}} = \frac{3}{2 \cdot \frac{\sqrt{3}}{2}} = \sqrt{3}$$

Equality holds for $a = b = c$.

2119. In ΔABC the following relationship holds:

$$\frac{\sin A}{\cot A} + \frac{\sin B}{\cot B} + \frac{\sin C}{\cot C} \geq \frac{9}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc} \frac{\sin A}{\cot A} = \sum_{cyc} \frac{\sin^2 A}{\cos A} = \sum_{cyc} \frac{1 - \cos^2 A}{\cos A} = \sum_{cyc} \frac{1}{\cos A} - \sum_{cyc} \cos A =$$

$$=\sum_{cyc} secA-\left(1+\frac{r}{R}\right)\overset{JENSEN}{\geq} 3sec\left(\frac{A+B+C}{3}\right)-1-\frac{r}{R}=$$

$$=3sec\frac{\pi}{3}-1-\frac{r}{R}\overset{EULER}{\geq} 3\cdot 2-1-\frac{1}{2}=\frac{9}{2}$$

Equality holds for $A=B=C$.

2120. In ΔABC the following relationship holds:

$$\frac{r}{R}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{16R^2}\leq\frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{r}{R}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{16R^2}=\frac{r}{R}+\frac{2(a^2+b^2+c^2-ab-bc-ca)}{16R^2}=$$

$$=\frac{r}{R}+\frac{(a^2+b^2+c^2)-(ab+bc+ca)}{8R^2}=$$

$$=\frac{r}{R}+\frac{(2s^2-2r^2-8Rr)-(s^2+r^2+4Rr)}{8R^2}=$$

$$=\frac{r}{R}+\frac{s^2-12Rr-3r^2}{8R^2}\overset{GERRETSEN}{\leq}\frac{r}{R}+\frac{4R^2+4Rr+3r^2-12Rr-3r^2}{8R^2}=$$

$$=\frac{r}{R}+\frac{4R^2-8Rr}{8R^2}=\frac{r}{R}+\frac{1}{2}-\frac{r}{R}=\frac{1}{2}$$

Equality holds for $a=b=c$.

2121. In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2+m_c^2}{3R^2}\leq\frac{9}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{m_a^2+m_b^2+m_c^2}{3R^2}=\frac{3}{4}\cdot\frac{a^2+b^2+c^2}{3R^2}=$$

$$=\frac{a^2+b^2+c^2}{4R^2}\overset{LEIBNIZ}{\leq}\frac{9R^2}{4R^2}=\frac{9}{4}$$

Equality holds for $a=b=c$.

2122. In ΔABC the following relationship holds:

$$\frac{a^2}{\cos^2\frac{A}{2}}+\frac{b^2}{\cos^2\frac{B}{2}}+\frac{c^2}{\cos^2\frac{C}{2}}\geq 12R^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{a^2}{\cos^2\frac{A}{2}}+\frac{b^2}{\cos^2\frac{B}{2}}+\frac{c^2}{\cos^2\frac{C}{2}}=\sum_{cyc}\frac{a^2}{\cos^2\frac{A}{2}}=\sum_{cyc}\frac{a^2}{\frac{s(s-a)}{bc}}=$$

$$=\frac{abc}{s}\sum_{cyc}\frac{a}{s-a}=\frac{abc}{s}\cdot\frac{2(2R-r)}{r}=\frac{4Rrs}{s}\cdot\frac{2(2R-r)}{r}=$$

$$=8R(2R-r)\overset{EULER}{\geq}8R\left(2R-\frac{R}{2}\right)=12R^2$$

Equality holds for $a=b=c$.

2123. In ΔABC the following relationship holds:

$$\sqrt{(a+b-c)(b+c-a)(c+a-b)}\leq\frac{3\sqrt{3}abc}{(a+b+c)\sqrt{a+b+c}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sqrt{(a+b-c)(b+c-a)(c+a-b)} \le \frac{3\sqrt{3}abc}{(a+b+c)\sqrt{a+b+c}}$$

$$\sqrt{(2s-2c)(2s-2a)(2s-2b)} \le \frac{3\sqrt{3}\cdot 4RF}{2s\sqrt{2s}}$$

$$2\sqrt{2}\cdot 2s\cdot\sqrt{s(s-a)(s-b)(s-c)} \le 12\sqrt{3}RF$$

$$2\sqrt{2}\cdot 2s\cdot F \le 12\sqrt{3}RF$$

$$4\sqrt{2}\cdot\sqrt{2}\cdot s \le 12\sqrt{3}R,\ \ s \le \frac{12\sqrt{3}R}{8}$$

$$s \le \frac{3\sqrt{3}R}{2}\ (MITRINOVICI)$$

Equality holds for $a=b=c$.

2124. In ΔABC the following relationship holds:

$$(b+c-a)(c+a-b)(a+b-c) \le 2R\sqrt{2Rh_ah_bh_c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$(b+c-a)(c+a-b)(a+b-c) \le 2R\sqrt{2Rh_ah_bh_c}$$

$$(2s-2a)(2s-2b)(2s-2c) \le 2R\sqrt{2R\cdot\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}$$

$$8(s-a)(s-b)(s-c) \le 2R\sqrt{\frac{4F^2\cdot 4RF}{abc}}$$

$$8s(s-a)(s-b)(s-c) \le 2Rs\cdot\sqrt{\frac{4F^2\cdot 4RF}{4RF}}$$

$$8F^2 \le 2Rs\cdot 2F,\quad 2F \le Rs,\quad 2rs \le Rs$$

$$R \ge 2r(EULER)$$

Equality holds for $a=b=c$.

2125. **If I −incenter in ΔABC then:**

$$IA^2\cdot IB^2\cdot IC^2\le\frac{8}{27}R^3h_ah_bh_c$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$IA^2\cdot IB^2\cdot IC^2\le\frac{8}{27}R^3h_ah_bh_c$$

$$\frac{r^2}{sin^2\frac{A}{2}}\cdot\frac{r^2}{sin^2\frac{A}{2}}\cdot\frac{r^2}{sin^2\frac{A}{2}}\le\frac{8}{27}R^3\cdot\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}$$

$$\frac{r^6}{\left(sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\right)^2}\le\frac{8}{27}\cdot\frac{4RF}{abc}\cdot 2(RF)^2$$

$$\frac{r^6}{\left(\frac{r}{4R}\right)^2}\le\frac{16}{27}\cdot\frac{4RF}{4RF}\cdot(RF)^2$$

$$16R^2r^4\le\frac{16}{27}\cdot R^2\cdot F^2$$

$$27r^4\le F^2,\quad F\ge 3\sqrt{3}r^2,\quad rs\ge 3\sqrt{3}r^2$$

$$s\ge 3\sqrt{3}r\ (MITRINOVICI)$$

Equality holds for $a=b=c$.

2126. **In ΔABC the following relationship holds:**

$$\frac{9R}{a^2+b^2+c^2}\le\frac{1}{h_a+\sqrt{h_bh_c}}+\frac{1}{h_b+\sqrt{h_ch_a}}+\frac{1}{h_c+\sqrt{h_ah_b}}\le\frac{1}{2r}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{h_a+\sqrt{h_bh_c}}\overset{AM-HM}{\le}\frac{1}{4}\left(\frac{1}{h_a}+\frac{1}{\sqrt{h_bh_c}}\right)=$$

$$=\frac{1}{4}\left(\frac{1}{h_a}+\sqrt{\frac{1}{h_b}\cdot\frac{1}{h_c}}\right)\overset{AM-GM}{\le}\frac{1}{4}\left(\frac{1}{h_a}+\frac{1}{2}\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\right)\ (1)$$

$$\frac{1}{h_a+\sqrt{h_bh_c}}+\frac{1}{h_b+\sqrt{h_ch_a}}+\frac{1}{h_c+\sqrt{h_ah_b}}=\sum\frac{1}{h_a+\sqrt{h_bh_c}}\overset{(1)}{\leq}$$

$$\leq\frac{1}{4}\sum\left(\frac{1}{h_a}+\frac{1}{2}\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\right)=\frac{2}{4}\sum\frac{1}{h_a}=\frac{1}{2r}$$

$$h_a+h_b+h_c=2F\frac{ab+bc+ca}{abc}=\frac{2F}{4RF}\sum ab\leq\frac{1}{2R}(a^2+b^2+c^2)\ (2)$$

$$\frac{1}{h_a+\sqrt{h_bh_c}}+\frac{1}{h_b+\sqrt{h_ch_a}}+\frac{1}{h_c+\sqrt{h_ah_b}}=\sum\frac{1}{h_a+\sqrt{h_bh_c}}\overset{AM-GM}{\geq}$$

$$\geq\sum\frac{1}{h_a+\frac{h_b+h_c}{2}}\overset{CBS}{\geq}\frac{(1+1+1)^2}{2(h_a+h_b+h_c)}\overset{(2)}{\geq}\frac{9R}{a^2+b^2+c^2}$$

Equality holds for $a=b=c$

2127. **In ΔABC the following relationship holds:**

$$\frac{\cot\frac{A}{2}}{a}+\frac{\cot\frac{B}{2}}{b}+\frac{\cot\frac{C}{2}}{c}\geq\frac{3}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{\cot\frac{A}{2}}{a}+\frac{\cot\frac{B}{2}}{b}+\frac{\cot\frac{C}{2}}{c}\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\cdot\frac{1}{abc}}=$$

$$=3\cdot\sqrt[3]{\frac{s}{r}\cdot\frac{1}{4RF}}=3\cdot\sqrt[3]{\frac{s}{4Rr\cdot rs}}=3\cdot\sqrt[3]{\frac{1}{R\cdot(2r)^2}}\geq$$

$$\overset{EULER}{\geq}3\cdot\sqrt[3]{\frac{1}{R\cdot\left(2\cdot\frac{R}{2}\right)^2}}=\frac{3}{R}$$

Equality holds for $a=b=c$.

2128. **If in ΔABC : $x=2R\sum_{cyc}\frac{1}{w_a}\cos\frac{B-C}{2}$ and $y=\frac{4}{x}$, then prove that :**

$$\frac{1}{h_a}\left(\frac{h_a}{h_b}\right)^y+\frac{1}{h_b}\left(\frac{h_b}{h_c}\right)^y+\frac{1}{h_c}\left(\frac{h_c}{h_a}\right)^y\leq\frac{1}{r}$$

Proposed by Tapas Das-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$x = 2R\sum_{cyc}\frac{1}{w_a}\cos\frac{B-C}{2} = 2R\sum_{cyc}\frac{b+c}{2bc\cos\frac{A}{2}}\cdot\frac{b+c}{4R\cos\frac{A}{2}} =$$

$$= \frac{1}{4}\sum_{cyc}\frac{(b+c)^2}{s(s-a)} = \frac{1}{4s}\sum_{cyc}\frac{\left(a+2(s-a)\right)^2}{s-a} \geq$$

$$\overset{AM-GM}{\geq} \frac{1}{4s}\sum_{cyc}\frac{4.2(s-a)a}{s-a} = 4 \Rightarrow y \leq 1.$$

By Bernoulli's inequality, we have :

$$\sum_{cyc}\frac{1}{h_a}\left(\frac{h_a}{h_b}\right)^y = \sum_{cyc}\frac{1}{h_a}\left(1+\left(\frac{h_a}{h_b}-1\right)\right)^y \leq \sum_{cyc}\frac{1}{h_a}\left(1+y\left(\frac{h_a}{h_b}-1\right)\right) = \sum_{cyc}\frac{1}{h_a} = \frac{1}{r}$$

Equality holds iff ΔABC is equilateral.

2129.

If in Δ ABC the following relationship holds : $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$

then prove that : $2\left(\frac{R}{r}\right) \leq \left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right) \leq \left(\frac{R}{r}\right)^2$

Proposed by Tapas Das-India

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2} \Rightarrow a.\frac{s(s-c)}{ab} + c.\frac{s(s-a)}{bc} = \frac{3b}{2}$$

$$\Rightarrow \frac{(a+b)^2 - c^2 + (b+c)^2 - a^2}{4b} = \frac{3b}{2}$$

$$\Rightarrow a^2 + b^2 + 2ab - c^2 + b^2 + c^2 + 2bc - a^2 = 6b^2$$

$$\Rightarrow 4b^2 = 2b(c+a) \Rightarrow 2b = c + a \Rightarrow 8R\sin\frac{B}{2}\cos\frac{B}{2} = 4R\cos\frac{B}{2}\cos\frac{C-A}{2}$$

$$\Rightarrow \boxed{\cos\frac{C-A}{2} \overset{(*)}{=} 2\sin\frac{B}{2}} \Rightarrow 2\sin\frac{B}{2} \leq 1 \left(\because \cos\frac{C-A}{2} \leq 1\right)$$

$$\Rightarrow \sin\frac{B}{2} \le \frac{1}{2} \Rightarrow \frac{B}{2} \le \frac{\pi}{6} \Rightarrow \boxed{B \overset{(**)}{\le} \frac{\pi}{3}}$$

$$\text{Now, } \left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right) = \frac{1}{s}.(r_a+r_c).s.\frac{r_a+r_c}{r_ar_c} =$$

$$\frac{4R\cos^2\frac{B}{2}.4R\cos^2\frac{B}{2}}{s(s-b)} = \frac{4Rs(s-b)}{cas(s-b)}.4R\cos^2\frac{B}{2} = \frac{16R^2\cos^2\frac{B}{2}}{16R^2\cos\frac{C}{2}\sin\frac{C}{2}\cos\frac{A}{2}\sin\frac{A}{2}} \le \left(\frac{R}{r}\right)^2$$

$$\Leftrightarrow \frac{\cos^2\frac{B}{2}}{\cos\frac{C}{2}\sin\frac{C}{2}\cos\frac{A}{2}\sin\frac{A}{2}} \le \frac{1}{16\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}$$

$$\Leftrightarrow 16\cos^2\frac{B}{2}\sin^2\frac{B}{2}.\left(2\sin\frac{A}{2}\sin\frac{C}{2}\right) \le 2\cos\frac{C}{2}\cos\frac{A}{2}$$

$$\Leftrightarrow (4\sin^2 B)\left(\cos\frac{C-A}{2}-\sin\frac{B}{2}\right) \le \sin\frac{B}{2}+\cos\frac{C-A}{2}$$

$$\overset{\text{via } (*)}{\Leftrightarrow} (4\sin^2 B)\left(2\sin\frac{B}{2}-\sin\frac{B}{2}\right) \le \sin\frac{B}{2}+2\sin\frac{B}{2} \Leftrightarrow \boxed{\boxed{\sin^2 B \le \frac{3}{4}}}$$

$$\to \text{true} \because B \overset{\text{via } (**)}{\le} \frac{\pi}{3} \Rightarrow \sin B \le \frac{\sqrt{3}}{2} \Rightarrow \sin^2 B \le \frac{3}{4}$$

$$\therefore \left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right) \le \left(\frac{R}{r}\right)^2$$

$$\text{Again, } \left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right) \ge 2\left(\frac{R}{r}\right)$$

$$\Leftrightarrow \frac{\cos^2\frac{B}{2}}{\cos\frac{C}{2}\sin\frac{C}{2}\cos\frac{A}{2}\sin\frac{A}{2}} \ge \frac{1}{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \Leftrightarrow 4\sin\frac{B}{2}\cos^2\frac{B}{2} \ge 2\cos\frac{C}{2}\cos\frac{A}{2}$$

$$= \sin\frac{B}{2}+\cos\frac{C-A}{2} \overset{\text{via } (*)}{=} 3\sin\frac{B}{2} \Leftrightarrow \boxed{\boxed{\cos^2\frac{B}{2} \ge \frac{3}{4}}} \to \text{true} \because \frac{B}{2} \overset{\text{via } (**)}{\le} \frac{\pi}{6}$$

$$\Rightarrow \cos^2\frac{B}{2} \ge \cos^2\frac{\pi}{6} = \frac{3}{4} \therefore \left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right) \ge 2\left(\frac{R}{r}\right) \text{ and so,}$$

$$2\left(\frac{R}{r}\right) \le \left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right) \le \left(\frac{R}{r}\right)^2$$

$$\text{whenever } a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2} = \frac{3b}{2}, \text{"="} \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have } \frac{3b}{2} = a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2} = a.\frac{s(s-c)}{ab}+c.\frac{s(s-a)}{bc} = s, \text{then } 2b = a+c$$

$$\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right) = \left(\frac{r}{s-a}+\frac{r}{s-c}\right)\left(\frac{s-a}{r}+\frac{s-c}{r}\right) = \frac{b^2}{(s-a)(s-c)} =$$

$$=\frac{abc}{(s-a)(s-b)(s-c)}\cdot\frac{b(c+a-b)}{2ca}=\frac{4Rsr}{sr^2}\cdot\frac{b^2}{2ca}=2\frac{R}{r}\cdot\frac{(a+c)^2}{4ca}\geq 2\frac{R}{r}$$

$$\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right)=\frac{b^2}{(s-a)(s-c)}=$$

$$=\left(\frac{abc}{(s-a)(s-b)(s-c)}\right)^2\cdot\frac{(s-a)(s-b).(s-b)(s-c)}{a^2c^2}\leq$$

$$\leq\left(\frac{4R}{r}\right)^2\cdot\frac{[(s-a)+(s-b)]^2.[(s-b)+(s-c)]^2}{16a^2c^2}=\left(\frac{R}{r}\right)^2.$$

which completes the proof. Equality holds iff ΔABC is equilateral.

2130. In acute ΔABC the following relationship holds:

$$tan\frac{C}{2}(a^2tanA+b^2tanB)\geq a^2+b^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

WLOG: $a\leq b\to tanA\leq tanB$

$$tan\frac{C}{2}(a^2tanA+b^2tanB)\overset{CEBYSHEV}{\geq}$$

$$\geq\frac{1}{2}tan\frac{C}{2}(a^2+b^2)(tanA+tanB)\geq a^2+b^2\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{2}tan\frac{C}{2}(tanA+tanB)\geq 1\Leftrightarrow$$

$$\Leftrightarrow tan\frac{C}{2}\cdot\frac{sin(A+B)}{cosAcosB}\geq 2\Leftrightarrow\frac{sin\frac{C}{2}}{cos\frac{C}{2}}\cdot sin(\pi-C)\geq 2cosAcosB\Leftrightarrow$$

$$\Leftrightarrow\frac{sin\frac{C}{2}}{cos\frac{C}{2}}\cdot 2sin\frac{C}{2}cos\frac{C}{2}\geq 2cosAcosB\Leftrightarrow$$

$$\Leftrightarrow 2sin^2\frac{C}{2}\geq 2cosAcosB\Leftrightarrow 1-cosC\geq 2cosAcosB\Leftrightarrow$$

$$\Leftrightarrow 1+cos(A+B)\geq 2cosAcosB\Leftrightarrow$$

$$2cosAcosB-cosAcosB+sinAsinB\leq 1\Leftrightarrow cos(A-B)\leq 1$$

Equality holds for $A = B$.

2131. In ΔABC the following relationship holds:

$$r_a cosA + r_b cosB + r_c cosC \le \frac{9R}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

WLOG: $a \le b \le c \Rightarrow -a \ge -b \ge -c \Rightarrow s-a \ge s-b \ge s-c \Rightarrow$

$$\frac{1}{s-a} \le \frac{1}{s-b} \le \frac{1}{s-c} \Rightarrow \frac{F}{s-a} \le \frac{F}{s-b} \le \frac{F}{s-c} \Rightarrow r_a \le r_b \le r_c$$

$$a \le b \le c \Rightarrow cosA \ge cosB \ge cosC$$

$$\sum_{cyc} r_a cosA \overset{CEBYSHEV}{\le} \frac{1}{3} \cdot \sum_{cyc} r_a \cdot \sum_{cyc} cosA \overset{KLAMKIN}{\le} \frac{1}{3} \cdot \frac{9R}{2} \cdot \sum_{cyc} cosA =$$

$$= \frac{3R}{2} \cdot \left(1 + \frac{r}{R}\right) \overset{EULER}{\le} \frac{3R}{2} \cdot \left(1 + \frac{1}{2}\right) = \frac{9R}{4}$$

Equality holds for $a = b = c$.

2132. In ΔABC the following relationship holds:

$$a^2 cosA + b^2 cosB + c^2 cosC \le \frac{9R^2}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

WLOG: $a \le b \le c \Rightarrow a^2 \le b^2 \le c^2, \ \ cosA \ge cosB \ge cosC$

$$\sum_{cyc} a^2 cosA \le \frac{1}{3} \cdot \sum_{cyc} a^2 \cdot \sum_{cyc} cosA = \frac{2}{3} \cdot (s^2 - r^2 - 4Rr) \cdot \left(1 + \frac{r}{R}\right) \le$$

$$\overset{GERRETSEN}{\leq}\frac{2}{3}\cdot(4R^2+4Rr+3r^2-r^2-4Rr)\cdot\left(1+\frac{r}{R}\right)\leq$$

$$\overset{EULER}{\leq}\frac{2}{3}\cdot(4R^2+2r^2)\cdot\left(1+\frac{1}{2}\right)=4R^2+2r^2\leq$$

$$\overset{EULER}{\leq}4R^2+2\cdot\frac{R^2}{4}=\frac{9R^2}{2}$$

Equality holds for $a=b=c$.

2133. In ΔABC the following relationship holds:

$$F\leq\frac{3abc}{4\sqrt{a^2+b^2+c^2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$F\leq\frac{3abc}{4\sqrt{a^2+b^2+c^2}}\Leftrightarrow 4F\sqrt{a^2+b^2+c^2}\leq 3\cdot 4RF\Leftrightarrow$$

$$\Leftrightarrow\sqrt{a^2+b^2+c^2}\leq 3R\Leftrightarrow a^2+b^2+c^2\leq 9R^2\ \ (LEIBNIZ)$$

Equality holds for $a=b=c$.

2134. In ΔABC the following relationship holds:

$$((a^2+b^2)cosC+(b^2+c^2)cosA+(c^2+a^2)cosB\leq 9R^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\text{WLOG: } a\leq b\leq c\rightarrow a^2\leq b^2\leq c^2\rightarrow\begin{cases}a^2+b^2\leq c^2+b^2\\a^2+c^2\leq b^2+c^2\\a^2+b^2\leq a^2+c^2\end{cases}\rightarrow$$

$$\rightarrow a^2+b^2\leq a^2+c^2\leq b^2+c^2$$

$$a\le b\le c\to cosA\ge cosB\ge cosC$$

$$\sum_{cyc}(a^2+b^2)\,cosC\le\frac{1}{3}\sum_{cyc}(a^2+b^2)\cdot\sum_{cyc}cosC=$$

$$=\frac{2}{3}\sum_{cyc}a^2\cdot\left(1+\frac{r}{R}\right)\overset{EULER}{\leq}\frac{2}{3}\cdot 2(s^2-r^2-4Rr)\left(1+\frac{1}{2}\right)\le$$

$$\overset{GERRETSEN}{\leq}\quad 2(4R^2+4Rr+3r^2-r^2-4Rr)=2(4R^2+2r^2)\le$$

$$\overset{EULER}{\leq}\ 2\left(4R^2+\frac{R^2}{2}\right)=9R^2$$

Equality holds for: $a=b=c$.

2135. **In ΔABC the following relationship holds:**

$$a\ sec\frac{A}{2}+b\ sec\frac{B}{2}+c sec\frac{C}{2}\ge 12r$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$a\le b\le c\to cos\frac{A}{2}\le cos\frac{B}{2}\le cos\frac{C}{2}\to\frac{1}{cos\frac{A}{2}}\ge\frac{1}{cos\frac{B}{2}}\ge\frac{1}{cos\frac{C}{2}}\to$$

$$\to sec\frac{A}{2}\le sec\frac{B}{2}\le sec\frac{C}{2}$$

$$\sum_{cyc}a\,sec\frac{A}{2}\overset{CEBYSHEV}{\leq}\frac{1}{3}\sum_{cyc}a\cdot\sum_{cyc}sec\frac{A}{2}\overset{JENSEN}{\leq}$$

$$\le\frac{2s}{3}\cdot 3sec\left(\frac{A+B+C}{6}\right)=2s\cdot sec\frac{\pi}{6}\overset{MITRINOVIC}{\geq}$$

$$\geq 2\cdot 3\sqrt{3}r\cdot\frac{1}{\cos\frac{\pi}{6}}=2\cdot 3\sqrt{3}r\cdot\frac{1}{\frac{\sqrt{3}}{2}}=12r$$

Equality holds for: $a=b=c$.

2136. In ΔABC the following relationship holds:

$$h_a\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+h_a\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+h_a\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)\geq 12\sqrt{3}r$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$a\leq b\leq c\to\frac{1}{a}\geq\frac{1}{b}\geq\frac{1}{c}\to\frac{2F}{a}\geq\frac{2F}{b}\geq\frac{2F}{c}\to h_a\geq h_b\geq h_c$$

$$a\leq b\leq c\to\cos\frac{A}{2}\geq\cos\frac{B}{2}\geq\cos\frac{C}{2}\to\frac{1}{\cos\frac{A}{2}}\leq\frac{1}{\cos\frac{B}{2}}\leq\frac{1}{\cos\frac{C}{2}}\to$$

$$\to\sec\frac{A}{2}\leq\sec\frac{B}{2}\leq\sec\frac{C}{2}\to\begin{cases}\sec\frac{A}{2}+\sec\frac{B}{2}\leq\sec\frac{B}{2}+\sec\frac{C}{2}\\ \sec\frac{A}{2}+\sec\frac{C}{2}\leq\sec\frac{B}{2}+\sec\frac{C}{2}\\ \sec\frac{A}{2}+\sec\frac{B}{2}\leq\sec\frac{A}{2}+\sec\frac{C}{2}\end{cases}\to$$

$$\sec\frac{B}{2}+\sec\frac{C}{2}\geq\sec\frac{A}{2}+\sec\frac{C}{2}\geq\sec\frac{A}{2}+\sec\frac{B}{2}$$

$$\sum_{cyc}h_a\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)\overset{CEBYSHEV}{\geq}\frac{1}{3}\sum_{cyc}h_a\cdot\sum_{cyc}\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)=$$

$$=\frac{1}{3}\sum_{cyc}\frac{2F}{a}\cdot 2\sum_{cyc}\sec\frac{A}{2}\overset{JENSEN}{\geq}\frac{4F}{3}\cdot\frac{ab+bc+ca}{abc}\cdot 3\sec\left(\frac{A+B+C}{6}\right)=$$

$$=\frac{4F}{3}\cdot\frac{s^2+r^2+4Rr}{4RF}\cdot 3\sec\frac{\pi}{6}\overset{GERRETSEN}{\geq}\frac{1}{R\cdot\cos\frac{\pi}{6}}\cdot(16Rr-5r^2+r^2+4Rr)=$$

$$=\frac{2}{\sqrt{3}R}\cdot(20Rr-4r^2)\geq 12\sqrt{3}r\Leftrightarrow 40Rr-8r^2\geq 36Rr\Leftrightarrow$$

$$\Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r\ (EULER)$$

Equality holds for $a = b = c$.

2137. In ΔABC the following relationship holds:

$$\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \geq \frac{2\sqrt{3}}{\sqrt{s^2 + r^2 - 8Rr}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$\sum w_a \leq \sum \sqrt{s(s-a)} \overset{CBS}{\leq} \sqrt{3s(s-a+s-b+s-c)} = s\sqrt{3}\ (1)$$

$$\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \overset{CBS}{\geq} \frac{(1+1+1)^2}{\sum w_a} \overset{(1)}{\geq} \frac{9}{s\sqrt{3}} = \frac{3\sqrt{3}}{s}$$

$$We\ need\ to\ show: \frac{3\sqrt{3}}{s} \geq \frac{2\sqrt{3}}{\sqrt{s^2 + r^2 - 8Rr}}$$

$$3\sqrt{s^2 + r^2 - 8Rr} \geq 2s$$

$$9s^2 + 9r^2 - 72Rr \geq 4s^2$$

$$5s^2 - 72Rr + 9r^2 \geq 0$$

$$5(16Rr - 5r^2) - 72Rr + 9r^2 \overset{Gerretsen}{\geq} 0$$

$$8Rr \geq 16r^2,\ R \geq 2r\ true\ Euler$$

Equality holds for an equilateral triangle.

