ROMANIAN MATHEMATICAL MAGAZINE CHALLENGES 1-500

CONTENT

PROBLEMS FOR JUNIORS

PROBLEM 1.001.

Let a, b, c, d be non-negative real numbers such that: $a+b+c+d=4$. Prove that:

$$2+\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d} \geq ab+ac+ad+bc+bd+cd$$

Proposed by Hung Nguyen Viet – Hanoi – Vietnam

PROBLEM 1.002.

Determine all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x+a-1)-x|f(x+a-1)| \leq x \leq f(x)-(x-a+1)|f(x)|+a-1$$

for all $x \in \mathbb{R}$, when $a \in \mathbb{R}$.

Proposed by Mihály Bencze – Romania

PROBLEM 1.003.

If $a, b > 0$ then:

$$4\sqrt{a^4+a^2b^2+b^4}+(a^2+b^2)\sqrt{3} \geq 2a\sqrt{2a^2+ab}+2b\sqrt{2b^2+ab}+$$
$$+a\sqrt{2a^2+b^2}+b\sqrt{a^2+2b^2}$$

Proposed by Mihály Bencze – Romania

PROBLEM 1.004.

Let be $n \in \mathbb{N}^* \setminus \{1\}$ și $a_k \in \mathbb{R}$; $k \in \overline{1,n}$. Prove that:

$$\sum_{k=1}^{n} \sqrt{a_k^2 - a_k a_{k+1} + a_{k+1}^2} \geq \sum_{k=1}^{n} a_k \,;\, a_{n+1} = a_1$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 1.005.

Prove that if $a, b, x, y, z \in (0, \infty)$ then:

$$\frac{yz(a^2y+b^2z)}{x}+\frac{zx(a^2z+b^2x)}{y}+\frac{xy(a^2x+b^2y)}{z} \geq \frac{2}{3}ab(x+y+z)^2$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 1.006.

Prove that if $a, b, c \in \mathbb{R}$, $a+b+c=2$ then:

$$2(a^4+b^4+c^4)+10(a^2+b^2+c^2) > 5(a^3+b^3+c^3)+1$$

Proposed by Daniel Sitaru-Romania

PROBLEM 1.007.

Prove that if: $a, b, c > 0$; $a+b+c=3$ then:

$$\sum a\left(\frac{1}{b^3}+\frac{1}{c^3}\right) \geq \frac{18}{a^3+b^3+c^3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.008.

If a, b, c are the length's sides in any triangle the following relationship doesn't holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{2}{3}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$

Proposed by Redwane El Mellas – Morroco

PROBLEM 1.009.

Prove that if $a,b,c\in\mathbb{R}; 0<c\le b\le a$ then:

$$(a+2b)(a+2c)(b+2c)\le 8\prod\left(\frac{a^2+ab+b^2}{a+b}\right)\le(2a+b)(2a+c)(2b+c)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.010.

Prove that:

$$\tan 78^\circ=\frac{4\cos 12^\circ+4\cos 36^\circ+1}{\sqrt{3}}$$

Proposed by Kevin Soto Palacios – Huarmey - Peru

PROBLEM 1.011.

If a,b,c are the length sides in any triangle ABC then:

$$\frac{a}{\sqrt{s-a}}+\frac{b}{\sqrt{s-b}}+\frac{c}{\sqrt{s-c}}\ge 3\sqrt{s}$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 1.012.

Prove that if: $a,b,c,d>0$ then: $a^2+b^2+c^2+d^2=1; abc+bcd+cda+dab=\frac{1}{2}$

$$\sum\frac{a^2}{1+2bcd}\ge\frac{4}{5}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.013.

Prove that if $a>0, a\ne 1$, then it does exists an infinity of pairs of numbers real strictly positive (x,y) such that:

a. $\log_a(x+y)=\log_a x+\log_a y$.

b. $\log_a(x+y)=(\log_a x)\cdot(\log_a y)$.

Proposed by Dana Heuberger – Romania

PROBLEM 1.014.

Let a,b,c be a non-negative real numbers such that: $a+b+c=3$. Prove that:

$$11+\frac{2}{3}\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)\ge 13abc$$

Proposed by Hung Nguyen Viet – Hanoi – Vietnam

PROBLEM 1.015.

Prove that if $a,b,c\in(0,\infty); \sqrt{a}+\sqrt{b}+\sqrt{c}=3$ then:

$$\frac{a\sqrt{b}+b\sqrt{a}}{a-\sqrt{ab}+b}+\frac{b\sqrt{c}+c\sqrt{b}}{b-\sqrt{bc}+c}+\frac{c\sqrt{a}+a\sqrt{c}}{c-\sqrt{ca}+a}\le 6$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.016.

Find all triplets (m,n,p) where m,n are two natural numbers and p is a prime number, satisfying the equation:

$$m^4=4(p^n-1)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.017.

Prove the following inequality holds for all positive real numbers a,b,c

$$a^2+b^2+c^2\geq\frac{1}{2}(ab+bc+ca)+\sqrt{\frac{2(a+b+c)(a^3b^3+b^3c^3+c^3a^3)}{(a+b)(b+c)(c+a)}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.018.

Let ABC be a triangle with the known normal notations. Prove that for any point P moving on the incircle,

$$5r\leq\frac{PA^2}{h_a}+\frac{PB^2}{h_b}+\frac{PC^2}{h_c}\leq\frac{5}{2}R$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.019.

If $a,b,c>0$ and $x,y,z\geq 1$ then:

$$x^{\frac{4a^3}{a^2+bc}}y^{\frac{4b^3}{b^2+ca}}z^{\frac{4c^3}{c^2+ab}}\geq\left(\frac{x^4}{yz}\right)^a\left(\frac{y^4}{zx}\right)^b\left(\frac{z^4}{xy}\right)^c$$

Proposed by Mihály Bencze – Romania

PROBLEM 1.020.

Given $x_1,x_2,\dots,x_n$ be positive real numbers such that: $\sum_{k=1}^{n}x_k=n$.
If $\alpha,\beta>0,4\alpha(n-1)\left(2\alpha n\sqrt{n}+\beta\right)>\beta^2\sqrt{n}$ then:

$$\alpha\sum_{k=1}^{n}\frac{1}{a_k}+\frac{\beta}{\sqrt{\sum_{k=1}^{n}a_k^2}}\geq n\alpha+\frac{\beta}{\sqrt{n}}$$

Proposed by Ngo Minh Ngoc Bao – Vietnam

PROBLEM 1.021.

Prove that if $x,y,z>0,xyz=8$ then:

$$x^3+y^3+z^3\geq 2x\sqrt{y+z}+2y\sqrt{z+x}+2z\sqrt{x+y}$$

Proposed by Iuliana Trașcă – Romania

PROBLEM 1.022.

Let ABC be an acute triangle with the orthocenter H, inradius r, and circumradius R. Prove that:

$$\frac{HA}{\sqrt{bc}}+\frac{HB}{\sqrt{ca}}+\frac{HC}{\sqrt{ab}}\leq\sqrt{2\left(1+\frac{r}{R}\right)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.023.

Prove that for all positive real numbers a,b,c,d:

$$\frac{a}{bc}+\frac{b}{cd}+\frac{c}{da}+\frac{d}{ab}\geq\frac{8}{\sqrt{a^2+b^2+c^2+d^2}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.024.

Given a triangle ABC and let P be any point in its plane. Prove that:

$$\frac{PB\cdot PC}{bc}+\frac{PC\cdot PA}{ca}+\frac{PA\cdot PB}{ab}\le\frac{1}{4}\left(\frac{PA}{h_a}+\frac{PB}{h_b}+\frac{PC}{h_c}\right)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.025.

Let $n\ge 2$ be an integer and let a,b,c be positive numbers such that $ab+bc+ca\le 1$. Prove that:

$$\frac{bc}{(2a^2+bc)^n}+\frac{ca}{(2b^2+ca)^n}+\frac{ab}{(2c^2+ab)^n}\ge 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.026.

Let a,b,c be non-negative real numbers and let x,y,z be real numbers different from 0,

such that $by+cz=x,\quad cz+ax=y, ax+by=z$. Prove that:

$$a.\ abc\le\frac{1}{8}\quad b.\ \frac{1}{2+a+b}+\frac{1}{2+b+c}+\frac{1}{2+c+a}\le 1\quad c.\ a+b+c\ge 2(ab+bc+ca)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.027.

Find all real numbers x satisfying the following equation:

$$(x+\{x\})^2-(x+\{x\})=6\lfloor x\rfloor\{x\}-1$$

where $\lfloor x\rfloor$ and $\{x\}$ denote the integer part and fractional part of x, respectively.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.028.

Prove that in any triangle ABC:

$$\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\ge\sqrt{1+\frac{4R}{r}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.029.

In acute angled $\Delta\ ABC$; $L-$ Nagel's point,$M,M'\in(AB)$; $N'\in(AC)$; (M,L,N); (M',L,N') - collinear points. Prove that:

$$(a+c-b)\left(\frac{MB}{MA}+\frac{M'B}{M'A}\right)+(a+b-c)\left(\frac{NC}{NA}+\frac{N'C}{N'A}\right)>b+c-a$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.030.

I$x_k\in[0,1]$ $(k=1,2,\dots,n)$ then:

$$3\sum_{k=1}^{n}x_k^2\le 2n+x_1x_2x_3+x_2x_3x_4+\dots+x_nx_1x_2$$

Proposed by Mihály Bencze – Romania

PROBLEM 1.031.

Let a, b, c be non-negative real numbers. Prove that:

$$9(a+b+c) \geq \sqrt[4]{\frac{a^4+b^4+c^4}{3}} + 26\sqrt{\frac{ab+bc+ca}{3}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.032.

Prove the following inequality holds for all non-negative real numbers a, b:

$$\frac{1}{4a+1} + \frac{1}{4b+1} + \frac{6}{2a+2b+1} \geq \frac{4}{3a+b+1} + \frac{4}{3b+a+1}$$

Proposed by Nguyen Viet Hung- Hanoi – Vietnam

PROBLEM 1.033.

Let a, b, c be positive real numbers such that: $a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} + 2abc = 1$.Prove that:

$$\frac{\sqrt{bc}}{a} + \frac{\sqrt{ca}}{b} + \frac{\sqrt{ab}}{c} \geq 2(a+b+c)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.034.

Find all pairs (x, y) of integers satisfying the equation

$$x^4 - (y+2)x^3 + (y-1)x^2 + (y^2+2)x + y = 2.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.035.

Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that:

$$5 + \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + abc \geq 3(ab+bc+ca)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.036.

Prove the following inequality:

$$[(x+y)(y+z)(z+x)]^4 \geq \frac{16^3}{27}(x+y+z)^3x^3y^3z^3$$

where x, y, z are positive real numbers.

Proposed by Andrei Bogdan Ungureanu – Romania

PROBLEM 1.037.

Let x, y, z be positive real numbers such that: $16(a^2+b^2+c^2) + 27 = 128abc$

Find the maximum value of the expression:

$$E = \frac{1}{a^3+b^3+\frac{27}{64}} + \frac{1}{b^3+c^3+\frac{27}{64}} + \frac{1}{c^3+a^3+\frac{27}{64}}$$

Proposed by Iuliana Trașcă; Neculai Stanciu – Romania

PROBLEM 1.038.

Let $a, b, c > 0$, prove that:

$$6\left(\sum ab\right)\left(\sum a^2\right) + 7abc\left(\sum a\right) \geq 23abc\sqrt{3\left(\sum a^2\right)}$$

Proposed by Soumitra Mandal–India

PROBLEM 1.039.

In ABC triangle the following relationship holds:

$$3\left(a^a b^b c^c\right)^{\frac{1}{2s}} \geq \sqrt[9]{4RS}\sum\left(a^a b^b c^c\right)^{\frac{1}{3s}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.040.

Prove that if $a,b,c,d \in (0,\infty)$; $\sqrt{3}(ad-bc)=ac+bd \neq 0$ then:

$$d\left(a+b\sqrt{3}\right)-c\left(b-a\sqrt{3}\right) > 4\sqrt[4]{abcd}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.041.

Prove that in an ABC acute-angled triangle the following relationship holds:

$$\cos\left(\frac{\pi}{4}-A\right)+\cos\left(\frac{\pi}{4}-B\right)+\cos\left(\frac{\pi}{4}-C\right) > \frac{2S}{R^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.042.

Prove that in $\Delta\, ABC$:

$$\sum\frac{a^2(b^2+c^2-a^2)^3}{b^2c^2} \geq 64S^2(1-\cos^2 A-\cos^2 B-\cos^2 C)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.043.

Let a,b,c,d be nonnegative real numbers such as $a+b+c+d=4$. Prove that:

$$a.\ ab+bc+cd+da \leq 4$$

$$b.\ a^2bc+b^2cd+c^2da+d^2ab \leq 4$$

$$c.\ abc+bcd+cda+dab \leq 4$$

$$d.\ ab\sqrt{c}+bc\sqrt{d}+cd\sqrt{a}+da\sqrt{b} \leq 4$$

Proposed by Nguyen Tuan Anh - Vietnam

PROBLEM 1.044.

Let a,b,c,d be nonnegative real numbers such as $a+b+c+d=4$.

$a.\ a\sqrt{bc}+b\sqrt{cd}+c\sqrt{da}+d\sqrt{ab} \leq 4$ $\quad b.\ \sqrt{abc}+\sqrt{bcd}+\sqrt{cda}+\sqrt{dab} \leq 4$

$$c.\ \sqrt[n]{abc}+\sqrt[n]{bcd}+\sqrt[n]{cda}+\sqrt[n]{dab} \leq 4;\ (n \in \mathbb{N})$$

$$d.\ ab\sqrt[n]{c}+bc\sqrt[n]{d}+cd\sqrt[n]{a}+da\sqrt[n]{b} \leq 4;\ (n \in \mathbb{N})$$

Proposed by Nguyen Tuan Anh – Vietnam

PROBLEM 1.045.

If $a,b,c \geq \frac{1}{3}$ then:

$$\prod\left(a^2+\sum a^3+\sum ab-3abc\right) \geq (a+b)^2(b+c)^2(c+a)^2$$

Proposed by Mihály Bencze – Romania

PROBLEM 1.046.

Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that:

$$\frac{a}{b(b+c+d)^2}+\frac{b}{c(c+d+a)^2}+\frac{c}{d(d+a+b)^2}+\frac{d}{a(a+b+c)^2} \geq \frac{4}{9}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.047.

Let a, b, c be positive real numbers such that $ab+bc+ca+abc\le 4$. Prove that

$$a.\,3+a+b+c\ge 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

$$b.\,3+\frac{5}{3}(a+b+c)\ge\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{b}+\sqrt[3]{c}\right)\left(\sqrt[3]{c}+\sqrt[3]{a}\right)$$

Proposed by Nguyen Hung Viet – Hanoi – Vietnam

PROBLEM 1.048.

Prove that for any positive real numbers a, b, c holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{1}{3}+\frac{(a+b)(b+c)(c+a)}{a^2b+b^2c+c^2a}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.049.

Let $x_1, x_2, \dots, x_n$ be positive real numbers such that:

$$\frac{1}{x_1}+\frac{2}{x_2}+\dots+\frac{n}{x_n}=\frac{n(n+1)}{2}.$$

Find the minimum possible value of $x_1+x_2^2+\dots+x_n^n$.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.050.