2138. In ΔABC the following relationship holds:

$$\frac{m_a}{a^2 + bc} + \frac{m_b}{b^2 + ca} + \frac{m_c}{c^2 + ab} \geq \frac{3}{4R}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$\sqrt{(a^2 + b^2)(a^2 + c^2)} \overset{C-S}{\geq} (a^2 + bc),\ m_a \overset{Tereshin}{\geq} \frac{b^2 + c^2}{4R}$$

$$\frac{m_a}{a^2+bc} \ge \frac{\frac{b^2+c^2}{4R}}{\sqrt{(a^2+b^2)(a^2+c^2)}} = \frac{1}{4R}\frac{b^2+c^2}{\sqrt{(a^2+b^2)(a^2+c^2)}} \quad (1)$$

$$\frac{m_a}{a^2+bc}+\frac{m_b}{b^2+ca}+\frac{m_c}{c^2+ab} = \sum \frac{m_a}{a^2+bc} \overset{(1)}{\ge}$$

$$\ge \frac{1}{4R}\sum \frac{b^2+c^2}{\sqrt{(a^2+b^2)(a^2+c^2)}} \overset{AM-GM}{\ge} \frac{3}{4R}\sqrt[3]{\prod \frac{b^2+c^2}{\sqrt{(a^2+b^2)(a^2+c^2)}}} = \frac{3}{4R}$$

Equality holds for an equilateral triangle.

2139. In any Δ ABC, the following relationship holds :

$$\frac{w_a+w_b+w_c}{6} \ge \sqrt[5]{\frac{s^2Rr^2(32R^2s^2r^2+16R^2r^4+40Rs^2r^3+8Rr^5+s^6+3s^4r^2+3s^2r^4+r^6)}{6(s^2+2Rr+r^2)^3}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{(b+c)^2} = \sum_{cyc}\frac{(a-2s)+2s}{(b+c)^2} = 2s.\frac{\sum_{cyc}(c+a)^2(a+b)^2}{\prod_{cyc}(b+c)^2} - \sum_{cyc}\frac{1}{b+c}$$

$$= \frac{\left(\sum_{cyc}(c+a)(a+b)\right)^2 - 2.2s(s^2+2Rr+r^2)(4s)}{2s(s^2+2Rr+r^2)^2} - \frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)}$$

$$= \frac{\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)^2 - 16s^2(s^2+2Rr+r^2) - (s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2} \Rightarrow$$

$$\sum_{cyc}\frac{a}{(b+c)^2} \overset{(\bullet)}{=} \frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\text{Now, } \sum_{cyc}w_a^2 = \sum_{cyc}\frac{4bcs(s-a)}{(b+c)^2} = \sum_{cyc}\frac{bc((b+c)^2-a^2)}{(b+c)^2} = \sum_{cyc}\left(bc-\frac{a^2bc}{(b+c)^2}\right) \overset{via\ (\bullet)}{=} s^2+4Rr+r^2$$

$$+2Rr.\frac{(s^2+2Rr+r^2)(5s^2+4Rr+r^2)+16s^2(s^2+2Rr+r^2)-(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$= \frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(16R^2+8Rr+r^2)}{(s^2+2Rr+r^2)^2}$$

$$\therefore \frac{32R^2s^2r^2+16R^2r^4+40Rs^2r^3+8Rr^5+s^6+3s^4r^2+3s^2r^4+r^6}{(s^2+2Rr+r^2)^2} \overset{(\bullet\bullet)}{=} \sum_{cyc} w_a^2$$

$$\text{Again, } w_aw_bw_c = \prod_{cyc}\frac{2bc\cos\frac{A}{2}}{b+c} = \frac{8.16R^2r^2s^2.\frac{s}{4R}}{2s(s^2+2Rr+r^2)} = \frac{16Rr^2s^2}{s^2+2Rr+r^2}$$

$$\therefore \frac{s^2Rr^2}{s^2+2Rr+r^2} \overset{(\bullet\bullet\bullet)}{=} \frac{w_aw_bw_c}{16} \therefore (\bullet\bullet) \text{ and } (\bullet\bullet\bullet) \Rightarrow$$

$$\sqrt[5]{\frac{s^2Rr^2(32R^2s^2r^2+16R^2r^4+40Rs^2r^3+8Rr^5+s^6+3s^4r^2+3s^2r^4+r^6)}{6(s^2+2Rr+r^2)^3}}$$

$$= \sqrt[5]{\frac{w_aw_bw_c}{96}.\sum_{cyc}w_a^2} \le \frac{w_a+w_b+w_c}{6} \Leftrightarrow \frac{1}{6^5}\left(\sum_{cyc}x\right)^5 \ge \frac{1}{96}.xyz\sum_{cyc}x^2$$

$$(x=w_a, y=w_b, z=w_c) \Leftrightarrow \left(\sum_{cyc}x\right)^5 \overset{(*)}{\ge} 81xyz\sum_{cyc}x^2$$

Assigning $y+z=M, z+x=N, x+y=P \Rightarrow M+N-P=2z>0, N+P-M=2x>0$ and $P+M-N=2y>0 \Rightarrow M+N>P, N+P>M, P+M>N \Rightarrow$ M, N, P form sides of a triangle with semiperimeter, circumradius and inradius $=$

$$s', R', r' \text{ (say) yielding } 2\sum_{cyc}x = \sum_{cyc}M = 2s' \Rightarrow \sum_{cyc}x = s' \to (1)$$

$\Rightarrow x = s'-M, y = s'-N, z = s'-P \Rightarrow xyz = r'^2s' \to (2)$ and via such substitutions, $\Rightarrow xyz = r'^2s' \to (2)$ and via such substitutions,

$$\sum_{cyc}x^2 = \left(\sum_{cyc}x\right)^2 - 2\sum_{cyc}xy = \left(\sum_{cyc}x\right)^2 - 2\sum_{cyc}(s-M)(s-N)$$

$$\overset{\text{via }(1)}{=} s'^2 - 2(4R'r'+r'^2) \therefore \sum_{cyc}x^2 = s'^2-8R'r'-2r'^2 \to (3)$$

$$\therefore (1),(2),(3) \Rightarrow (*) \Leftrightarrow s'^5 \ge 81r'^2s'(s'^2-8R'r'-2r'^2)$$

$$\Leftrightarrow s'^4 \overset{(**)}{\ge} 81r'^2(s'^2-8R'r'-2r'^2)$$

Since $(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore$ in order to prove : $s^4-81r^2(s^2-8Rr-2r^2) \ge 0$, it suffices to prove : LHS $\ge (s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (32R-91r)s^2 \overset{(***)}{\ge} r(256R^2-808Rr-137r^2)$$

Case 1 $32R-91r \ge 0$ and then : LHS of $(***) \overset{\text{Gerretsen}}{\ge} (32R-91r)(16Rr-5r^2) \overset{?}{\ge} r(256R^2-808Rr-137r^2) \Leftrightarrow 32R^2-101Rr+74r^2 \overset{?}{\ge} 0$

$$\Leftrightarrow (R-2r)(32R-37r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \geq \frac{91r}{32} > 2r$$

$$\Rightarrow (***) \text{ is true (strict inequality)}$$

$\boxed{Case\ 2}$ $32R - 91r < 0$ and then : LHS of $(***) \overset{\text{Gerretsen}}{\geq}$

$$(32R-91r)(4R^2+4Rr+3r^2) \overset{?}{\geq} r(256R^2-808Rr-137r^2)$$

$$\Leftrightarrow 32t^3-123t^2+135t-34 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(30t(t-2)+2t^2+t+17) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2r$$

$\Rightarrow (***)$ is true and combining both cases, $(***)$ is true $\forall \Delta ABC$

$$\therefore s^4 \geq 81r^2(s^2-8Rr-2r^2) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{w_a+w_b+w_c}{6}$$

$$\geq \sqrt[5]{\frac{s^2Rr^2(32R^2s^2r^2+16R^2r^4+40Rs^2r^3+8Rr^5+s^6+3s^4r^2+3s^2r^4+r^6)}{6(s^2+2Rr+r^2)^3}}$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)

2140. In any ΔABC, the following relationship holds :

$$3 \leq \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b} < 3+\frac{7+\sqrt{5}}{10}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$845 > 841 \Rightarrow \sqrt{5*169} > 29 \Rightarrow \frac{\sqrt{5}}{10} > \frac{29}{130} = \frac{12}{13} - \frac{7}{10} \Rightarrow \frac{7+\sqrt{5}}{10} > \frac{12}{13}$$

$$\Rightarrow 3+\frac{7+\sqrt{5}}{10} > \frac{51}{13} \overset{?}{>} \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}$$

$$\Leftrightarrow \frac{51}{13} \overset{?}{>} \frac{1}{2s(s^2+2Rr+r^2)} \cdot \sum_{cyc}\left((a+b)\left(\sum_{cyc} ab + a^2\right)\right)$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \left(\left(\sum_{cyc} ab\right)\left(\sum_{cyc}(a+b)\right)+\sum_{cyc} a^3+\sum_{cyc} a^2b\right)$$

$$= \frac{(s^2+4Rr+r^2)(4s)+2s(s^2-6Rr-3r^2)+\sum_{cyc} a^2b}{2s(s^2+2Rr+r^2)}$$

$$\Leftrightarrow \frac{51}{13}-\frac{3s^2+2Rr-r^2}{s^2+2Rr+r^2} \overset{?}{>} \frac{\sum_{cyc} a^2b}{2s(s^2+2Rr+r^2)}$$

$$\Leftrightarrow \frac{12s^2+76Rr+64r^2}{13(s^2+2Rr+r^2)} \overset{?}{\underset{(*)}{\geq}} \frac{\sum_{cyc} a^2b}{2s(s^2+2Rr+r^2)}$$

$$\text{Now, } \frac{\sum_{cyc} a^2b}{2s(s^2+2Rr+r^2)} \overset{CBS}{\leq} \frac{\sqrt{(\sum_{cyc} a^2b^2)(\sum_{cyc} a^2)}}{2s(s^2+2Rr+r^2)} \overset{?}{<} \frac{12s^2+76Rr+64r^2}{13(s^2+2Rr+r^2)}$$

$$\Leftrightarrow 2s^2(12s^2+76Rr+64r^2)^2 \overset{?}{>} 169(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)$$

$$\Leftrightarrow 119s^6+(5676Rr+2903r^2)s^4+r^2(3440R^2+18104Rr+8361r^2)s^2$$

$$+r^3(10816R^3+8112R^2r+2028Rr^2+169r^3) \overset{?}{>} 0 \to \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b} < 3+\frac{7+\sqrt{5}}{10} \text{ and via AM}-\text{GM}, \sum_{cyc}\frac{a+b}{b+c} \geq 3 \text{ and so,}$$

$$3 \leq \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b} < 3+\frac{7+\sqrt{5}}{10}\ \forall\, \Delta\, ABC,$$

$$''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

2141. In any ΔABC prove that :

$$\frac{1}{\sqrt{r_a}}+\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}} \leq \frac{\sqrt{6}}{9}\cdot\frac{w_a+w_b+w_c}{r\sqrt{R}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$w_a = \frac{2\sqrt{bcs(s-a)}}{b+c} = \sqrt{\frac{8abc.s(s-a)}{2a(b+c)^2}} \overset{AM-GM}{\underset{(3)}{\geq}} \sqrt{\frac{27.8abc.s(s-a)}{[2a+(b+c)+(b+c)]^3}} =$$

$$= \frac{9}{\sqrt{6}}.\sqrt{\frac{Rr(s-a)}{s}} = \frac{9r\sqrt{R}}{\sqrt{6}}.\frac{1}{\sqrt{r_a}} \Rightarrow \frac{\sqrt{6}}{9}.\frac{w_a}{r\sqrt{R}} \geq \frac{1}{\sqrt{r_a}} \text{ (and analogs)}$$

Adding this inequality with similar ones yields the desired result.

Equality holds iff ΔABC is equilateral.

2142. In ΔABC the following relationship holds:

$$(a+b)m_cw_c+(b+c)m_aw_a+(c+a)m_bw_b \geq 2s(s^2-r^2-4Rr)$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$We\ \ know\ that\ m_c \geq \frac{a+b}{2}\cos\frac{C}{2}$$

$$(a+b)m_c w_c \geq (a+b)\frac{a+b}{2}\cos\frac{C}{2}\cdot\frac{2ab}{a+b}\cos\frac{C}{2} = (a+b)ab\cos^2\frac{C}{2} =$$

$$= (a+b)ab\frac{s(s-c)}{ab} = s(s-c)(a+b) = s(s-c)(2s-c) = s(2s^2-3cs+c^2)$$

$$(a+b)m_c w_c + (b+c)m_a w_a + (c+a)m_b w_b = \sum(a+b)m_c w_c \geq$$

$$\geq s\sum(2s^2-3cs+c^2) = s\left(6s^2-3s(a+b+c)+(a^2+b^2+c^2)\right) =$$

$$= s\left(6s^2-6s^2+2(s^2-r^2-4Rr)\right) = 2s(s^2-r^2-4Rr)$$

$$Equality\ holds\ for\ a=b=c$$

2143. In ΔABC the following relationship holds:

$$h_a+h_b+h_c \geq \sqrt{\frac{2r}{R}}\,(w_a+w_b+w_c)$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$\frac{w_a}{h_a} = \frac{2\sqrt{bc.s(s-a)}}{b+c}\frac{2R}{bc} = \frac{4R\sqrt{s(s-a)}}{\sqrt{bc}(2s-a)} = \frac{4R\sqrt{s(s-a)}}{\sqrt{bc}(2(s-a)+a)} \overset{AM-GM}{\leq}$$

$$\leq \frac{4R\sqrt{s(s-a)}}{\sqrt{bc}2\sqrt{2(s-a).a}} = \frac{2R\sqrt{s}}{\sqrt{2abc}} = \frac{2R\sqrt{s}}{\sqrt{8Rrs}} = \sqrt{\frac{R}{2r}}\ (1)$$

$$\sqrt{\frac{2r}{R}}\,(w_a+w_b+w_c) = \sqrt{\frac{2r}{R}}\,w_a + \sqrt{\frac{2r}{R}}\,w_b + \sqrt{\frac{2r}{R}}\,w_c \overset{(1)}{\leq}$$

$$\leq \frac{h_a}{w_a}.w_a + \frac{h_b}{w_b}.w_b + \frac{h_c}{w_c}.w_c = h_a+h_b+h_c$$

$$Equality\ holds\ for\ a=b=c$$

2144. In any Δ ABC, the following relationship holds :

$$\frac{h_a\sqrt{h_a}}{w_a^2}+\frac{h_b\sqrt{h_b}}{w_b^2}+\frac{h_c\sqrt{h_c}}{w_c^2}\le\frac{2(R+r)}{R\sqrt{3r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, }\sum_{cyc}\left((s-b)(s-c)(b+c)^2\right)=\sum_{cyc}\left((-s^2+sa+bc)(b^2+c^2+2bc)\right)=$$

$$=-2s^2\sum_{cyc}a^2-2s^2\sum_{cyc}ab+s\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)+6sabc+$$

$$+\sum_{cyc}\left(bc\left(\sum_{cyc}a^2-a^2\right)\right)+2\sum_{cyc}a^2b^2$$

$$=-4s^2(s^2-4Rr-r^2)+12Rrs^2+2(s^2-4Rr-r^2)(s^2+4Rr+r^2)-8Rrs^2$$

$$+2(s^2+4Rr+r^2)^2-32Rrs^2=4r(R+2r)s^2\overset{(i)}{=}\sum_{cyc}\left((s-b)(s-c)(b+c)^2\right)$$

$$\sum_{cyc}(bc(s-b)(s-c)(b+c)^2)=\sum_{cyc}\left(bc(s-b)(s-c)\left(\sum_{cyc}a^2-a^2+2bc\right)\right)=$$

$$=\left(\sum_{cyc}a^2\right)r^2s^2\sum_{cyc}\frac{bc}{s(s-a)}-4Rrs.\sum_{cyc}\left(a(-s^2+sa+bc)\right)+2\sum_{cyc}\left(b^2c^2(-s^2+sa+bc)\right)=$$

$$=2r^2s^2(s^2-4Rr-r^2).\frac{s^2+(4R+r)^2}{s^2}$$

$$-4Rrs.(-s^2(2s)+2s(s^2-4Rr-r^2)+12Rrs)-2s^2((s^2+4Rr+r^2)^2-16Rrs^2)$$

$$+8Rrs^2(s^2+4Rr+r^2)+2(s^2+4Rr+r^2)^3-48Rrs^2(s^2+2Rr+r^2)$$

$$=4r^2s^2(4R^2+2Rr+r^2+s^2)\overset{(ii)}{=}\sum_{cyc}(bc(s-b)(s-c)(b+c)^2)$$

$$\text{Now, }\frac{h_a\sqrt{h_a}}{w_a^2}+\frac{h_b\sqrt{h_b}}{w_b^2}+\frac{h_c\sqrt{h_c}}{w_c^2}=\sum_{cyc}\left(\frac{bc}{2R}.\sqrt{\frac{bc}{2R}}.\frac{(b+c)^2}{4bc}.\frac{(s-b)(s-c)}{r^2s^2}\right)$$

$$=\frac{1}{2R.\sqrt{2R}.4r^2s^2}.\sum_{cyc}(\sqrt{bc}.(s-b)(s-c)(b+c)^2)$$

$$=\frac{1}{2R.\sqrt{2R}.4r^2s^2}.\sum_{cyc}\left(\sqrt{(s-b)(s-c)(b+c)^2}.\sqrt{bc(s-b)(s-c)(b+c)^2}\right)\overset{CBS}{\le}$$

$$\frac{1}{2R.\sqrt{2R}.4r^2s^2}.\sqrt{\sum_{cyc}((s-b)(s-c)(b+c)^2)}.\sqrt{\sum_{cyc}(bc(s-b)(s-c)(b+c)^2)}$$

$$\overset{via\ (i)\ and\ (ii)}{=}\frac{1}{2R.\sqrt{2R}.4r^2s^2}.\sqrt{4r(R+2r)s^2}.\sqrt{4r^2s^2(4R^2+2Rr+r^2+s^2)}=$$

$$\frac{1}{2R}.\sqrt{\frac{(R+2r)(4R^2+2Rr+r^2+s^2)}{2Rr}}\overset{Gerretsen}{\le}\frac{1}{2R}.\sqrt{\frac{(R+2r)(8R^2+6Rr+4r^2)}{2Rr}}\overset{?}{\le}\frac{2(R+r)}{R.\sqrt{3r}}$$

$$\Leftrightarrow 4t^3-t^2-8t-12\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(4t^2+7t+6)\overset{?}{\ge}0\to true\because t\overset{Euler}{\ge}2$$

$$\therefore\frac{h_a\sqrt{h_a}}{w_a^2}+\frac{h_b\sqrt{h_b}}{w_b^2}+\frac{h_c\sqrt{h_c}}{w_c^2}\le\frac{2(R+r)}{R.\sqrt{3r}}\ \forall\Delta ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)

2145. In any Δ ABC, the following relationship holds :

$$(w_a+w_b+w_c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 3s.\sqrt{\frac{r}{2R}}\left(\frac{1}{\sqrt{h_ar_a}}+\frac{1}{\sqrt{h_br_b}}+\frac{1}{\sqrt{h_cr_c}}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2h_ar_a=\frac{4rs^2\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}=\frac{rs^2}{R\cos^2\frac{A}{2}}\Rightarrow\frac{1}{\sqrt{h_ar_a}}=\frac{1}{s}.\sqrt{\frac{2R}{r}}\cos\frac{A}{2}\ and\ analogs$$

$$\Rightarrow 3s.\sqrt{\frac{r}{2R}}\left(\frac{1}{\sqrt{h_ar_a}}+\frac{1}{\sqrt{h_br_b}}+\frac{1}{\sqrt{h_cr_c}}\right)=3\sum_{cyc}\cos\frac{A}{2}\to(i)$$

$$Now, w_a+w_b+w_c=\sum_{cyc}\left(\frac{2bc}{b+c}.\cos\frac{A}{2}\right)\overset{Chebyshev}{\ge}\frac{2}{3}.\sum_{cyc}\frac{bc}{b+c}.\sum_{cyc}\cos\frac{A}{2}$$

$$\left(\begin{array}{c}\because WLOG\ assuming\ a\ge b\ge c\Rightarrow\frac{bc}{b+c}\le\frac{ca}{c+a}\le\frac{ab}{a+b}\ and\\ \cos\frac{A}{2}\le\cos\frac{B}{2}\le\cos\frac{C}{2}\end{array}\right)$$

$$=\frac{2}{3}.\frac{1}{2s(s^2+2Rr+r^2)}.\sum_{cyc}\left(bc\left(a^2+\sum_{cyc}ab\right)\right).\sum_{cyc}\cos\frac{A}{2}$$

$$=\frac{(s^2+4Rr+r^2)^2+8Rrs^2}{3s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\cos\frac{A}{2}\Rightarrow(w_a+w_b+w_c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge$$

$$\frac{((s^2+4Rr+r^2)^2+8Rrs^2)(s^2+4Rr+r^2)}{3s(s^2+2Rr+r^2).4Rrs}\cdot\sum_{cyc}\cos\frac{A}{2}\overset{?}{\ge}$$

$$3s.\sqrt{\frac{r}{2R}}\left(\frac{1}{\sqrt{h_ar_a}}+\frac{1}{\sqrt{h_br_b}}+\frac{1}{\sqrt{h_cr_c}}\right)\overset{\text{via (i)}}{=}3\sum_{cyc}\cos\frac{A}{2}$$

$$\Leftrightarrow((s^2+4Rr+r^2)^2+8Rrs^2)(s^2+4Rr+r^2)\overset{?}{\ge}36Rrs^2(s^2+2Rr+r^2)$$

$$\Leftrightarrow s^6-(16Rr-3r^2)s^4+r^2(8R^2-4Rr+3r^2)s^2+r^3(4R+r)^3\overset{?}{\underset{(1)}{\ge}}0\text{ and }\because$$

$(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\ge}0$ ∴ in order to prove (1), it suffices to prove :

LHS of (1) $\ge(s^2-16Rr+5r^2)^3\Leftrightarrow(8R-3r)s^4-r(190R^2-119Rr+18r^2)s^2$

$$+r^2(1040R^3-948R^2r+303Rr^2-31r^3)\underset{(2)}{\ge}0$$

and $\because(8R-3r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0$ ∴ in order to prove (2),

it suffices to prove : LHS of (2) $\ge(8R-3r)(s^2-16Rr+5r^2)^2$

$\Leftrightarrow(66R^2-57Rr+12r^2)s^2\underset{(3)}{\ge}r(1008R^3-1100R^2r+377Rr^2-44r^3)$ *and*

finally, $(66R^2-57Rr+12r^2)s^2\overset{\text{Gerretsen}}{\ge}(66R^2-57Rr+12r^2)(16Rr-5r^2)$

$\overset{?}{\ge}r(1008R^3-1100R^2r+377Rr^2-44r^3)\Leftrightarrow24t^3-71t^2+50t-8\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$

$\Leftrightarrow(t-2)(12t^2+12t(t-2)+t+4)\overset{?}{\ge}0\to$ true $\because t\overset{\text{Euler}}{\ge}2\Rightarrow(3)\Rightarrow(2)\Rightarrow(1)$

is true ∴ $(w_a+w_b+w_c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge$

$3s.\sqrt{\frac{r}{2R}}\left(\frac{1}{\sqrt{h_ar_a}}+\frac{1}{\sqrt{h_br_b}}+\frac{1}{\sqrt{h_cr_c}}\right)\ \forall\,\Delta\,ABC$," = " iff $\Delta\,ABC$ is equilateral (QED)

2146. In any Δ ABC, the following relationship holds :

$$\frac{h_a}{\cos\frac{A}{2}}+\frac{h_b}{\cos\frac{B}{2}}+\frac{h_c}{\cos\frac{C}{2}}\ge\frac{r(5R-2r)(4R+r)}{4R^2}\cdot\sqrt{\frac{2(5R-2r)}{2R-r}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{\cos\frac{A}{2}}+\frac{h_b}{\cos\frac{B}{2}}+\frac{h_c}{\cos\frac{C}{2}}=\sum_{cyc}\frac{2rs}{a\cos\frac{A}{2}}\overset{Bergstrom}{\geq}\frac{18rs}{\sum_{cyc}a\cos\frac{A}{2}}$$

$$=\frac{18rs}{\sum_{cyc}\left(a.\sqrt{\frac{sa(s-a)}{4Rrs}}\right)}=\frac{36rs.\sqrt{Rr}}{\sum_{cyc}\left(\sqrt{a}.\sqrt{a^2(s-a)}\right)}$$

$$\overset{CBS}{\geq}\frac{36rs.\sqrt{Rr}}{\sqrt{2s}.\sqrt{2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}}=\frac{18r.\sqrt{Rr}}{\sqrt{2Rr+2r^2}}=\frac{9r.\sqrt{2R}}{\sqrt{R+r}}$$

$$\overset{?}{\geq}\frac{r(5R-2r)(4R+r)}{4R^2}.\sqrt{\frac{2(5R-2r)}{2R-r}}$$

$$\Leftrightarrow 1296R^5(2R-r)\overset{?}{\geq}(R+r)(4R+r)^2(5R-2r)^3$$

$$\Leftrightarrow 592t^6-1896t^5+1515t^4-87t^3-198t^2+12t+8\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\begin{pmatrix}236t^5+356t^4(t-2)+91t^3++90t^2+4t(t-2)\\+(t-2)(t+2)\end{pmatrix}\overset{?}{\geq}0$$

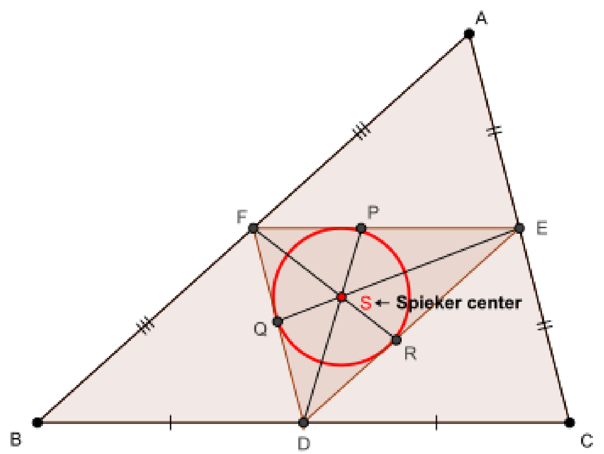
$$\rightarrow \text{true} \because t\overset{Euler}{\geq}2 \therefore \frac{h_a}{\cos\frac{A}{2}}+\frac{h_b}{\cos\frac{B}{2}}+\frac{h_c}{\cos\frac{C}{2}}\geq\frac{r(5R-2r)(4R+r)}{4R^2}.\sqrt{\frac{2(5R-2r)}{2R-r}}$$

$$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

2147. In any acute $\Delta\,ABC$ with $p_a, p_b, p_c \rightarrow$ Spieker cevians, the following relationship holds : $p_a+p_b+p_c\geq\frac{23R}{10}+\frac{22r}{5}$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say)

$$and\ inradius\ of\ \Delta\ DEF = r'(say)$$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\ DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2}\ and\ m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

$$Via\ (1),(2)\ and\ using\ cosine\ law\ on\ \Delta\ AFS\ and\ \Delta\ AES, we\ arrive\ at:$$

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Now, $p_a\overset{?}{\geq}h_a+\frac{2}{3}.\frac{(b-c)^2}{a}\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$

$$\overset{?}{\geq}s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}+\frac{4}{9}.\frac{(b-c)^4}{a^2}+\frac{4h_a}{3}.\frac{(b-c)^2}{a}$$

$$\Leftrightarrow\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2}\overset{?}{\underset{(\blacksquare)}{\geq}}\frac{4h_a}{3a}\ (\because (b-c)^2\geq 0)$$

We have : $\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2}\overset{\because a^2>(b-c)^2}{>}\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4}{9}$

$$=\frac{9s(3s+a)a^2+9s(s-a)(2s+a)^2-4a^2(2s+a)^2}{9a^2(2s+a)^2}$$

$$=\frac{4(s-a)(9s^3+9s^2a+5sa^2+a^3)}{9a^2(2s+a)^2}\overset{s>a}{>}0\Rightarrow \text{LHS of }(\blacksquare)>0\therefore(\blacksquare)\Leftrightarrow$$

$$\frac{T^2}{a^4(2s+a)^4}+\frac{16(b-c)^4}{81a^4}-\frac{8T(b-c)^2}{9a^4(2s+a)^2}\overset{?}{\geq}\frac{16}{9a^2}\cdot\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$(T=s(3s+a)a^2+s(s-a)(2s+a)^2)$$

$$\Leftrightarrow\frac{16(b-c)^4}{81a^4}-\frac{8(b-c)^2}{9a^4}\left(\frac{T}{(2s+a)^2}-2s(s-a)\right)+\frac{T^2}{a^4(2s+a)^4}-\frac{16s(s-a)}{9a^2}$$

$$\Leftrightarrow\left(\frac{4(b-c)^2}{9}\right)^2+\frac{4(b-c)^2}{9}\cdot\frac{4s(2s^3-3sa^2-a^3)}{(2s+a)^2}$$

$$+\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}\boxed{\overset{?}{\underset{(■■)}{\geq}}}0$$

Now, LHS of (■■) is a quadratic polynomial in $''\frac{4(b-c)^2}{9}''$ whose $\boxed{\text{discriminant}}$

$$=\frac{16s^2(2s^3-3sa^2-a^3)^2}{(2s+a)^4}-4.\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}$$

$$=-\frac{16sa^7}{9(2s+a)^4}.(44t^5-28t^4-49t^3+10t^2+19t+4)\ \left(t=\frac{s}{a}\right)$$

$$=-\frac{16sa^7}{9(2s+a)^4}.(t-1)^2(44t^3+60t^2+27t+4)<0\Rightarrow \text{LHS of }(■■)>0$$

$$\Rightarrow(■■)\Rightarrow(■)\text{ is true}\therefore p_a\geq h_a+\frac{2}{3}.\frac{(b-c)^2}{a}\to(m)$$

$$\text{Now, }\sum_{cyc}\frac{(b-c)^2}{a}=\sum_{cyc}\frac{b^2+c^2+a^2}{a}-\sum_{cyc}a-\frac{2}{4Rrs}.\sum_{yc}b^2c^2$$

$$=\frac{1}{4Rrs}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)-\frac{8Rrs^2}{4Rrs}-\frac{2}{4Rrs}\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)$$

$$=\frac{1}{4Rrs}\left(\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2-2\sum_{cyc}ab\right)+24Rrs^2\right)$$

$$=\frac{2(s^2+4Rr+r^2)\left(s^2-4Rr-r^2-(s^2+4Rr+r^2)\right)+24Rrs^2}{4Rrs}$$

$$=\frac{(2R-r)s^2-r(4R+r)^2}{Rs}\boxed{\boxed{\overset{(n)}{=}}}\sum_{cyc}\frac{(b-c)^2}{a}$$

$$\text{We have : }p_a+p_b+p_c\overset{\text{via }(m)}{\geq}\sum_{cyc}h_a+\frac{2}{3}.\sum_{cyc}\frac{(b-c)^2}{a}\overset{\text{via }(n)}{=}$$

$$\frac{s^2+4Rr+r^2}{2R}+\frac{2}{3}.\frac{(2R-r)s^2-r(4R+r)^2}{Rs}$$

$$\overset{Walker}{\geq} \frac{R^2+6Rr+2r^2}{R}+\frac{2}{3}\cdot\frac{(2R-r)(2R^2+8Rr+3r^2)-r(4R+r)^2}{Rs} \overset{Gerretsen}{\geq}$$

$$\frac{R^2+6Rr+2r^2}{R}+\frac{4}{3}\cdot\frac{2R^3-R^2r-5Rr^2-2r^3}{R.\sqrt{4R^2+4Rr+3r^2}}$$

$$\left(\because 2R^3-R^2r-5Rr^2-2r^3=(R-2r)(2R^2+3Rr+r^2)\overset{Euler}{\geq}0\right)\overset{?}{\geq}\frac{23R}{10}+\frac{22r}{5}$$

$$=\frac{23R+44r}{10}\Leftrightarrow\frac{4}{3}\cdot\frac{2R^3-R^2r-5Rr^2-2r^3}{R.\sqrt{4R^2+4Rr+3r^2}}\boxed{\overset{?}{\geq}}\frac{13R^2-16Rr-20r^2}{10R}$$

$$\Leftrightarrow\frac{16}{9}\cdot\frac{(2R^3-R^2r-5Rr^2-2r^3)^2}{4R^2+4Rr+3r^2}\overset{?}{\geq}\frac{(13R^2-16Rr-20r^2)^2}{100}$$

$$\left(\because 13R^2-16Rr-20r^2=(13R+10r)(R-2r)\overset{Euler}{\geq}0\right)$$

$$\Leftrightarrow 316t^6+2492t^5-10483t^4+896t^3+16088t^2+320t-4400\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)^2\left(316t^4+3756t^3+2492t^2+510t(t-2)+275(t^2-4)\right)\overset{?}{\geq}0$$

$$\to \text{true} \because t\overset{Euler}{\geq}2 \therefore p_a+p_b+p_c\geq\frac{23R}{10}+\frac{22r}{5}\ \forall\ acute\ \Delta\ ABC,$$

with equality iff Δ ABC is equilateral (QED)

2148. In ΔABC the following relationship holds:

$$\frac{1}{w_a sin\frac{A}{2}}+\frac{1}{w_b sin\frac{B}{2}}+\frac{1}{w_c sin\frac{C}{2}}=4\left(\frac{sin\frac{A}{2}}{w_a}+\frac{sin\frac{B}{2}}{w_b}+\frac{sin\frac{C}{2}}{w_c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{1}{w_a\sin\left(\frac{A}{2}\right)}=\frac{1}{\frac{2bc}{b+c}.\cos\left(\frac{A}{2}\right).\sin\left(\frac{A}{2}\right)}=\frac{b+c}{bcsin(A)}=\frac{2R(sinB+sinC)}{4R^2.\sin(A).\sin(B).\sin(C)}=$$

$$=\frac{\sin(B)+\sin(C)}{2R.\frac{F}{2R^2}}=\frac{R.\sin(B)+R.\sin(C)}{F}=\frac{b+c}{2F}$$

$$\sum_{cyc}\frac{1}{w_a sin\frac{A}{2}}=\sum_{cyc}\frac{b+c}{2F}=\frac{4s}{2F}=\frac{2s}{sr}=\frac{2}{r}\ (LHS)$$

$$w_a=\frac{2bc}{b+c}.cos\frac{A}{2}\ ;\ h_a=\frac{2F}{a}=\frac{2}{a}.\frac{1}{2}bcsinA=\frac{bc}{a}.sinA$$

$$\frac{w_a}{h_a}=\frac{\frac{2bc}{b+c}.cos\frac{A}{2}}{\frac{bc}{a}.sinA}=\frac{a}{(b+c).sin\frac{A}{2}}\to w_a=\frac{a.h_a}{(b+c).sin\frac{A}{2}}$$

$$4.\frac{sin\frac{A}{2}}{w_a}=4.\frac{(b+c).sin^2\frac{A}{2}}{a.h_a}=4.\frac{b+c}{a}.\frac{sin^2\frac{A}{2}}{h_a}=4.\frac{cos\frac{B-C}{2}}{sin\frac{A}{2}}.\frac{sin^2\frac{A}{2}}{h_a}$$

$$=4.\frac{cos\frac{B-C}{2}.sin\frac{A}{2}}{h_a}$$

$$Here\quad \boxed{\frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)}=\frac{b+c}{a}}\quad (Mollweides\ \ formula)$$

$$\cos\left(\frac{B-C}{2}\right).\sin\left(\frac{A}{2}\right)=\frac{1}{2}\left(\sin\left(\frac{A}{2}+\frac{B-C}{2}\right)+\sin\left(\frac{A}{2}-\frac{B-C}{2}\right)\right)=$$

$$\frac{1}{2}\left(\sin\left(\frac{\pi}{2}-C\right)+\sin\left(\frac{\pi}{2}-B\right)\right)=\frac{1}{2}(\cos(B)+\cos(C))$$

$$4.\frac{\sin\left(\frac{A}{2}\right)}{w_a}=4.\frac{\cos\left(\frac{B-C}{2}\right).\sin\left(\frac{A}{2}\right)}{h_a}=\frac{2}{h_a}(\cos(B)+\cos(C))$$

$$4\sum\frac{sin\frac{A}{2}}{w_a}=\frac{2}{h_a}(\cos(B)+\cos(C))+\frac{2}{h_b}(\cos(A)+\cos(C))+\frac{2}{h_C}(\cos(B)+\cos(A))=$$

$$=\frac{a}{F}(\cos(B)+\cos(C))+\frac{b}{F}(\cos(A)+\cos(C))+\frac{C}{F}(\cos(B)+\cos(A))=$$

$$=\frac{1}{F}\big((a.\cos(A)+b.\cos(A))+(c.\cos(A)+a.\cos(C))+(b.\cos(C)+c.\cos(B))\big)=$$

$$=\frac{1}{F}(a+b+c)=\frac{2S}{F}=\frac{2S}{Sr}=\frac{2}{r}(RHS)\quad Proved$$

$$Here\quad \boxed{\begin{array}{l}a=b.\cos(C)+c.\cos(B)\\ b=c.\cos(A)+a.\cos(C)\\ c=a.\cos(B)+b.\cos(A)\end{array}}$$

2149. In any Δ ABC the following relationship holds :

$$w_ah_bh_c+h_aw_bh_c+h_ah_bw_c\le w_aw_bw_c+2h_ah_bh_c\le h_aw_bw_c+w_ah_bw_c+w_aw_bh_c$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$w_ah_bh_c+h_aw_bh_c+h_ah_bw_c\le w_aw_bw_c+2h_ah_bh_c$$

$$\Leftrightarrow\sum_{cyc}\frac{w_a}{h_a}\le\frac{16Rr^2s^2}{s^2+2Rr+r^2}\cdot\frac{R}{2r^2s^2}+2$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{2bc\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}\cdot\frac{2R}{bc}\right) \le \frac{8R^2}{s^2+2Rr+r^2}+2$$

$$\Leftrightarrow \frac{1}{\prod_{cyc}\cos\frac{B-C}{2}}\cdot\sum_{cyc}\left(\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right) \overset{(1)}{\le} \frac{8R^2}{s^2+2Rr+r^2}+2$$

$$\text{Now, }\cos\frac{B-C}{2}+\cos\frac{B+C}{2} = 2\cos\frac{B}{2}\cos\frac{C}{2} = \frac{2s}{4R}\sec\frac{A}{2} \Rightarrow$$

$$\cos^2\frac{B-C}{2} = \left(\frac{s}{2R}\sec\frac{A}{2}-\sin\frac{A}{2}\right)^2 \Rightarrow$$

$$\cos^2\frac{B-C}{2} = \frac{s^2}{4R^2}\sec^2\frac{A}{2}+\sin^2\frac{A}{2}-\frac{s}{R}\tan\frac{A}{2} \text{ and analogs}$$

$$\therefore \sum_{cyc}\cos^2\frac{B-C}{2} = \frac{s^2}{4R^2}\cdot\sum_{cyc}\sec^2\frac{A}{2}+\sum_{cyc}\sin^2\frac{A}{2}-\frac{1}{R}\sum_{cyc}s\tan\frac{A}{2}$$

$$= \frac{s^2}{4R^2}\cdot\frac{s^2+(4R+r)^2}{s^2}+\frac{2R-r}{2R}-\frac{4R+r}{R}$$

$$= \frac{s^2+(4R+r)^2+2R(2R-r)-4R(4R+r)}{4R^2}$$

$$\Rightarrow \sum_{cyc}\cos^2\frac{B-C}{2} = \frac{s^2+4R^2+2Rr+r^2}{4R^2} \to \text{(m) and}$$

$$\prod_{cyc}\cos\frac{B-C}{2} = \prod_{cyc}\frac{b+c}{a}\cdot\prod_{cyc}\sin\frac{A}{2} = \frac{2s(s^2+2Rr+r^2)}{4Rrs}\cdot\frac{r}{4R}$$

$$\Rightarrow \prod_{cyc}\cos\frac{B-C}{2} = \frac{s^2+2Rr+r^2}{8R^2} \to \text{(n)}$$

$$\text{Now, LHS of } (1) \le \frac{1}{\prod_{cyc}\cos\frac{B-C}{2}}\cdot\sum_{cyc}\cos^2\frac{B-C}{2} \overset{\text{via (m) and (n)}}{=}$$

$$\frac{8R^2}{s^2+2Rr+r^2}\cdot\frac{s^2+4R^2+2Rr+r^2}{4R^2} = \frac{2(s^2+2Rr+r^2)+8R^2}{s^2+2Rr+r^2}$$

$$= \frac{8R^2}{s^2+2Rr+r^2}+2 \Rightarrow (1) \text{ is true}$$

$$\therefore w_a h_b h_c + h_a w_b h_c + h_a h_b w_c \le w_a w_b w_c + 2h_a h_b h_c$$
$$\text{Again, } w_a w_b w_c + 2h_a h_b h_c \le h_a w_b w_c + w_a h_b w_c + w_a w_b h_c$$

$$\Leftrightarrow \sum_{cyc} \frac{h_a}{w_a} \ge \frac{4r^2s^2}{R} \cdot \frac{s^2+2Rr+r^2}{16Rr^2s^2} + 1$$

$$\Leftrightarrow \sum_{cyc} \left(\frac{bc}{2R} \cdot \frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2bc\cos\frac{A}{2}} \right) \ge \frac{s^2+2Rr+r^2+4R^2}{4R^2}$$

$$\Leftrightarrow \sum_{cyc} \cos^2\frac{B-C}{2} + 2\sum_{cyc}\left(\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right) \ge \frac{(s^2+4R^2+2Rr+r^2)^2}{16R^4}$$

$$\overset{\text{via (m) and (n)}}{\Leftrightarrow} \frac{s^2+4R^2+2Rr+r^2}{4R^2} + \frac{s^2+2Rr+r^2}{4R^2} \cdot \sum_{cyc} \frac{1}{\cos\frac{B-C}{2}}$$

$$\overset{②}{\ge} \frac{(s^2+4R^2+2Rr+r^2)^2}{16R^4}$$

Now, $\because 0 < \cos\frac{B-C}{2} \le 1 \therefore$ LHS of ② $\ge \frac{s^2+4R^2+2Rr+r^2}{4R^2} + \frac{3s^2+6Rr+3r^2}{4R^2}$

$$= \frac{s^2+R^2+2Rr+r^2}{R^2} \overset{?}{\ge} \frac{(s^2+4R^2+2Rr+r^2)^2}{16R^4}$$

$$\Leftrightarrow \frac{s^2+2Rr+r^2}{R^2} + 1 \overset{?}{\ge} \frac{(s^2+2Rr+r^2)^2}{16R^4} + \frac{8R^2(s^2+2Rr+r^2)}{16R^4} + 1$$

$$\Leftrightarrow \frac{1}{2R^2} \overset{?}{\ge} \frac{s^2+2Rr+r^2}{16R^4} \Leftrightarrow 8R^2 - 2Rr - r^2 \overset{?}{\ge} s^2 \to \text{true}$$

$$\because 8R^2 - 2Rr - r^2 = 4R^2 + 4Rr + 3r^2 + 2(R-2r)(2R+r) \overset{\text{Gerretsen and Euler}}{\ge} s^2$$

$\Rightarrow$ ② is true $\therefore w_a w_b w_c + 2h_a h_b h_c \le h_a w_b w_c + w_a h_b w_c + w_a w_b h_c$ *and* so, $w_a h_b h_c + h_a w_b h_c + h_a h_b w_c \le w_a w_b w_c + 2h_a h_b h_c \le$ $h_a w_b w_c + w_a h_b w_c + w_a w_b h_c \ \forall \Delta ABC$," = " iff ΔABC is equi*lateral* (QED)

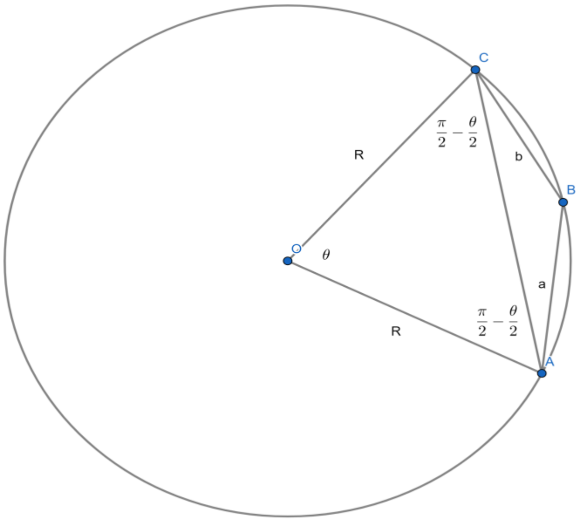
2150.

$$[OABC] = \frac{1}{8}\left[(a+b)^2\cot\left(\frac{\theta}{4}\right) - (a-b)^2\tan\left(\frac{\theta}{4}\right)\right]$$

$$R = \frac{\sqrt{a^2+b^2+2ab\cos\left(\frac{\theta}{2}\right)}}{2\sin\left(\frac{\theta}{2}\right)}$$

Proposed by Thanasis Gakopoulos-Greece

Solution by Mirsadix Muzefferov-Azerbaijan



$$\text{In } AOC \text{ rule sine} \frac{R}{\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)} = \frac{AC}{\sin(\theta)} \;=>\; AC = 2R\sin\left(\frac{\theta}{2}\right) \quad (1)$$

$$\text{In } ABC \text{ rule cosine } AC^2 = a^2+b^2-2ab\cos\left(\pi-\frac{\theta}{2}\right) = a^2+b^2+2abcos\left(\frac{\theta}{2}\right) \ (2)$$

$$\text{Using } (1) \text{ and } (2):$$

$$4R^2\sin^2\left(\frac{\theta}{2}\right) = a^2+b^2+2ab\cos\left(\frac{\theta}{2}\right) \;=>\; R = \frac{\sqrt{a^2+b^2+2ab\cos\left(\frac{\theta}{2}\right)}}{2\sin\left(\frac{\theta}{2}\right)} \ (3)$$

$$[OABC] = \frac{1}{2}R^2\sin(\theta)+\frac{1}{2}AB\times BC\sin\left(\pi-\frac{\theta}{2}\right) =$$

$$\stackrel{(3)}{=} \frac{1}{2}\frac{a^2+b^2+2ab\cos\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)} * \sin(\theta)+\frac{1}{2}ab\sin\left(\frac{\theta}{2}\right) =$$

$$= \frac{1}{4}\frac{a^2+b^2+2ab\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} * \cos\left(\frac{\theta}{2}\right)+\frac{1}{2}ab\sin\left(\frac{\theta}{2}\right) =$$

$$= \frac{1}{4}\cot\frac{\theta}{2}\left(a^2+b^2+2ab\cos\frac{\theta}{2}\right)+\frac{1}{2}ab\sin\frac{\theta}{2} =$$

$$= \frac{1}{4}\cot\frac{\theta}{2}(a^2+b^2)+\frac{1}{2}ab\sin\frac{\theta}{2}+\frac{1}{2}ab\cos\frac{\theta}{2}\cdot\cot\frac{\theta}{2} =$$

$$= \frac{\frac{1}{4}\cos\frac{\theta}{2}(a^2+b^2)+\frac{1}{2}absin^2\frac{\theta}{2}+\frac{1}{2}abcos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \frac{1}{2}ab\frac{1}{\sin\frac{\theta}{2}}+\frac{1}{4}\cot\frac{\theta}{2}(a^2+b^2) =$$

$$=\frac{1}{8}\cdot\frac{4ab+2\cos\frac{\theta}{2}(a^2+b^2)}{\sin\frac{\theta}{2}}=\frac{1}{8}=\frac{1}{8}\cdot\frac{\left((a+b)^2-(a-b)^2+\left((a+b)^2+(a-b)^2\right)\cos\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$$=$$

$$=\frac{1}{8}\cdot\frac{\left((a+b)^2\left(1+\cos\frac{\theta}{2}\right)-(a-b)^2\left(1-\cos\frac{\theta}{2}\right)\right)}{\sin\frac{\theta}{2}}=$$

$$=\frac{1}{8}\cdot\frac{\left((a+b)^2\cdot 2\,cos^2\frac{\theta}{4}-(a-b)^2\cdot 2\,sin^2\frac{\theta}{4}\right)}{2cos^2\frac{\theta}{4}\cdot\tan\frac{\theta}{4}}=\frac{1}{8}\cdot\frac{\left((a+b)^2-(a-b)^2\,tan^2\frac{\theta}{4}\right)}{\tan\frac{\theta}{4}}=$$

$$=\frac{1}{8}\cdot\left((a+b)^2\cot\frac{\theta}{4}-(a-b)^2\ \tan\frac{\theta}{4}\right)$$

$$[OABC]=\frac{1}{8}\cdot\left((a+b)^2\cot\frac{\theta}{4}-(a-b)^2\ \tan\frac{\theta}{4}\right)\ (proved)$$

2151. In any acute Δ ABC, the following relationship holds :

$$\frac{R}{s+r}+\frac{s}{r+R}+\frac{r}{R+s}>\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{R}{s+r}+\frac{s}{r+R}+\frac{r}{R+s}>\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$$\Leftrightarrow (R+r+s)\left(\frac{1}{s+r}+\frac{1}{r+R}+\frac{1}{R+s}\right)>2s\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$$

$$\Leftrightarrow \frac{(R+r+s)(s^2+3s(R+r)+R^2+3Rr+r^2)}{(R+r)(s^2+s(R+r)+Rr)}>\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2}$$

$$\Leftrightarrow s^5-(R+r)s^4-(R^2-Rr)s^3+(R^3+3R^2r+6Rr^2+4r^3)s^2$$

$$+r(R^3+13R^2r+11Rr^2+3r^3)s+r(2R^4+5R^3r+7R^2r^2+5Rr^3+r^4)\overset{①}{>}0$$

Now, $\because$ Δ ABC is acute $\therefore$ $s>2R+r$ and so:

$$P=(s-2R-r)^5+(9R+4r)(s-2R-r)^4+(31R^2+29Rr+6r^2)(s-2R-r)^3+$$
$$+(51R^3+78R^2r+39Rr^2+8r^3)(s-2R-r)^2$$
$$+2(20R^4+49R^3r+50R^2r^2+27Rr^3+6r^4)(s-2R-r)>0$$

$\therefore$ in order to prove ①, it suffices to prove : LHS of ① $>$ P

$$\Leftrightarrow 6R^5 + 27R^4r + 51R^3r^2 + 49R^2r^3 + 23Rr^4 + 4r^5 > 0 \to \text{true}$$

$$\therefore \frac{R}{s+r} + \frac{s}{r+R} + \frac{r}{R+s} > \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\ \forall \Delta ABC\ (QED)$$

2152. In any ΔABC, the following relationship holds :

$$h_a + w_a + m_a - r_a - r_b - r_c \le 4\sqrt{R(R-2r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$w_a = \frac{2bc}{b+c}.\cos\frac{A}{2} = \frac{4R^2(\cos(B-C)+\cos A).\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}$$

$$= \frac{R\left(2\cos^2\frac{B-C}{2} - 1 + 1 - 2\sin^2\frac{A}{2}\right)}{\cos\frac{B-C}{2}} = \frac{2R(c^2-s^2)}{c}\ \left(c = \cos\frac{B-C}{2}\ \textit{and}\ s = \sin\frac{A}{2}\right)$$

$$\overset{?}{\le} R + r + \sqrt{R(R-2r)}$$

$$= R + 2R\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right) + R.\sqrt{1-4sc+4s^2}$$

$$\Leftrightarrow 2c - \frac{2s^2}{c} \overset{?}{\le} 1 + 2sc - 2s^2 + \sqrt{1-4sc+4s^2}$$

$$\Leftrightarrow 1 + 2sc - 2c + \frac{2s^2}{c} - 2s^2 + \sqrt{1-4sc+4s^2}\ \boxed{\overset{?}{\underset{(1)}{\ge}}}\ 0$$

$$\text{Now}, \frac{2s^2}{c} - 2s^2 = \frac{2s^2\left(1-\cos\frac{B-C}{2}\right)}{\cos\frac{B-C}{2}} \ge 0\ \textit{and}\ 1 - 4sc + 4s^2 \overset{0<c\le 1}{\ge} 1 - 4s + 4s^2$$

$$= (1-2s)^2 \Rightarrow \sqrt{1-4sc+4s^2} \ge |1-2s|\ \textit{and so, in order to prove (1)},$$

$$\text{it suffices to prove : } 1 + 2sc - 2c + |1-2s|\ \boxed{\overset{(2)}{\ge}}\ 0$$

$\boxed{\textit{Case 1}}$ $1 - 2s \ge 0$ *and then* : LHS of (2) $= 1 + 2sc - 2c + 1 - 2s$

$$= 1 - c - c(1-2s) + 1 - 2s = (1-2s)(1-c) + (1-c) = 2(1-c)(1-s) \ge 0$$

$$\because c = \cos\frac{B-C}{2} \le 1\ \textit{and}\ s = \sin\frac{A}{2} < 1 \Rightarrow (2)\ \text{is true}$$

$\boxed{\textit{Case 2}}$ $1 - 2s < 0$ *and then* : LHS of (2) $= 1 + 2sc - 2c + 2s - 1$

$$= 1 - c + c(2s-1) + (2s-1) \overset{c\le 1}{\ge} (2s-1)(1+c) > 0 \because 1 - 2s < 0$$

$$\Rightarrow (2)\ \text{is true (strict inequality)} \therefore \text{combining both cases},$$

$$(2)\ \text{is true}\ \forall \Delta ABC \therefore \boxed{w_a \le R + r + \sqrt{R(R-2r)}}\ \forall \Delta ABC \to (m)$$

$$\text{We shall now prove that}: m_a \le 2R - r + 2.\sqrt{R(R-2r)}\ \forall\ \Delta ABC$$

$$\boxed{\text{Case 1}}\ \hat{A}\ \text{is acute and then}: m_a \le 2R\cos^2\frac{A}{2} \overset{?}{\le} 2R - r + 2.\sqrt{R(R-2r)}$$

$$\Leftrightarrow 2Rs^2 - 2Rs(c-s) + 2R.\sqrt{1-4sc+4s^2} \overset{?}{\ge} 0 \Leftrightarrow \sqrt{1-4sc+4s^2} \boxed{\underset{③}{\overset{?}{\ge}}} sc - 2s^2$$

which is trivially true if $sc - 2s^2 < 0$ and so, we now focus on the scenario when : $sc - 2s^2 \ge 0$ and then : ③ $\Leftrightarrow 1 - 4sc + 4s^2 \underset{④}{\overset{?}{\ge}} s^2c^2 + 4s^4 - 4cs^3$ and

$\because c \le 1 \therefore$ in order to prove ④, it suffices to prove :

$$1 - 4sc + 4s^2 \overset{?}{\ge} s^2 + 4s^4 - 4cs^3 \Leftrightarrow 1 - s^2 + 4s^2(1-s^2) - 4sc(1-s^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow (1-s^2)(1-4sc+4s^2) \boxed{\overset{?}{\ge}} 0 \Leftrightarrow \cos^2\frac{A}{2}.\frac{R-2r}{R} \overset{?}{\ge} 0 \to \text{true}$$

$$\Rightarrow ④ \Rightarrow ③\ \text{is true} \therefore m_a \le 2R - r + 2.\sqrt{R(R-2r)}$$

$$\boxed{\text{Case 2}}\ \hat{A} \ge \frac{\pi}{2}\ \text{and then}: 4m_a^2 = 2b^2 + 2c^2 - 2a^2 + a^2 = 4bc\cos A + a^2 \le a^2$$

$$\Rightarrow m_a \le \frac{a}{2} = R\sin A \le R \overset{?}{\le} 2R - r + 2.\sqrt{R(R-2r)} \Leftrightarrow R - r + 2.\sqrt{R(R-2r)} \overset{?}{\ge} 0$$