Let a, b, c be non – negative real numbers such that $a+b+c=3$. Prove that

$$a^3+b^3+c^3+8(ab+bc+ca)\le 27$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.051.

Prove that in any triangle the following relationship holds:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}\le\frac{R}{nR+(1-n)\cdot 2r},\ 0\le n\le\frac{1}{2}$$

Proposed by Marin Chirciu – Romania

PROBLEM 1.052.

Given $a, b, c>0$ and $a^2+b^2+c^2=6$, prove:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+a+b+c\ge 6$$

Proposed by Nguyen Phuc Tang-Vietnam

PROBLEM 1.053.

If $a, b, c>0$ and $a+b+c=3$ prove that:

$$\sum a\left(\frac{1}{b^n}+\frac{1}{c^n}\right)\ge\frac{18}{a^n+b^n+c^n}$$

where $n\ge 0$.

Proposed by Marin Chirciu – Romania

PROBLEM 1.054.

Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC. Prove that:

$$\frac{9}{4R+r}\le\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\le\frac{1}{r},$$

Proposed by Martin Lukarevski – Stip – Macedonia

PROBLEM 1.055.

Let $ABCD$ be an inscriptible and circumscriptible quadrilateral, p its semi perimeter. R and r the radii of circumcenter, respectively incenter, a, b, c, d its sides (a and c are the opposite sides). Prove that:

$$a.\ 2\frac{R^2}{r^2} \geq \frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b} \geq 2\sqrt{2}\frac{R}{r}$$

$$b.\ 4\frac{R^2}{r^2}-4 \geq \left(\frac{a}{c}+\frac{c}{a}\right)\left(\frac{b}{d}+\frac{d}{b}\right)$$

Proposed by Vasile Jiglău – Romania

PROBLEM 1.056.

Let s_a is symedian and r_a, r are exradius and inradius triangle of ABC respectively. Prove that:

$$\frac{r_a}{s_a+r}+\frac{r_b}{s_b+r}+\frac{r_c}{s_c+r} \geq \left(\frac{3r}{R}\right)^2$$

Proposed by Mehmet Şahin – Ankara – Turkey

PROBLEM 1.057.

Let ABC be an arbitrary triangle and I_a, I_b, I_c are excenters of ABC. I_aBC, I_bCA, I_cAB are the extriangles of ABC. Let h_i $(i = 1, 2, 3, \dots, 9)$ the altitudes of extriangles. Prove that

$$\prod_{i=1}^{9} h_i = \left(\prod_{a,b,c} r_a\right)^3$$

where r_a, r_b, r_c are exradii of ABC.

Proposed by Mehmet Şahin – Ankara – Turkey

PROBLEM 1.058.

Prove that for all $x \in \mathbb{R}$ we have:

$$\cos(\sin x) > |\sin(\cos x)|$$

Proposed by Abdallah El Farissi – Bechar – Algerie

PROBLEM 1.059.

Let a, b, c be the side lengths of a triangle ABC with inradius r. Prove that:

$$\frac{1}{a^3}\tan\frac{A}{2}+\frac{1}{b^3}\tan\frac{B}{2}+\frac{1}{c^3}\tan\frac{C}{2} \leq \frac{R}{48r^4}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.060.

Let a, b and c be the lengths of the sides of a triangle with circumradius R. Prove that:

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a} \leq \frac{3\sqrt{3}}{2}R\,.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.061.

Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that:

$$\frac{1}{a^3+b^3+c^3}+\frac{8}{ab+bc+ca} \geq 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.062.

Prove that if: $a,b,c,d\in[1,\infty)$ then:

$$3a+3b+2c+d\le 6+ab(1+c+cd)$$

Proposed by Daniel Sitaru-Romania

PROBLEM 1.063.

Let x,y,z be non-negative real numbers satisfying $\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}=3$. Find the minimum possible value of

$$\sqrt[3]{x+2y+5z}+\sqrt[3]{y+2z+5x}+\sqrt[3]{z+2x+5y}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.064.

Let a,b,c be non-negative real numbers. Prove that:

$$\sqrt[3]{1+a^3}+\sqrt[3]{1+b^3}+\sqrt[3]{1+c^3}\ge\frac{\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}}{\sqrt[6]{2}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.065.

Let ABC be an equilateral triangle inscribed in the circle (O) whose radius R. Prove that for an arbitrary point P lies on (O):

$$6\sqrt{2}<\frac{PA^3+PB^3+PC^3}{R^3}<3\sqrt[4]{216}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.066.

Prove that in any triangle

$$\sum\frac{m_a^2}{h_bh_c}\ge n\cdot\frac{R}{r}+(3-2n)\cdot\frac{s}{3r\sqrt{3}},n\le\frac{3}{2}$$

Proposed by Marin Chirciu – Romania

PROBLEM 1.067.

Prove that in any triangle:

$$n\cdot\frac{s^2+r^2}{Rr}+\sqrt[k]{\frac{2r}{R}}\ge 14n+1,n\ge\frac{1}{2},k\in N,k\ge 2\,.$$

Proposed by Marin Chirciu – Romania

PROBLEM 1.068.

Let a,b and c be the side lengths of a triangle ABC, with circumradius R and inradius r. Prove that:

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{a+b}+\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{b+c}+\frac{\tan\frac{C}{2}+\tan\frac{A}{2}}{c+a}\le\frac{1}{r}-\frac{1}{R}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.069.

Let a, b be positive real numbers such that $a^2 + ab + b^2 = 9$. Find the maximal value of expression:

$$(a+b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.070.

Let a, b be positive real numbers with $a^2 + ab + b^2 = k^2, k > 0$.
Prove that:

$$\sqrt{a+b} + \sqrt[4]{ab} \le \frac{\sqrt{2}+1}{\sqrt[4]{3}} \cdot \sqrt{k}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.071.

Let ABC be a triangle with circumradius R and inradius r, and let w_a, w_b, w_c be the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c, respectively.
Prove that:

$$\left(\frac{w_a}{a}\right)^2 \cdot \tan\frac{A}{2} + \left(\frac{w_b}{b}\right)^2 \cdot \tan\frac{B}{2} + \left(\frac{w_c}{c}\right)^2 \cdot \tan\frac{C}{2} \le \frac{3\sqrt{3}}{8} \cdot \frac{R}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.072.

Let a, b and c denote, as usual, the lengths of the sides BC, CA, and AB, respectively, in $\Delta\, ABC$. Let R be the circumradius, r the inradius of $\Delta\, ABC$, and r_a, r_b and r_c the exradii to A, B and C, respectively. Prove that:

$$a.\ \frac{r_a}{a^3} + \frac{r_b}{b^3} + \frac{r_c}{c^3} \le \frac{\sqrt{3}}{8r^2} \qquad b.\ \frac{3R}{2r} \ge \sqrt{\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 6}$$

Proposed by George Apostolopoulos – Messolonghi - Greece

PROBLEM 1.073.

If $a, b, c > 0; n \ge 1$ then:

$$\frac{3n(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \ge n+1$$

Proposed by Marin Chirciu – Romania

PROBLEM 1.074.

If $a, b, c, n > 0; n(ab+bc+ca) + 2abc = n^3$ then:

$$\frac{1}{a+b+2n} + \frac{1}{b+c+2n} + \frac{1}{c+a+2n} \le \frac{1}{n}$$

Proposed by Marin Chirciu – Romania

PROBLEM 1.075.

Let R and r be the circumradius and the inradius of a triangle ABC respectively. Prove that:

$$\csc A + \csc B + \csc C \ge 3\sqrt{3}\,\frac{R}{R+r}$$

Proposed by Martin Lukarevski – Stip – Macedonia

PROBLEM 1.076.

Let ABC be an acute triangle. Prove that: $(a\cot A)^a(b\cot B)^b(c\cot C)^c \le (2r)^{a+b+c}$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.077.

Let $a_1, a_2, \dots, a_9$ be non-negative real numbers such that $a_1 + a_2 + \dots + a_9 = 1$. Prove that for all $\lambda \ge 4$, the following inequality holds:

$$\sqrt{\sum_{1\le i\le 9} a_i^2} + \lambda\sqrt{\sum_{1\le i<j\le 9} a_i a_j} \le \frac{2\lambda+1}{3}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

PROBLEM 1.078.

Let a, b, c be positive real numbers such that $a^2 = b^2 + c^2$. Prove that:

$$ab + bc + ca + \left(\sqrt{2}-1\right)\frac{abc}{a+b+c} \le 2a^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.079.

Prove the inequality holds for all positive real numbers a, b, c

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{4}{2a+2b+3c} + \frac{4}{2b+2c+3a} + \frac{4}{2c+2a+3b}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.080.

Prove that in any triangle ABC:

$$\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}\right) \ge 2\sqrt{3}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

PROBLEM 1.081.

If $x, y, z > 0$ then:

$$\sqrt{\frac{13x}{6x+7y}} + \sqrt{\frac{13y}{6y+7z}} + \sqrt{\frac{13z}{6z+7x}} \le 3$$

Proposed by Marin Chirciu-Romania

PROBLEM 1.082.

If $a, b, c > 0; a + b + c = 3$ then:

$$\frac{a}{1+3b^4} + \frac{b}{1+3c^4} + \frac{c}{1+3a^4} \ge \frac{3}{4}$$

Proposed by Marin Chirciu – Romania

PROBLEM 1.083.

In $\Delta\ ABC$ then following relationship holds:

$$(a^{2m} + b^{2m} + c^{2m})\left(\frac{1}{(a+b)^{2n}} + \frac{1}{(b+c)^{2n}} + \frac{1}{(c+a)^{2n}}\right) \ge$$
$$\ge 3^{m-n+2}\cdot 4^{m-2n}\cdot r^{2(m-n)};\ m, n \ge 1$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu - Romania

PROBLEM 1.084.

In $\Delta\ ABC$ the following relationship holds:

$$\sum\left((a+b)\tan\frac{C}{2}\right)^m\cdot\sum\frac{1}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{2n}}\geq 3^{n+2}\cdot 4^{m-n}\cdot r^n;m\geq n\geq 1$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu - Romania

PROBLEM 1.085.

Let ABC denote a triangle, I its incenter, R its circumradius, r its inradius and x,y and z the inradii of triangles $IBC, ICA,$ and IAB respectively. Prove that:

$$\frac{\sin A}{x}+\frac{\sin B}{y}+\frac{\sin C}{z}\leq\frac{4+3\sqrt{3}}{2r}+\frac{2}{R}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.086.

Let a,b,c be the side lengths of a triangle ABC with incentre I, circumradius R and inradius r. Prove that:

$$\frac{\sqrt{AI}}{a}+\frac{\sqrt{BI}}{b}+\frac{\sqrt{CI}}{c}\leq\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{R+r}}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.087.

Let ABC be an acute triangle. Prove that:

$$\sqrt{\cos A\cdot\sin B\cdot\sin C}+\sqrt{\sin A\cdot\cos B\cdot\sin C}+\sqrt{\sin A\cdot\sin B\cdot\sin C}\leq\frac{3}{2}\sqrt{\frac{3}{2}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.088.

Let a,b,c be positive real numbers. Prove that:

$$\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca}\geq\frac{9abc}{ab+bc+ca}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

PROBLEM 1.089.

Let a,b,c be positive real numbers, take: $X=\frac{a}{b}+\frac{b}{a}, Y=\frac{b}{c}+\frac{c}{b}, Z=\frac{c}{a}+\frac{a}{c}$.

Prove that:

$$X+Y+Z\geq 2\sqrt[4]{(X^2+Y^2+Z^2-3)(X+Y+Z+3)}$$

Proposed by Nguyen Ngoc Tu – Ha Giang –Vietnam

PROBLEM 1.090.

In $\Delta\ ABC$ the following relationship holds:

$$\frac{1+\cos A\cos B\cos B}{\sin A\sin B\sin A}\geq\frac{s}{3r}$$

Proposed by Martin Lukarevski-Skopje-Macedonia

PROBLEM 1.091.

Prove that the following inequalities hold for all positive real numbers:

$$a.\frac{a^3}{ab+c^2}+\frac{b^3}{bc+a^2}+\frac{c^3}{ca+b^2}\ge\frac{3}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c}$$

$$b.\frac{1}{a(b+c)}+\frac{1}{b(c+a)}+\frac{1}{c(a+b)}\ge\frac{3}{2}\cdot\frac{a+b+c}{a^3+b^3+c^3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.092.

Prove that the following inequalities holds for all positive real numbers a,b,c

$$a.\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\ge\frac{3(a+b+c)}{a^2+b^2+c^2}$$

$$b.\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\ge\frac{3(a^2+b^2+c^2)}{a+b+c}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.093.

Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that:

$$a.\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}\le\frac{1}{4abc}\quad b.\frac{\sqrt{a}}{a+\sqrt{bc}}+\frac{\sqrt{b}}{b+\sqrt{ca}}+\frac{\sqrt{c}}{c+\sqrt{ab}}\le\frac{1}{2\sqrt{abc}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.094.

Let a,b,c be positive real numbers such that $ab+bc+ca=1$. Prove that:

$$bc\sqrt{a^2+2b}+ca\sqrt{b^2+2ca}+ab\sqrt{c^2+2ab}\ge 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.095.

Prove that for all positive real numbers a,b,c :

$$\frac{a(b^2+c^2)}{2a^2+bc}+\frac{b(c^2+a^2)}{2b^2+ca}+\frac{c(a^2+b^2)}{2c^2+ab}\ge\frac{6abc}{ab+bc+ca}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

PROBLEM 1.096.

Let a,b,c positive numbers such that $a^4+b^4+c^4=3$. Prove that:

$$\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\ge 9$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

PROBLEM 1.097.

Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=8$. Prove that:

$$\frac{a}{a+1}+\sqrt{\frac{2b}{b+1}}+2\sqrt[4]{\frac{2c}{c+1}}\le\frac{7}{2}$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

PROBLEM 1.098.

Let a, b and c be the side lengths of a triangle ABC with incenter I. Prove that:

$$\frac{1}{IA^2}+\frac{1}{IB^2}+\frac{1}{IC^2}\geq 3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.099.

If $x, y, z > 0$ and $b \geq a > 0$ then:

$$\int_a^b \frac{x\,dy}{3x^2+2y^2+z^2}+\int_a^b \frac{y\,dz}{3y^2+2z^2+x^2}+\int_a^b \frac{z\,dx}{3z^2+2x^2+y^2}\leq \frac{1}{3}\ln\frac{b}{a}+\frac{b-a}{18}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Proposed by Mihály Bencze – Romania

PROBLEM 1.100.

Let in triangle w_a, w_b, w_c be the angle bisectors and R, r the circumradius and inradius respectively. Prove the inequality:

$$\frac{3}{R+r}\leq\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\leq\frac{1}{r}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

PROBLEM 1.101.

Let x, y, z be positive real numbers with $xyz = 1$. Prove that:

$$\frac{\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1}}{x^2+y^2+z^2}\leq\sqrt{2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

PROBLEM 1.102.