$\to$ true (strict inequality) $\therefore$ combining both cases,

$$\boxed{m_a \le 2R - r + 2.\sqrt{R(R-2r)}}\ \forall\ \Delta ABC \to \text{(n)}$$

$$\text{So}, h_a + w_a + m_a - r_a - r_b - r_c \le 2w_a + m_a - 4R - r \overset{\text{via (m) and (n)}}{\le}$$

$$2R + 2r + 2\sqrt{R(R-2r)} + 2R - r + 2.\sqrt{R(R-2r)} - 4R - r = 4.\sqrt{R(R-2r)}$$

$$\therefore h_a + w_a + m_a - r_a - r_b - r_c \le 4.\sqrt{R(R-2r)}\ \forall\ \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)

2153. In ΔABC the following relationship holds:

$$\frac{1}{w_aw_bw_c} + \frac{1}{w_aw_bh_c} + \frac{1}{w_ah_bw_c} + \frac{1}{h_aw_bw_c} \ge \frac{2(R+2r)}{s^2Rr}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$w_aw_bw_c \overset{w_a \le \sqrt{s(s-a)}}{\le} \sqrt{s(s-a)s(s-b)s(s-c)} \le \sqrt{s^4r^2} = s^2r\ (1)$$

We need to show:

$$\frac{1}{w_aw_bw_c} + \frac{1}{w_aw_bh_c} + \frac{1}{w_ah_bw_c} + \frac{1}{h_aw_bw_c} \ge \frac{2(R+2r)}{s^2Rr}$$

$$or, \frac{s^2r}{w_aw_bw_c}+\frac{s^2r}{w_aw_bh_c}+\frac{s^2r}{w_ah_bw_c}+\frac{s^2r}{h_aw_bw_c}\geq\frac{2(R+2r)}{R}$$

$$\frac{w_aw_bw_c}{w_aw_bw_c}+\frac{w_aw_bw_c}{w_aw_bh_c}+\frac{w_aw_bw_c}{w_ah_bw_c}+\frac{w_aw_bw_c}{h_aw_bw_c}\geq\frac{2(R+2r)}{R}\ (using\ (1))$$

$$1+\frac{w_c}{h_c}+\frac{w_b}{h_b}+\frac{w_a}{h_a}\overset{w_a\geq h_a\ or,\frac{w_a}{h_a}\geq 1}{\geq}\frac{2(R+2r)}{R}$$

$$1+1+1+1\geq\frac{2(R+2r)}{R}$$

$$4R\geq 2R+4r\ or\ R\geq 2r\ Euler$$

$$Equality\ holds\ for\ a=b=c.$$

2154. In ΔABC the following relationship holds:

$$w_a+w_b+w_c\leq\ s^2r\left(\frac{1}{w_a\sqrt{r_ah_a}}+\frac{1}{w_b\sqrt{r_bh_b}}+\frac{1}{w_c\sqrt{r_ch_c}}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sum w_a=\sum\frac{2bc}{b+c}\cos\frac{A}{2}\overset{AM-GM}{\leq}\sum\frac{2bc}{2\sqrt{bc}}\cos\frac{A}{2}=$$

$$=\sum\sqrt{bc}\cos\frac{A}{2}\overset{CBS}{\leq}\sqrt{\left(\sum bc\right)\left(\sum\cos^2\frac{A}{2}\right)}=\sqrt{\left(\sum bc\right)\left(\frac{4R+r}{2R}\right)}\ (1)$$

$$w_a\sqrt{r_ah_a}=\frac{2\sqrt{bcs(s-a)}}{b+c}\cdot\sqrt{\frac{rs}{s-a}\cdot\frac{2rs}{a}}=\frac{2\sqrt{2}rs\sqrt{bcs}}{(b+c)\sqrt{a}}=\frac{2\sqrt{2}rs\sqrt{abcs}}{(b+ç)a}=$$

$$=\frac{2\sqrt{2}rs\sqrt{4Rrs^2}}{(ab+aç)}=\frac{2\sqrt{2}s^2r\sqrt{4Rr}}{ab+ac}=\frac{2s^2r\sqrt{8Rr}}{ab+ac}\ (2)$$

$$s^2r\left(\frac{1}{w_a\sqrt{r_ah_a}}+\frac{1}{w_b\sqrt{r_bh_b}}+\frac{1}{w_c\sqrt{r_ch_c}}\right)=s^2r\sum\frac{1}{w_a\sqrt{r_ah_a}}\overset{(2)}{\geq}$$

$$\geq\ s^2r\sum\frac{ab+ac}{2s^2r\sqrt{8Rr}}=\frac{2\sum ab}{2\sqrt{8Rr}}\ (3)$$

$$From\ (1)\&(3)\ we\ need\ to\ show:$$

$$\frac{2\sum ab}{2\sqrt{8Rr}} \ge \sqrt{\left(\sum bc\right)\left(\frac{4R+r}{2R}\right)} \text{ or } \sqrt{\sum ab} \ge \sqrt{8Rr\left(\frac{4R+r}{2R}\right)}$$

$$\left(\sqrt{\sum ab}\right)^2 \ge \left(\sqrt{8Rr\left(\frac{4R+r}{2R}\right)}\right)^2 \text{ or } \sum ab \ge 4r(4R+r)$$

$$or, s^2 + r^2 + 4Rr \ge 16Rr + 4r^2$$

$$16Rr - 5r^2 + r^2 + 4Rr \ge 16Rr + 4r^2 (Gerretsen)$$

$$4Rr \ge 8r^2 \text{ or}, R \ge 2r \text{ (Euler) true}$$

$$\text{Equality holds for } a = b = c$$

2155. In any Δ ABC, the following relationship holds :

$$\frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}} + \frac{1}{\frac{1}{r_b}+\frac{1}{r_c}} + \frac{1}{\frac{1}{r_c}+\frac{1}{r_a}} + R - \frac{r}{2}\right) \ge \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \ge 1 + \frac{9r}{2R} - \frac{r^2}{R^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a}{w_a} = \sum_{cyc}\left(\frac{bc}{2R}\cdot\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2bc\cos\frac{A}{2}}\right) \therefore \sum_{cyc}\frac{h_a}{w_a} = \sum_{cyc}\cos\frac{B-C}{2} \to (m)$$

$$\text{Now, } \frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}} + \frac{1}{\frac{1}{r_b}+\frac{1}{r_c}} + \frac{1}{\frac{1}{r_c}+\frac{1}{r_a}} R - \frac{r}{2}\right) = \frac{1}{\sqrt{2Rr}}\cdot\left(\sum_{cyc}\frac{s(s-a)}{r_b+r_c} + \frac{2R-r}{2}\right)$$

$$= \frac{1}{\sqrt{2Rr}}\cdot\left(\sum_{cyc}\frac{bc\cos^2\frac{A}{2}}{4R\cos^2\frac{A}{2}} + \frac{2R-r}{2}\right) = \frac{1}{\sqrt{2Rr}}\cdot\left(\frac{s^2+4Rr+r^2}{4R} + \frac{2R-r}{2}\right)$$

$$\therefore \frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}} + \frac{1}{\frac{1}{r_b}+\frac{1}{r_c}} + \frac{1}{\frac{1}{r_c}+\frac{1}{r_a}} R - \frac{r}{2}\right) = \frac{1}{\sqrt{2Rr}}\cdot\frac{s^2+4R^2+2Rr+r^2}{4R} \to (n)$$

$$\text{Now, } \cos\frac{B-C}{2} + \cos\frac{B+C}{2} = 2\cos\frac{B}{2}\cos\frac{C}{2} = \frac{2s}{4R}\sec\frac{A}{2} \Rightarrow \cos^2\frac{B-C}{2}$$

$$= \left(\frac{s}{2R}\sec\frac{A}{2} - \sin\frac{A}{2}\right)^2 \Rightarrow \cos^2\frac{B-C}{2} = \frac{s^2}{4R^2}\sec^2\frac{A}{2} + \sin^2\frac{A}{2} - \frac{s}{R}\tan\frac{A}{2} \text{ and analogs}$$

$$\therefore \sum_{cyc}\cos^2\frac{B-C}{2}=\frac{s^2}{4R^2}\cdot\sum_{cyc}\sec^2\frac{A}{2}+\sum_{cyc}\sin^2\frac{A}{2}-\frac{1}{R}\sum_{cyc}s\tan\frac{A}{2}$$

$$=\frac{s^2}{4R^2}\cdot\frac{s^2+(4R+r)^2}{s^2}+\frac{2R-r}{2R}-\frac{4R+r}{R}$$

$$=\frac{s^2+(4R+r)^2+2R(2R-r)-4R(4R+r)}{4R^2}$$

$$\Rightarrow \sum_{cyc}\cos^2\frac{B-C}{2}=\frac{s^2+4R^2+2Rr+r^2}{4R^2}\to (r)\ \textit{and}\ \prod_{cyc}\cos\frac{B-C}{2}=$$

$$\prod_{cyc}\frac{b+c}{a}\cdot\prod_{cyc}\sin\frac{A}{2}=\frac{2s(s^2+2Rr+r^2)}{4Rrs}\cdot\frac{r}{4R}\Rightarrow \prod_{cyc}\cos\frac{B-C}{2}\overset{(s)}{=}\frac{s^2+2Rr+r^2}{8R^2}$$

$$\text{We have}: \frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c}\overset{\text{via (m)}}{=}\sum_{cyc}\cos\frac{B-C}{2}\overset{CBS}{\leq}\sqrt{3\sum_{cyc}\cos^2\frac{B-C}{2}}\overset{\text{via (r)}}{=}$$

$$\sqrt{\frac{3(s^2+4R^2+2Rr+r^2)}{4R^2}}\overset{?}{\leq}\frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}+R-\frac{r}{2}\right)$$

$$\overset{\text{via (n)}}{=}\frac{1}{\sqrt{2Rr}}\cdot\frac{s^2+4R^2+2Rr+r^2}{4R}\Leftrightarrow s^2+4R^2+2Rr+r^2\ \overset{?}{\underset{(1)}{\geq}}\ 24Rr$$

$$\text{Via Gerretsen, LHS of (1)}\geq 4R^2+18Rr-4r^2\overset{?}{\geq}24Rr\Leftrightarrow 2R^2-3Rr-2r^2\overset{?}{\geq}0$$

$$\Leftrightarrow (2R-r)(R-2r)\overset{?}{\geq}0\to \text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow (1)\text{ is true}$$

$$\therefore \frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}+R-\frac{r}{2}\right)\geq\frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c}$$

$$\text{Again},\left(\frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c}\right)^2\overset{\text{via (m)}}{=}\sum_{cyc}\cos^2\frac{B-C}{2}+2\sum_{cyc}\left(\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right)$$

$$\overset{\text{via (r)}}{=}\frac{s^2+4R^2+2Rr+r^2}{4R^2}+2\prod_{cyc}\cos\frac{B-C}{2}\cdot\sum_{cyc}\frac{1}{\cos\frac{B-C}{2}}\overset{\text{via (r)}}{\geq}$$

$$\frac{s^2+4R^2+2Rr+r^2}{4R^2}+\frac{s^2+2Rr+r^2}{4R^2}.(1+1+1)$$

$$\left(\because 0<\cos\frac{B-C}{2}\leq 1\ \textit{and analogs}\right)=\frac{s^2+R^2+2Rr+r^2}{R^2}\overset{\text{Gerretsen}}{\geq}$$

$$\frac{R^2+18Rr-4r^2}{R^2}\overset{?}{\geq}\left(1+\frac{9r}{2R}-\frac{r^2}{R^2}\right)^2=\frac{(2R^2+9Rr-2r^2)^2}{4R^4}$$

$$\Leftrightarrow 36t^3-89t^2+36t-4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow (t-2)(27t^2+9t(t-2)+t+2)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c} \geq 1+\frac{9r}{2R}-\frac{r^2}{R^2} \text{ and so,}$$

$$\frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}+R-\frac{r}{2}\right) \geq \frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c} \geq$$

$$1+\frac{9r}{2R}-\frac{r^2}{R^2}\ \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

2156. In any ΔABC, the following relationship holds :

$$\frac{3\sqrt{6}r}{\sqrt{R}} \leq \frac{h_a \cdot \sqrt{h_a}}{w_a}+\frac{h_b \cdot \sqrt{h_b}}{w_b}+\frac{h_c \cdot \sqrt{h_c}}{w_c} \leq \frac{2s^2}{3\sqrt{3r} \cdot R}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{w_a}=\frac{bc}{2R} \cdot \frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2bc\cos\frac{A}{2}} \therefore \frac{h_a}{w_a}=\cos\frac{B-C}{2} \text{ and analogs} \rightarrow (m)$$

$$\text{Now, } \cos\frac{B-C}{2}+\cos\frac{B+C}{2}=2\cos\frac{B}{2}\cos\frac{C}{2}=\frac{2s}{4R}\sec\frac{A}{2}$$

$$\Rightarrow \cos^2\frac{B-C}{2}=\left(\frac{s}{2R}\sec\frac{A}{2}-\sin\frac{A}{2}\right)^2$$

$$\Rightarrow \cos^2\frac{B-C}{2}=\frac{s^2}{4R^2}\sec^2\frac{A}{2}+\sin^2\frac{A}{2}-\frac{s}{R}\tan\frac{A}{2} \text{ and analogs}$$

$$\therefore \sum_{cyc}\cos^2\frac{B-C}{2}=\frac{s^2}{4R^2} \cdot \sum_{cyc}\sec^2\frac{A}{2}+\sum_{cyc}\sin^2\frac{A}{2}-\frac{1}{R}\sum_{cyc}s\tan\frac{A}{2}$$

$$=\frac{s^2}{4R^2} \cdot \frac{s^2+(4R+r)^2}{s^2}+\frac{2R-r}{2R}-\frac{4R+r}{R}$$

$$=\frac{s^2+(4R+r)^2+2R(2R-r)-4R(4R+r)}{4R^2}$$

$$\Rightarrow \sum_{cyc}\cos^2\frac{B-C}{2}=\frac{s^2+4R^2+2Rr+r^2}{4R^2} \rightarrow (n)$$

$$\text{We have : } \frac{h_a \cdot \sqrt{h_a}}{w_a}+\frac{h_b \cdot \sqrt{h_b}}{w_b}+\frac{h_c \cdot \sqrt{h_c}}{w_c} \overset{\text{via (m)}}{=} \sum_{cyc}\left(\cos\frac{B-C}{2} \cdot \sqrt{h_a}\right) \overset{\text{CBS}}{\leq}$$

$$\sqrt{\sum_{cyc}\cos^2\frac{B-C}{2}} \cdot \sqrt{\frac{1}{2R} \cdot \sum_{cyc}ab} \overset{\text{via (n)}}{=} \sqrt{\frac{s^2+4R^2+2Rr+r^2}{4R^2} \cdot \frac{s^2+4Rr+r^2}{2R}} \overset{?}{\leq} \frac{2s^2}{3\sqrt{3r}.R}$$

$$\Leftrightarrow 32Rs^4 \overset{?}{\geq} 27r(s^2+4R^2+2Rr+r^2)(s^2+4Rr+r^2)$$

$$\Leftrightarrow (32R-27r)s^4 - r(108R^2+162Rr+54r^2)s^2$$

$$-27r^2(16R^3+12R^2r+6Rr^2+r^3) \underset{①}{\overset{?}{\geq}} 0 \text{ and } \because (32R-27r)(s^2-16Rr+5r^2)^2$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove ①, it suffices to prove :}$$

$$\text{LHS of ①} \geq (32R-27r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (458R^2-673Rr+108r^2)s^2 \overset{②}{\geq} r(4312R^3-5854R^2r+2641Rr^2-324r^3)$$

$$\text{Now, LHS of ②} \overset{\text{Gerretsen}}{\geq} (458R^2-673Rr+108r^2)(16Rr-5r^2) \overset{?}{\geq}$$

$$r(4312R^3-5854R^2r+2641Rr^2-324r^3) \Leftrightarrow 754t^3-1801t^2+613t-54 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)(754t^2-293t+27) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$\therefore \frac{h_a.\sqrt{h_a}}{w_a}+\frac{h_b.\sqrt{h_b}}{w_b}+\frac{h_c.\sqrt{h_c}}{w_c} \leq \frac{2s^2}{3\sqrt{3r}.R}$$

$$\text{Again, } \frac{h_a.\sqrt{h_a}}{w_a}+\frac{h_b.\sqrt{h_b}}{w_b}+\frac{h_c.\sqrt{h_c}}{w_c} = \sum_{cyc}\frac{h_a^2}{\sqrt{h_a}.w_a} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} h_a\right)^2}{\sqrt{\sum_{cyc} h_a}.\sqrt{\sum_{cyc} w_a^2}} \geq$$

$$\frac{\left(\sum_{cyc} h_a\right).\sqrt{\sum_{cyc} h_a}}{\sqrt{\sum_{cyc} s(s-a)}} \overset{?}{\geq} \frac{3\sqrt{6}r}{\sqrt{R}} \Leftrightarrow \frac{(s^2+4Rr+r^2)^3}{8R^3} \overset{?}{\geq} \frac{s^2.54r^2}{R}$$

$$\Leftrightarrow (s^2+4Rr+r^2)^3 \underset{③}{\overset{?}{\geq}} 432R^2r^2s^2$$

$$\text{Now, } (s^2+4Rr+r^2)^2 \overset{?}{\geq} 24Rrs^2 \Leftrightarrow s^4-(16Rr-2r^2)s^2+r^2(4R+r)^2 \underset{(i)}{\overset{?}{\geq}} 0$$

$$\text{We have : LHS of (i)} \overset{\text{Gerretsen}}{\geq} -3r^2s^2+r^2(4R+r)^2 = r^2((4R+r)^2-3s^2)$$

$$\overset{\text{Doucet or Trucht}}{\geq} 0 \Rightarrow \text{(i) is true} \Rightarrow (s^2+4Rr+r^2)^2 \geq 24Rrs^2 \to (a)$$

$$\text{Also, } s^2+4Rr+r^2 = 18Rr+s^2-14Rr+r^2$$

$$= 18Rr+s^2-16Rr+5r^2+2r(R-2r) \overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\geq} 18Rr \Rightarrow s^2+4Rr+r^2 \geq 18Rr$$

$$\to (b) \therefore (a)\bullet(b) \Rightarrow ③ \text{ is true} \therefore \frac{h_a.\sqrt{h_a}}{w_a}+\frac{h_b.\sqrt{h_b}}{w_b}+\frac{h_c.\sqrt{h_c}}{w_c} \geq \frac{3\sqrt{6}r}{\sqrt{R}} \text{ and so,}$$

$$\frac{3\sqrt{6}r}{\sqrt{R}} \leq \sum_{cyc}\frac{h_a.\sqrt{h_a}}{w_a} \leq \frac{2s^2}{3\sqrt{3r}.R} \quad \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

2157. In any Δ ABC, the following relationship holds :

$$p^2r\left(2R-r+2\sqrt{R(R-2r)}\right)\ge m_aw_ah_ar_a\ge\frac{4p^2r^3\left(2R-r-2\sqrt{R(R-2r)}\right)}{R^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

We shall first prove that : $m_a\le 2R-r+2.\sqrt{R(R-2r)}\ \forall\ \Delta\ ABC$

$\boxed{Case\ 1}$ $\hat{A}$ is acute and then : $m_a\le 2R\cos^2\frac{A}{2}\overset{?}{\le}2R-r+2.\sqrt{R(R-2r)}\Leftrightarrow$

$2Rs^2-2Rs(c-s)+2R.\sqrt{1-4sc+4s^2}\overset{?}{\ge}0\ \left(c=\cos\frac{B-C}{2}\text{ and } s=\sin\frac{A}{2}\right)$

$\Leftrightarrow\sqrt{1-4sc+4s^2}\ \boxed{\underset{①}{\overset{?}{\ge}}}\ sc-2s^2$ which is trivially true if $sc-2s^2<0$ and

so, we now focus on the scenario when : $sc-2s^2\ge 0$ and then :

$① \Leftrightarrow 1-4sc+4s^2\underset{②}{\overset{?}{\ge}}s^2c^2+4s^4-4cs^3$ and $\because c\le 1\ \therefore$ in order to prove ②,

it suffices to prove : $1-4sc+4s^2\overset{?}{\ge}s^2+4s^4-4cs^3$

$\Leftrightarrow 1-s^2+4s^2(1-s^2)-4sc(1-s^2)\overset{?}{\ge}0\Leftrightarrow(1-s^2)(1-4sc+4s^2)\ \boxed{\overset{?}{\ge}}\ 0$

$\Leftrightarrow\cos^2\frac{A}{2}.\frac{R-2r}{R}\overset{?}{\ge}0\to$ true $\Rightarrow ② \Rightarrow ①$ is true $\therefore m_a\le 2R-r+2.\sqrt{R(R-2r)}$

$\boxed{Case\ 2}$ $\hat{A}\ge\frac{\pi}{2}$ and then : $4m_a^2=2b^2+2c^2-2a^2+a^2=4bc\cos A+a^2\le a^2$

$\Rightarrow m_a\le\frac{a}{2}=R\sin A\le R\overset{?}{\le}2R-r+2.\sqrt{R(R-2r)}\Leftrightarrow R-r+2.\sqrt{R(R-2r)}\overset{?}{\ge}0$

$\to$ true (strict inequality) $\therefore$ combining both cases,

$m_a\le 2R-r+2.\sqrt{R(R-2r)}\ \forall\ \Delta\ ABC\to$ (m)

We shall now prove that : $h_a\ge R+r-\sqrt{R^2-4r^2}\ \forall\ \Delta\ ABC$

Now, $\sqrt{R^2-4r^2}=\sqrt{(R-2r)(R+2r)}=\sqrt{R(1-4sc+4s^2)\left(R+4Rs(c-s)\right)}$

$=R.\sqrt{(1-4sc+4s^2)\left(1+4s(c-s)\right)}\ge R.\sqrt{1-4sc+4s^2}$

$$\left(\because c = \cos\frac{B-C}{2} = \frac{b+c}{a}.\sin\frac{A}{2} > \sin\frac{A}{2} = s\right) \therefore h_a + \sqrt{R^2-4r^2} \geq$$

$$2R(c^2-s^2) + R.\sqrt{1-4sc+4s^2} \overset{?}{\geq} R + r = R + 2Rs(c-s)$$

$$\Leftrightarrow \sqrt{1-4sc+4s^2} \boxed{\overset{?}{\underset{③}{\geq}}} 1 + 2sc - 2c^2 \text{ and its's trivially true when } 1 + 2sc - 2c^2$$

< 0 and so we now focus on the scenario when : $1 + 2sc - 2c^2 \geq 0$ and then :

$$③ \Leftrightarrow 1 - 4sc + 4s^2 \geq (1 + 2sc - 2c^2)^2 \Leftrightarrow -c^4 + 2c^3s - c^2s^2 + c^2 - 2cs + s^2$$

$$\geq 0 \Leftrightarrow -c^2(c-s)^2 + (c-s)^2 \geq 0 \Leftrightarrow (c-s)^2(1-c^2) \geq 0 \to \text{true}$$

$$\because 1 \geq \cos^2\frac{B-C}{2} \Rightarrow ③ \text{ is true } \therefore h_a \geq R + r - \sqrt{R^2-4r^2}\ \forall\ \Delta\ ABC \to (n)$$

We shall denote the semi – perimeter by "p" and via Lascu + A – G, $m_a w_a h_a r_a$

$$\geq r_b r_c r_a h_a \overset{\text{via (n)}}{\geq} rp^2\left(R + r - \sqrt{R^2-4r^2}\right) \overset{?}{\geq} \frac{4p^2r^3\left(2R - r - 2.\sqrt{R(R-2r)}\right)}{R^2}$$

$$\Leftrightarrow R^3 + R^2r - 8Rr^2 + 4r^3 \overset{?}{\geq} R^2.\sqrt{R^2-4r^2} - 8r^2.\sqrt{R(R-2r)}$$

$$\Leftrightarrow t^3 + t^2 - 8t + 4 - t^2.\sqrt{t^2-4} + 8.\sqrt{t^2-2t} \overset{?}{\geq} 0\ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow \left(t^3 + t^2 - 8t + 4 + 8.\sqrt{t^2-2t}\right)^2 \overset{?}{\geq} t^4(t^2-4)$$

$$\left(\because t^3 + t^2 - 8t + 4 = (t-2)(t^2+3t-2) \overset{\text{Euler}}{\geq} 0\right)$$

$$\Leftrightarrow (t-2)(2t^4 - 7t^3 - 22t^2 + 92t - 8) + 16(t-2)(t^2+3t-2).\sqrt{t^2-2t} \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{(t-2)}{128}\left((16t^2 + 48t - 33)(4t-13)^2 + 232t + 4553\right) +$$

$$16(t-2)(t^2+3t-2).\sqrt{t^2-2t} \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore m_a w_a h_a r_a \boxed{\geq} \frac{4p^2r^3\left(2R - r - 2.\sqrt{R(R-2r)}\right)}{R^2}$$

$$\text{Again, } m_a w_a h_a r_a \overset{\text{via (m)}}{\leq} \left(2R - r + 2.\sqrt{R(R-2r)}\right).\frac{2bc}{b+c}.\cos\frac{A}{2}.\frac{rp^2}{2R}.\sec^2\frac{A}{2}$$

$$\overset{?}{\leq} p^2r\left(2R - r + 2.\sqrt{R(R-2r)}\right) \Leftrightarrow \frac{4R^2.\sin B\sin C}{4R^2.\cos\frac{B-C}{2}} \overset{?}{\leq} \cos^2\frac{A}{2}$$

$$\Leftrightarrow \cos(B-C)+\cos A \overset{?}{\leq} 2\cos^2\frac{A}{2}\cos\frac{B-C}{2} \Leftrightarrow c(1-s^2) \overset{?}{\geq} c^2-s^2$$

$$\Leftrightarrow c(1-c)+s^2(1-c) \overset{?}{\geq} 0 \Leftrightarrow (1-c)(c+s^2) \overset{?}{\geq} 0 \to \text{true} \because 1 \geq \cos\frac{B-C}{2}$$

$$\therefore m_a w_a h_a r_a \boxed{\leq} p^2 r\left(2R-r+2.\sqrt{R(R-2r)}\right) \textit{ and so,}$$

$$p^2 r\left(2R-r+2.\sqrt{R(R-2r)}\right) \geq m_a w_a h_a r_a \geq \frac{4p^2 r^3\left(2R-r-2.\sqrt{R(R-2r)}\right)}{R^2}$$

$$\forall \Delta ABC,'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

2158. In any Δ ABC, the following relationship holds :

$$\frac{m_a}{R}+\frac{w_b}{s}+\frac{h_c}{r} \leq \frac{(27+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)}{12r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall first prove that : } w_a \leq R+r+\sqrt{R(R-2r)} \;\forall \Delta ABC$$

$$w_a = \frac{2bc}{b+c}.\cos\frac{A}{2} = \frac{4R^2(\cos(B-C)+\cos A).\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}$$

$$= \frac{R\left(2\cos^2\frac{B-C}{2}-1+1-2\sin^2\frac{A}{2}\right)}{\cos\frac{B-C}{2}} = \frac{2R(c^2-s^2)}{c}\left(c=\cos\frac{B-C}{2} \textit{ and } s=\sin\frac{A}{2}\right)$$

$$\overset{?}{\leq} R+r+\sqrt{R(R-2r)}$$

$$= R+2R\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)+R.\sqrt{1-4sc+4s^2}$$

$$\Leftrightarrow 2c-\frac{2s^2}{c} \overset{?}{\leq} 1+2sc-2s^2+\sqrt{1-4sc+4s^2}$$

$$\Leftrightarrow 1+2sc-2c+\frac{2s^2}{c}-2s^2+\sqrt{1-4sc+4s^2} \boxed{\overset{?}{\underset{①}{\geq}}} 0$$

$$\text{Now, } \frac{2s^2}{c}-2s^2 = \frac{2s^2\left(1-\cos\frac{B-C}{2}\right)}{\cos\frac{B-C}{2}} \geq 0 \textit{ and } 1-4sc+4s^2 \overset{0<c\leq 1}{\geq} 1-4s+4s^2$$

$$= (1-2s)^2 \Rightarrow \sqrt{1-4sc+4s^2} \geq |1-2s| \textit{ and so, in order to prove ①,}$$

$$\text{it suffices to prove : } 1+2sc-2c+|1-2s| \boxed{\overset{②}{\geq}} 0$$

$\boxed{\text{Case 1}}$ $1-2s \geq 0$ and then : LHS of ② $= 1+2sc-2c+1-2s$

$= 1-c-c(1-2s)+1-2s = (1-2s)(1-c)+(1-c) = 2(1-c)(1-s) \geq 0$

$\because c = \cos\frac{B-C}{2} \leq 1$ and $s = \sin\frac{A}{2} < 1 \Rightarrow$ ② is true

$\boxed{\text{Case 2}}$ $1-2s < 0$ and then : LHS of ② $= 1+2sc-2c+2s-1$

$= 1-c+c(2s-1)+(2s-1) \overset{c\leq 1}{\geq} (2s-1)(1+c) > 0 \because 1-2s < 0$

$\Rightarrow$ ② is true (strict inequality) $\therefore$ combining both cases,

② is true $\forall\, \Delta\, ABC \therefore w_a \leq R+r+\sqrt{R(R-2r)}\ \forall\, \Delta\, ABC \to$ (m)

We shall now prove that : $m_a \leq 2R-r+2.\sqrt{R(R-2r)}\ \forall\, \Delta\, ABC$

$\boxed{\text{Case 1}}$ $\hat{A}$ is acute and then : $m_a \leq 2R\cos^2\frac{A}{2} \overset{?}{\leq} 2R-r+2.\sqrt{R(R-2r)}$

$\Leftrightarrow 2Rs^2-2Rs(c-s)+2R.\sqrt{1-4sc+4s^2} \overset{?}{\geq} 0 \Leftrightarrow \sqrt{1-4sc+4s^2} \underset{③}{\overset{?}{\geq}} sc-2s^2$

which is trivially true if $sc-2s^2 < 0$ and so, we now focus on the scenario

when : $sc-2s^2 \geq 0$ and then : ③ $\Leftrightarrow 1-4sc+4s^2 \underset{④}{\overset{?}{\geq}} s^2c^2+4s^4-4cs^3$ and

$\because c \leq 1 \therefore$ in order to prove ④, it suffices to prove :

$1-4sc+4s^2 \overset{?}{\geq} s^2+4s^4-4cs^3 \Leftrightarrow 1-s^2+4s^2(1-s^2)-4sc(1-s^2) \overset{?}{\geq} 0$

$\Leftrightarrow (1-s^2)(1-4sc+4s^2) \overset{?}{\geq} 0 \Leftrightarrow \cos^2\frac{A}{2}.\frac{R-2r}{R} \overset{?}{\geq} 0 \to$ true

$\Rightarrow$ ④ $\Rightarrow$ ③ is true $\therefore m_a \leq 2R-r+2.\sqrt{R(R-2r)}$

$\boxed{\text{Case 2}}$ $\hat{A} \geq \frac{\pi}{2}$ and then : $4m_a^2 = 2b^2+2c^2-2a^2+a^2 = 4bc\cos A+a^2 \leq a^2$

$\Rightarrow m_a \leq \frac{a}{2} = R\sin A \leq R \overset{?}{\leq} 2R-r+2.\sqrt{R(R-2r)} \Leftrightarrow R-r+2.\sqrt{R(R-2r)} \overset{?}{\geq} 0$

$\to$ true (strict inequality) $\therefore$ combining both cases,

$m_a \leq 2R-r+2.\sqrt{R(R-2r)}\ \forall\, \Delta\, ABC \to$ (n)

We have : $\frac{m_a}{R}+\frac{w_b}{s}+\frac{h_c}{r} \leq \frac{m_a}{R}+\frac{w_b}{s}+\frac{w_c}{r} \overset{\text{via (m) and (n)}}{\leq}$

$\frac{2R-r+2.\sqrt{R(R-2r)}}{R}+\frac{R+r+\sqrt{R(R-2r)}}{s}+\frac{R+r+\sqrt{R(R-2r)}}{r} \overset{\text{Euler and Mitrinovic}}{\leq}$

$\frac{2R-r+2.\sqrt{R(R-2r)}+2R+2r+2.\sqrt{R(R-2r)}}{2r}+\frac{\sqrt{3}\left(R+r+\sqrt{R(R-2r)}\right)}{9r}$

$= \frac{(36+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)+(9+2\sqrt{3})r}{18r} \overset{?}{\leq} \frac{(27+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)}{12r}$

$\Leftrightarrow (81+6\sqrt{3}-72-4\sqrt{3})\left(R+\sqrt{R(R-2r)}\right) \overset{?}{\geq} 2r(9+2\sqrt{3})$

$\Leftrightarrow (9+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right) \overset{?}{\geq} 2r(9+2\sqrt{3}) \Leftrightarrow R-2r+\sqrt{R(R-2r)} \overset{?}{\geq} 0$

$$\rightarrow \text{true} \because R \overset{Euler}{\geq} 2r \therefore \frac{m_a}{R}+\frac{w_b}{s}+\frac{h_c}{r} \leq \frac{(27+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)}{12r}$$

$\forall \Delta ABC, "=" $ iff ΔABC is equilateral (QED)

2159. In ΔABC the following relationship holds:

$$w_a^3 r_a + w_b^3 r_b + w_c^3 r_c \leq 3^5\left(\frac{R}{2}\right)^4$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$w_a^3 r_a \overset{w_a \leq \sqrt{s(s-a)}}{\leq} s(s-a)\sqrt{s(s-a)}\frac{F}{s-a} = Fs\sqrt{s}\sqrt{s-a}$$

$$w_a^3 r_a + w_b^3 r_b + w_c^3 r_c = \sum w_a^3 r_a \leq Fs\sqrt{s}\sum\sqrt{s-a} \overset{CBS}{\leq}$$

$$\leq rs^2\sqrt{s}\sqrt{3(s-a+s-b+s-c)} = rs^3\sqrt{3} \overset{Euler\ \&\ Mitrinovic}{\leq}$$

$$\leq \frac{R}{2}\cdot\frac{27}{4}R^2\frac{3\sqrt{3}}{2}R\sqrt{3} = 3^5\left(\frac{R}{2}\right)^4$$

Equality holds for an equilateral triangle.

2160. In ΔABC the following relationship holds:

$$6 \leq \sum\left(\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\right) \leq \frac{3R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\sqrt{\frac{h_a}{r_a}} \overset{CBS}{\leq} \sqrt{\left(\sum h_a\right)\left(\sum\frac{1}{r_a}\right)} \overset{h_a \leq m_a}{\leq} \sqrt{\left(\sum m_a\right)\left(\sum\frac{1}{r_a}\right)} \overset{Leunberger}{\leq}$$

$$\leq \sqrt{(4R+r)\frac{1}{r}} \overset{Euler}{\leq} \sqrt{\frac{9R}{2r}} = \sqrt{\frac{9R^2}{2Rr}} \overset{Euler}{\leq} \sqrt{\frac{9R^2}{4r^2}} = \frac{3R}{2r}$$

$$\sum\sqrt{\frac{r_a}{h_a}}\overset{CBS}{\le}\sqrt{\left(\sum r_a\right)\left(\sum\frac{1}{h_a}\right)}=\sqrt{(4R+r)\frac{1}{r}}\overset{Euler}{\le}\sqrt{\frac{9R}{2r}}=\sqrt{\frac{9R^2}{2Rr}}\overset{Euler}{\le}\sqrt{\frac{9R^2}{4r^2}}=\frac{3R}{2r}$$

$$\sum\left(\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\right)=\sum\sqrt{\frac{h_a}{r_a}}+\sum\sqrt{\frac{r_a}{h_a}}\le\frac{3R}{2r}+\frac{3R}{2r}=\frac{3R}{r}\ and$$

$$\sum\left(\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\right)\overset{AM-GM}{\ge}2+2+2=6$$

Equality holds for $a=b=c$.

2161. **In ΔABC the following relationship holds:**

$$\sum\sqrt{\frac{r}{r_a(s^2+r_a^2)}}\le\frac{3}{2s}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$s^2+r_a^2=r_ar_b+r_br_c+r_cr_a+r_a^2=(r_a+r_b)(r_a+r_c)\ (1)$$

$$\sum\frac{1}{s^2+r_a^2}\overset{(1)}{=}\sum\frac{1}{(r_a+r_b)(r_a+r_c)}=\frac{\sum(r_a+r_b)}{(r_a+r_b)(r_a+r_c)(r_b+r_c)}=$$

$$=\frac{2\sum r_a}{(\sum r_a)(\sum r_ar_b)-r_ar_br_c}=\frac{2(4R+r)}{(4R+r)s^2-s^2r}=\frac{2(4R+r)}{4Rs^2}=$$

$$=\frac{2}{4s^2}\left(4+\frac{r}{R}\right)\overset{Euler}{\le}\frac{2}{4s^2}\left(4+\frac{1}{2}\right)=\frac{9}{4s^2}\ (2)$$

$$\sum\sqrt{\frac{r}{r_a(s^2+r_a^2)}}=\sqrt{r}\sum\sqrt{\frac{1}{r_a}}\sqrt{\frac{1}{s^2+r_a^2}}\overset{CBS}{\le}$$

$$\le\sqrt{r}\sqrt{\left(\sum\frac{1}{r_a}\right)\left(\sum\frac{1}{s^2+r_a^2}\right)}\overset{(2)}{\le}\sqrt{r}\sqrt{\frac{1}{r}\frac{9}{4s^2}}=\frac{3}{2s}$$

Equality holds for $a=b=c$

2162. **In ΔABC the following relationship holds:**

$$\sum\sqrt{\frac{4r}{r_a}+1}\le\sqrt{21}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\sqrt{\frac{4r}{r_a}+1}=\sum\sqrt{\frac{4r}{\frac{rs}{s-a}}+1}=\sum\sqrt{4\frac{s-a}{s}+1}\overset{CBS}{\le}$$

$$\le\sqrt{3\sum\left(4\frac{s-a}{s}+1\right)}=\sqrt{3\left(\frac{4s}{s}+3\right)}=\sqrt{21}$$

Equality holds for an equilateral triangle

2163. In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{4r}{h_a}+1}\le\sqrt{21}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\sqrt{4\cdot\frac{r}{h_a}+1}=\sum\sqrt{4\cdot\frac{r}{\frac{2rs}{a}}+1}=\sum\sqrt{\frac{2a}{s}+1}\overset{CBS}{\le}\sqrt{3\left(\sum\left(\frac{2a}{s}+1\right)\right)}=$$

$$=\sqrt{3\left(\frac{2(a+b+c)}{s}+3\right)}=\sqrt{3\left(\frac{4s}{s}+3\right)}=\sqrt{21}$$

Equality holds for an equilateral triangle

2164. In ΔABC the following relationship holds:

$$\sum\frac{\sin\frac{A}{2}}{w_a}\cdot\sum\frac{1}{w_a\sin\frac{A}{2}}\ge\frac{4}{R^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{\sin\frac{A}{2}}{w_a}\cdot\sum\frac{1}{w_a\sin\frac{A}{2}}=\sum\left(\sqrt{\frac{\sin\frac{A}{2}}{w_a}}\right)^2\cdot\sum\left(\sqrt{\frac{1}{w_a\sin\frac{A}{2}}}\right)^2\overset{C-S}{\geq}\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)^2\geq$$

$$\overset{w_a\leq m_a}{\geq}\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)^2\overset{CBS}{\geq}\left(\frac{(1+1+1)^2}{m_a+m_b+m_c}\right)^2\overset{Gotman\ II}{\geq}\left(\frac{9}{\frac{9R}{2}}\right)^2=\frac{4}{R^2}$$

Equality holds for an equilateral triangle.

2165. In ΔABC the following relationship holds:

$$243r^4\leq w_a^3r_a+w_b^3r_b+w_c^3r_c\leq\left(\frac{3Rs}{2}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$w_a^3r_a\overset{w_a\leq\sqrt{s(s-a)}}{\leq}s(s-a)\sqrt{s(s-a)}\frac{F}{s-a}=Fs\sqrt{s}\sqrt{s-a}$$

$$w_a^3r_a+w_b^3r_b+w_c^3r_c=\sum w_a^3r_a\leq Fs\sqrt{s}\sum\sqrt{s-a}\overset{CBS}{\leq}$$

$$\leq rs^2\sqrt{s}\sqrt{3(s-a+s-b+s-c)}=rs^3\sqrt{3}\overset{Euler\ \&\ Mitrinovic}{\leq}\frac{R}{2}s^2\frac{3\sqrt{3}}{2}R\sqrt{3}=\left(\frac{3Rs}{2}\right)^2$$

$$w_aw_bw_c\geq h_ah_bh_c\overset{GM-HM}{\geq}\left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^2=(3r)^3=27r^3$$

$$w_a^3r_a+w_b^3r_b+w_c^3r_c\overset{AM-GM}{\geq}3w_aw_bw_c\sqrt[3]{r_ar_br_c}\geq$$

$$\geq 3.27r^3\sqrt[3]{s^2r}\overset{Mitrinovic}{\geq}81r^3\sqrt[3]{27r^3}=243r^4$$

Equality holds for an equilateral triangle.

2166. In ΔABC the following relationship holds:

$$\sum \cot^2\frac{A}{2} \geq \sqrt{3}\sum \cot\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \cot^2\frac{A}{2} = \sum \frac{s^2}{r_a^2} = s^2\left(\left(\sum \frac{1}{r_a}\right)^2 - 2\sum \frac{1}{r_a r_b}\right) = s^2\left(\frac{1}{r^2} - 2\cdot\frac{4R+r}{s^2 r}\right) =$$

$$= \frac{s^2 - 8Rr - 2r^2}{r^2} \overset{Gerretsen}{\geq} \frac{16Rr - 5r^2 - 8Rr - 2r^2}{r^2} =$$

$$= \frac{8R}{r} - 7 = \frac{9R}{2r} + \frac{7R}{2r} - 7 \geq$$

$$\overset{Euler}{\geq} \frac{9R}{2r} + 7 - 7 = \frac{9R}{2r} \overset{Mitrinovic}{\geq} \frac{9}{2r}\frac{2s}{3\sqrt{3}} = \sqrt{3}\frac{s}{r} = \sqrt{3}\sum \cot\frac{A}{2}$$

Equality holds for an equilateral triangle.

2167. In ΔABC the following relationship holds:

$$r_a^2 + r_b^2 + r_c^2 \geq m_a^2 + m_b^2 + m_c^2 + 2(R^2 - 4r^2)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$r_a^2 + r_b^2 + r_c^2 = \left(\sum r_a\right)^2 - 2\sum r_a r_b = 2(4R+r)^2 - 2s^2, \sum m_a^2 = \frac{3}{2}(s^2 - r^2 - 4Rr)$$

$$We\ \ need\ to\ show: r_a^2 + r_b^2 + r_c^2 \geq m_a^2 + m_b^2 + m_c^2 + 2(R^2 - 4r^2)$$

$$(4R+r)^2 - 2s^2 \geq \frac{3}{2}(s^2 - r^2 - 4Rr) + 2(R^2 - 4r^2)$$

$$2(4R+r)^2 - 4s^2 \geq 3(s^2 - r^2 - 4Rr) + 4(R^2 - 4r^2)$$

$$2(16R^2 + 8Rr + r^2) - 4s^2 \geq 3(s^2 - r^2 - 4Rr) + 4(R^2 - 4r^2)$$

$$28R^2 + 28Rr + 21r^2 \geq 7s^2$$

$$7(4R^2+4Rr+3r^2)\ge 7s^2 \ true\ (Gerretsen)$$

Equality holds for an equilateral triangle.

2168. In ΔABC the following relationship holds:

$$r_a^2+r_b^2+r_c^2\ge m_a^2+m_b^2+m_c^2+\frac{1}{4}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das

$$r_a^2+r_b^2+r_c^2=\left(\sum r_a\right)^2-2\sum r_ar_b=2(4R+r)^2-2s^2,\ \sum m_a^2=\frac{3}{2}(s^2-r^2-4Rr)$$

$$We\ \ will\ show:$$

$$r_a^2+r_b^2+r_c^2\ge m_a^2+m_b^2+m_c^2+2(R^2-4r^2)\ (A)$$

$$(4R+r)^2-2s^2\ge\frac{3}{2}(s^2-r^2-4Rr)+2(R^2-4r^2)$$

$$2(4R+r)^2-4s^2\ge 3(s^2-r^2-4Rr)+4(R^2-4r^2)$$

$$2(16R^2+8Rr+r^2)-4s^2\ge 3(s^2-r^2-4Rr)+4(R^2-4r^2)$$

$$28R^2+28Rr+21r^2\ge 7s^2\ or, 7(4R^2+4Rr+3r^2)\ge 7s^2\ true\ (Gerretsen)$$

$$\frac{1}{4}\left((a-b)^2+(b-c)^2+(c-a)^2\right)=\frac{1}{2}(a^2+b^2+c^2-ab-bc-ca)$$

$$=\frac{1}{2}(2s^2-2r^2-8Rr-s^2-r^2-4Rr)=\frac{1}{2}(s^2-12Rr-3r^2)\overset{(Gerretsen)}{\le}$$

$$\le\frac{1}{2}(4R^2+4Rr+3r^2-12Rr-3r^2)=2(R^2-2Rr)\overset{Euler}{\le}2(R^2-4r^2)\ (B)$$

$$From\ (A)\&(B)\ we\ \ get:$$

$$r_a^2+r_b^2+r_c^2\ge m_a^2+m_b^2+m_c^2+\frac{1}{4}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

Equality holds for an equilateral triangle.

2169.

In any Δ ABC, the following relationship holds :

$$\frac{\sin\frac{A}{2}}{m_a}+\frac{\sin\frac{B}{2}}{m_b}+\frac{\sin\frac{C}{2}}{m_c}\geq 2\sqrt{\frac{2R-r}{R(9R^2-8Rr+4r^2)}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(a\sin\frac{A}{2}\right)=4R\sum_{cyc}\left(\sin^2\frac{A}{2}\cos\frac{A}{2}\right)=R\sum_{cyc}\left(2\sin^2\frac{A}{2}\cdot\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}\right)\geq$$

$$\overset{0<\cos\frac{B-C}{2}\leq 1 \text{ and analogs}}{\geq} R\sum_{cyc}\left((1-\cos A)\left(2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\right)\right)=$$

$$=R\sum_{cyc}\big((1-\cos A)(\sin B+\sin C)\big)=$$

$$=R\sum_{cyc}(\sin B+\sin C)-R\sum_{cyc}\left(\cos A\left(\sum_{cyc}\sin A-\sin A\right)\right)$$

$$=2R\cdot\frac{s}{R}-R.\left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\sin A\right)+\frac{R}{2}.\sum_{cyc}\sin 2A=$$

$$=2s-R\left(\frac{R+r}{R}\right)\left(\frac{s}{R}\right)+2R.\frac{4Rrs}{8R^3}=2s-s-\frac{rs}{R}+\frac{rs}{R}\Rightarrow\sum_{cyc}\left(a\sin\frac{A}{2}\right)\geq s \text{ and so,}$$

$$\frac{\sin\frac{A}{2}}{m_a}+\frac{\sin\frac{B}{2}}{m_b}+\frac{\sin\frac{C}{2}}{m_c}\overset{Panaitopol}{\geq}\sum_{cyc}\frac{a\sin\frac{A}{2}}{Rs}\geq\frac{s}{Rs}=\frac{1}{R}\overset{?}{\geq}2.\sqrt{\frac{2R-r}{R(9R^2-8Rr+4r^2)}}$$

$$\Leftrightarrow R^2-4Rr+4r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)^2\overset{?}{\geq}0\rightarrow$$

$$\text{true}\therefore\frac{\sin\frac{A}{2}}{m_a}+\frac{\sin\frac{B}{2}}{m_b}+\frac{\sin\frac{C}{2}}{m_c}\geq 2\sqrt{\frac{2R-r}{R(9R^2-8Rr+4r^2)}}\ \forall\Delta\text{ ABC}$$

" = " iff Δ ABC is equilateral (QED)

2170. **In any Δ ABC, the following relationship holds :**

$$\frac{a^2+b^2+c^2}{2R} \le \frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a \le \frac{9R}{2}$$

Proposed by Tapas Das-India

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a = \sum_{cyc}\left(\frac{2bc}{b^2+c^2}.\frac{m_a m_b}{\left(\frac{bc}{2R}\right)}\right) = 4R\sum_{cyc}\frac{m_a m_b}{b^2+c^2}$$

$$\therefore \frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a \le \frac{9R}{2} \Leftrightarrow \sum_{cyc}\frac{m_a m_b}{b^2+c^2} \le \frac{9}{8} \to (1)$$

We shall now prove that : $\sum_{cyc}\frac{ab}{m_b^2+m_c^2} \le 2$ and indeed, $\sum_{cyc}\frac{ab}{m_b^2+m_c^2} =$

$$4.\sum_{cyc}\frac{ab}{\sum_{cyc}a^2+3a^2} \overset{A-G}{\le} 4.\sum_{cyc}\frac{ab}{2a.\sqrt{3\sum_{cyc}a^2}} = \frac{2}{\sqrt{3\sum_{cyc}a^2}}.\sum_{cyc}b = 2.\frac{\sum_{cyc}a}{\sqrt{3\sum_{cyc}a^2}}$$

≤ 2 and implementing $\sum_{cyc}\frac{ab}{m_b^2+m_c^2} \le 2$ on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$

whose medians as a consequence of trivial calculations $= \frac{a}{2},\frac{b}{2},\frac{c}{2}$, we get :

$$\sum_{cyc}\frac{\frac{4}{9}m_a m_b}{\left(\frac{b^2+c^2}{4}\right)} \le 2 \Rightarrow \sum_{cyc}\frac{m_a m_b}{b^2+c^2} \le \frac{9}{8} \Rightarrow (1) \text{ is true } \therefore \frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a \le \frac{9R}{2}$$

$$\text{and } \frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a \ge \sum_{cyc}m_a \overset{\text{Tereshin}}{\ge} \sum_{cyc}\frac{b^2+c^2}{4R} = \frac{a^2+b^2+c^2}{2R}$$

$$\text{and so, } \frac{a^2+b^2+c^2}{2R} \le \frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a \le \frac{9R}{2}\ \forall\ \Delta\ ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)

2171. **In ΔABC the following relationship holds:**

$$\frac{\cot^n\frac{A}{2}}{a}+\frac{\cot^n\frac{B}{2}}{b}+\frac{\cot^n\frac{C}{2}}{c} \ge \frac{3^{\frac{n+1}{2}}}{R},\ n \in N$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$abc=4Rrs\overset{Euler\ \&\ Mitrinovic}{\leq}4R\frac{R}{2}\frac{3\sqrt{3}R}{2}=3\sqrt{3}R^3\ and$$

$$\prod\cot\frac{A}{2}=\frac{s}{r}\overset{Mitrinovic}{\geq}3\sqrt{3}=3^{\frac{3}{2}}$$

$$\frac{\cot^n\frac{A}{2}}{a}+\frac{\cot^n\frac{B}{2}}{b}+\frac{\cot^n\frac{C}{2}}{c}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{\left(\prod\cot\frac{A}{2}\right)^n}{abc}}\geq 3\sqrt[3]{\frac{3^{\frac{3n}{2}}}{R3^{\frac{3}{2}}}}=\frac{3.3^{\frac{n-1}{2}}}{R}=\frac{3^{\frac{n+1}{2}}}{R}$$

$$Equality\ holds\ for\ a=b=c.$$

2172. In ΔABC the following relationship holds:

$$\frac{\left(r_a^3+r_b^3\right)^3}{\sin^2 A}+\frac{\left(r_b^3+r_c^3\right)^3}{\sin^2 B}+\frac{(r_c^3+r_a^3)^3}{\sin^2 C}\geq 32(3r)^9$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum r_a^3\overset{AM-GM}{\geq}3r_ar_br_c=3s^2r\overset{Mitrinovic}{\geq}3.27r^3=81r^3\ (1)$$

$$\sum\sin A\overset{Jensen}{\leq}3\sin\frac{A+B+C}{3}=3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}\ (2)$$

$$\frac{\left(r_a^3+r_b^3\right)^3}{\sin^2 A}+\frac{\left(r_b^3+r_c^3\right)^3}{\sin^2 B}+\frac{(r_c^3+r_a^3)^3}{\sin^2 C}=\sum\frac{\left(r_a^3+r_b^3\right)^3}{\sin^2 A}\overset{Radon}{\geq}$$

$$\geq\frac{(2\sum r_a^3)^3}{(\sum\sin A)^2}\overset{(1)\&(2)}{\geq}8\frac{(81\,r^3)^3}{\left(\frac{3\sqrt{3}}{2}\right)^2}=\frac{32r^9 3^{12}}{3^3}=32(3r)^9$$

$$Equality\ holds\ for\ a=b=c$$

2173. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)h_a\right) \le 3\sqrt{3} \le \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right) = \sum_{cyc}\left(\left(\sum_{cyc}\frac{1}{a}-\frac{1}{a}\right)r_a\right) = \left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}r_a\right) - \sum_{cyc}\frac{r_a}{a}$$

$$= \frac{(s^2+4Rr+r^2)(4R+r)}{4Rrs} - \sum_{cyc}\frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}$$

$$= \frac{(s^2+4Rr+r^2)(4R+r)}{4Rrs} - \frac{s}{4R}\cdot\frac{s^2+(4R+r)^2}{s^2} = \frac{4Rs^2}{4Rrs}$$

$$\Rightarrow \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right) = \frac{s}{r} \to (1)$$

$$\text{Now,} \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)h_a\right) = \sum_{cyc}\left(\left(\frac{b+c}{bc}\right)\left(\frac{bc}{2R}\right)\right) = \frac{4s}{2R} \Rightarrow \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)h_a\right) = \frac{2s}{R}$$

$$\overset{\text{Mitrinovic}}{\le} 3\sqrt{3} \overset{\text{Mitrinovic}}{\le} \frac{s}{r} \overset{\text{via (1)}}{=} \frac{s}{r} \text{ and so, } \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)h_a\right) \le 3\sqrt{3}$$

$$\le \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right) \ \forall\, \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

2174. In the acute ΔABC, $k = \frac{\sqrt[4]{3}(673\sqrt{3}+441)}{598}$. Prove that :

$$\frac{1}{s-R}+\frac{1}{R-r}+\frac{1}{s-r} \le \frac{k}{\sqrt{F}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$s-r = \sqrt{s^2}-\sqrt{r^2} \overset{Mitrinovic}{\ge} \sqrt{3\sqrt{3}sr} - \sqrt{\frac{sr}{3\sqrt{3}}} = (3\sqrt{3}-1)\sqrt{\frac{F}{3\sqrt{3}}}$$

$$R-r\overset{Mitrinovic}{\gtrsim}\frac{2s}{3\sqrt{3}}-r=2\sqrt{\frac{s^2}{27}}-\sqrt{r^2}\overset{Mitrinovic}{\gtrsim}2\sqrt{\frac{sr}{3\sqrt{3}}}-\sqrt{\frac{sr}{3\sqrt{3}}}=\sqrt{\frac{F}{3\sqrt{3}}}$$

$$s-R=\sqrt{s}\left(\sqrt{s}-\frac{R}{\sqrt{s}}\right)\overset{Walker}{\gtrsim}\sqrt{s}\left(\sqrt[4]{2R^2+8Rr+3r^2}-\frac{R}{\sqrt[4]{2R^2+8Rr+3r^2}}\right)$$
$$=\sqrt{F}.f\left(\frac{R}{r}\right)$$

where $f(x)=\sqrt[4]{2x^2+8x+3}-\frac{x}{\sqrt[4]{2x^2+8x+3}}, x\ge 0.$ It is easy to find that

$$f'(x)=\frac{(x+2)\sqrt{2x^2+8x+3}-(x^2+6x+3)}{\sqrt[4]{2x^2+8x+3}}$$
$$=\frac{x^4+4x^3+x^2+8x+3}{\sqrt[4]{2x^2+8x+3}\left((x+2)\sqrt{2x^2+8x+3}+x^2+6x+3\right)}>0,$$

$$\Rightarrow s-R\ge f\left(\frac{R}{r}\right).\sqrt{F}\overset{Euler}{\gtrsim}f(2).\sqrt{F}=\left(3\sqrt{3}-2\right)\sqrt{\frac{F}{3\sqrt{3}}}.$$

Therefore

$$\frac{1}{s-R}+\frac{1}{R-r}+\frac{1}{s-r}\le\frac{\sqrt{3\sqrt{3}}}{(3\sqrt{3}-2)\sqrt{F}}+\sqrt{\frac{3\sqrt{3}}{F}}+\frac{\sqrt{3\sqrt{3}}}{(3\sqrt{3}-1)\sqrt{F}}=\frac{k}{\sqrt{F}}.$$

Equality holds iff ΔABC is equilateral.