Let $x, y, z > 0$ be positive real numbers. Then:

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\geq\frac{4\sqrt{3xyz(x+y+z)}}{(x+y)(y+z)(z+x)}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

PROBLEM 1.103.

Let $x, y, z > 0$ be positive real numbers. Then in triangle ABC with semiperimeter s and inradius r.

$$\frac{x}{y+z}cot^2\frac{A}{2}+\frac{y}{z+x}cot^2\frac{B}{2}+\frac{z}{x+y}cot^2\frac{C}{2}\geq 18-\frac{s^2}{2r^2}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

PROBLEM 1.104.

Let r_a, r_b, r_c be the exradii, h_a, h_b, h_c the altitudes and m_a, m_b, m_c the medians of a triangle ABC with semiperimeter s, circumradius R and inradius r. Then

$$\frac{r_a^2}{h_bm_c}+\frac{r_b^2}{h_cm_a}+\frac{r_c^2}{h_am_b}\geq\frac{54r^2}{s^2-r^2-4Rr}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

PROBLEM 1.105.

Let $m > 0$ and F be the area of the triangle ABC. Then:

$$\frac{a^{m+2}}{b^m + c^m} + \frac{b^{m+2}}{c^m + a^m} + \frac{c^{m+2}}{a^m + b^m} \geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

PROBLEM 1.106.

Let $a, b, c > 0$. Prove that:

$$\frac{8(a+b+c)^3}{(a+b)(b+c)(c+a)} + 5(ab+bc+ca) \geq 12 + 10(a+b+c)$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

PROBLEM 1.107.

Prove that in any triangle ABC with incentre I the following relationship holds:

$$AI \cdot \frac{a^2}{w_a} + BI \cdot \frac{b^2}{w_b} + CI \cdot \frac{c^2}{w_c} \leq 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)},$$

where R is the circumradius, r is the inradius of triangle ABC and w_a, w_b, w_c are the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c, respectively.

Proposed by George Apostolopoulos-Messolonghi-Greece

PROBLEM 1.108.

If $a, b > 0$, then:

$$4\sqrt{ab} \cdot \frac{\sin x}{x} + b\left(\frac{\tan x}{x}\right)^2 + a > 6\sqrt{ab}, \forall x \in \left(0, \frac{\pi}{2}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 1.109.

If $a, b > 0$, then:

$$(a+b) \cdot \frac{\sin x}{x} + \frac{2ab}{a+b} \cdot \frac{\tan x}{x} > \frac{6ab}{a+b}, \forall x \in \left(0, \frac{\pi}{2}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 1.110.

If $x, y, z \in (0, 1)$ and ABC is a triangle, then prove that:

$$\frac{\sin^2 \frac{A}{2}}{x(1-x^3)} + \frac{\sin^2 \frac{B}{2}}{y(1-y^3)} + \frac{\sin^2 \frac{C}{2}}{z(1-z^3)} \geq \frac{2\sqrt[3]{4}}{3R}(2R-r)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 1.111.

Prove that: (i) If $a, b, c, d, x, y, z, t \in \mathbb{R}_+^*$, then:

$$4(a^2+b^2+c^2+d^2+x^2+y^2+z^2+t^2) + 8\sqrt{(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+t^2)} \geq$$
$$\geq (a+b+c+d+x+y+z+t)^2;$$

(ii) If $a, b, c, m, n, p, x, y, z \in \mathbb{R}_+^*$ then:

$$5(a^3+b^3+c^3+m^3+n^3+p^3+x^3+y^3+z^3)$$
$$+ 3\sqrt[3]{(a^3+b^3+c^3)(m^3+n^3+p^3)(x^3+y^3+z^3)} \geq$$
$$\geq 2(a+b+c)(m+n+p)(x+y+z)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 1.112.

Prove that if $a,b\in R_+, x,y,u,v\in R_+^*$ then:

$$\left(a\cdot\frac{x}{y}+b\cdot\frac{u}{v}\right)^2+\left(a\cdot\frac{y}{x}+b\cdot\frac{v}{u}\right)^2\geq 2(a+b)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 1.113.

Let x,y,z be real numbers such that:

$$x^2y+y^2z+z^2x+4(xy^2+yz^2+zx^2)+13xyz=5$$

Find the minium value of the expression:

$$P=(x^2+4xy+5y^2)(y^2+4yz+5z^2)(z^2+4zx+5x^2)+6x^2y^2z^2$$

Proposed Do Quoc Chinh – Vinh Phuc – Viet Nam

PROBLEM 1.114.

Let a,b,c be positive real numbers. Prove that:

$$a.\ ln^3\left(\frac{ab}{c}\right)+ln^3\left(\frac{bc}{a}\right)+ln^3\left(\frac{ca}{b}\right)+24\ln a\cdot\ln b\cdot\ln c=ln^3(abc)$$

$$b.\ ln(ab)\cdot ln(bc)\cdot ln(ca)=\frac{ln^3(abc)-ln^3 a-ln^3 b-ln^3 c}{3}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.115.

If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$ then:

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\geq 2+\frac{18}{a+b+c}$$

Proposed by Pham Quoc Sang – Ho Chi Minh – Vietnam

PROBLEM 1.116.

If a,b,c are positive real numbers such that $abc=1$ then:

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\leq\frac{3}{2}$$

Proposed by Pham Quoc Sang – Ho Chi Minh – Vietnam

PROBLEM 1.117.

If a,b,c are positive real numbers then:

$$\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc}\geq\frac{3}{2}+\frac{3}{2(a+b+c)^2}\cdot\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

PROBLEM 1.118.

Let a,b,c be the three sides of a triangle. Prove that:

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\geq\frac{3(a^3+b^3+c^3)}{a+b+c}$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

PROBLEM 1.119.

Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a+b}{c^2(c^3+a+b)}+\frac{b+c}{a^2(a^3+b+c)}+\frac{c+a}{b^2(b^3+c+a)}\geq\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{\sqrt[3]{abc}}$$

Proposed by Do Quoc Chinh – Vinh Phuc – Vietnam

PROBLEM 1.120.

Let a, b, c be positive real numbers and $k\in[1;3]$. Prove that:

$$\frac{1}{a^2+ab+ca+kbc}+\frac{1}{b^2+bc+ab+kca}+\frac{1}{c^2+ca+bc+kab}\leq\frac{9}{(k+3)(ab+bc+ca)}$$

Proposed by Do Quoc Chinh – Vinh Phuc – Vietnam

PROBLEM 1.121.

Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\sum\frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B}\leq\left(\frac{R}{r}\right)^2$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.122.

Prove that in $\Delta\, ABC$ the following relationship holds:

$$\min\left(\frac{a}{s-a},\frac{b}{s-b},\frac{c}{s-c}\right)\leq 2\left(\frac{R}{r}-1\right)\leq\max\left(\frac{a}{s-a},\frac{b}{s-b},\frac{c}{s-c}\right)$$

Proposed by Marian Ursărescu – Romania

PROBLEM 1.123.

Solve for real numbers: $\log_a(b^x+a-b)=\log_b(a^x+b-a),\ b>a>1$

Proposed by Marian Ursărescu – Romania

PROBLEM 1.124.

Let a, b, c be the lengths of the sides of a triangle ABC with inradius r. Prove that:

$$\frac{(2r)^{a+b+c}}{a^a\cdot b^b\cdot c^c}\leq\left(\tan\frac{A}{2}\right)^b\cdot\left(\tan\frac{B}{2}\right)^c\cdot\left(\tan\frac{C}{2}\right)^a\leq\left(\frac{\sqrt{3}}{36}\right)^{a+b+c}\cdot\frac{a^{2b}\cdot b^{2c}\cdot c^{2a}}{r^{2(a+b+c)}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.125.

Prove that in $\Delta\, ABC$ the following relationship holds:

$$\frac{4}{3}\left(r_a^2+r_b^2+r_c^2\right)\geq 4\sqrt{3}S+(a-b)^2+(b-c)^2+(c-a)^2$$

Proposed by Marian Ursărescu – Romania

PROBLEM 1.126.

Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\cot A + \cot B + \cot C \geq \sqrt{\frac{3R}{2r}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.127.

Prove that in any non-equilateral triangle the following inequality holds:

$$tan\left(\frac{\widehat{OIH}}{2}\right) < 3\sqrt{\frac{R+2r}{R-2r}}$$

Proposed by Marian Ursărescu – Romania

PROBLEM 1.128.

In ΔABC the following relationship holds:

$$\frac{1}{m_a+m_b}+\frac{1}{m_b+m_c}+\frac{1}{m_c+m_a} \leq \frac{1}{2r}$$

Proposed by Marian Ursărescu – Romania

PROBLEM 1.129.

Let a, b and c be the lengths of the sides of a triangle ABC with inradius r and circumradius R. Prove that:

$$a.\ \frac{a^3}{b^2+c^2}+\frac{b^3}{c^2+a^2}+\frac{c^3}{a^2+b^2} \leq \frac{3\sqrt{3}}{2}\cdot\frac{R(R-r)}{r}$$

$$b.\ \frac{a}{\sqrt{b^2+c^2}}+\frac{b}{\sqrt{c^2+a^2}}+\frac{c}{\sqrt{a^2+b^2}} \leq \sqrt{\frac{6R-3r}{2r}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.130.

Solve the equation in real numbers:

$$3\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{x^8+1}{2}} = 2(x^4-3x+4)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.131.

Solve the system of equation in positive real numbers:

$$\begin{cases} 3\left(\sqrt[3]{x^2}+\sqrt[3]{y^2}+\sqrt[3]{z^2}\right)+21 = 10(xy+yz+zx) \\ x+y+z=3 \end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.132.

Let $x, y \in \left(0, \frac{\pi}{2}\right)$. Denote $k = 2 - min\{sin^2 x, cos^2 x\}$. Prove that:

$$\left(\frac{\sin^2 x}{1-\cos y}+\frac{\cos^2 x}{1-\sin y}\right)\left(\frac{\sin^2 x}{1+\cos y}+\frac{\cos^2 x}{1+\sin y}\right)\le\left(\frac{\pi^2}{4}+\frac{\cos^2 x}{\cos^2 y}\right)^k$$

Proposed by Stefan Andrei Mihalcea-Romania

PROBLEM 1. 133.

Let a, b, c be positive real numbers. Prove that:

$$\sqrt[8]{\frac{a^8+b^8}{2}}+\sqrt[8]{\frac{b^8+c^8}{2}}+\sqrt[8]{\frac{c^8+a^8}{2}}\le(a+b+c)^{10}\left(\frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c}\right)^9$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.134.

Let $a, b, c > 0$ such that: $a^2+b^2+c^2=3abc$. Find the maximum value of:

$$P=\frac{ab}{2a^6-a^5+b^4+a^2+1}+\frac{bc}{2b^6-b^5+c^4+b^2+1}+\frac{ca}{2c^6-c^5+a^4+c^2+1}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.135.

Let x, y, z be positive real numbers such that: $x+y+z=3$. Prove that:

$$\frac{x^3y^3}{x^4+y^3-x+2}+\frac{y^3z^3}{y^4+z^3+y+2}+\frac{z^3x^3}{z^4+x^3-z+2}\le\frac{x^4+y^4+z^4+3xyz}{6}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.136.

Let x, y, z be positive real numbers such that: $xyz=1$. Find the maximum of the expression:

$$Q=\frac{1}{\sqrt[3]{2x^5+y^4-x^2+4}}+\frac{1}{\sqrt[3]{2y^5+z^4-y^2+4}}+\frac{1}{\sqrt[3]{2z^5+x^4-z^2+4}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.137.

Let $x, y \ge 1$. Prove that:

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\ge 2+\frac{4(x-y)^2}{(2x+xy+1)(2y+xy+1)}$$

Proposed by Andrei Ștefan Mihalcea – Romania

PROBLEM 1.138.

Let $a, b, c > 0$, with $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Prove that: $(4a-3)(4b-3)(4c-3)\ge 243\sqrt[3]{abc}$

Proposed by Andrei Ștefan Mihalcea – Romania

PROBLEM 1.139.

Let x, y, z be positive real numbers such that: $x^2+y^2+z^2=3$. Find the minimum of the expression:

$$P=\frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz}+\frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}}+3zx}+\frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}}+3xy}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.140.

Let $a,b,c>0$. Prove that:

$$\sum\frac{\sqrt{a+b}}{a}\le\left(\sum\frac{1}{a}\right)\sqrt{\sum a-\frac{\sum ab}{\sum a}}$$

Proposed by Andrei Ștefan Mihalcea – Romania

PROBLEM 1.141.

Let $a,b,c>0$. Prove that:

$$\left(\sum\sqrt{\frac{b+c}{a}}\right)^2\le\frac{2(\sum ab)^3}{3a^2b^2c^2}$$

Proposed by Andrei Ștefan Mihalcea – Romania

PROBLEM 1.142.

Let $a,b,c\ge 1$. Prove that:

$$\sum\sqrt{\frac{a-1}{bc}}\le\left(\sum\frac{1}{ab}\right)\sqrt{abc-\frac{\sum a}{3}}$$

Proposed by Andrei Ștefan Mihalcea – Romania

PROBLEM 1.143.

In any ABC triangle the following relationship holds:

$$\frac{w_a^2}{h_b\cdot h_c}+\frac{w_b^2}{h_c\cdot h_a}+\frac{w_c^2}{h_a\cdot h_b}\le\left(\frac{R}{r}\right)^2-1$$

Proposed by Mehmet Șahin – Ankara – Turkey

PROBLEM 1.144.

In any ABC triangle the following relationship holds:

$$\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}\ge\frac{4s}{3R}$$

Proposed by Mehmet Șahin – Ankara – Turkey

PROBLEM 1.145.

In any ABC triangle the following relationship holds:

$$\frac{m_a}{r+r_a}+\frac{m_b}{r+r_b}+\frac{m_c}{r+r_c}\le\frac{s}{2r}$$

all notations are usual sense.

Proposed by Mehmet Șahin – Ankara – Turkey

PROBLEM 1.146.

Let x, y, z be positive real numbers such that: $xyz = 1$. Find the maximum of the expression:

$$P = \frac{1}{\sqrt[3]{2(x^5 - x^3 + 4)}} + \frac{1}{\sqrt[3]{2(y^5 - y^3 + 4)}} + \frac{1}{\sqrt[3]{2(z^5 - z^3 + 4)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.147.

Let abc be positive real numbers such that: $a^2 + b^2 + c^2 = 3abc$. Find the minimum of the expression:

$$P = \frac{a^2}{\sqrt[3]{4(b^3 + c^3)}} + \frac{b^2}{\sqrt[3]{4(c^3 + a^3)}} + \frac{c^2}{\sqrt[3]{4(a^3 + b^3)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.148.

Let a, b, c be positive real numbers such that: $ab + bc + ca = 12$. Prove that:

$$\frac{a^3 + b^3}{2b^2 - bc + 2c^2} + \frac{b^3 + c^3}{2c^2 - ca + 2a^2} + \frac{c^3 + a^3}{2a^2 - ab + 2b^2} \geq 4$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.149.

Find all functions: $f: (0, +\infty) \to \mathbb{R}$ which verify the relationship:

$$ln(xy) \leq xf(x) + yf(y) \leq xyf(xy), \forall x, y > 0$$

Proposed by Marian Ursărescu – Romania

PROBLEM 1.150.

Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that: $|z_1| = |z_2| = |z_3|$. If $(z_1 + z_2)(z_2 + z_3)(z_3 + z_1) + z_1z_2z_3 = 0$, then z_1, z_2, z_3 are the affixes of an equilateral triangle.

Proposed by Marian Ursărescu – Romania

PROBLEM 1.151.

Let a, b be positive real numbers. Find the maximum of k such that inequality is true:

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{k}{a^4 + b^4} \geq \frac{8k + 32}{(a + b)^4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.152.

Let ABC; h_a, h_b, h_c denote the lengths of altitudes from A, B, C; l_a, l_b, l_c are the lengths of the symmetric divergence lines from A, B, C; r_a, r_b, r_c are radii of the circle next to the corners A, B, C. Prove that:

$$\frac{h_ar_a}{l_a^2} + \frac{h_br_b}{l_b^2} + \frac{h_cr_c}{l_c^2} \geq 3$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.153.

In ΔABC the lengths BC, CA, AB are a, b, c. Let l_a, l_b, l_c be the lengths of the bisectors from the vertices A, B, C in triangle ABC. Prove that:

$$\frac{l_a l_b}{\sin\frac{C}{2}}+\frac{l_b l_c}{\sin\frac{A}{2}}+\frac{l_c l_a}{\sin\frac{B}{2}}\le\frac{3}{2}\sqrt{3abc(a+b+c)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.154.

In triangle ABC with sides $BC=a, CA=b, AB=c. r_a, r_b, r_c$ are exradii, h_a, h_b, h_c are distances from A, B, C *to* BC, CA, AB. Prove that:

$$\frac{r_a r_b}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{r_b r_c}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{r_c r_a}{\sin\frac{C}{2}\sin\frac{A}{2}}\ge\frac{h_a h_b}{\sin^2\frac{C}{2}}+\frac{h_b h_c}{\sin^2\frac{A}{2}}+\frac{h_c h_a}{\sin^2\frac{B}{2}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.155.

Let a, b, c be positive real numbers such that: $a+b+c=3$. Prove that:

$$\frac{a^2}{b^4c\sqrt[3]{4(b^6+1)}}+\frac{b^2}{c^4a\sqrt[3]{4(b^6+1)}}+\frac{c^2}{a^4b\sqrt[3]{4(a^6+1)}}\ge\frac{3}{2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.156.

Let ABC be a triangle having the area S. Let be $A'\in(BC)$ such that the incircles of $\Delta AA'B, \Delta AA'C$ have the same radius. Analogous, we obtain the points $B'\in(AC), C'\in(AB)$. Prove that:

$$S=\frac{AA'\cdot BB'\cdot CC'}{s}$$

where s is the semiperimeter of ΔABC.

Proposed by Marian Ursărescu – Romania

PROBLEM 1.157.

Prove that in ΔABC, the following inequality holds:

$$\sum_{cyc}\left(a\cot\frac{A}{2}\cdot\sin^2 B\cdot\sin^2 C\right)\ge\frac{81r^4}{2R^2(2R-r)}$$

Proposed by Marian Ursărescu – Romania

PROBLEM 1.158.

Let $a, b, c>0$. Prove that:

$$\sum_{cyc}\frac{1}{a}+\sum\frac{a}{b^2+c^2}\ge 3\sum_{cyc}\frac{1}{b+c}$$

Proposed by Andrei Ștefan Mihalcea – Romania

PROBLEM 1.159.

Prove that in any ΔABC the following inequality holds:

$$a^2h_bh_c+b^2h_ah_c+c^2h_ah_b\le 4(R+r)^4$$

Proposed by Marian Ursărescu – Romania

PROBLEM 1.160.

Prove that for all non-negative real numbers x, y, z the following inequality holds:

$$\frac{y+z}{(1+x)^2}+\frac{z+x}{(1+y)^2}+\frac{x+y}{(1+z)^2}\ge\frac{x}{1+yz}+\frac{y}{1+zx}+\frac{z}{1+xy}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.161.

Let a, b, c, d be positive real numbers such that $abcd \ge 1$. Prove that:

$$\frac{a^2+b^2+c^2+1}{a^3+b^3+c^3+1}+\frac{b^2+c^2+d^2+1}{b^3+c^3+d^3+1}+\frac{c^2+d^2+a^2+1}{c^3+d^3+a^3+1}+\frac{d^2+a^2+b^2+1}{d^3+a^3+b^3+1}\le 4$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.162.

Let x, y, z be non-negative real numbers such that $x+y+z=1$. Prove that:

$$a.\ \frac{3}{\sqrt{10}}\le\frac{x}{\sqrt{1+yz}}+\frac{y}{\sqrt{1+zx}}+\frac{z}{\sqrt{1+xy}}\le 1 \quad b.\ \sqrt{\frac{3}{5}}\le\frac{x}{\sqrt{1+y+z}}+\frac{y}{\sqrt{1+z+x}}+\frac{z}{\sqrt{1+x+y}}\le 1$$

When do the equalities occur?

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.163.

Let ABC be an acute triangle with standard notations. Prove that:

$$\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}\le\frac{a+b+c}{4r}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 1.164.

If $a, b \ge 0$ then: $\left(a+b+\sqrt{a^2+b^2}\right)^2\ge 6\sqrt{3}ab$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.165.

If $a, b, c \ge 0$ then:

$$4(a+b+c)\le\left(3\sqrt{3}-2\right)\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEMS FROM SENIORS

PROBLEM 2.001

Prove that if $a,b,c\in(0,\infty)$ then:

$$\sum\frac{2a+3c}{a+2b+5c}<\frac{273\sum ab+87\sum a^2}{64\left(\sum\sqrt{ab}\right)^2}$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.002

Prove that in any acute-angled ΔABC the following relationship holds:

$$cosA+cosB+cosC+ln\big((A+1)(B+1)(C+1)\big)\le 3+\pi$$

Proposed by Daniel Sitaru-Romania

PROBLEM 2.003

Let be $f:[0,1]\to(0,\infty)$ a differentiable function, convex and $a,b,c\in[0,1]$ such that:

$$f'(a)+f'(b)+f'(c)=1;\ af'(a)+bf'(b)+cf'(c)=2$$

Prove that:

$$\frac{1}{2}+\int_0^1 f(x)\,dx\ge\frac{1}{3}\big(f(a)+f(b)+f(c)\big)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.004.

If $x,y,z\in(0,\infty)$ then:

$$x+y+z+\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\ge\frac{12}{\sqrt{3\sqrt{3}}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.005.

Prove that:

$$\sum_{k=1}^{\infty}\left(\frac{1+(k^2-1)^{\frac{1}{2}}}{1+(k^2-1)^{\frac{3}{4}}}\right)^4\le\frac{\pi^2}{3}$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.006.

If $f:\mathbb{R}\to\mathbb{R}$ continuous and convexe then:

$$\int_0^e f(x)\,dx\ge\int_0^1(x^3+nx^{n-1})f\left(\frac{e^{2x}+nx^{2n-1}}{e^x+nx^{n-1}}\right)dx$$

where $n\ge 1$.

Proposed by Mihály Bencze – Romania

PROBLEM 2.007.

If $x_k > 1$ $(k = 1, 2, \dots, n)$ and $S = \sum_{k=1}^{n} x_k$ then:

$$\prod_{k=1}^{n} \log_{x_k} \frac{s-k}{n-1} \geq 1$$

for all $n \geq 3$.

Proposed by Mihály Bencze – Romania

PROBLEM 2.008.

Prove that if $a, b, c \in (0, \infty)$ then:

$$12 \sum \frac{c}{a^2 + b^2 + 9} \leq \frac{1}{abc} \sum c^2 \sqrt{a^2 + b^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.009.

Let be $a, b, c \in (0, \infty)$; $a < b < c$; $f: [0, a] \to [0, b]$; $g: [0, b] \to [0, c]$ continuous, bijectifs and strictly increasing functions. Prove that:

$$\frac{1}{c} \int_0^a (g \circ f)^2 (x) dx + \frac{1}{a} \int_0^c (f^{-1} \circ g^{-1})^2 (x) dx < ac$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.010.

In all acute – angle triangle ABC holds:

$$\sum \left(\frac{\cosh \frac{A}{2}}{\cos \frac{A}{2}} \right)^2 \leq \frac{s^2 + r^2 - 4R^2}{s^2 - (2R + r)^2}$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.011.

Prove that for all $n \geq 1$ positive integers:

$$\frac{\pi^2}{6} - \frac{1}{n} < \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6} - \frac{1}{n+1}$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.012.

If $A, B \in M_2(C)$ then:

$$\sum_{k=1}^{n} (\det(A + kB) + \det(A - kB)) = 2n \det A + \frac{n(n+1)(2n+1)}{3} \det B$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.013.

Let be $a, b \in \mathbb{R}$, $a < b$. Find:

$$\lim_{n \to \infty} \int_a^b \sin x \cdot \arctan(nx)\, dx$$

Proposed by Dan Nedeianu – Romania

PROBLEM 2.014.

Prove that if $a,b,c\in(0,\infty)$ and $b\ge a$, then:

$$2\sqrt{2}\left(e^{bc}-e^{ac}\right)\le c(b-a)\sqrt{e^{2ac}+e^{2bc}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.015.

Prove that if $a,b,c\in[0,\infty)$ then:

$$25\sum a^2\sqrt[5]{a}+11\sum ab^6\ge 33\sum a^2b$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.016.

Let a,b,c,s,t,u be positive real numbers such that $a+b+c=1$. Prove that:

$$\frac{sa^2+tb^2+uc^2}{sa+tb+uc}+\frac{sb^2+tc^2+ua^2}{sb+tc+ua}+\frac{sc^2+ta^2+ub^2}{sc+ta+ub}\ge 1$$

Proposed by Kunihiko Chikaya – Tokyo – Japan

PROBLEM 2.017.

Let $a_k(k=1,2,\dots,n)$ be a positive real numbers such that

$$\sum_{k=1}^{n}a_k=\frac{n(n+1)}{2}$$

Prove that:

$$\sum_{k=1}^{n}\frac{(k^2-1)a_k+k^2+2k}{a_k^2+a_k+1}\ge\frac{n(n+1)}{2}$$

Proposed by Kunihiko Chikaya – Tokyo – Japan

PROBLEM 2.018.

If $a,b,c>0$ and $x,y,z\ge 1$ then:

$$x^{\frac{8a^3}{a+b}}y^{\frac{8b^3}{b+c}}z^{\frac{8c^3}{c+a}}\ge\left(\frac{x^5}{z}\right)^{a^2}\left(\frac{y^5}{x}\right)^{b^2}\left(\frac{z^5}{y}\right)^{c^2}$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.019.

Prove that:

$$1.\ \sum_{k=1}^{n}(2k+1)\,(2k^2+2k+5)(k^2+k)^4=\frac{1}{3}(n^3+3n^2+2n)^4$$

$$2.\ \sum_{k=1}^{n}(2k+1)(k^2+k+1)(k^2+k+7)\,(k^2+k)^6=\frac{1}{9}(n^3+3n^2+2n)^6$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.020.

If $x,y,z\in\left(0,\frac{\pi}{2}\right)$, then prove that:

$$\frac{\tan^2 x}{(y+x)^2}+\frac{\tan^2 y}{(z+x)^2}+\frac{\tan^2 x}{(x+y)^2}>\frac{3}{4}$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 2.021.

If $x, y, z > 0$, then prove that:

$$(x^3y^3 + y^3z^3 + z^3x^3)\left(\frac{1}{(x+y)^5z} + \frac{1}{(y+z)^5x} + \frac{1}{(z+x)^5y}\right) \geq \frac{9}{32}$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 2.022.

Prove that if $n \in \mathbb{N}; n \geq 2; 0 < a \leq b$ then:

$$\frac{b^{n+1} - a^{n+1}}{n+1} + \frac{ab(b^{n-1} - a^{n-1})}{n-1} \leq (b-a)\sqrt{2(a^{2n} + b^{2n})}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.023.

Let $A, B \in M_n(C)$ such that $\det A = \det B \neq 0$. Prove that:

$$\det(AB + xy(AB)^{-1} + (x+y)I_n) = \det(BA + xy(BA)^{-1} + (x+y)I_n)$$

for all $x, y \in C$.

Proposed by Mihály Bencze – Romania

PROBLEM 2.024.

Let ABC be a triangle with the centroid G and denote by S_{ABC} its area. Prove that for any point P in the plane:

$$\frac{PA \cdot GA^2}{BC} + \frac{PB \cdot GB^2}{CA} + \frac{PC \cdot GC^2}{AB} \geq \frac{4}{3}S_{ABC}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.025.

If $a_i > 0 \ (i = 1, 2, \dots, n)$ and $k \geq 1$ then:

$$\left(\sum_{i=1}^{n} a_i\right)^k \leq \sum_{i_1=1,\dots,i_k=1}^{n} \frac{i_1 \dots i_k}{i_i + \dots + i_k - k + 1} a_{i_1} \dots a_{i_k}$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.026.

Let be $n \in \mathbb{N}^*$. Compute:

$$I = \int_{n}^{n+1} \frac{\sqrt{n+1-x}}{\sqrt{n+1-x} + \sqrt{x-n}e^{2x-2n-1}} dx$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.027.

Solve the following equation in set of real numbers:

$$8^x + 27^{\frac{1}{x}} + 2^{x+1} \cdot 3^{\frac{x+1}{x}} + 2^x \cdot 3^{\frac{2x+1}{x}} = 125$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.028.

Compute:

$$L=\lim_{n\to\infty}\frac{1}{n}\int_1^n \frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2. 029.

Compute:

$$L=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_0^1 \frac{x\sin\pi x}{x+(1-x)k^{1-2x}}dx$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.030.

Prove that:

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\cos^2(2015x)-\cos^2(2016x)}{\sin x}dx>0.0001$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.031.

If $(a_n)_{n\ge1}\subset(0,\infty)$ is a sequence such that $\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a\in(0,\infty)$ then find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[2n+2]{a_{n+1}}-\sqrt[2n]{a_n}\right)\sqrt{n}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

PROBLEM 2.032.

If $(a_n)_{n\ge1};(b_n)_{n\ge1}\subset(0,\infty)$ such that

$$\lim_{n\to\infty}\frac{a_n+1}{na_n}=a\in(0,\infty);\ \lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b\in(0,\infty)$$

then find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[2n+2]{a_{n+1}\cdot b_{n+1}}-\sqrt[2n]{a_n\cdot b_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 2.033.

Let be: $r,s\in[0,\infty);(a_n)_{n\ge1};(b_n)_{n\ge1}\subset(0,\infty)$ such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^r}=a\in(0,\infty);\ \lim_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^{s+1}}=b\in(0,\infty);\ x_n=\sum_{k=1}^{n}\frac{1}{k}$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}-\sqrt[n]{a_nb_n}\right)e^{-(r+s)x_n}\right)$$

Proposed by D. M. Bătinețu – Giurgiu – Romania

PROBLEM 2.034.

Let $f:(0,\infty)\to(0,\infty)$ be a continuous function such that: $lim_{x\to\infty}\frac{f(x+1)}{xf(x)}=a\in(0,\infty)$ **and it does exists:** $lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x}$**.Find:**

$$\Omega=\lim_{x\to\infty}\left(\frac{x+1}{\left(f(x+1)\right)^{\frac{1}{2x+2}}}-\frac{x}{\left(f(x)\right)^{\frac{1}{2x}}}\right)\cdot\sqrt{x}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

PROBLEM 2.035.

Evaluate:

$$\lim_{n\to\infty}\frac{\lfloor\sqrt{44}\rfloor+\lfloor\sqrt{4444}\rfloor+\cdots+\lfloor\sqrt{\overbrace{44\ldots44}^{2n\text{ digits }4}}\rfloor}{10^n}$$

where $\lfloor x\rfloor$ denotes the integer part of x.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.036.

Let a,b,c be positive real numbers such that:

$$3(a+b)(b+c)(c+a)\geq\frac{8}{\sqrt[8]{a^3+b^3+c^3}}$$

Prove that :

$$a+b+c\geq\sqrt[3]{9}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.037.

Compute the limit

$$\lim_{n\to\infty}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}n\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cos\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta}d\theta$$

Proposed by Kunihiko Chikaya – Tokyo – Japan

PROBLEM 2.038.

If $x,y,z\in\mathbb{R}\setminus\{1\}$ and $n\in\mathbb{N}$ then:

$$\frac{1}{3}\sum_{cyclic}(x-2y-2z)\left(\frac{x^{n+1}-1}{x-1}\right)+\sqrt{x^2+y^2+z^2}\sum_{k=0}^{n}\sqrt{x^{2k}+y^{2k}+z^{2k}}\geq 0$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.039.

Prove that if $a,b,c\in(1,\infty)$ then:

$$e^{\left|\ln\frac{ab}{c}\right|}\cdot e^{\left|\ln\frac{ac}{b}\right|}\cdot e^{\left|\ln\frac{bc}{a}\right|}\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3\geq 27$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.040.

Prove that if $a,b,c\in(\sqrt{3},\infty)$ then:

$$\frac{\ln(bc)}{\ln(ea^2)}+\frac{\ln(ac)}{\ln(eb^2)}+\frac{\ln(ab)}{\ln(ec^2)}\geq 2\sum\frac{\ln c}{1+2\sqrt{\ln a\ln b}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.041.

Let be $f:[0,1]\to\mathbb{R}, f$ continuous on $[0,1]$. Compute:

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left((n-k+1)^2\sum_{j=1}^{k}f\left(\frac{j}{n}\right)\right)}{n^2(n+1)(2n+1)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.042.

If $A,B\in M_2(R)$ then:

$$\det(xI_2+yAB+zBA)+\det(yI_2+zAB+xBA)+\det(zI_2+xAB+yBA)\geq$$
$$\geq(xy+yz+zx)\left(\left(1+Tr(AB)\right)^2+2\det(AB)-Tr(A^2B^2)\right)$$

for any $x,y,z\in\mathbb{R}$

Proposed by Mihály Bencze – Romania

PROBLEM 2.043.

If $x,y,z,a,b,c>0$ then:

$$x^3y+y^3z+z^3x\geq\left(x^{3a+c}y^{3b+a}z^{3c+b}\right)^{\frac{1}{a+b+c}}+\left(y^{3a+c}z^{3b+a}x^{3c+b}\right)^{\frac{1}{a+b+c}}+$$
$$+\left(z^{3a+c}x^{3b+a}y^{3c+b}\right)^{\frac{1}{a+b+c}}$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.044.

In all convex quadrilateral $ABCD$ we have:

$$(-a+b+c+d)^\alpha+(a-b+c+d)^\alpha+(a+b-c+d)^\alpha+(a+b+c-d)^\alpha\geq$$
$$\geq\left(\frac{a+b+c}{3}+d\right)^\alpha+\left(\frac{b+c+d}{3}+a\right)^\alpha+\left(\frac{c+d+a}{3}+b\right)^\alpha+\left(\frac{d+a+b}{3}+c\right)^\alpha$$

for all $\alpha\geq 1$.

Proposed by Mihály Bencze – Romania

PROBLEM 2.045.

If $a,b,c\in(0,1)$ then:

$$\frac{1}{\left(a(1-a^4)\right)^{4n}}+\frac{1}{\left(b(1-b^4)\right)^{4n}}+\frac{1}{\left(c(1-c^4)\right)^{4n}}\geq 3\left(\frac{3125}{256}\right)^n$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.046.

Prove that for every positive integer n:

$$\ln\frac{n+1}{2}<\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\log_2\frac{n+1}{2}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.047.
Evaluate without calculator:

$$\sum_{k=1}^{17}\cos^4\frac{k\pi}{36}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.048.
Prove that the following inequality holds for all non-negative real numbers a,b,c

$$(a^4+b^4+c^4)(ab^3+bc^3+ca^3)\ge(a^3b+b^3c+c^3a)(a^2b^2+b^2c^2+c^2a^2)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.049.
Prove that the following inequality holds for all positive real numbers x,y:

$$x^{y-x}\cdot y^{x-y}\le 1.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.050.
Let $a\ge b\ge c>0$. Prove that:

$$a^{a-b}b^{b-c}c^{c-a}\ge 1.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.051.
If $a,b,x,y\in(0,\infty)$ and $m\in[0,\infty)$ then:

$$\frac{x}{(ay+bz)^{m+1}}+\frac{y}{(az+bx)^{m+1}}+\frac{z}{(ax+by)^{m+1}}\ge\frac{3^{m+1}}{(a+b)^{m+1}(x+y+z)^m}$$

Proposed by D.M. Bătinețu-Giurgiu; Neculai Stanciu – Romania

PROBLEM 2.052.
In any triangle ABC the following relationship holds:

$$\frac{1}{(\cos A+\cos B)^2}+\frac{1}{(\cos B+\cos C)^2}+\frac{1}{(\cos C+\cos A)^2}\ge 3$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 2.053.
If $x,y,z\in(0,\infty)$ then:

$$\frac{x}{(y+z)^3}+\frac{y}{(z+x)^3}+\frac{z}{(x+y)^3}\ge\frac{27}{8(x+y+z)^2}$$

Proposed by D. M. Bătinețu – Giurgiu – Romania

PROBLEM 2.054.
Let $a\in\left(0,\frac{\pi}{2}\right)$, $b\in[1,\infty)$, $m,n\in\mathbb{R}_+^*$ and $f,g,h,k:[-a,a]\to\mathbb{R}$ be continuous functions such that: $f(-x)=-f(x)$, $g(-x)=-g(x)$,
$h(-x)=h(x)$, $k(-x)=k(x)$. Evaluate:

$$\int_{-a}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}\,dx.$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 2.055.

Let m_a, m_b, m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 4r$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.056.

Let ABC be a triangle such that:

$$\left(\frac{1}{\sin B} + \frac{1}{\sin C}\right)(-\sin A + \sin B + \sin C) = 2$$

Prove that $\measuredangle A \leq \frac{\pi}{3}$.

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.057.

If $a, b, c, d \in \mathbb{R}_+^*, a < b$ and $f: \mathbb{R} \to \mathbb{R}_+^*$ is a continuous function such that $f(a + b - x) = f(x), \forall\, x \in \mathbb{R}$, then evaluate

$$\int_a^b \frac{f(x-a)\big(c + df(b-x)\big)}{c\big(f(x-a) + f(b-x)\big) + 2df(x-a)f(b-x)}\,dx\,.$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 2.058.

Find:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(tan\frac{1}{n+i} - tan\frac{1}{n+i+1}\right)\left(cos\frac{1}{n+1} + cos\frac{1}{n+2} + \cdots + cos\frac{1}{n+i}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.059.

Compute:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(1^p + 2^p + \cdots + k^p)^2}{n^{p+1}(1^p + 2^p + \cdots + n^p)}; p \in \mathbb{N}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.060.

Prove that if $a, b, c, d \in \mathbb{R}; a^2 + b^2 \neq 0; c^2 + d^2 \neq 0$ then:

$$\frac{(ad - bc)(3(a^2 + b^2)(c^2 + d^2) - 4(ad - bc)^2)}{\big((a^2 + b^2)(c^2 + d^2)\big)^{\frac{3}{2}}} \leq 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.061.

Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers satisfying

$$\frac{x_1}{1 + x_1} + \frac{2x_2}{1 + x_2} + \cdots + \frac{nx_n}{1 + x_n} = 1$$

Find the maximum possible value of $P = x_1 x_1^2 \ldots x_n^n$.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.062.

If $a,b,c\in\mathbb{C}$ then:

$$\left|a^3+b^3+c^3-3abc\right|\le|a+b+c|(|a|+|b|+|c|)^2$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.063.

Prove that for any triangle ABC holds:

$$\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\ge$$
$$\ge\sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\frac{1}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right).$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.064.

If $x,y,z\in(0,1)$ then:

$$\left(\frac{yz}{1-x^2}\right)^{2n}+\left(\frac{zx}{1-y^2}\right)^{2n}+\left(\frac{xy}{1-z^2}\right)^{2n}\ge\frac{3^{3n+1}}{4^n}(xyz)^{2n}$$

for all $n\in\mathbb{N}$.

Proposed by Mihály Bencze – Romania

PROBLEM 2.065.

If $a,b,c>0$ and $n\in\mathbb{N}^*$ then:

$$2^n(a^n+b^n+c^n)\ge(a+b)^{n-1}(a+c)+(b+c)^{n-1}(b+a)+(c+a)^{n-1}(c+b)$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.066.

Let $t\in\mathbb{R}^+$ and $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ such that $\lim_{x\to\infty}\frac{f(x+1)}{x^t f(x)}\in\mathbb{R}_+$ and

$\lim_{x\to\infty}\left(f(x)\right)^{\frac{1}{x}}\cdot\frac{1}{x^t}\in\mathbb{R}_+^*$ .Prove that:

$$\lim_{x\to\infty}\left(f(x)\right)^{\frac{1}{x}}\cdot\frac{1}{x^t}=\frac{1}{e^t}\cdot\lim_{x\to\infty}\frac{f(x+1)}{x^t f(x)}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 2.067.

If $x,y,z>0$ then:

$$\frac{1}{(x^2+yz)(3x^2+2y^2+z^2)}+\frac{1}{(y^2+zx)(3y^2+2z^2+x^2)}+$$
$$+\frac{1}{(z^2+xy)(3z^2+2x^2+y^2)}\le\frac{x^2+y^2+z^2+xy+yz+zx}{24x^2y^2z^2}$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.068.

Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[mn+m]{(2n+1)!!}-\sqrt[mn]{(2n-1)!!}\right)\cdot n^{\frac{m-1}{m}}$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 2.069.

If $x, y, z > 0$ and $b \geq a > 0$ then:

$$\ln\frac{b^2+xy}{a^2+xy}+\ln\frac{b^2+yz}{a^2+yz}+\ln\frac{b^2+zx}{a^2+zx}\leq(b-a)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.070.

Prove that if $a, b, c \in \mathbb{R}$ then:

$$(2-a-b-c+abc)^2\leq(a^2+2)(b^2+2)(c^2+2)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.071.

Prove that if $a, b, c \in \mathbb{R}$ then:

$$(3abc-a^3-b^3-c^3)^2\leq(a^2+b^2+c^2)^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.072.

If $a, b, c > 0; n \in \mathbb{N}^*$ then:

$$\left(\frac{2na}{b+(2n-1)c}\right)^{\frac{2}{3}}+\left(\frac{2nb}{c+(2n-1)a}\right)^{\frac{2}{3}}+\left(\frac{2nc}{a+(2n-1)b}\right)^{\frac{2}{3}}\geq 3$$

Proposed by Marin Chirciu – Romania

PROBLEM 2.073.

If $x, y, z > 0$ then:

$$log\left(1+\frac{1}{x}\right)+log\left(1+\frac{1}{y}\right)+log\left(1+\frac{1}{z}\right)\geq 3\,log\left(1+\frac{3}{x+y+z}\right)$$

Proposed by Marin Chirciu – Romania

PROBLEM 2.074.

Let x, y, z be positive real numbers such that: $x+y+z=3$. Find the minimum value of:

$$P=\frac{x^4}{y^4\cdot\sqrt[3]{4z(x^5+1)}}+\frac{y^4}{z^4\cdot\sqrt[3]{4x(y^5+1)}}+\frac{z^4}{x^4\cdot\sqrt[3]{4y(z^5+1)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.075.

Let x, y, z be positive real numbers such that: $xyz=1$. Find the minimum of expression:

$$P=2(x+y+z)+\frac{x}{y^3+z^3+1}+\frac{y}{z^3+x^3+1}+\frac{z}{x^3+y^3+1}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.076.

Let a, b, c be the side – lengths of an acute triangle with perimeter 1. Prove that

$$E_1\geq a^ab^bc^c\geq E_2$$

where

$$E_1=\frac{(b+c-a)(c+a-b)(a+b-c)}{(b^2+c^2-a^2)^a(c^2+a^2-b^2)^b(a^2+b^2-c^2)^c},$$

and

$$E_2=\frac{(b^2+c^2-a^2)^{b+c}(c^2+a^2-b^2)^{c+a}(a^2+b^2-c^2)^{a+b}}{(b+c-a)(c+a-b)(a+b-c)}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.077.

Prove that in any acute triangle ABC the following inequality holds:

$$\frac{m_a}{h_a}\cos A+\frac{m_a}{h_b}\cos B+\frac{m_c}{h_c}\cos C\ge\frac{3}{2}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

PROBLEM 2.078.

Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that:

$$a^{-a}b^{-b}c^{-c}+a^{-b}b^{-c}c^{-a}+a^{-c}b^{-a}c^{-b}\le a^{-1}+b^{-1}+c^{-1}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.079.

Prove that for all positive real numbers a,b,c and integer $n\ge 3$, the following inequality holds:

$$\frac{a^n+b^n+c^n}{9}\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)\ge\left(\frac{b+c}{6a}+\frac{c+a}{6b}+\frac{a+b}{6c}\right)^n$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

PROBLEM 2.080.

Prove that for all positive real numbers a,b,c the following inequality holds:

$$\frac{(a+b)^2}{a^2-ab+b^2}+\frac{(b+c)^2}{b^2-bc+c^2}+\frac{(c+a)^2}{c^2-ca+a^2}\ge\frac{9(a^2b+b^2c+c^2a+abc)}{a^3+b^3+c^3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.081.

Let a,b,c be positive real numbers and $k\ge 2$. Prove that:

$$\sqrt{\frac{bc}{(b+ka)(c+ka)}}+\sqrt{\frac{ca}{(c+kb)(a+kb)}}+\sqrt{\frac{ab}{(a+kc)(b+kc)}}\ge\frac{3}{k+1}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.082.

Let ABC be an equilateral triangle with side-length a and let M be any point inside the triangle. Prove that:

$$\frac{a^2}{2}\ge xMA+yMB+zMC\ge 2(xy+yz+zx)$$

where x,y,z denote the distances from M to the sides BC,CA,AB, respectively.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.083.

Let m_a, m_b, m_c be the lengths of the medians of a triangle with circumradius R. Prove that:

$$\left(1+\frac{1}{m_a}\right)\left(1+\frac{1}{m_b}\right)\left(1+\frac{1}{m_c}\right) \geq \left(1+\frac{2}{3R}\right)^3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

PROBLEM 2.084.