2175. In any Δ ABC, the following relationship holds :

$$4\sqrt{2Rsr}\prod_{cyc}\sqrt{m_a-h_a}\ge|(a-b)(b-c)(c-a)|\ge 4R\sqrt{r}\prod_{cyc}\sqrt{w_a-h_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$
$$=\frac{1}{64}\left(-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right)\rightarrow(1)$$

$$\sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \to (2)$$

$$\text{and,} \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \to (3)$$

$$\therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2 =$$

$$\frac{1}{64}\begin{pmatrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{pmatrix}$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{\begin{matrix}-32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 - 576Rrs^2(s^2-4Rr-r^2) \\ -432R^2r^2s^2\end{matrix}}{64}$$

$$\Rightarrow m_a^2m_b^2m_c^2 =$$

$$\frac{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3}{16} \to (m)$$

$$\text{Now,} \left(4.\sqrt{2Rsr}.\prod_{cyc}\sqrt{m_a-h_a}\right)^2 = 32Rsr.\prod_{cyc}\frac{m_a^2-h_a^2}{m_a+h_a} \geq$$

$$\frac{4Rsr}{m_am_bm_c}.\prod_{cyc}\left(\frac{(b-c)^2}{4} + \frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$= \frac{4Rsr}{m_am_bm_c}.\frac{1}{64.16R^2r^2s^2}.\prod_{cyc}(b+c)^2.\prod_{cyc}(b-c)^2 \overset{?}{\geq} \prod_{cyc}(b-c)^2$$

$$\Leftrightarrow s(s^2+2Rr+r^2)^2 \overset{?}{\geq} 64Rr\prod_{cyc} m_a \overset{\text{via (m)}}{\Leftrightarrow} s^2(s^2+2Rr+r^2)^4 \geq$$

$$256R^2r^2(s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3)$$

$$\Leftrightarrow s^2(s^2+2Rr+r^2)^4$$
$$-256R^2r^2(s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3)$$
$$\overset{?}{\underset{(*)}{\geq}} 0\ and\ \because P=(s^2-16Rr+5r^2)^5+(88Rr-21r^2)(s^2-16Rr+5r^2)^4$$
$$+8r^2(355R^2-185Rr+22r^2)(s^2-16Rr+5r^2)^3$$
$$+8r^3(5652R^3-5463R^2r+1164Rr^2-92r^3)(s^2-16Rr+5r^2)^2$$
$$+16r^4(24705R^4-37890R^3r+14868R^2r^2-1624Rr^3+96r^4)(s^2-16Rr+5r^2)$$
$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \geq P$$
$$\Leftrightarrow 104976t^5-203877t^4+129384t^3-28152t^2+1696t-80\geq 0\ \left(t=\frac{R}{r}\right)\Leftrightarrow$$
$$(t-2)(104976t^4+6075t^3+141534t^2+254916t+511528)+1022976\geq 0$$
$$\rightarrow \text{true (strict inequality)} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$
$$\therefore 4.\sqrt{2Rsr}.\prod_{cyc}\sqrt{m_a-h_a}\geq |(a-b)(b-c)(c-a)|$$

$$\text{Again,}\left(4R.\sqrt{r}.\prod_{cyc}\sqrt{w_a-h_a}\right)^2=16R^2r.\prod_{cyc}\frac{w_a^2-h_a^2}{w_a+h_a}\leq$$
$$\frac{2R^2r}{h_ah_bh_c}.\prod_{cyc}\left(\frac{s(s-a)(b-c)^2}{a^2}-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$
$$=\frac{2R^3r}{2r^2s^2}.\frac{\prod_{cyc}(4s^2(s-a)^2)}{16R^2r^2s^2.\prod_{cyc}(b+c)^2}.\prod_{cyc}(b-c)^2$$
$$=\frac{R^3}{rs^2}.\frac{64s^6.r^4s^2}{16R^2r^2s^2.4s^2(s^2+2Rr+r^2)^2}.\prod_{cyc}(b-c)^2\overset{?}{\leq}\prod_{cyc}(b-c)^2$$
$$\Leftrightarrow (s^2+2Rr+r^2)^2 \overset{?}{\geq} Rrs^2 \rightarrow \text{true (strict inequality)} \because s^2+2Rr+r^2>s^2, Rr$$
$$\therefore 4R.\sqrt{r}.\prod_{cyc}\sqrt{w_a-h_a}\leq |(a-b)(b-c)(c-a)|\ and\ so,$$
$$4.\sqrt{2Rsr}.\prod_{cyc}\sqrt{m_a-h_a}\geq |(a-b)(b-c)(c-a)|\geq 4R.\sqrt{r}.\prod_{cyc}\sqrt{w_a-h_a}$$
$$\forall \Delta ABC,''='' \text{ iff } \Delta ABC \text{ is isosceles (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will prove that

$$2a(m_a-h_a)\geq (b-c)^2\geq \frac{a^2(w_a-h_a)}{\sqrt{s(s-a)}}.\quad (1)$$

We have

$$2a(m_a-h_a)\overset{?}{\geq}(b-c)^2 \Leftrightarrow 2am_a\geq 4F+(b-c)^2$$

$$\Leftrightarrow a^2(2b^2+2c^2-a^2)\geq 16F^2+8F(b-c)^2+(b-c)^4$$

$$\Leftrightarrow a^2(2b^2+2c^2-a^2)-2(a^2b^2+b^2c^2+c^2a^2)+(a^4+b^4+c^4)\geq$$

$$\geq 8F(b-c)^2+(b-c)^4$$

$$\Leftrightarrow (b^2-c^2)^2\geq 8F(b-c)^2+(b-c)^4 \Leftrightarrow 4(b-c)^2(bc-2F)\geq 0$$
$$\Leftrightarrow 4(b-c)^2bc(1-\sin A)\geq 0,$$

which is true. So the proof of the left side of inequality (1) is complete.

$$(b-c)^2\overset{?}{\geq}\frac{a^2(w_a-h_a)}{\sqrt{s(s-a)}} \Leftrightarrow (b-c)^2\geq a^2\left(\frac{2\sqrt{bc}}{b+c}-\frac{2\sqrt{(s-b)(s-c)}}{a}\right)$$

$$\Leftrightarrow (b-c)^2\geq a.\frac{4a^2bc-(b+c)^2[a^2-(b-c)^2]}{2(b+c)\left(a\sqrt{bc}+(b+c)\sqrt{(s-b)(s-c)}\right)}$$

$$\Leftrightarrow 1\geq \frac{a.4s(s-a)}{2(b+c)\left(a\sqrt{bc}+(b+c)\sqrt{(s-b)(s-c)}\right)},$$

which is true because $b+c=s+(s-a)\geq 2\sqrt{s(s-a)}$,

$$bc=s(s-a)+(s-b)(s-c)\geq s(s-a).$$

So the proof of the right side of inequality (1) is complete.

Using the inequality (1), we have

$$\prod_{cyc}\sqrt{2a}.\sqrt{m_a-h_a}\geq |(a-b)(b-c)(c-a)|\geq \prod_{cyc}\frac{a}{\sqrt[4]{s(s-a)}}.\sqrt{w_a-h_a}$$

$$\Leftrightarrow 4\sqrt{2Rsr}.\prod_{cyc}\sqrt{m_a-h_a}\geq |(a-b)(b-c)(c-a)|\geq 4R\sqrt{r}.\prod_{cyc}\sqrt{w_a-h_a},$$

as desired. Equality holds iff ΔABC is isosceles.

2176. In any ΔABC the following relationship holds :

$$3-\sqrt{2}+\frac{R}{\sqrt{2}r}\ge\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\ge 3+\frac{3(R-2r)}{2(13R-2r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{m_a}{w_a}=\sum_{cyc}\frac{(b+c)m_a}{2\sqrt{bc.s(s-a)}}\overset{CBS}{\le}\frac{1}{2}\sqrt{\sum_{cyc}\frac{(b+c)^2}{bc}.\sum_{cyc}\frac{{m_a}^2}{s(s-a)}},$$

with

$$\sum_{cyc}\frac{(b+c)^2}{bc}=\frac{\sum_{cyc}a(b+c)^2}{abc}=\frac{2s(s^2+r^2+10Rr)}{4Rrs}\overset{Gerretsen}{\le}\frac{4R^2+14Rr+4r^2}{2Rr}$$

$$\sum_{cyc}\frac{{m_a}^2}{s(s-a)}=\sum_{cyc}\frac{2(b^2+c^2)-a^2}{4s(s-a)}=\frac{1}{4s}\sum_{cyc}\left(\frac{2(a^2+b^2+c^2)-3s^2}{s-a}+3(s+a)\right)=$$

$$=\frac{1}{4s}\left(\frac{[2(a^2+b^2+c^2)-3s^2](4R+r)}{sr}+15s\right)$$

$$=\frac{1}{4s}\left(\frac{(s^2-4r^2-16Rr)(4R+r)}{sr}+15s\right)=$$

$$=\frac{R+4r}{r}-\frac{(4R+r)^2}{s^2}\overset{Gerretsen-Blundon}{\le}\frac{R+4r}{r}-\frac{2(2R-r)}{R}=\frac{R^2+2r^2}{Rr}$$

then

$$\sum_{cyc}\frac{m_a}{w_a}\le\frac{\sqrt{(R^2+2r^2)(2R^2+7Rr+2r^2)}}{2Rr}\overset{?}{\le}3-\sqrt{2}+\frac{R}{\sqrt{2}r}$$

$$\overset{squaring}{\Leftrightarrow}\frac{(R-2r)\left(3(4\sqrt{2}-5)R^2+8Rr+2r^2\right)}{4R^2r}\ge 0$$

which is true and the proof of the left side of the desired inequality is complete.

$$\sum_{cyc}\frac{m_a}{w_a}=\sum_{cyc}\frac{m_a w_a}{{w_a}^2}\ge\sum_{cyc}\frac{s(s-a)}{{w_a}^2}=\sum_{cyc}\frac{(b+c)^2}{4bc}=\frac{\sum_{cyc}a(b+c)^2}{4abc}$$

$$=\frac{2s(s^2+r^2+10Rr)}{16Rrs}$$

$$\overset{Gerretsen}{\ge}\frac{26Rr-4r^2}{8Rr}=3+\frac{R-2r}{4R}\overset{Euler}{\ge}3+\frac{3(R-2r)}{2(13R-2r)}$$

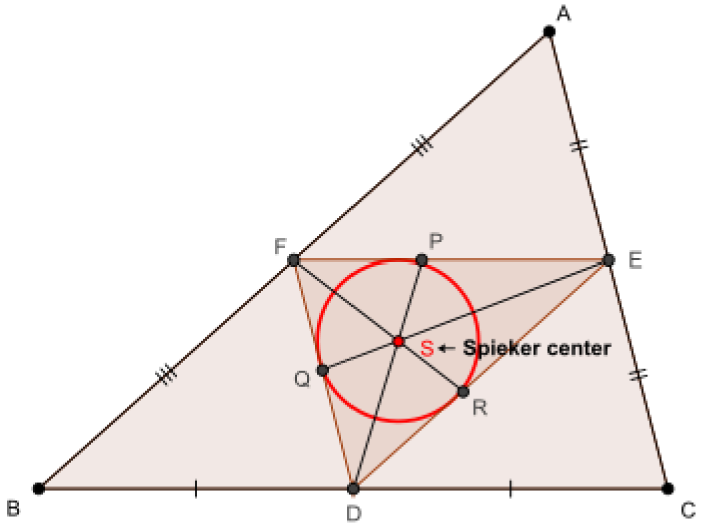
So the proof is complete. Equality holds iff ΔABC is equilateral.

2177. In any Δ ABC, the following relationship holds :

$$\frac{h_a}{m_a}+\frac{w_a}{g_a}+\frac{p_a}{n_a}\le\frac{g_a}{h_a}+\frac{m_a}{p_a}+\frac{n_a}{w_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, }16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow[DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{ Spieker center is incenter of }\Delta\text{ DEF},\therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2} \stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2 s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \stackrel{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow \mathrm{c}\sin\alpha \overset{(***)}{=} \frac{\mathrm{r}(a+\mathrm{b})}{2\mathrm{AS}} \text{ and via sine law on } \Delta\,\mathrm{AES}, \mathrm{b}\sin\beta \overset{(****)}{=} \frac{\mathrm{r}(a+\mathrm{c})}{2\mathrm{AS}}$$

$$\text{Now, } [\mathrm{BAX}] + [\mathrm{BAX}] = [\mathrm{ABC}] \Rightarrow \frac{1}{2}\mathrm{p}_a \mathrm{c}\sin\alpha + \frac{1}{2}\mathrm{p}_a \mathrm{b}\sin\beta = \mathrm{rs}$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow} \frac{\mathrm{p}_a(a+\mathrm{b}+a+\mathrm{c})}{4\mathrm{AS}} = \mathrm{s} \Rightarrow \mathrm{p}_a = \frac{4\mathrm{s}}{2\mathrm{s}+a}\mathrm{AS}$$

$$\Rightarrow \mathrm{p}_a^2 \overset{\text{via (ii)}}{=} \frac{16\mathrm{s}^2}{(2\mathrm{s}+a)^2} \cdot \frac{\mathrm{b}^3+\mathrm{c}^3-a\mathrm{bc}+a(4\mathrm{m}_a^2)}{8\mathrm{s}}$$

$$\therefore \mathrm{p}_a^2 \overset{(\bullet)}{=} \frac{2\mathrm{s}}{(2\mathrm{s}+a)^2}\left(\mathrm{b}^3+\mathrm{c}^3-a\mathrm{bc}+a(4\mathrm{m}_a^2)\right)$$

$$\text{Now, } \mathrm{b}^3+\mathrm{c}^3-a\mathrm{bc}+a(4\mathrm{m}_a^2) = \mathrm{b}^3+\mathrm{c}^3-a\mathrm{bc}+a(2\mathrm{b}^2+2\mathrm{c}^2-a^2)$$

$$= (\mathrm{b}+\mathrm{c})(\mathrm{b}^2-\mathrm{bc}+\mathrm{c}^2) + a(\mathrm{b}^2-\mathrm{bc}+\mathrm{c}^2) + a(\mathrm{b}^2+\mathrm{c}^2-a^2)$$

$$= 2\mathrm{s}(\mathrm{b}^2-\mathrm{bc}+\mathrm{c}^2) + a(\mathrm{b}^2-\mathrm{bc}+\mathrm{c}^2+\mathrm{bc}-a^2)$$

$$= (2\mathrm{s}+a)(\mathrm{b}^2-\mathrm{bc}+\mathrm{c}^2) + a\left(\frac{(\mathrm{b}+\mathrm{c})^2-(\mathrm{b}-\mathrm{c})^2}{4} - a^2\right)$$

$$= (2\mathrm{s}+a)(\mathrm{b}^2-\mathrm{bc}+\mathrm{c}^2) + \frac{a(\mathrm{b}+\mathrm{c}+2a)(\mathrm{b}+\mathrm{c}-2a)}{4} - \frac{a(\mathrm{b}-\mathrm{c})^2}{4}$$

$$= (2\mathrm{s}+a)(\mathrm{b}^2-\mathrm{bc}+\mathrm{c}^2) + \frac{a(2\mathrm{s}-a+2a)(\mathrm{b}+\mathrm{c}-2a)}{4} - \frac{a(\mathrm{b}-\mathrm{c})^2}{4}$$

$$= (2\mathrm{s}+a).\frac{4\mathrm{b}^2+4\mathrm{c}^2-4\mathrm{bc}+a(\mathrm{b}+\mathrm{c}-2a)}{4} - \frac{a(\mathrm{b}-\mathrm{c})^2}{4}$$

$$= (2\mathrm{s}+a).$$

$$\frac{4(\mathrm{z}+x)^2+4(x+\mathrm{y})^2-4(\mathrm{z}+x)(x+\mathrm{y})+(\mathrm{y}+\mathrm{z})\left((\mathrm{z}+x)+(x+\mathrm{y})-2(\mathrm{y}+\mathrm{z})\right)}{4}$$

$$-\frac{a(\mathrm{b}-\mathrm{c})^2}{4}\ (a=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+x, \mathrm{c}=x+\mathrm{y})$$

$$= (2\mathrm{s}+a).\frac{4x(x+\mathrm{y}+\mathrm{z})+2x(\mathrm{y}+\mathrm{z})+3(\mathrm{y}-\mathrm{z})^2}{4} - \frac{a(\mathrm{b}-\mathrm{c})^2}{4}$$

$$= (2\mathrm{s}+a)\left(\mathrm{s}(\mathrm{s}-a)+\frac{3}{4}(\mathrm{b}-\mathrm{c})^2+\frac{a(\mathrm{s}-a)}{2}\right) - \frac{a(\mathrm{b}-\mathrm{c})^2}{4}$$

$$= (2\mathrm{s}+a)\left(\mathrm{s}(\mathrm{s}-a)+\frac{3}{4}(\mathrm{b}-\mathrm{c})^2+\frac{a(\mathrm{s}-a)}{2}\right) - \frac{(a+2\mathrm{s}-2\mathrm{s})(\mathrm{b}-\mathrm{c})^2}{4}$$

$$= (2\mathrm{s}+a)\left(\mathrm{s}(\mathrm{s}-a)+\frac{(\mathrm{b}-\mathrm{c})^2}{2}+\frac{a(\mathrm{s}-a)}{2}\right) + \frac{\mathrm{s}(\mathrm{b}-\mathrm{c})^2}{2}$$

$$\therefore \mathrm{b}^3+\mathrm{c}^3-a\mathrm{bc}+a(4\mathrm{m}_a^2) \overset{(\bullet\bullet)}{=} (2\mathrm{s}+a)\left(\frac{(\mathrm{s}-a)(2\mathrm{s}+a)}{2}+\frac{(\mathrm{b}-\mathrm{c})^2}{2}\right) + \frac{\mathrm{s}(\mathrm{b}-\mathrm{c})^2}{2}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow \mathrm{p}_a^2 = \frac{2\mathrm{s}}{(2\mathrm{s}+a)^2}\left(\frac{(\mathrm{s}-a)(2\mathrm{s}+a)^2}{2}+\frac{(2\mathrm{s}+a)(\mathrm{b}-\mathrm{c})^2}{2}+\frac{\mathrm{s}(\mathrm{b}-\mathrm{c})^2}{2}\right)$$

$$= \mathrm{s}(\mathrm{s}-a)+(\mathrm{b}-\mathrm{c})^2\left(\left(\frac{\mathrm{s}}{2\mathrm{s}+a}\right)^2+\frac{\mathrm{s}}{2\mathrm{s}+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= \mathrm{s}(\mathrm{s}-a)-\frac{(\mathrm{b}-\mathrm{c})^2}{4}+(\mathrm{b}-\mathrm{c})^2.\left(\frac{\mathrm{s}}{2\mathrm{s}+a}+\frac{1}{2}\right)^2$$

$$= \mathrm{s}(\mathrm{s}-a)+\frac{(\mathrm{b}-\mathrm{c})^2}{4}\left(\frac{(4\mathrm{s}+a)^2}{(2\mathrm{s}+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\geq} 0 \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{s(s-a)\left(144s^3-52s^2a-16sa^2+5a^3\right)}{36a(2s+a)^2}$$

$$+\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)\left(144s^2+92sa+76a^2\right)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)\left(1296s^3+1820s^2a+1736sa^2+1700a^3\right)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2+\frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow \frac{m_a}{p_a} \geq \frac{p_a}{n_a} \rightarrow (3)$$

$$\text{Finally, } an_a^2.ag_a^2 \geq a^2s^2(s-a)^2 \Leftrightarrow$$

$$\left(b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\right)\begin{pmatrix} b^2(s-b)+c^2(s-c) \\ -a(s-b)(s-c)\end{pmatrix} \overset{(a)}{\geq} a^2s^2(s-a)^2$$

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z$,

$b=z+x$ and $c=x+y$

Via such substitutions, $(a) \Leftrightarrow$

$$\left(z(z+x)^2+y(x+y)^2-yz(y+z)\right)\left(y(z+x)^2+z(x+y)^2-yz(y+z)\right)$$

$$\geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2+xz^2+y^3+z^3 \geq 2xyz+yz(y+z) \Leftrightarrow x(y-z)^2+(y+z)(y-z)^2 \geq 0$$

$$\rightarrow \text{true} \Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \geq s(s-a) \geq w_a^2 \Rightarrow \frac{n_a}{w_a} \geq \frac{w_a}{g_a} \rightarrow (4)$$

$$\text{Also, } m_a g_a \geq h_a^2 \Rightarrow \frac{g_a}{h_a} \geq \frac{h_a}{m_a} \rightarrow (5) \therefore (3)+(4)+(5) \Rightarrow \frac{h_a}{m_a}+\frac{w_a}{g_a}+\frac{p_a}{n_a} \leq$$

$$\frac{g_a}{h_a}+\frac{m_a}{p_a}+\frac{n_a}{w_a} \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

2178. In any Δ ABC, the following relationship holds :

$$4-\frac{2r}{R}\le\frac{m_a h_a}{w_a r_a}+\frac{m_b h_b}{w_b r_b}+\frac{m_c h_c}{w_c r_c}\le\frac{2\sqrt{2}R^2+2(3-4\sqrt{2})r^2}{Rr}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a h_a}{w_a r_a}+\frac{m_b h_b}{w_b r_b}+\frac{m_c h_c}{w_c r_c}=\sum_{cyc}\frac{m_a bc(b+c)}{2R.2bc\cos\frac{A}{2}.s\tan\frac{A}{2}}$$

$$=\sum_{cyc}\frac{m_a(b+c).\sqrt{bc}.\sqrt{s-a}}{4Rs.\sqrt{(s-b)(s-c)(s-a)}}=\frac{1}{4Rrs.\sqrt{s}}.\sum_{cyc}\left(m_a.\sqrt{bc}.(b+c).\sqrt{s-a}\right)$$

$$\overset{CBS}{\le}\frac{1}{4Rrs.\sqrt{s}}.\sqrt{\sum_{cyc}(bc.m_a^2)}.\sqrt{\sum_{cyc}\left((s-a)(b+c)^2\right)}$$

$$=\frac{1}{4Rrs.\sqrt{s}}.\sqrt{\frac{1}{4}\sum_{cyc}\left(bc\left(2\sum_{cyc}a^2-3a^2\right)\right)}.\sqrt{\sum_{cyc}\left((s-a)(4s^2-4sa+a^2)\right)}$$

$$=\frac{1}{4Rrs.\sqrt{s}}\sqrt{\frac{1}{4}\begin{pmatrix}4(s^2-4Rr-r^2)(s^2+4Rr+r^2)\\-24Rrs^2\end{pmatrix}}\sqrt{\begin{matrix}4s(s^2-2(4Rr+r^2))+\\2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)\end{matrix}}$$

$$=\frac{1}{2Rrs}.\sqrt{s^4-6Rrs^2-r^2(4R+r)^2}.\sqrt{s^2-7Rr-r^2}\overset{?}{\le}\frac{2\sqrt{2}R^2+2(3-4\sqrt{2})r^2}{Rr}$$

$$\Leftrightarrow(s^4-6Rrs^2-r^2(4R+r)^2)(s^2-7Rr-r^2)$$

$$\overset{?}{\underset{①}{\le}}4s^2\left(8(R^2-4r^2)^2+36r^4+24\sqrt{2}r^2(R^2-4r^2)\right)$$

$$\text{Now, }R^2-4r^2\overset{Euler}{\ge}0\text{ and }24\sqrt{2}>33\therefore\text{RHS of ①}\ge$$

$$4s^2\left(8(R^2-4r^2)^2+36r^4+33r^2(R^2-4r^2)\right)$$

$$\overset{?}{\ge}(s^4-6Rrs^2-r^2(4R+r)^2)(s^2-7Rr-r^2)$$

$$\Leftrightarrow s^6-(13Rr+r^2)s^4-(32R^4-150R^2r^2+2Rr^3+129r^4)s^2$$

$$+r^3(112R^3+72R^2r+15Rr^2+r^3)\overset{?}{\underset{②}{\le}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge0$ and $s^2-(m+n)\le0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le0$$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \overset{(*)}{\le} 0$$

$$\Rightarrow P = s^2(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) +$$

$$(4R^2 + 7Rr - 3r^2)(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) \overset{\text{via }(*)}{\le} 0$$

$\therefore$ in order to prove ②, it suffices to prove : LHS of ② $\le$ P

$$\Leftrightarrow (8R^4 - 22R^3r - 111R^2r^2 + 44Rr^3 + 62r^4)s^2$$

$$+r(128R^5 + 320R^4r + 40R^3r^2 - 64R^2r^3 - 22Rr^4 - 2r^5) \overset{③}{\ge} 0$$

Case 1 $8R^4 - 22R^3r - 111R^2r^2 + 44Rr^3 + 62r^4 \ge 0$ and then : LHS of ③ $\ge$ $r(128R^5 + 320R^4r + 40R^3r^2 - 64R^2r^3 - 22Rr^4 - 2r^5) > 0$ $(\because R \ge 2r)$

Case 2 $8R^4 - 22R^3r - 111R^2r^2 + 44Rr^3 + 62r^4 < 0$ and then : LHS of ③

$$\overset{\text{Gerretsen}}{\ge} (8R^4 - 22R^3r - 111R^2r^2 + 44Rr^3 + 62r^4)(4R^2 + 4Rr + 3r^2)$$

$$+r(128R^5 + 320R^4r + 40R^3r^2 - 64R^2r^3 - 22Rr^4 - 2r^5) \overset{?}{\ge} 0$$

$$\Leftrightarrow 32t^6 + 72t^5 - 188t^4 - 294t^3 + 27t^2 + 358t + 184 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(32t^4 + 200t^3 + 484t^2 + 842t + 1459) + 2826\right) \overset{?}{\ge} 0 \to \text{true}$$

$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow$ ③ $\Rightarrow$ ② $\Rightarrow$ ① is true

$$\therefore \frac{m_a h_a}{w_a r_a} + \frac{m_b h_b}{w_b r_b} + \frac{m_c h_c}{w_c r_c} \le \frac{2\sqrt{2}R^2 + 2(3 - 4\sqrt{2})r^2}{Rr}$$

$$\text{Again, } \frac{m_a h_a}{w_a r_a} + \frac{m_b h_b}{w_b r_b} + \frac{m_c h_c}{w_c r_c} \ge \sum_{cyc} \frac{h_a}{r_a} = \sum_{cyc} \frac{2rs}{4Rs\cos^2\frac{A}{2}\tan^2\frac{A}{2}} = \frac{r}{2R} \cdot \sum_{cyc} \frac{bc(s-a)}{r^2 s}$$

$$= \frac{s(s^2 - 8Rr + r^2)}{2Rrs} \overset{\text{Gerretsen}}{\ge} \frac{8R - 4r}{2R} = 4 - \frac{2r}{R} \text{ and so,}$$

$$4 - \frac{2r}{R} \le \frac{m_a h_a}{w_a r_a} + \frac{m_b h_b}{w_b r_b} + \frac{m_c h_c}{w_c r_c} \le \frac{2\sqrt{2}R^2 + 2(3 - 4\sqrt{2})r^2}{Rr} \ \forall \Delta ABC,$$

'' ='' iff Δ ABC is equilateral (QED)

2179. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2}{h_a\sqrt{h_a}} + \frac{m_b^2}{h_b\sqrt{h_b}} + \frac{m_c^2}{h_c\sqrt{h_c}} \le \frac{R^2 + Rr + (3\sqrt{6} - 6)r^2}{r\sqrt{2r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2}{h_a \cdot \sqrt{h_a}} + \frac{m_b^2}{h_b \cdot \sqrt{h_b}} + \frac{m_c^2}{h_c \cdot \sqrt{h_c}} \le \frac{R^2 + Rr + (3\sqrt{6} - 6)r^2}{r.\sqrt{2r}}$$

$$\Leftrightarrow \sum_{cyc} \frac{a.\sqrt{a}.m_a^2}{2rs.\sqrt{2rs}} \le \frac{R^2+Rr+(3\sqrt{6}-6)r^2}{r.\sqrt{2r}}$$

$$\Leftrightarrow \sum_{cyc} \frac{a.\sqrt{a}.m_a^2}{2s.\sqrt{s}} \le R^2+Rr+(3\sqrt{6}-6)r^2 \to (m)$$

$$\text{Now, } \sum_{cyc} \frac{a.\sqrt{a}.m_a^2}{2s.\sqrt{s}} \overset{CBS}{\le} \frac{1}{2s.\sqrt{s}}.\sqrt{\sum_{cyc} a^2m_a^2}.\sqrt{\sum_{cyc} am_a^2}$$

$$= \frac{1}{4s.\sqrt{s}}.\sqrt{\sum_{cyc} a^2(2b^2+2c^2-a^2)}.\sqrt{s\sum_{cyc} a(s-a)+\frac{1}{4}\sum_{cyc} a(b^2+c^2-2bc)}$$

$$= \frac{1}{4s.\sqrt{s}}.\sqrt{4\sum_{cyc} a^2b^2+16r^2s^2-2\sum_{cyc} a^2b^2}.\sqrt{\begin{array}{c} s(2s^2-2(s^2-4Rr-r^2)) \\ +\dfrac{2s(s^2+4Rr+r^2)-36Rrs}{4}\end{array}}$$

$$= \frac{1}{4s}.\sqrt{(s^2+4Rr+r^2)^2-16Rrs^2+8r^2s^2}.\sqrt{s^2+2Rr+5r^2}$$

$$\overset{?}{\le} R^2+Rr+(3\sqrt{6}-6)r^2$$

$$\Leftrightarrow \frac{((s^2+4Rr+r^2)^2-16Rrs^2+8r^2s^2)(s^2+2Rr+5r^2)}{16s^2}$$

$$\overset{?}{\underset{(1)}{\le}} (R^2+Rr-6r^2)^2+54r^2+6\sqrt{6}r^2(R^2+Rr-6r^2)$$

$$\text{Now, } R^2+Rr-6r^2=(R-2r)(R+3r) \overset{Euler}{\ge} 0 \text{ and } 6\sqrt{6}>14 \therefore \text{RHS of } (1) \ge$$

$$(R^2+Rr-6r^2)^2+54r^2+14r^2(R^2+Rr-6r^2)$$

$$\overset{?}{\ge} \frac{((s^2+4Rr+r^2)^2-16Rrs^2+8r^2s^2)(s^2+2Rr+5r^2)}{16s^2}$$

$$\Leftrightarrow s^6-(6Rr-15r^2)s^4-r(32R^3+48R^2r+44Rr^2+45r^3)s^2$$

$$+r^3(32R^3+96R^2r+42Rr^2+5r^3) \overset{?}{\underset{(2)}{\le}} 0$$

$$\text{Now, Rouche} \Rightarrow s^2-(m-n)\ge 0 \text{ and } s^2-(m+n)\le 0, \text{where } m=$$

$$2R^2+10Rr-r^2 \text{ and } n=2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0 \Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \overset{(*)}{\le} 0$$

$$\Rightarrow P = s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)+$$

$$(4R^2+14Rr+13r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)+$$

$$4r(10R^3+57R^2r+44Rr^2-18r^3)(s^2-4R^2-4Rr-3r^2) \overset{\text{via } (*) \text{ and Gerretsen}}{\le} 0$$

$$\therefore \text{in order to prove } (2), \text{it suffices to prove : LHS of } (2) \le P$$

$$\Leftrightarrow 12t^5+2t^4-27t^3-50t^2-14t+28\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(12t^4+26t^3+25t^2-14)\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow ②\Rightarrow ①$$

$$\text{is true} \therefore \frac{m_a^2}{h_a.\sqrt{h_a}}+\frac{m_b^2}{h_b.\sqrt{h_b}}+\frac{m_c^2}{h_c.\sqrt{h_c}}\leq\frac{R^2+Rr+(3\sqrt{6}-6)r^2}{r.\sqrt{2r}}\ \forall\Delta ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)

2180. In ΔABC the following relationship holds:

$$3\sqrt{3}\cdot\frac{2r}{R}\leq\sum\left(\frac{1}{b}+\frac{1}{c}\right)h_a\leq 3\sqrt{3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\left(\frac{1}{b}+\frac{1}{c}\right)h_a=\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{bc}{2R}=\frac{1}{2R}\sum(b+c)=\frac{2(a+b+c)}{2R}=$$

$$=\frac{2s}{R}\overset{\text{Mitrinovic}}{\leq}2\cdot\frac{3\sqrt{3}R}{2R}=3\sqrt{3}$$

$$\sum\left(\frac{1}{b}+\frac{1}{c}\right)h_a=\sum\left(\frac{1}{b}+\frac{1}{c}\right)\frac{bc}{2R}=\frac{1}{2R}\sum(b+c)=\frac{2(a+b+c)}{2R}=$$

$$=\frac{2s}{R}\overset{\text{Mitrinovic}}{\geq}2\cdot\frac{3\sqrt{3}r}{R}=3\sqrt{3}\frac{2r}{R}$$

Equality holds for an equilateral triangle

2181. In any Δ ABC, the following relationship holds :

$$3\sqrt{3}\leq\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)\leq 3\sqrt{3}\cdot\frac{R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)=\sum_{cyc}\left(\left(\sum_{cyc}\frac{1}{a}-\frac{1}{a}\right)r_a\right)=\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}r_a\right)-\sum_{cyc}\frac{r_a}{a}=$$

$$=\frac{(s^2+4Rr+r^2)(4R+r)}{4Rrs}-\sum_{cyc}\frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}=$$

$$=\frac{(s^2+4Rr+r^2)(4R+r)}{4Rrs}-\frac{s}{4R}\cdot\frac{s^2+(4R+r)^2}{s^2}=\frac{4Rs^2}{4Rrs}\Rightarrow\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)$$

$$=\frac{s}{r}\overset{\text{Mitrinovic}}{\leq}3\sqrt{3}.\frac{R}{2r}\text{ and also},\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)=\frac{s}{r}\overset{\text{Mitrinovic}}{\geq}3\sqrt{3}$$

$$\text{and so},3\sqrt{3}\leq\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)\leq 3\sqrt{3}.\frac{R}{2r}\ \forall\Delta ABC,$$

$''=''$ iff ΔABC is equilateral (QED)

2182. **In ΔABC the following relationship holds:**

$$\cos A+\cos B+\cos C+\sqrt{3}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\geq\frac{15}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sqrt{3}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)=\sqrt{3}\,2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=$$

$$=\sqrt{3}\,2R\frac{s^2+r^2+4Rr}{4Rrs}\overset{\text{Gerretsen \& Mitrinovic}}{\geq}\sqrt{3}\cdot\frac{16Rr-5r^2+r^2+4Rr}{\frac{2r3\sqrt{3}R}{2}}=$$

$$=\frac{(20Rr-4r^2)}{3Rr}=\frac{20}{3}-\frac{4r}{R}$$

$$\cos A+\cos B+\cos C+\sqrt{3}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\geq 1+\frac{r}{R}+\frac{20}{3}-\frac{4r}{R}=$$

$$=\frac{23}{3}-\frac{r}{3R}\overset{\text{Euler}}{\geq}\frac{23}{3}-\frac{1}{6}=\frac{45}{6}=\frac{15}{2}$$

Equality holds for an equilateral triangle.

2183. In ΔABC the following relationship holds:

$$\sin A+\sin B+\sin C+\frac{2}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\geq\frac{59\sqrt{3}}{18}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\,\frac{R}{s}=x\overset{Mitrinovic}{\geq}\frac{2s}{3\sqrt{3}}.\frac{1}{s}=\frac{2}{3\sqrt{3}}$$

We need to show:

$$\sin A+\sin B+\sin C+\frac{2}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\geq\frac{59\sqrt{3}}{18}$$

$$or,\frac{s}{R}+\frac{8R}{s}\geq\frac{59\sqrt{3}}{18}$$

$$or, 18(8R^2+s^2)\geq 59\sqrt{3}Rs\ or,$$

$$144R^2-59\sqrt{3}Rs+18s^2\geq 0\ or, 144x^2-59\sqrt{3}x+18\overset{\frac{R}{s}=x}{\geq}0$$

$$We\ take\ f(x)=144x^2-59\sqrt{3}x+18\,,f'(x)=288x-59\sqrt{3}>0$$

$$\left(\ as\ \ x\geq\frac{2}{3\sqrt{3}}\ and\ 288\cdot\frac{2}{3\sqrt{3}}-59\sqrt{3}=64\sqrt{3}-59\sqrt{3}>0\right)$$

$$so\ f(x)\ is\ an\ increasing\ function$$

$$and\ f\left(\frac{2}{3\sqrt{3}}\right)=144\left(\frac{2}{3\sqrt{3}}\right)^2-59\sqrt{3}.\frac{2}{3\sqrt{3}}+18=\frac{64}{3}-\frac{118}{3}+18=0$$

$$We\ can\ say\ f(x)\geq f\left(\frac{2}{3\sqrt{3}}\right)\ or, f(x)\geq 0\ or,$$

$$144x^2-59\sqrt{3}x+18\geq 0$$

Equality holds for $A=B=C$.

2184. In ΔABC the following relationship holds:

$$r_a sec\frac{A}{2}+r_b sec\frac{B}{2}+r_c sec\frac{C}{2}\geq 6\sqrt{3}r$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$In\quad \Delta ABC\quad wlog\quad a\geq b\geq c\geq\ \rightarrow\ \frac{A}{2}\geq\frac{B}{2}\geq\frac{C}{2}\ \rightarrow cos\frac{A}{2}\leq cos\frac{B}{2}\leq cos\frac{C}{2}$$

$$\boxed{sec\frac{A}{2} \geq sec\frac{B}{2} \geq sec\frac{C}{2}} \quad (1)$$

$$and\ \ a \geq b \geq c \quad \boxed{r_a \geq r_b \geq r_c} \quad (2)$$

Then, according to Chebyshev's theorem:

$$r_a sec\frac{A}{2} + r_b sec\frac{B}{2} + r_c sec\frac{C}{2} \geq \frac{1}{3}(r_a + r_b + r_c)\left(sec\frac{A}{2} + sec\frac{B}{2} + sec\frac{C}{2}\right) =$$

$$= \frac{1}{3}s\left(tan\frac{A}{2} + tan\frac{B}{2} + tan\frac{C}{2}\right)\left(sec\frac{A}{2} + sec\frac{B}{2} + sec\frac{C}{2}\right) \overset{A-G}{\geq}$$

$$\geq \frac{1}{3}s \cdot 3\left(\prod_{cyc} tan\frac{A}{2}\right)^{\frac{1}{3}} \cdot 3\left(\prod_{cyc}\frac{1}{cos\frac{A}{2}}\right)^{\frac{1}{3}} = 3s \cdot \left(\prod_{cyc}\frac{1}{cos\frac{A}{2}} \cdot \prod_{cyc} tan\frac{A}{2}\right)^{\frac{1}{3}} =$$

$$= 3s \cdot \left(\frac{r}{s} \cdot \frac{1}{\frac{s}{4R}}\right)^{\frac{1}{3}} = 3s\left(\frac{4Rr}{s^2}\right)^{\frac{1}{3}} = 3\left(\frac{4Rr.s^3}{s^2}\right)^{\frac{1}{3}} = 3(4Rrs)^{\frac{1}{3}} \overset{Euler \& Mitrinovici}{\geq}$$

$$\geq 3\left(4 \cdot 2r \cdot r.3\sqrt{3} \cdot r\right)^{\frac{1}{3}} = 3 \cdot 2r \cdot \left(3\sqrt{3}\right)^{\frac{1}{3}} = 6\sqrt{3}r$$

$$r_a sec\frac{A}{2} + r_b sec\frac{B}{2} + r_c sec\frac{C}{2} \geq 6\sqrt{3}r \ \ (Proved)$$

2185. In ΔABC the following relationship holds:

$$\cos A + \cos B + \cos C + 2\left(\sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2}\right) \geq \frac{19}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ x = \cos A, y = \cos B, z = \cos C$$

$$\cos A + \cos B + \cos C = x + y + z = 1 + \frac{r}{R} \overset{Euler}{\leq} \frac{3}{2}$$

$$\cos A + \cos B + \cos C + 2\left(\sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2}\right) = \sum\left(\cos A + 2\sec^2\frac{A}{2}\right)$$

$$= \sum\left(\cos A + \frac{4}{2\cos^2\frac{A}{2}}\right) = \sum\left(\cos A + \frac{4}{1 + \cos A}\right) = \sum\left(x + \frac{4}{1 + x}\right)$$

$$Lemma: x + \frac{4}{1+x} \geq \frac{(32 - 7x)}{9} \ \forall x \in \left(0, \frac{3}{2}\right)$$

Proof:

$$9x + 9x^2 + 36 \geq 32 + 32x - 7x - 7x^2$$

$$16x^2 - 16x + 4 \geq 0\ or, (4x - 2)^2 \geq 0\ true$$

$$\cos A+\cos B+\cos C+2\left(\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\right)=\sum\left(\cos A+2\sec^2\frac{A}{2}\right)=$$

$$=\sum\left(\cos A+\frac{4}{2\cos^2\frac{A}{2}}\right)=\sum\left(\cos A+\frac{4}{1+\cos A}\right)=\sum\left(x+\frac{4}{1+x}\right)\ge\sum\frac{32-7x}{9}=$$

$$=\frac{96-7(x+y+z)}{9}\ge\frac{96-\frac{7.3}{2}}{9}=\frac{192-21}{18}=\frac{171}{18}=\frac{19}{2}$$

$$\textit{Equality holds for } x=y=z=\frac{1}{2} \textit{ or } A=B=C=\frac{\pi}{3}.$$

2186. In any Δ ABC, the following relationship holds :

$$\frac{w_a w_b}{\sqrt{r_c}}+\frac{w_b w_c}{\sqrt{r_a}}+\frac{w_c w_a}{\sqrt{r_b}}\ge 4\sqrt{2}r(r+4R)^2.\sqrt{\frac{R}{128R^4+32R^3r-117R^2r^2-52Rr^3-4r^4}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{w_a w_b}{\sqrt{r_c}}+\frac{w_b w_c}{\sqrt{r_a}}+\frac{w_c w_a}{\sqrt{r_b}}=w_a w_b w_c.\sum_{cyc}\frac{\sqrt{s-a}.(b+c)}{\sqrt{rs}.2\sqrt{bc}.\sqrt{s(s-a)}}=$$

$$\frac{8Rr^2s}{s^2+2Rr+r^2}.\frac{1}{\sqrt{4Rr^2s}}.\sum_{cyc}\frac{(b+c)\sqrt{a}.a.\sqrt{b+c}}{\sqrt{a^2(b+c)}}=\frac{4r.\sqrt{Rs}}{s^2+2Rr+r^2}.\sum_{cyc}\frac{\left(a(b+c)\right)^{\frac{3}{2}}}{\sqrt{a^2(b+c)}}$$

$$\overset{\text{Radon}}{\ge}\frac{4r.\sqrt{Rs}}{s^2+2Rr+r^2}.\frac{\left(2(s^2+4Rr+r^2)\right)^{\frac{3}{2}}}{\sqrt{2s(s^2+4Rr+r^2)-12Rrs}}\overset{?}{\ge}$$

$$4\sqrt{2}r(r+4R)^2.\sqrt{\frac{R}{128R^4+32R^3r-117R^2r^2-52Rr^3-4r^4}}$$

$$\Leftrightarrow\frac{8Rs(s^2+4Rr+r^2)^3}{(s^2+2Rr+r^2)^2.2s(s^2-2Rr+r^2)}\overset{?}{\ge}\frac{2R(r+4R)^4}{128R^4+32R^3r-117R^2r^2-52Rr^3-4r^4}$$

$$\Leftrightarrow-(192R^3+330R^2r+120Rr^2+9r^3)s^6+$$

$$(2560R^5+256R^4r-3576R^3r^2-2270R^2r^3-458Rr^4-27r^5)s^4$$

$$+r\left(\begin{matrix}13312R^6+9216R^5r-10336R^4r^2-11504R^3r^3-3930R^2r^4\\-556Rr^5-27r^6\end{matrix}\right)s^2+$$

$$r^2\left(\begin{matrix}18432R^7+19456R^6r-7552R^5r^2-17120R^4r^3-8624R^3r^4\\-1990R^2r^5-218Rr^6-9r^7\end{matrix}\right)\underset{①}{\overset{?}{\ge}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$, where $m=$

$2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(*)}{\le}0$$

$$\Rightarrow -(192R^3+330R^2r+120Rr^2+9r^3)s^2\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\+r(4R+r)^3\end{pmatrix}\ge 0$$

and so, in order to prove ①, it suffices to prove : LHS of ① $\ge$ $-(192R^3+330R^2r+120Rr^2+9r^3)s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$

$$\Leftrightarrow (1792R^5-4904R^4r-10272R^3r^2-4046R^2r^3-398Rr^4-9r^5)s^4+$$
$$r\begin{pmatrix}25600R^6+39552R^5r+15488R^4r^2-1016R^3r^3-1728R^2r^4\\-328Rr^5-18r^6\end{pmatrix}s^2+$$
$$r^2\left((R-2r)\begin{pmatrix}18432R^6+56320R^5r+105088R^4r^2+193056R^3r^3\\+377488R^2r^4+752986Rr^5+1505754r^6\end{pmatrix}\right.$$
$$\left.+3011499r^7\right)\overset{②}{\ge}0$$

and it's trivi*ally* true if :

$$1792R^5-4904R^4r-10272R^3r^2-4046R^2r^3-398Rr^4-9r^5\ge 0\ \left(\because R\overset{Euler}{\ge}2r\right)$$

and so, we now focus on the *case* when :

$1792R^5-4904R^4r-10272R^3r^2-4046R^2r^3-398Rr^4-9r^5<0$ *and* then :

$$\begin{pmatrix}1792R^5-4904R^4r-10272R^3r^2\\-4046R^2r^3-398Rr^4-9r^5\end{pmatrix}\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\+r(4R+r)^3\end{pmatrix}\overset{via\ (*)}{\ge}0$$

$\therefore$ in order to prove ②, it suffices to prove : LHS of ② $\ge 0\Leftrightarrow$

$$(896R^6+5228R^5r-12900R^4r^2-24541R^3r^3-7873R^2r^4-204Rr^5+36r^6)s^2$$
$$\overset{③}{\ge}r\begin{pmatrix}14336R^7-30784R^6r-111344R^5r^2-100188R^4r^3-41341R^3r^4\\-8735R^2r^5-908Rr^6-36r^7\end{pmatrix}$$

Case 1 $896R^6+5228R^5r-12900R^4r^2-24541R^3r^3-7873R^2r^4-204Rr^5$ $+36r^6\ge 0$ *and* then : LHS of ③ $\overset{Gerretsen}{\ge}$

$$\begin{pmatrix}896R^6+5228R^5r-12900R^4r^2-24541R^3r^3-7873R^2r^4\\-204Rr^5+36r^6\end{pmatrix}(16Rr-5r^2)$$

$$\overset{?}{\ge}\text{RHS of ③}\Leftrightarrow 54976t^6-60598t^5-113984t^4+19039t^3+22418t^2$$
$$+1252t-72\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\begin{pmatrix}(t-2)(54976t^4+159306t^3+303336t^2+595159t+1189710)\\+2379456\end{pmatrix}$$

$\overset{?}{\ge}0\to$ true $\because t\overset{Euler}{\ge}2\Rightarrow$ ③ is true

Case 2 $896R^6+5228R^5r-12900R^4r^2-24541R^3r^3-7873R^2r^4$ $-204Rr^5+36r^6<0$ *and* then : LHS of ③ $\overset{Gerretsen}{\ge}$

$$\begin{pmatrix}896R^6+5228R^5r-12900R^4r^2-24541R^3r^3-7873R^2r^4\\-204Rr^5+36r^6\end{pmatrix}(4R^2+4Rr+3r^2)$$

$$\overset{?}{\ge}\text{RHS of ③}\Leftrightarrow 1792t^8+5080t^7+1392t^6-11368t^5-34084t^4-32295t^3$$

$$-7778t^2+220t+72\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(\begin{matrix}1792t^7+8664t^6+18720t^5+26072t^4+18060t^3+3825t^2\\-128t-36\end{matrix}\right)\overset{?}{\geq}0$$

$\rightarrow$ true $\because t\overset{Euler}{\geq}2\Rightarrow$ ③ is true $\therefore$ combining both cases, ③ $\Rightarrow$ ② $\Rightarrow$ ① is true

$$\forall\,\Delta\,ABC\therefore\frac{w_aw_b}{\sqrt{r_c}}+\frac{w_bw_c}{\sqrt{r_a}}+\frac{w_cw_a}{\sqrt{r_b}}\geq$$

$$4\sqrt{2}r(r+4R)^2.\sqrt{\frac{R}{128R^4+32R^3r-117R^2r^2-52Rr^3-4r^4}}\ \forall\,\Delta\,ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)

2187. In any Δ ABC, the following relationship holds :

$$\frac{h_a}{w_a\sqrt{r_a}}+\frac{h_b}{w_b\sqrt{r_b}}+\frac{h_c}{w_c\sqrt{r_c}}\geq 4\sqrt{\frac{2R-r}{R(5R-2r)}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{w_a\sqrt{r_a}}=\frac{bc.\,4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2R.\,2bc\cos\frac{A}{2}}.\frac{\sqrt{s-a}}{\sqrt{rs}}=\frac{b+c}{a}\sin\frac{A}{2}.\frac{\sqrt{s-a}}{\sqrt{rs}}$$

$$=\frac{b+c}{abc}.\frac{bc}{\sqrt{bc}}.\sqrt{\frac{(s-a)(s-b)(s-c)}{rs}}=\frac{b+c}{4Rrs}.\sqrt{bc}.\sqrt{r}$$

$$\Rightarrow\frac{h_a}{w_a\sqrt{r_a}}=\frac{\sqrt{r}}{4Rrs}.\sqrt{bc}(b+c)\text{ and analogs}\Rightarrow\left(\sum_{cyc}\frac{h_a}{w_a\sqrt{r_a}}\right)^2=$$

$$\frac{1}{16R^2rs^2}.\left(\sum_{cyc}\left(bc(b^2+c^2+a^2+2bc-a^2)\right)+2.\sum_{cyc}\left(\sqrt{bc}.\sqrt{ca}.(b+c)(c+a)\right)\right)$$

$$\overset{GM-HM}{\geq}\frac{1}{16R^2rs^2}.\left(\begin{matrix}2(s^2-4Rr-r^2)(s^2+4Rr+r^2)+2(s^2+4Rr+r^2)^2-32Rrs^2\\-8Rrs^2+8\sum_{cyc}\left(\frac{bc}{b+c}.\frac{ca}{c+a}.(b+c)(c+a)\right)\end{matrix}\right)$$

$$=\frac{1}{16R^2rs^2}.\left(\begin{matrix}2(s^2-4Rr-r^2)(s^2+4Rr+r^2)+2(s^2+4Rr+r^2)^2-32Rrs^2\\-8Rrs^2+8(4Rrs)(2s)\end{matrix}\right)$$

$$=\frac{s^2+10Rr+r^2}{4R^2r}\overset{Gerretsen}{\geq}\frac{13R-2r}{2R^2}\overset{?}{\geq}16.\frac{2R-r}{R(5R-2r)}\Leftrightarrow R^2-4Rr+4r^2\overset{?}{\geq}0$$

$$\Leftrightarrow (R-2r)^2 \overset{?}{\geq} 0 \to \text{true} \therefore \frac{h_a}{w_a\sqrt{r_a}}+\frac{h_b}{w_b\sqrt{r_b}}+\frac{h_c}{w_c\sqrt{r_c}} \geq 4.\sqrt{\frac{2R-r}{R(5R-2r)}}\ \forall\, \Delta\, ABC,$$

$'' =''$ iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{h_a}{w_a\sqrt{r_a}}=\sum_{cyc}\frac{(b+c)\sqrt{r}}{a\sqrt{bc}}=\sqrt{\frac{r}{abc}}\cdot\sum_{cyc}\frac{b+c}{\sqrt{a}} \overset{H\ddot{o}lder}{\geq} \sqrt{\frac{1}{4Rs}\cdot\frac{\left(\sum_{cyc}(b+c)\right)^3}{\sum_{cyc}a(b+c)}}=$$

$$=\frac{4s}{\sqrt{2R(s^2+r^2+4Rr)}} \overset{?}{\geq} 4\sqrt{\frac{2R-r}{R(5R-2r)}} \Leftrightarrow s^2 \geq \frac{2r(2R-r)(4R+r)}{R},$$

which is Gerretsen – Blundon inequality.

So the proof is complete. Equality holds iff ΔABC is equilateral.

2188. In ΔABC the following relationship holds:

$$\frac{R}{r} \geq 2+\frac{(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)}{4abc}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$\frac{(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)}{4abc}=2s\cdot\frac{2(a^2+b^2+c^2-ab-bc-ca)}{4.4Rrs}=$$

$$=\frac{1}{4Rr}(s^2-3r^2-12Rr) \overset{Gerretsen}{\leq} \frac{1}{4Rr}(4R^2+4Rr+3r^2-3r^2-12Rr)=$$

$$=\frac{1}{4Rr}(4R^2-8Rr)=\frac{R}{r}-2$$

$$2+\frac{(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)}{4abc} \leq 2+\frac{R}{r}-2=\frac{R}{r}$$

Equality holds for an equilateral triangle.