Prove that if $n \in \mathbb{N}^*$ then:

$$2\int_0^1 \arctan(x^{n-1})\arctan(x^n)\,dx \leq \int_0^1 \arctan^2(x^n)\,dx + \frac{1}{2n-1}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.085.

Prove that if $a, b \in (0,\infty)$; $n \in \mathbb{N}^*$ then:

$$\left(\frac{a}{b^n}+\frac{b}{a^n}\right)\left(\frac{a^n}{b}+\frac{b^n}{a}\right)\left(\frac{a^n}{b^n}+\frac{b}{a}\right)\left(\frac{b^n}{a^n}+\frac{a}{b}\right) \geq 8\left(\sqrt{\left(\frac{a}{b}\right)^{n-1}}+\sqrt{\left(\frac{b}{a}\right)^{n-1}}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.086.

Prove that if a, b, c are the length's sides in $\Delta\ ABC$ then:

$$\sin^2 a + \sin^2 b + \sin^2 c \geq 4\sin s \cdot \sin(s-a)\cdot \sin(s-b)\cdot \sin(s-c)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.087.

Let z_1, z_2, z_3 be the affixes of A, B respectively C in acute-angled $\Delta\ ABC$.
Prove that:

$$\prod\left(\left|\frac{z_2-z_3}{z_2+z_3}\right|+\left|\frac{z_3-z_1}{z_3+z_1}\right|\right) \geq \frac{32sr^3}{(s^2-(2R+r)^2)^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.088.

Let $a, b, c > 0$ such that $ab+bc+ca+abc=4$.Prove that:

$$(a+1)\sqrt{(b+1)(c+1)}+(b+1)\sqrt{(c+1)(a+1)}+(c+1)\sqrt{(a+1)(b+1)} \geq$$
$$\geq a+b+c+9$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

PROBLEM 2.089.

Let r_a, r_b, r_c be the exradii of a triangle ABC, h_a, h_b, h_c the altitudes and let R, r, s denote the circumradius, inradius and semiperimeter respectively. Prove that

$$\frac{r_a^2}{h_a}+\frac{r_b^2}{h_b}+\frac{r_c^2}{h_c} \geq \frac{2s^2}{3}\left(\frac{1}{r}-\frac{1}{R}\right)$$

Proposed by Martin Lukarevski-Skopje-Macedonia

PROBLEM 2.090.

If $u, v > 0$, with $2u - v > 0$ and α, β, γ are the measures of the angles of triangle ABC, then

$$\sum_{cyc} \frac{\sin\alpha}{u\sin\beta + v\sqrt{\sin\alpha\sin\beta}} \geq \frac{3}{u+v}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 2.091.

Prove that for all positive real numbers a, b, c, d:

$$\frac{a^2}{a+b+c} + \frac{b^2}{b+c+d} + \frac{c^2}{c+d+a} + \frac{d^2}{d+a+b} \geq \frac{a+b+c+d}{3} + \frac{4(2a+b-2c-d)^2}{27(a+b+c+d)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.092.

Prove that for all positive real numbers a, b, c:

$$a.\ \frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \geq \frac{a+b+c}{2} + \frac{(b-c)^2}{2(a+b+c)}$$

$$b.\ \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2} + \frac{(a+b-2c)^2}{2(a+b+c)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.093.

Prove that in any triangle ABC the following inequality holds:

$$\frac{(b+c)a}{m_a^2} + \frac{(c+a)b}{m_b^2} + \frac{(a+b)c}{m_c^2} \geq 8$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.094.

Prove that in any acute triangle ABC the following inequality holds

$$\frac{\cos B\cos C}{\sin A} + \frac{\cos C\cos A}{\sin B} + \frac{\cos A\cos B}{\sin C} \leq \frac{\sqrt{3}}{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.095.

Let a, b, c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that:

$$(b^4+c^4)\sin^2 A + (c^4+a^4)\sin^2 B + (a^4+b^4)\sin^2 C \leq \frac{81}{4}(3R^4 - 16r^4)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.096.

Let ABC be a triangle and w_a, w_b, w_c are bisectors of ABC. Prove that:

$$\frac{1}{aw_a^2} + \frac{1}{bw_b^2} + \frac{1}{cw_c^2} \geq \frac{1}{R\Delta}$$

where R is the circumradius of ABC, Δ is area of ABC.

Proposed by Mehmet Şahin – Ankara – Turkey

PROBLEM 2.097.

Let a, b, c be the side lengths of a triangle ABC with incentre I, circumradius R and inradius r. Prove that:

$$\frac{\sqrt{AI}}{a}+\frac{\sqrt{BI}}{b}+\frac{\sqrt{CI}}{c}\le\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{R+r}}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.098.

Let ABC be an acute triangle with orthocenter H. Prove that:

$$AH\cdot BH+BH\cdot CH+CH\cdot AH\le 6Rr$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.099.

Let a, b, c be non-negative such that $a+b+c=3$. Prove that:

$|(a-b)(b-c)(c-a)|\le\frac{3\sqrt{3}}{2}$. Equality occurs when?

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

PROBLEM 2.100.

Let a, b, c be the lengths of the sides of a triangle with perimeter 3 and inradius r. Prove that:

$$288r^2\le\frac{(a+b)^4}{a^2+b^2}+\frac{(b+c)^4}{b^2+c^2}+\frac{(c+a)^4}{c^2+a^2}\le\frac{2}{r^2}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.101.

Let a, b and c be the side lengths of a triangle with inradius r. Prove that:

$$\sqrt[4]{\frac{1}{a^4+2b^2c^2}+\frac{1}{b^4+2c^2a^2}+\frac{1}{c^4+2a^2b^2}}\le\frac{\sqrt{3}}{6r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.102.

Let ABC be a triangle with circumradius R and inradius r. Prove that:

$$4\le sec^2\frac{A}{2}+sec^2\frac{B}{2}+sec^2\frac{C}{2}\le\frac{2R}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.103.

Let m, n be positive real numbers. Prove that:

$$\left(\frac{1}{m}+\frac{1}{n}\right)^{-1}\le\frac{4034-2015m}{m+2017}+\frac{4034-2015n}{n+2017}+\frac{m+n+2009}{2}$$

Proposed by Iuliana Trașcă – Romania

PROBLEM 2.104.

Prove that in any triangle ABC the following relationship holds:

$$r\sum\frac{1}{\sin\frac{A}{2}}+\frac{abc}{2}\sum\frac{1}{\sqrt{abs(s-c)}}\leq 6R$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.105.

Let G be the centroid in $\Delta\ ABC$. Prove that:

$$cot(\widehat{GBA})+cot(\widehat{GCB})+cot(\widehat{GAC})>\cot A+\cot B+\cot C+3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.106.

In ΔABC the following relationship holds:

$$(a\cot 20^\circ+b\cot 40^\circ+c\cot 80^\circ)^3>9\sqrt{3}r\left(\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.107.

Prove that:

$$\left(\int_0^1 arctan^2 x\,dx\right)\left(\int_0^1\frac{dx}{arctan^2\left(\frac{1}{x^2-x+1}\right)}\right)>\frac{1}{4}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.108.

If $a,b,c>0, a+b+c=abc$ then:

$$\frac{4(a+b)(a+c)}{(b+c)^2}+\frac{4(b+c)(b+a)}{(c+a)^2}+\frac{4(c+a)(c+b)}{(a+b)^2}\leq 3+a^2+b^2+c^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.109.

If $a,b,c\geq 0;\Omega(a)=\int_0^a\sin\left(\frac{x}{x^2+1}\right)dx$ then:

$$e^{\pi(b\Omega(a)+c\Omega(b)+a\Omega(c))}\geq(a^2+1)^b(b^2+1)^c(c^2+1)^a$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.110.

Let $m,x,y,z>0$ be positive real numbers and F be the area of the triangle ABC. Prove that:

$$\frac{a^{2m+2}x^{m+1}}{(y+z)^{m+1}}+\frac{b^{2m+2}y^{m+1}}{(z+x)^{m+1}}+\frac{c^{2m+2}z^{m+1}}{(x+y)^{m+1}}\geq\frac{2^{m+1}}{(\sqrt{3})^{m-1}}F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

PROBLEM 2.111.

Let $x,y,z>0$ be positive real numbers and F the area of the triangle ABC. Prove that:

$$\frac{(y+z)^2a^4}{x^2}+\frac{(z+x)^2b^4}{y^2}+\frac{(z+x)^2c^4}{z^2}\geq 64F^2$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

PROBLEM 2.112.

Let $x, y, z > 0$ be positive real numbers and F be the area of the triangle ABC with circumradius R. Prove that:

$$\frac{x}{y+z}\sin^2\frac{A}{2}+\frac{y}{z+x}\sin^2\frac{B}{2}+\frac{z}{x+y}\sin^2\frac{C}{2}\geq\frac{2\sqrt{3}F}{R^2}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania, Martin Lukarevski-Skopje-Macedonia

PROBLEM 2.113.

If $x, y, z > 0$ then:

$$4xyz(x^3+y^3+z^3)\leq(x^2+y^2)(x^4+y^4)+(y^2+z^2)(y^4+z^4)+(z^2+x^2)(z^4+x^4)$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.114.

If $x, y, z > 0$ then:

$$2\left(\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2+\left(z+\frac{1}{z}\right)^2\right)\leq\frac{3}{2}+\frac{1}{xyz}\left((x^4+y^4)z+(y^4+z^4)x+(z^4+x^4)y\right)+$$
$$+\frac{x^2+y^2}{z}+\frac{y^2+z^2}{x}+\frac{z^2+x^2}{y}$$

Proposed by Mihály Bencze – Romania

PROBLEM 2.115.

Let a, b, c be the lengths of the sides of a triangle with inradius r and circumradius R. Let r_a, r_b, r_c be the exradii of triangle. Prove that:

$$1728\cdot r^5\leq\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_c}\leq 108R^4(R-r)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.116.

A triangle with side lenghts a, b, c has perimeter equal to 3.Prove that:

$$a^3+b^3+c^3+a^4+b^4+c^4\geq 2(a^2b^2+b^2c^2+c^2a^2)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.117.

Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$a.\ \frac{8\sqrt{3}}{3}\leq\frac{1}{\cos^3\frac{A}{2}}+\frac{1}{\cos^3\frac{B}{2}}+\frac{1}{\cos^3\frac{C}{2}}\leq\frac{2\sqrt{3}}{3}\left(\frac{R}{2}\right)^2$$

$$b.\ 9\left(\frac{r}{R}\right)^2\leq\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\leq\frac{9}{4}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.118.

Let $a, b, c > 0$ such that: $a^2+b^2+c^2=3$. Find the minimum of the expression:

$$P=\frac{a^3}{\sqrt[4]{\frac{b^8+c^8}{2}}+5bc}+\frac{b^3}{\sqrt[4]{\frac{c^8+a^8}{2}}+5ca}+\frac{c^3}{\sqrt[4]{\frac{a^8+b^8}{2}}+5ab}+\frac{(a+b)(b+c)(c+a)}{16}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.119.

Let $a, b, c > 0$ such that: $a + b + c = 3$. Find the minimum of the expression:

$$P = \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc} + \frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca} + \frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab} + \frac{(a+b)(b+c)(c+a)}{24}$$

Proposed by Hoang Le Nhat Tung – Hanoi - VietNam

PROBLEM 2.120.

In $\Delta\ ABC$ the following relationship holds:

$$\sqrt[3]{a^2B} + \sqrt[3]{b^2C} + \sqrt[3]{c^2A} \le \sqrt[3]{4\pi s^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.121.

Let x, y, z be positive real numbers such that: $x + y + z = 3$. Prove that:

$$\frac{x^4}{5-3\sqrt[3]{y}} + \frac{y^4}{5-3\sqrt[3]{z}} + \frac{z^4}{5-3\sqrt[3]{x}} + \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2} \ge 3$$

Proposed by Hoang Le Nhat Tung – Hanoi – VietNam

PROBLEM 2.122.

If $z_1, z_2, z_3 \in \mathbb{C}$ are different in pairs and $|z_1| = |z_2| = |z_3| = 1$ then:

$$|z_1 - z_3| + |z_2 - z_3| \le 3 + |z_1 + z_2|$$

Proposed by Marian Ursărescu – Romania

PROBLEM 2.123.

Let $A \in M_n(\mathbb{R})$ be a symmetric and invertible matrix. Prove that: $det(A^2 + A^{-2} + 2I_n) \ge 4^n$

Proposed by Marian Ursărescu – Romania

PROBLEM 2.124.

Let a, b, c be the side lenghts of a triangle ABC with inradius r and circumradius R. Prove:

$$\frac{3}{2} \le \frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \le \frac{2R-r}{2r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.125.

Let triangle ABC have exradii r_a, r_b, r_c, altitudes h_a, h_b, h_c and a, b, c be the lengths of the sides. Prove that:

$$\left(\frac{h_a}{r_a}\right)^2 + \left(\frac{h_b}{r_b}\right)^2 + \left(\frac{h_c}{r_c}\right)^2 \le \frac{1}{2}\left(\frac{a^4+b^4}{c^4} + \frac{b^4+c^4}{a^4} + \frac{c^4+a^4}{b^4}\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.126.

Let m_a, m_b, m_c the lenghts of the medians of a triangle ABC with circumradius R. Prove that:

$$\frac{1}{m_am_b} + \frac{1}{m_bm_c} + \frac{1}{m_cm_a} + \frac{3}{m_am_b+m_bm_c+m_cm_a} \ge 4 \cdot \frac{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}{m_a+m_b+m_c}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 2.127.

Let be $A, B \in M_n(\mathbb{R})$ such that: $A^2 + B^2 = 2\sin\frac{\pi}{x}AB$. If $AB - BA$ is invertible, then nx is an even integer.

Proposed by Marian Ursărescu – Romania

PROBLEM 2.128.

Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum of the expression:

$$P = \frac{x^3}{y\left(\sqrt{2(y^4+z^4)}+yz\right)^2} + \frac{y^3}{z\left(\sqrt{2(z^4+x^4)}+zx\right)^2} + \frac{z^3}{x\left(\sqrt{2(x^4+y^4)}+xy\right)^2} + \frac{\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}}{27}$$

Proposed by Hoang Le Nhat Tung – Hanoi – VietNam

PROBLEM 2.129.

Let a, b, c be positive real numbers such that: $a + b + c = 3$. Prove that:

$$\frac{a^3}{b^2\left(\sqrt[4]{\frac{b^8+c^8}{2}}+5bc\right)} + \frac{b^3}{c^2\left(\sqrt[4]{\frac{c^8+a^8}{2}}+5ca\right)} + \frac{c^3}{a^2\left(\sqrt[4]{\frac{a^8+b^8}{2}}+5ab\right)} + \frac{ab+bc+ca}{2(a^2+b^2+c^2)} \geq 1$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.130.

Let a, b, c be positive real numers such that: $a + b + c = 3$. Prove that:

$$\frac{a^3}{b^2(b^2+bc+c^2)} + \frac{b^3}{c^2(c^2+ca+a^2)} + \frac{c^3}{a^2(a^2+ab+b^2)} + \frac{ab+bc+ca}{a^2+b^2+c^2} \geq 2$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.131.

Let $a, b, c > 0$, with sum 1. Prove that:

$$\sum ab^2 \geq \left(\sum ab\right)^2 - 10\sum ab + 12abc$$

Proposed by Mihalcea Andrei Stefan-Romania

PROBLEM 2.132.

Let a, b be two positive numbers.Prove that:

$$\frac{(1+ab)(ab-a-b-1)}{(a^2+1)(b^2+1)} + \frac{4(a^2+b^2+a+b)}{(2+a+b)(a^2+b^2+a+b)-2(a+b)^2} \leq 1$$

Proposed by Mihalcea Andrei Ștefan – Romania

PROBLEM 2.133.

Let $a, b, c > 0$, with sum 1. Prove that:

$$2\sqrt{abc}\sum\frac{a}{1+a^2} \leq \frac{1+\sum a^2}{3+\sum a^2}$$

Proposed by Mihalcea Andrei Ștefan – Romania

PROBLEM 2.134.

Let $ABCD$ be a cyclic quadrilateral with perimeter 2.Denote $AB = a, BC = b, CD = c$, $DA = d$.Prove that:

$$4\le\sum tan\frac{A}{2}<\frac{2(a+c)(b+d)}{\sqrt{\prod(1-a)}}$$

Proposed by Mihalcea Andrei Ștefan – Romania

PROBLEM 2.135.

If $a,b,c>1$ then:

$$\frac{log_a b}{a+b+c}+\frac{log_b c}{b+c+d}+\frac{log_c d}{c+d+a}+\frac{log_d a}{d+a+b}\ge\frac{16}{3(a+b+c+d)}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 2.136.

Let x,y,z be positive real numbers such that: $x^4+y^4+z^4=xy+yz+zx$. Find the maximum of the expression:

$$P=\sqrt[3]{\frac{x^6+y^6}{2}}+\sqrt[3]{\frac{y^6+z^6}{2}}+\sqrt[3]{\frac{z^6+x^6}{2}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.137.

Let $a,b,c>0$ such that $a+b+c=3$. Prove that:

$$\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{7}{12}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.138.

Let a,b,c be positive real numbers such that: $a+b+c=3$. Prove that:

$$\frac{a^2}{\sqrt{5(b^4+4)}}+\frac{b^2}{\sqrt{5(c^4+4)}}+\frac{c^2}{\sqrt{5(a^4+4)}}\ge\frac{3}{5}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.139.

In ABC triangle the lengths of sides BC,CA,AB are a,b,c. Let h_a,h_b,h_c be the distances from A,B,C to BC,CA,AB; l_a,l_b,l_c are the lengths of the bisectors A,B,C. Prove that:

$$\frac{l_al_b}{l_c}+\frac{l_bl_c}{l_a}+\frac{l_cl_a}{l_b}\ge\frac{h_ah_b}{h_c}+\frac{h_bh_c}{h_a}+\frac{h_ch_a}{h_b}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.140.

Let a,b,c be positive real numbers. Prove that:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\ge\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

PROBLEM 2.141.

Let $a,b,c>0$ such that: $a+b+c=3$. Prove that:

$$\frac{a^4}{b^4(2ab-\sqrt{c}+2)}+\frac{b^4}{c^4(2bc-\sqrt{a}+2)}+\frac{c^4}{a^4(2ca-\sqrt{b}+2)}\geq\frac{a^2+b^2+c^2}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.142.

Let a,b,c be positive real numbers such that: $abc=1$. Prove that:

$$\frac{a^2b^2}{a^2-2a+b^2+2}+\frac{b^2c^2}{b^4-2b+c^2+2}+\frac{c^2a^2}{c^4-2c+a^2+2}\leq\frac{a^2+b^2+c^2+3}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.143.

Let x,y,z be non-negative real numbers. Prove that:

$$x\sqrt{3x^2+yz}+y\sqrt{3y^2+zx}+z\sqrt{3z^2+xy}\geq x^2+y^2+z^2+xy+yz+zx$$

Proposed by Do Quoc Chinh – Ho Chi Minh – Vietnam

PROBLEM 2.144.

Let A,B,C be the corners in a triangle ABC. Prove that:

$$\left(\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}}\right)^2+\left(\frac{\sin\frac{B}{2}}{\tan\frac{C}{2}}\right)^2+\left(\frac{\sin\frac{C}{2}}{\tan\frac{A}{2}}\right)^2\geq\frac{9}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.145.

If $1<a\leq b$ then:

$$\int_a^b\int_a^b\int_a^b\frac{dx\,dy\,dz}{1+\sqrt[3]{xyz}}\leq\log\left(\sqrt[3]{\frac{b+1}{a+1}}\right)^{(b-a)^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.146.

Let be $A,B\in M_3(\mathbb{R})$ such that:

$$AB=\begin{pmatrix}2&1&1\\0&-1&1\\0&0&-1\end{pmatrix}$$

Find: $det((BA)^2-3I_3)$

Proposed by Marian Ursărescu – Romania

PROBLEM 2.147.

Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ having the property:

$$f(x)+2f(2x)+f(4x)=25x^2+9x+4,\forall x\in\mathbb{R}$$

Proposed by Marian Ursărescu – Romania

PROBLEM 2.148.

Let be $x_0 > 0$ and $x_{n+1} = \arctan\frac{x_n}{1+x_n}, \forall n \in \mathbb{N}$. Find:

$$\Omega = \lim_{n\to\infty}(n \cdot x_n)$$

Proposed by Marian Ursărescu – Romania

PROBLEM 2.149.

Let be the sequence $(x_n)_{n\in\mathbb{N}}$: $x_0 > 1$ and $x_{n+1} = 1 + \ln\left(\frac{2x_n}{1+x_n}\right), \forall n \in \mathbb{N}$. Find:

$$\lim_{n\to\infty}(n \ln x_n)$$

Proposed by Marian Ursărescu – Romania

PROBLEM 2.150.

Let be $f \in \mathbb{Z}, f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, such that $a_1, a_2, \dots, a_n \in \{\pm 1, \pm 2, \dots, \pm n\}$. If a_0 is a prime number, $a_0 > n^2$ then f is irreducible over $\mathbb{Z}$.

Proposed by Marian Ursărescu – Romania

PROBLEM 2.151.

Let a, b, c be positive real numbers such that: $a + b + c = 3$.
Prove that:

$$\frac{a}{\sqrt{2(b^4+c^4)+7bc}} + \frac{b}{\sqrt{2(c^4+a^4)+7ca}} + \frac{c}{\sqrt{2(a^4+b^4)+7ab}} \geq \frac{1}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.152.

Let a, b, c be positive real numbers. Find the minimum value of:

$$P = \frac{1}{\sqrt{2(a^4+b^4)}} + \frac{1}{\sqrt{2(b^4+c^4)}} + \frac{1}{\sqrt{2(c^4+a^4)}} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.153.

Solve the system of equations:

$$\begin{cases} 2\left(\frac{x^3}{y^2} + \frac{y^3}{x^2}\right) = \sqrt[4]{8(x^4+y^4)} + 2\sqrt{xy} \\ 16x^5 - 20x^3 + 5\sqrt{xy} = \sqrt{\frac{y+1}{2}} \end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.154.

Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum value of:

$$P = \frac{\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} + 1}{4(\sqrt{x} + \sqrt{y} + \sqrt{z})} + \frac{8}{3}(x^2 + y^2 + z^2)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.155.

Let x, y, z be positive real numbers such that: $xyz = 1$. Find the minimum value of:

$$P = \frac{x^3}{(2y^2 - yz + 2z^2)^2} + \frac{y^3}{(2z^2 - zx + 2x^2)^2} + \frac{z^3}{(2x^2 - xy + 2y^2)^2} + \frac{xy + yz + zx}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 2.156.

Find all the polynomials $P \in \mathbb{R}[x]$ having the property

$$P(x) = P\left(x + \sqrt{x^2 + 1}\right), \forall x \in \mathbb{R}$$

Proposed by Marian Ursărescu – Romania

PROBLEM 2.157.

Let $f: [0, +\infty) \to [0, +\infty)$ a derivable function and $a > 1$. If $f'(x)(f(x) + x^2 + 2x + a) = 1, \forall x \geq 0$ then: $\lim_{x\to\infty} f(x)$ exists and it is finite.

Proposed by Marian Ursărescu – Romania

PROBLEM 2.158.

Prove that for any real numbers $a_1, a_2, \dots, a_n$, the inequality holds:

$$\sum_{i=1}^{n} a_i(a_i + a_{i+1})\left(a_i^2 + a_{i+1}^2\right) \dots \left(a_i^{2^k} + a_{i+1}^{2^k}\right) \geq 0$$

Where k is a positive integer and $a_{n+1} = a_1$. When does the equality occur?

Proposed by Nguyen Viet Hung - Hanoi – Vietnam

PROBLEM 2.159.

Prove that in any triangle ABC with standard notations, the inequality holds:

$$(a + b + c)(a^2 + b^2 + c^2) \geq 2(b + c)h_a^2 + 2(c + a)h_b^2 + 2(a + b)h_c^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.160.

Let a, b, c be positive real numbers such that $a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that:

$$3(a^3b + b^3c + c^3a) \geq (a + b + c)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.161.

Prove that the following inequalities holds for all real numbers $a, b, c \in [0, 1]$

$$a.\ (1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \leq (1 - abc + a^2b^2c^2)$$

$$b.\ (1 - a + a^2)^2(1 - b + b^2)^2(1 - c + c^2)^2 \leq$$
$$\leq (1 - ab + a^2b^2)(1 - bc + b^2c^2)(1 - ca + c^2a^2)$$

When does the equality occur?

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.162.

If $m \in [0, \infty), x, y, z, t \in (0, \infty)$, then in any triangle ABC, with usual notations holds:

$$\sum_{cyclic} \frac{(xa^2 + ym_b^2)^{m+1}}{(zw_b^2 + th_c^2)^m} \geq \frac{(4x + 3y)^{m+1}}{3^m(z + t)^m} \sqrt{3} S$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 2.163.

If $m \in [0, \infty), x, y, z, t \in (0, \infty)$, then in any triangle ABC, with usual notations holds:

$$\sum_{cyclic} \frac{(xa^2 + yb^2)^{m+1}}{(zw_b^2 + tw_c^2)^m} \geq \frac{4^{m+1}(x + y)^{m+1}}{3^{m-\frac{1}{2}}(z + t)^m} S$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 2.164.

If $a, b, c > 0$ then:

$$(a + b)\sqrt{a^2 + b^2 - ab} + (b + c)\sqrt{b^2 + c^2 - bc} + (c + a)\sqrt{c^2 + a^2 - ca} \geq$$
$$\geq 2(ab + bc + ca)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.165.

If $a, b, c \geq 0$ then:

$$(a + b)\sqrt{a^2 + b^2} + (b + c)\sqrt{b^2 + c^2} + (c + a)\sqrt{c^2 + a^2} \geq$$
$$\geq (2\sqrt{3} - 1)(ab + bc + ca)$$

Proposed by Daniel Sitaru – Romania

UNDERGRADUATE PROBLEMS

PROBLEM 3.001.

Prove that if $\alpha \in [2,7]$ then:

$$\int_2^{\alpha} \arctan^5 x \cdot dx \leq \frac{\alpha - 2}{5} \int_2^{7} \arctan^5 x \cdot dx$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.002.

Let a, b, c be positive real numbers. Prove that:

$$\sum \frac{1}{5a} + \sum \frac{1}{3a+b+c} \geq \sum \left(\frac{1}{4a+b} + \frac{1}{4a+c}\right)$$

Proposed by Soumitra Mandal - Chandar Nagore – India

PROBLEM 3.003.

If $A, B \in M_2(C)$ then:

$$(\det(A+B))^2 + (\det(A-B))^2 \geq 2\det(AB+BA)$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.004.

Let $f: [a,c] \to \mathbb{R}, 0 < a < c$ be a continuous and convex function on $[a,c]$. Prove that if $b \in [a,c]$ then:

$$2\int_a^c f(x)\,dx \leq (b-a)[f(b)+f(a)] + (c-b)[f(b)+f(c)]$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.005.

If $x \geq 1$ then:

$$ex \ln\left(1+\frac{1}{x}\right) \leq \left(1+\frac{1}{x}\right)^x \leq \frac{2x}{\ln 2} \ln\left(1+\frac{1}{x}\right)$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.006.

If $a, b, c > 0$ and $x, y, z \geq 1$ then:

$$\left(\frac{xz}{y}\right)^{2a} \left(\frac{yx}{z}\right)^{2b} \left(\frac{zy}{x}\right)^{2c} \leq x^{\frac{a^2+b^2}{c}} y^{\frac{b^2+c^2}{a}} z^{\frac{c^2+a^2}{b}}$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.007.

Let be $a, r \in (0,\infty)$; $(a_n)_{n\geq 1}$; $a_1 = a$; $a_{n+1} = a_n + r, n \in \mathbb{N}^*$;

$$b_n = \prod_{k=1}^{n} a_k, c_n = \prod_{k=1}^{n} b_k^2.$$

Find:

$$\lim_{n\to\infty}\frac{n}{\sqrt[n^2]{c_n}}.$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.008.

Find:

$$\int\frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1}dx$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.009.

Prove that if $n\in\mathbb{N}; n\geq 3$ then:

$$\left(\frac{n!}{2}\right)^{2e}\leq e^{n^2+n-6}$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.010.

Find:

$$\int\frac{e^x\ln(1+e^x)-e^{2x}}{(1+e^x)^2\ln^2(1+e^x)}dx;\ x\in\mathbb{R}$$

Proposed by Daniel Sitaru-Romania

PROBLEM 3.011.

If $a,b,c>0$ and $x,y,z\geq 1$ then:

$$x^{\frac{18a}{3a^2+2b^2+c^2}}\cdot y^{\frac{18b}{3b^2+2c^2+a^2}}\cdot z^{\frac{18c}{3c^2+2a^2+b^2}}\leq\left(\frac{z^2}{y}\right)^{\frac{1}{a}}\cdot\left(\frac{x^2}{z}\right)^{\frac{1}{b}}\cdot\left(\frac{y^2}{x}\right)^{\frac{1}{c}}$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.012.

If $x>0$ then compute:

$$\int\frac{2e^x+\sin x+1003}{e^x+2\sinh x+\sin x-\cos x+2006}dx$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.013.

Let $(A,+,\cdot)$ be a ring with $1\neq 0$. If $x,y\in A$ such that $x+y=1$ and $x^{2016}=x$ prove that the elements $1-xy$ and $1-yx$ are invertible.

Proposed by Nicolae Papacu –Romania

PROBLEM 3.014.

Prove that:

$$\lim_{p\to\infty}\sum_{n=1}^{p}\left(\sum_{m=1}^{p}\frac{1}{mn(m+n)}\right)<\frac{\pi^3}{6}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.015.

Let be $(A,+,\cdot)$. If it does exists $k\in\mathbb{N}^*$ such that for any $a,b\in A$ we have $(a+b)^{2k+1}=a^{2k}+b^{2k}$ and $(a+b)^{2k+3}=a^{2k+2}+b^{2k+2}$, then prove that the ring is commutative.

Proposed by Dana Heuberger – Romania

PROBLEM 3.016.

Compute the limit:

$$\lim_{n\to\infty}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta}d\theta$$

Proposed by Kunihiko Chikaya - Tokyo – Japan

PROBLEM 3.017.

Let $\{a_n\}$ be a sequence defined inductively by

$$a_1=1,a_{n+1}=\frac{1}{2}a_n+\frac{n^2-2n-1}{n^2(n+1)^2}\quad(n=1,2,3,\dots).$$

Find the greatest value of n such that $a_1+a_2+\cdots+a_n$ is minimized.

Proposed by Kunihiko Chikaya – Tokyo – Japan

PROBLEM 3.018.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{\sqrt{\cos x}}{x}dx\le\frac{\pi}{24}\left(e^{\frac{\pi}{12}}-1\right)^2+\frac{\sqrt{2}-1}{4}+\frac{152}{\pi^3}$$

Proposed by Soumitra Mandal – Chandar Nagore – India

PROBLEM 3.019.

Let $E=(n,k,p)$ be the total number of $(x_1,x_2,\dots,x_p)$ for which $x_1+x_2+\cdots x_p$ is a perfect k power when the integers $x_1,x_2,\dots,x_p$ are selected independently at random from the set $\{1,2,\dots,n\}$. Compute: $\lim_{n\to\infty}\frac{E(n,k,p)}{n\sqrt[k]{n}}$ for $p=2;\ ^*p\ge 3$.

Proposed by Mihály Bencze – Romania

PROBLEM 3.020.

If $x_k\in\mathbb{R}\ \ (k=1,2,\dots,n)$ then:

$$\left(\sum_{k=1}^{n}\frac{x_k}{k(k+1)}\right)^2\le\frac{n}{n+1}\sum_{k=1}^{n}\frac{x_k^2}{k(k+1)}$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.021.

Prove that:

$$1\le\int_0^1\frac{dx}{\sqrt{1-x^2+x^{2015}-x^{2016}}}\le\frac{\pi}{2}$$

Proposed by Soumitra Mandal – Chandar Nagore – India

PROBLEM 3.022.

Let ABC be a triangle with the area S and denote by r, r_a, r_b, r_c inradius, exradii respectively. Prove that:

$$(r^2 + r_a r_b)(r^2 + r_b r_c)(r^2 + r_c r_a) \geq \left(\frac{10}{3}\right)^3 (rS)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietam

PROBLEM 3.023.

If $x, y, z, z, b, c > 0$ then:

$$\left(\frac{x+y}{2x}\right)^{b+c}\left(\frac{y+z}{2y}\right)^{c+a}\left(\frac{z+x}{2z}\right)^{a+b} \geq (x+y)^{b-c}(y+z)^{c-a}(z+x)^{a-b}$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.024.

Calculate:

$$\sum_{n=2}^{\infty} \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right),$$

where F_n is the n th Fibonacci number.

Proposed by Cornel Ioan Vălean – Romania

PROBLEM 3.025.

Compute:

$$\lim_{n\to\infty}\left(\sqrt[3n+3]{(n+1)!} - \sqrt[3n]{n!}\right)\cdot\sqrt[3]{n^2}$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.026.

Compute:

$$\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[2n+2]{(2n+1)!!}} - \frac{n}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}}$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.027.

If $\Gamma: (0,\infty) \to (0,\infty)$ is Euler's function, compute:

$$\lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{2x+2}}} - \frac{x}{(\Gamma(x+1))^{\frac{1}{2x}}}\right)^{\sqrt{x}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.028.

Let be $f: (0,\infty) \to (0,\infty), f(x) = (x+1)^{\frac{(m+1)(x+2)}{x+1}} - x^{\frac{(m+1)(x+1)}{x}}; m \in [0,\infty)$. Compute:

$$lim_{x\to\infty}\frac{f(x)}{x^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.029.

Let $x, y \in (0, \infty)$. Prove that $\frac{2}{\pi}\arctan(x+y)\arctan\left(\frac{1}{x+y}\right) < \frac{x+y}{4xy+1}$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.030.

Prove that:

$$\sum_{k=1}^{n} \frac{k^{m-1}+(k+1)^{m-1}+(k+2)^{m-1}}{(2k+3)k^{m+1}+2(k+1)^{m+2}+(2k+1)(k+2)^{m+1}} \le \frac{1}{8} - \frac{1}{4(n+1)(n+2)}$$

for all $n, m \in \mathbb{N}^*$.

Proposed by Mihály Bencze – Romania

PROBLEM 3.031.

If $A, B, C \in M_n(\mathbb{C})$; $n \ge 2$; $\det A \neq 0$; $AB = BA$; $AC = CA$; $A^2B + C = ABC$ then $BC = CB$.

Proposed by D.M. Bătinețu – Giurgiu – Romania

PROBLEM 3.032.

If $x, y, z \in \mathbb{C}^*$; $A, B, C \in M_n(\mathbb{C})$; $n \ge 2$; $x^2A + B = xAB$; $y^2B + C = yBC$;
$z^2C + A = zCA$ then:

$$\frac{xy(yz+z)+1}{yz}A + \frac{yz(zx+2)+1}{zx}B + \frac{zx(xy+2)+1}{xy}C = 3ABC$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.033.

If $a, b > 0$ then:

$$2\left(\sqrt{a}+\sqrt{b}\right)^2 + \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le 3\sqrt[3]{\frac{a+b}{2}}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{\frac{2a+b}{3}}+\sqrt[3]{\frac{a+2b}{3}}\right)$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.034.

Find the numbers $a, b, c \in \mathbb{N}^*$ knowing that: $\frac{a+1}{b} \in \mathbb{N}, \frac{b+1}{c} \in \mathbb{N}$ and $\frac{c+1}{a} \in \mathbb{N}$.

Proposed by Gheorghe Alexe; George – Florin Șerban – Romania

PROBLEM 3.035.

Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!} - \sqrt[3n]{(2n-1)!!}\right)\sqrt[3]{n^2}$$

Proposed by D. M. Bătinețu – Giurgiu – Romania

PROBLEM 3.036.

Let $(a_n)_{n\ge1}$ be a positive real sequence such that:

$$\lim_{n\to\infty}(a_{n+1}-a_n) = r \in R_+^*, u, v \in R, u+v = 1.$$

We denote $a_n! = a_1a_2 \dots a_n$, $G_n = (a_n!)^{\frac{1}{n}}, \forall\, n \in N^*$. Compute:

$$\lim_{n\to\infty}\left((n+1)^u \sqrt[n+1]{(G_{n+1}!)^v} - n^u\sqrt[n]{(G_n!)^v}\right)$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.037.

Let $(a_n)_{n\geq1}, (b_n)_{n\geq1}$ be positive real sequence such that
$\lim_{n\to\infty}(a_{n+1}-a_n)=a\in\mathbb{R}_+^*, \lim_{n\to\infty}(b_{n+1}-b_n)=b\in\mathbb{R}_+^*, u,v\in\mathbb{R}$ with $u+v=1$.
Calculate:

$$\lim_{n\to\infty}\left(a_{n+1}^u\sqrt[n+1]{(b_1b_2\dots b_nb_{n+1})^v}-a_n^u\sqrt[n]{(b_1b_2\dots b_n)^v}\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.038.

Let $(a_n)_{n\geq1}$ be a positive real sequence such that: $\lim_{n\to\infty}(a_{n+1}-a_n)=r\in\mathbb{R}_+^*$
We denote $a_n!=a_1a_2\dots a_n, G_n=(a_n!)^{\frac{1}{n}}, \forall\, n\in\mathbb{N}^*$. Evaluate:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{G_{n+1}!}}-\frac{n^2}{\sqrt[n]{G_n!}}\right)$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.039.

Let $(a_n)_{n\geq1}, (b_n)_{n\geq1}$ be positive real sequences with:

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a\in R_+^*, \lim_{n\to\infty}(b_{n+1}-b_n)=b\in R_+^*,$$

$$P_n=\sqrt{\frac{a_1^2+a_2^2+\dots+a_n^2}{n}}, P_n!=P_1P_2\dots P_n, \forall n\in N^*, u,v\in R, u+v=1.\text{ Find}$$

$$\lim_{n\to\infty}\left(b_{n+1}^u\sqrt[n+1]{(P_{n+1}!)^v}-b_n^u\sqrt[n]{(P_n!)^v}\right)$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.040.

Let $(a_n)_{n\geq1}$ be a positive real sequence such that $\lim_{n\to\infty}(a_{n+1}-a_n)=r\in R_+^*$.
For any $x\in R_+^*$ we denote $M_n^{[x]}=\left(\frac{a_1^x+a_2^x+\dots+a_n^x}{n}\right)^{\frac{1}{x}}$ and $M_n^{[x]}!=M_1^{[x]}M_2^{[x]}\dots M_n^{[x]}, \forall\, n\in N^*$.
Find:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{M_{n+1}^{[x]}!}}-\frac{n^2}{\sqrt[n]{M_n^{[x]}!}}\right)$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.041.

Prove that:

$$\frac{3^{\frac{3}{2}}}{2}\cdot\sum_0^\infty\frac{(n!)^2}{(2n+1)!}=\pi$$

Proposed by Francis Fregeau – Quebec – Canada

PROBLEM 3.042.

Let $ABCD$ be a trapeze where $AB \parallel CD; AB = a; CD = b; AD = c; BC = d; a > b$. Prove that :

$$\text{Area } [ABCD] < \frac{(a+b)(a-b+c+d)^2}{16(a-b)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.043.

Prove that in any $\Delta\ ABC$ we have:

$$2s + \sqrt{\sum (a^2 + 2ab\cos(A-B))} \geq \sum \sqrt{a^2 + 2ab\cos(A-B) + b^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.044.

For all $n \in \mathbb{N}^*$ holds:

$$[\sqrt{n} + \sqrt{n+1}] + [\sqrt{n} + \sqrt{n+2}] + [\sqrt{n} + \sqrt{n+3}] =$$
$$= [\sqrt{4n+1}] + [\sqrt{4n+3}] + [\sqrt{4n+5}]$$

where $[\cdot]$ denote the integer part.

Proposed by Mihály Bencze – Romania

PROBLEM 3.045.

Calculate:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \dots \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sqrt[2]{\tan(x_1)}\sqrt[3]{\tan(x_2)} \dots \sqrt[n+1]{\tan(x_n)}} dx_1 dx_2 \dots dx_n$$

Proposed by Cornel Ioan Vălean – Romania

PROBLEM 3.046.

Let a, b, c be positive real numbers such that $a + b + c = 1$.Prove that:

$$a^{a^2} b^{b^2} c^{c^2} \geq (a^2 + b^2 + c^2)^{a^2+b^2+c^2}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.047.

Let a, b, c be distinct rational numbers such that

$$\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0.$$

Prove that:

$$\sqrt{\frac{(b-c)^4}{a^2} + \frac{(c-a)^4}{b^2} + \frac{(a-b)^4}{c^2}}$$

is a rational number.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.048.

Let a, b, c be non-negative real numbers such that $a + b + c = 1$.Prove that:

$$a^4 + b^4 + c^4 + 26abc \leq 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.049.

Prove that the following inequality holds for any triangle ABC:

$$a^2(5m_a - m_b - m_c) + b^2(5m_b - m_c - m_a) + c^2(5m_c - m_a - m_b) \le 12m_am_bm_c$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.050.

Let a, b, c be positive real numbers such that $a^2b + b^2c + c^2a = 3$.Prove that:

$$\frac{1}{a(a+b)^2} + \frac{1}{b(b+c)^2} + \frac{1}{c(c+a)^2} \ge \frac{3}{4}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.051.

Let be $a \in [0, \infty)$; $f: (0, \infty) \to (0, \infty)$, $f(x) = \left(\Gamma(x+1)\right)^{\frac{1}{x}}$. Find:

$$\Omega = \lim_{x\to\infty}\left(\left(f(x+1)\right)^a - \left(f(x)\right)^a\right) \cdot x^{1-a}$$

Proposed by D.M. Bătinețu-Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.052.

Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{a^6}{a^2+b} + \frac{b^2}{b^2+c} + \frac{c^6}{c^2+a} \ge \frac{3}{2}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 3.053.

If $x, y, z \in \mathbb{C}^*$; $A, B, C \in M_n(\mathbb{C})$; $n \ge 2$ are such that $x^2A + B = xAB$; $y^2B + C = yBC$; $z^2C + A = zCA$ then:

$$\left((y^2+1)x + \frac{x^2+1}{x}\right)A + \left((z^2+1)y + \frac{y^2+1}{x}\right)B + \left((x^2+1)z + \frac{z^2+1}{y}\right)C =$$
$$= (x+y+z)ABC$$

Proposed by D.M. Bătinețu-Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.054.

If $x > 0$ then:

$$2\sqrt{x} \le \left(\frac{\Gamma(\sqrt{x}+1)}{\Gamma\left(\sqrt{x}+\frac{1}{2}\right)}\right)^2 + \left(\frac{\Gamma(\sqrt[4]{x}+1)}{\Gamma\left(\sqrt[4]{x}+\frac{1}{2}\right)}\right)^4 \le 2\sqrt{x} + \sqrt[4]{x} + \frac{1}{4}$$

where Γ denote the Euler's gamma function.

Proposed by Mihály Bencze – Romania

PROBLEM 3.055.

Evaluate:

$$I = \int_0^1 \frac{\ln^3 x}{2-x}dx$$

Proposed by Shivam Sharma – New Delhi – India

PROBLEM 3.056.

Let ABC and be a triangle and Ω is first Brocard point of ABC. Let D, E, F be on the sides BC, CA, AB of ABC respectively. If $m(B\Omega D) = m(C\Omega E) = m(A\Omega F) = 90°$ then prove that

$$\frac{|BD|}{|BC|} + \frac{|CE|}{|CA|} + \frac{|AF|}{|AB|} = 2$$

Proposed by Mehmet Şahin – Ankara – Turkey

PROBLEM 3.057.

Let $a, b \in \mathbb{R}$ such that $a + b > 0$ then:

$$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{n+1}\sum_{k=0}^{n} a^k b^{n-k} \leq \frac{a^n + b^n}{2}$$

Proposed by Abdallah El Farissi – Bechar – Algerie

PROBLEM 3.058.

Let ABC be an arbitrary triangle and XYZ is the Kiepert triangle of ABC. If $K(\theta)$ is a Kiepert perspector and ω is first Brocard angle then prove that

$$a. \frac{Area(XYZ)}{Area(ABC)} = \frac{1}{4}(3\tan^2\theta + 2\tan\theta \cdot \cot\omega + 1)$$

If $\theta = \omega$ then XYZ is Gallatly – Kiepert triangle takes the name.Prove that

$$\frac{Area\ (XYZ)}{Area\ (ABC)} = 3 \cdot \frac{a^2b^2 + b^2c^2 + c^2a^2}{(a^2 + b^2 + c^2)^2}$$

Proposed by Mehmet Şahin – Ankara – Turkey

PROBLEM 3.059.

Let a, b, c be positive real numbers. Prove that

$$\frac{(a^2 - ab + b^2)^2}{(a+b)^4} + \frac{(b^2 - bc