2189. In ΔABC the following relationship holds:

$$(i)\ \sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{R^4}{r^4}\geq 16+\sum\frac{m_a}{m_b+m_c}$$

$$(ii)\ \sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{R^5}{r^5}\geq 32+\sum\frac{w_a}{w_b+w_c}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$(i)\ \sum\frac{m_a}{m_b+m_c}\overset{AM-GM}{\leq}\sum\frac{m_a}{2\sqrt{m_bm_c}}\overset{CBS}{\leq}\frac{1}{2}\sqrt{\left(\sum m_a^2\right)\left(\sum\frac{1}{m_bm_c}\right)}\overset{Leibniz}{\leq}$$

$$\leq\frac{1}{2}\sqrt{\left(\frac{3}{4}9R^2\right)\left(\frac{m_a+m_b+m_c}{m_am_bm_c}\right)}\overset{Leunberger\ \&\ m_a\geq\sqrt{s(s-a)}}{\leq}$$

$$\leq\frac{3R}{4}\sqrt{\frac{3(4R+r)}{s^2r}}\overset{Doucet}{\leq}\frac{3R}{4}\sqrt{\frac{3(4R+r)}{3r(4R+r)r}}=\frac{3R}{4r}\ and\ \sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}\overset{Nestbitt}{\geq}\frac{3}{2}$$

$$We\ need\ to\ show:\ \sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{R^4}{r^4}\geq 16+\sum\frac{m_a}{m_b+m_c}$$

$$\frac{3}{2}+\frac{R^4}{r^4}\geq 16+\frac{3R}{4r}$$

$$4R^4-3Rr^3-58r^4\geq 0$$

$$(R-2r)(4R^3+8R^2r+16Rr^2+29r^3)\geq 0\ true\ (Euler)$$

$$(ii)\ \sum\frac{w_a}{w_b+w_c}\overset{AM-HM}{\leq}\frac{1}{4}\sum\left(\frac{w_a}{w_b}+\frac{w_a}{w_c}\right)\overset{CBS}{\leq}\frac{1}{4}.2\sqrt{\left(\sum w_a^2\right)\left(\sum\frac{1}{w_a^2}\right)}\overset{h_a\leq w_a\leq\sqrt{s(s-a)}}{\leq}$$

$$\leq\frac{1}{2}\sqrt{\left(\sum s(s-a)\right)\left(\sum\frac{1}{h_a^2}\right)}=\frac{1}{4r}\sqrt{a^2+b^2+c^2}\overset{Leibniz}{\leq}\frac{3R}{2r}$$

$$We\ need\ to\ show:$$

$$\sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{R^5}{r^5}\geq 32+\sum\frac{w_a}{w_b+w_c}$$

$$\frac{3}{2}+\frac{R^5}{r^5}\geq 32+\frac{3R}{4r}$$

$$4R^5-3Rr^4-122r^5\geq 0$$

$$(R-2r)(4R^4+8R^3r+16R^2r^2+32Rr^3+61r^4)\geq 0\ true$$

Equality holds for an equilateral triangle.

2190. **In any ΔABC the following relationship holds:**

$$\frac{a^2+b^2+c^2}{2R} \le \frac{s_a}{h_a}m_b + \frac{s_b}{h_b}m_c + \frac{s_c}{h_c}m_a \le \frac{9R}{2}$$

Proposed by Tapas Das-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{s_a}{h_a}m_b + \frac{s_b}{h_b}m_c + \frac{s_c}{h_c}m_a = \sum_{cyc}\frac{s_a}{h_a}m_b = \sum_{cyc}\frac{2bcm_a}{b^2+c^2}\cdot\frac{a}{2F}\cdot m_b = 4R\sum_{cyc}\frac{m_am_b}{b^2+c^2} \ge$$

$$\overset{Tereshin}{\ge} 4R\sum_{cyc}\frac{(b^2+c^2)(c^2+a^2)}{16R^2(b^2+c^2)} = \frac{a^2+b^2+c^2}{2R}$$

$$\bullet \frac{s_a}{h_a}m_b + \frac{s_b}{h_b}m_c + \frac{s_c}{h_c}m_a = 4R\sum_{cyc}\frac{m_am_b}{b^2+c^2} = \frac{R}{\sqrt{a^2+b^2+c^2}}\sum_{cyc}\frac{4\sqrt{a^2+b^2+c^2}m_a}{b^2+c^2}m_b$$

$$\overset{AM-GM}{\le} \frac{R}{\sqrt{a^2+b^2+c^2}}\sum_{cyc}\frac{a^2+b^2+c^2+4{m_a}^2}{b^2+c^2}m_b =$$

$$= \frac{3R}{\sqrt{a^2+b^2+c^2}}\sum_{cyc}m_b \overset{CBS}{\le} \frac{3R\sqrt{3({m_a}^2+{m_a}^2+{m_a}^2)}}{\sqrt{a^2+b^2+c^2}} = \frac{9R}{2}.$$

Equality holds iff ΔABC is equilateral.

2191. **In any ΔABC the following relationship holds:**

$$a(m_bw_b + m_cw_c) + b(m_cw_c + m_aw_a) + c(m_aw_a + m_bw_b) \ge \frac{2s^3}{\sqrt{2+\frac{r}{2R}}}$$

Proposed by Tapas Das-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Using the known inequality $m_a \ge \frac{b+c}{2}\cos\frac{A}{2}$, we have:

$$m_a w_a \geq s(s-a) \ \text{(and analogs), then:}$$

$$\sum_{cyc} a(m_b w_b + m_c w_c) \geq \sum_{cyc} a\big(s(s-b)+s(s-c)\big) = s\sum_{cyc} a^2 = 2s(s^2-4Rr-r^2) =$$

$$= 2s^3\left(1-\frac{4Rr+r^2}{s^2}\right) \overset{Gerretsen}{\geq} 2s^3\left(1-\frac{4Rr+r^2}{16Rr-5r^2}\right) = \frac{2s^3(12R-6r)}{16R-5r} \overset{?}{\geq} \frac{2s^3}{\sqrt{2+\frac{r}{2R}}}$$

$$\Leftrightarrow 6(2R-r)\sqrt{4R+r} \geq (16R-5r)\sqrt{2R} \overset{squaring}{\Leftrightarrow} (R-2r)(64R^2+16Rr-18r^2) \geq 0,$$

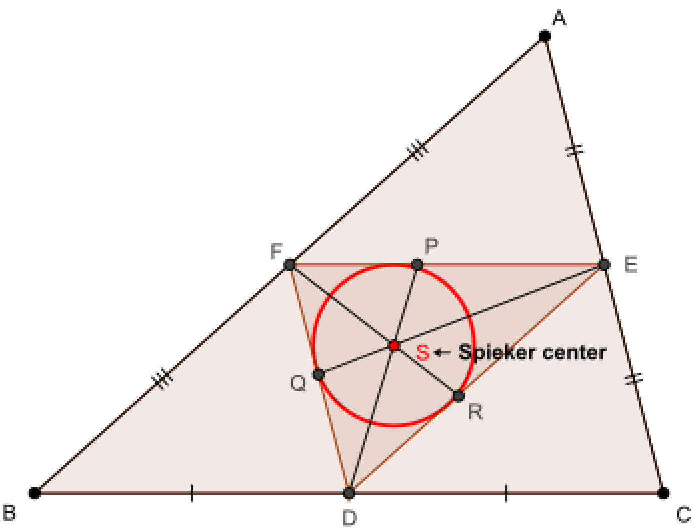
which is true and the proof is complete. Equality holds iff ΔABC is equilateral.

2192. In any Δ ABC the following relationship holds :

$$\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c} \leq 1+\frac{R}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and m(∡BAX) = α and m(∡CAX) = β (say) and inradius of Δ DEF = r′(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$

$$=\frac{\pi}{2}-\frac{A-B}{2}\ and\ m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$

$$=\sum_{\text{cyc}}a^3-abc+2a.2bc\cos A=2s(s^2-6Rr-3r^2)-4Rrs+16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}2s(s^2-8Rr-3r^2+8Rr\cos A)$$

$$=2s\left(s^2-8Rr-3r^2+8Rr\left(1-2\sin^2\frac{A}{2}\right)\right)\therefore(\bullet),(\bullet\bullet)\Rightarrow$$

$$p_a\overset{(\bullet\bullet\bullet)}{=}\frac{2s}{2s+a}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}$$

$$\text{We have : }\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}$$

$$=\sum_{cyc}\left(\frac{2s}{2s+a}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}\cdot\frac{a(b+c).\sqrt{bc}}{2abc.\sqrt{s(s-a)}}\right)$$

$$=\frac{2s}{2s(9s^2+6Rr+r^2).8Rrs}\cdot\sum_{cyc}\left(\frac{(2s+b)(2s+c)a(b+c).\sqrt{bc(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c)}}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}\right)$$

$$=\frac{1}{(9s^2+6Rr+r^2).8Rr^2s^2}\cdot\sum_{cyc}\left(\sqrt{(2s+b)(2s+c)a(b+c)bc(s-b)(s-c).}\sqrt{(2s+b)(2s+c)a(b+c)\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)}\right)$$

$$\overset{CBS}{\leq}\frac{1}{(9s^2+6Rr+r^2).8Rr^2s^2}\cdot\sqrt{x}.\sqrt{y}$$

$$\left(\begin{array}{l}\text{where } x=4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)\text{ and}\\ y=\sum_{cyc}\left((2s+b)(2s+c)a(b+c)\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)\right)\end{array}\right)$$

$$\therefore\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\overset{(1)}{\leq}\frac{1}{(9s^2+6Rr+r^2).8Rr^2s^2}\cdot\sqrt{x}.\sqrt{y}$$

Now, $\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)=\sum_{cyc}\left((8s^2-2sa+bc)(s-b)(s-c)\right)$

$$=r^2s.\sum_{cyc}\left(\frac{2s(s-a)+6s^2+bc}{s-a}\right)=r^2s\left(6s+\frac{6s^2(4Rr+r^2)}{r^2s}+s.\frac{s^2+(4R+r)^2}{s^2}\right)$$

$$\Rightarrow\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)\overset{(\blacksquare)}{=}6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2$$

and also,

$$\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right)=r^2s.\sum_{cyc}\frac{a(2s(s-a)+6s^2+bc)}{s-a}$$

$$=r^2s.\left(2s(2s)+6s^2.\sum_{cyc}\frac{a-s+s}{s-a}+\frac{4Rrs(4Rr+r^2)}{r^2s}\right)$$

$$=r^2s.\left(4s^2+6s^2.\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)+4R(4R+r)\right)$$

$$\Rightarrow\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right)\overset{(\blacksquare\blacksquare)}{=}r^2s\left(-8s^2+\frac{24Rs^2}{r}+16R^2+4Rr\right)$$

$$\textit{and moreover, } 4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)$$

$$=4Rrs.r^2s.\sum_{cyc}\frac{(2s+b)(2s+c)(s+s-a)}{s-a}$$

$$=\frac{4Rrs.r^2s^2}{r^2s}.\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)+$$

$$4Rrs.r^2s.\sum_{cyc}(8s^2-2sa+bc)\overset{via\,(\blacksquare)}{=}4Rrs^2\left(6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2\right)$$

$$+4Rr^3s^2(21s^2+4Rr+r^2)\Rightarrow 4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)$$

$$=x\ \boxed{\overset{(\blacksquare\blacksquare\blacksquare)}{=}}\ 8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right)$$

$$\text{Now, } y=\sum_{cyc}\left((s^2-3r^2)(2s+b)(2s+c)a(b+c)\right)$$

$$-\frac{16Rr}{4Rrs}.\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\left(\sum_{cyc}ab-bc\right)\right)$$

$$=(s^2-3r^2)\sum_{cyc}\left(\left(\sum_{cyc}ab-bc\right)(8s^2-2sa+bc)\right)-$$

$$\frac{4}{s}\left(\begin{matrix}(s^2+4Rr+r^2).\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right)\\ -4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)\end{matrix}\right)$$

$$\overset{via\,(\blacksquare)\,and\,(\blacksquare\blacksquare)}{=}(s^2-3r^2)(s^2+4Rr+r^2)(21s^2+4Rr+r^2)-$$

$$(s^2-3r^2)\left(8s^2(s^2+4Rr+r^2)-24Rrs^2+(s^2+4Rr+r^2)^2-16Rrs^2\right)$$

$$-\frac{4}{s}\left(\begin{matrix}(s^2+4Rr+r^2).r^2s\left(-8s^2+\frac{24Rs^2}{r}+16R^2+4Rr\right)\\ -4Rrs.\left(6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2\right)\end{matrix}\right)$$

$$\Rightarrow y\ \boxed{\overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=}}\ 4s^2\left(3s^4-(2Rr-2r^2)s^2-r^2(16R^2+10Rr+r^2)\right)$$

$$\therefore \textcircled{1},(\blacksquare\blacksquare\blacksquare),(\blacksquare\blacksquare\blacksquare\blacksquare)\Rightarrow\left(\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\right)^2\le$$

$$\frac{8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right).4s^2\left(\begin{matrix}3s^4-(2Rr-2r^2)s^2\\-r^2(16R^2+10Rr+r^2)\end{matrix}\right)}{(9s^2+6Rr+r^2)^2.64R^2r^4s^4}\overset{?}{\le}\frac{(R+r)^2}{r^2}$$

$$\Leftrightarrow -(36R+51r)s^6+(162R^3+324R^2r+154Rr^2-37r^3)s^4$$

$$+r(216R^4+676R^3r+676R^2r^2+208Rr^3+15r^4)s^2$$

$$+r^2(72R^5+296R^4r+298R^3r^2+112R^2r^3+18Rr^4+r^5)\overset{?}{\underset{②}{\geq}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ *and* $s^2-(m+n)\leq 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(a)}{\leq}0$$

$\Rightarrow -(36R+51r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0$ *and* so, in order to prove ②, it suffices to prove : LHS of ② $\geq$

$$-(36R+51r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\Leftrightarrow (18R^3-600R^2r-794Rr^2+65r^3)s^4$$

$$+r(2520R^4+5668R^3r+3556R^2r^2+856Rr^3+66r^4)s^2$$

$$+r^2(72R^5+296R^4r+298R^3r^2+112R^2r^3+18Rr^4+r^5)\overset{③}{\geq}0$$

We note *that* ③ is trivi*ally* true if : $18R^3-600R^2r-794Rr^2+65r^3\geq 0$ *and* so we now focus on the *case* when : $18R^3-600R^2r-794Rr^2+65r^3<0$

and then : $(18R^3-600R^2r-794Rr^2+65r^3)\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\+r(4R+r)^3\end{pmatrix}$

$\overset{\text{via }(a)}{\geq}$ 0 *and* so, in order to prove ③, it suffices to prove :

$$\text{LHS of ③}\geq(18R^3-600R^2r-794Rr^2+65r^3)\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\+r(4R+r)^3\end{pmatrix}$$

$$\Leftrightarrow (9R^5+60R^4r-1193R^3r^2-1358R^2r^3+468Rr^4-8r^5)s^2\overset{④}{\geq}$$

$$r(144R^6-4701R^5r-9962R^4r^2-5179R^3r^3-890R^2r^4-4Rr^5+8r^6)$$

Case 1 $9R^5+60R^4r-1193R^3r^2-1358R^2r^3+468Rr^4-8r^5\geq 0$ *and* then :

$$\text{LHS of ④}\overset{\text{Gerretsen}}{\geq}\begin{pmatrix}9R^5+60R^4r-1193R^3r^2\\-1358R^2r^3+468Rr^4-8r^5\end{pmatrix}(16Rr-5r^2)$$

$$\overset{?}{\geq} r(144R^6-4701R^5r-9962R^4r^2-5179R^3r^3-890R^2r^4-4Rr^5+8r^6)$$

$$\Leftrightarrow 2808t^5-4713t^4-5292t^3+7584t^2-1232t+16\overset{?}{\geq}0\quad\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2808t^4+903t^3-3486t^2+612t-8)\overset{?}{\geq}0\rightarrow \text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow \text{④ is true}$$

Case 2 $9R^5+60R^4r-1193R^3r^2-1358R^2r^3+468Rr^4-8r^5<0$ *and* then :

$$\text{LHS of ④}\overset{\text{Gerretsen}}{\geq}\begin{pmatrix}9R^5+60R^4r-1193R^3r^2\\-1358R^2r^3+468Rr^4-8r^5\end{pmatrix}(4R^2+4Rr+3r^2)$$

$$\overset{?}{\geq} r(144R^6-4701R^5r-9962R^4r^2-5179R^3r^3-890R^2r^4-4Rr^5+8r^6)$$

$$\Leftrightarrow 18t^7+66t^6+98t^5-31t^4-980t^3-672t^2+688t-16\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)\left(18t^6+102t^5+302t^4+573t^3+166t^2-340t+8\right)\overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ④ \text{ is true} \therefore \text{combining both cases}, ④ \Rightarrow ③ \Rightarrow ② \text{ is true } \forall \Delta ABC$$

$$\therefore \frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c} \leq 1+\frac{R}{r} \ \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

2193. **If $x,y,z>0$ then in ΔABC the following relationship holds:**

$$(x^2+y^2+z^2)^2(a^8+2)(b^8+2)(c^8+2) \geq 768(xy+yz+zx)^2F^4$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Tapas Das-India

$$(a^8+2)(b^8+2)(c^8+2) \overset{Holder}{\geq} \left((abc)^{\frac{8}{3}}+2\right)^3 =$$

$$=\left(((abc)^2)^{\frac{4}{3}}+1+1\right)^3 \overset{Carlitz}{\geq} \left(\left(\frac{4F}{\sqrt{3}}\right)^4+1+1\right)^3 \overset{AM-GM}{\geq}$$

$$\geq \left(3\sqrt[3]{\left(\frac{4F}{\sqrt{3}}\right)^4}\right)^3 = 27\left(\frac{4F}{\sqrt{3}}\right)^4 = 3\cdot 256\,F^4 = 768F^4$$

$$(x^2+y^2+z^2)^2(a^8+2)(b^8+2)(c^8+2) \geq (xy+yz+zx)^2.768F^4 =$$

$$=768(xy+yz+zx)^2F^4$$

Equality holds for an equilateral triangle

2194.

Let $x,y>0$ and M an interior point in ΔABC.
d_a,d_b,d_c the distances of
point M to the sides BC,CA,AB respectively. Prove that:

$$\frac{a^3b^4}{xr+yd_a}+\frac{b^3c^4}{xr+yd_b}+\frac{c^3a^4}{xr+yd_c} \geq \frac{128}{x+y}F^3$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Tapas Das-India

$$[ABC]=[MAB]+[MBC]+[MCA]\ or, F=\frac{1}{2}(ad_a+bd_b+cd_c)$$
$$2F=(ad_a+bd_b+cd_c)$$

$$\frac{a^3b^4}{xr+yd_a}+\frac{b^3c^4}{xr+yd_b}+\frac{c^3a^4}{xr+yd_c}=\sum\frac{a^3b^4}{xr+yd_a}=\sum\frac{a^4b^4}{xar+yad_a}=$$

$$=\sum\frac{(a^2b^2)^2}{xar+yad_a}\overset{CBS}{\geq}\frac{(\sum a^2b^2)^2}{xr(a+b+c)+y(ad_a+bd_b+cd_c)}\overset{Goldner\ II}{\geq}$$

$$\geq\frac{(16F^2)^2}{2Fx+2Fy}=\frac{256F^4}{2F(x+y)}=\frac{128}{x+y}F^3$$

Equality holds for an equilateral triangle.

2195. If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\left(\frac{x^2}{yz}a^2b^2+2\right)\left(\frac{y^2}{zx}b^2c^2+2\right)\left(\frac{z^2}{xy}c^2a^2+2\right)\geq 144F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Mihaly Bencze-Romania

Solution by Tapas Das-India

$$\left(\frac{x^2}{yz}a^2b^2+2\right)\left(\frac{y^2}{zx}b^2c^2+2\right)\left(\frac{z^2}{xy}c^2a^2+2\right)\overset{Holder}{\geq}\left((a^4b^4c^4)^{\frac{1}{3}}+(8)^{\frac{1}{3}}\right)^3=$$

$$=\left(((abc)^2)^{\frac{2}{3}}+2\right)^3\overset{Carlitz}{\geq}\left(\frac{16F^2}{3}+1+1\right)^3\overset{AM-GM}{\geq}\left(3\sqrt[3]{\frac{16F^2}{3}}\right)^3=27.\frac{16F^2}{3}$$
$$=144F^2$$

Equality holds for an equilateral triangle

2196.

Let $m\geq 0$, M an interior point in ΔABC with area F and $F_a=[MBC], F_b=[MCA], F_c=[MAB]$. Prove that:

$$\frac{a^{2m+2}}{F_b^m}+\frac{b^{2m+2}}{F_c^m}+\frac{c^{2m+2}}{F_a^m}\ge 2^{2m+2}\left(\sqrt{3}\right)^{m+1}F$$

Proposed by D.M.Bătinețu-Giurgiu, Mihaly Bencze-Romania

Solution by Tapas Das-India

$$[ABC]=[MBC]+[MCA]+[MAB]\ or\ F=F_a+F_b+F_c$$

$$\frac{a^{2m+2}}{F_b^m}+\frac{b^{2m+2}}{F_c^m}+\frac{c^{2m+2}}{F_a^m}=\frac{(a^2)^{m+1}}{F_b^m}+\frac{(b^2)^{m+1}}{F_c^m}+\frac{(c^2)^{m+1}}{F_a^m}\overset{Radon}{\ge}$$

$$\ge\frac{(a^2+b^2+c^2)^{m+1}}{(F_a+F_b+F_c)^m}\overset{Ionescu-Weitzenbock}{\ge}\frac{\left(4\sqrt{3}F\right)^{m+1}}{F^m}=2^{2m+2}\left(\sqrt{3}\right)^{m+1}F$$

Equality holds for an equilateral triangle.

2197. In any Δ ABC, the following relationship holds :

$$\frac{1}{\sqrt{w_a}}+\frac{1}{\sqrt{w_b}}+\frac{1}{\sqrt{w_c}}\le\sqrt{2R}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}w_a=\sum_{cyc}\left(2\sqrt{bc}.\frac{\sqrt{s(s-a)}}{b+c}\right)\overset{CBS}{\le}2\sqrt{\sum_{cyc}bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}}\overset{A-G}{\le}$$

$$\sqrt{\sum_{cyc}bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{bc}}=\sqrt{\sum_{cyc}bc}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}}$$

$$=\sqrt{\sum_{cyc}bc}.\sqrt{\frac{4R+r}{2R}}\Rightarrow\sum_{cyc}w_a\le\sqrt{\sum_{cyc}ab}.\sqrt{\frac{4R+r}{2R}}\to(a)$$

$$\text{Now, }\frac{1}{\sqrt{w_a}}+\frac{1}{\sqrt{w_b}}+\frac{1}{\sqrt{w_c}}=\frac{1}{\sqrt{w_aw_bw_c}}.\sum_{cyc}\sqrt{w_bw_c}\overset{CBS}{\le}$$

$$\frac{\sqrt{s^2+2Rr+r^2}}{\sqrt{16Rr^2s^2}}.\sqrt{\sum_{cyc}w_b}\sqrt{\sum_{cyc}w_c}\overset{via\ (a)}{\le}\frac{\sqrt{s^2+2Rr+r^2}}{\sqrt{16Rr^2s^2}}.\sqrt{\sum_{cyc}ab}.\sqrt{\frac{4R+r}{2R}}$$

$$\overset{?}{\leq}\sqrt{2R}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{\sqrt{2R}}{4Rrs}\left(\sum_{cyc}ab\right)$$

$$\Leftrightarrow 4R(s^2+4Rr+r^2)\overset{?}{\geq}(4R+r)(s^2+2Rr+r^2)\Leftrightarrow 8R^2-2Rr-r^2\overset{?}{\geq}s^2\to \text{true}$$

$$\because 8R^2-2Rr-r^2-s^2=2(R-2r)(2R+r)+4R^2+4Rr+3r^2-s^2\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Gerretsen}}}{\geq}0$$

$$\therefore \frac{1}{\sqrt{w_a}}+\frac{1}{\sqrt{w_b}}+\frac{1}{\sqrt{w_c}}\leq\sqrt{2R}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

2198. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{h_a}{\sqrt{9r^2+3h_a^2}}\leq\frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a}{\sqrt{9r^2+3h_a^2}}=\sum_{cyc}\frac{1}{\sqrt{\frac{9r^2a^2}{4r^2s^2}+3}}=\sum_{cyc}\frac{1}{\sqrt{\left(\frac{3a}{2s}\right)^2+3}}$$

$$\therefore \sum_{cyc}\frac{h_a}{\sqrt{9r^2+3h_a^2}}=\sum_{cyc}\frac{1}{\sqrt{x^2+3}}\ \left(x=\frac{3a}{2s},y=\frac{3b}{2s},z=\frac{3c}{2s}\right)\to (1)$$

$$\text{Now, }\frac{1}{\sqrt{x^2+3}}\overset{?}{\leq}\frac{5-x}{8}\Leftrightarrow(x^2+3)(5-x)^2\overset{?}{\geq}64\ \left(\because x=\frac{3a}{2s}<\frac{3}{2}<5\right)$$

$$\Leftrightarrow x^4-10x^3+28x^2-30x+11\overset{?}{\geq}0\Leftrightarrow\frac{(x-1)^2}{4}.\big((2x-3)(2x-13)+5\big)\overset{?}{\geq}0$$

$$\to \text{true} \because x<\frac{3}{2}\Rightarrow(2x-3),(2x-13)<0\Rightarrow(2x-3)(2x-13)>0$$

$$\therefore \frac{1}{\sqrt{x^2+3}}\leq\frac{5-x}{8}\text{ and analogs}\Rightarrow\sum_{cyc}\frac{1}{\sqrt{x^2+3}}\leq\frac{1}{8}\left(15-\sum_{cyc}\frac{3a}{2s}\right)=\frac{12}{8}=\frac{3}{2}$$

$$\overset{\text{via }(1)}{\Rightarrow}\sum_{cyc}\frac{h_a}{\sqrt{9r^2+3h_a^2}}\leq\frac{3}{2}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

2199. **In acute ΔABC the following relationship holds:**

$$\frac{6}{\pi}<\frac{1}{f(A)}+\frac{1}{f(B)}+\frac{1}{f(C)}<3,\qquad \int_0^{\frac{\pi}{2}}\frac{dx}{1+\cos\alpha\cos x}=f(\alpha)$$

Proposed by Tapas Das-India

Solution by Kartick Chandra Betal-India

$$f(\alpha)=\int_0^{\frac{\pi}{2}}\frac{dx}{1+\cos\alpha\cos x}=2\int_0^{\frac{\pi}{4}}\frac{dx}{1+\cos\alpha\left\{\frac{1-\tan^2 x}{1+\tan^2 x}\right\}}=$$

$$=2\int_0^{\frac{\pi}{4}}\frac{\sec^2 x\,dx}{1+\cos\alpha+(1-\cos\alpha)\tan^2 x}=$$

$$=\frac{2}{\sqrt{1-\cos^2\alpha}}\left(\tan^{-1}\left(\tan\frac{\alpha}{2}\tan x\right)\right)_0^{\frac{\pi}{4}}=\frac{\alpha}{\sin\alpha}$$

$$\frac{1}{f(A)}+\frac{1}{f(B)}+\frac{1}{f(C)}=\frac{\sin A}{A}+\frac{\sin B}{B}+\frac{\sin C}{C}$$

$$\frac{2}{\pi}<\sin x<1\ for\ 0<x<\frac{\pi}{2}$$

$$\frac{6}{\pi}<\frac{1}{f(A)}+\frac{1}{f(B)}+\frac{1}{f(C)}<3$$

2200. **In ΔABC the following relationship holds:**

$$\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\le\frac{3}{2}+\frac{3R}{4r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\frac{w_a}{h_a}=\frac{2\sqrt{bc.s(s-a)}}{b+c}\frac{2R}{bc}=\frac{4R\sqrt{s(s-a)}}{\sqrt{bc}(2s-a)}=\frac{4R\sqrt{s(s-a)}}{\sqrt{bc}(2(s-a)+a)}\overset{AM-GM}{\le}$$

$$\le\frac{4R\sqrt{s(s-a)}}{\sqrt{bc}2\sqrt{2(s-a).a}}=\frac{2R\sqrt{s}}{\sqrt{2abc}}=\frac{2R\sqrt{s}}{\sqrt{8Rrs}}=\sqrt{\frac{R}{2r}}\ (1)$$

$$\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\overset{(1)}{\leq} 3\sqrt{\frac{R}{2r}}= 3\sqrt{1.\frac{R}{2r}}\overset{AM-GM}{\leq} 3.\frac{1+\frac{R}{2r}}{2}=\frac{3}{2}+\frac{3R}{4r}$$

Equality holds for a = b = c.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru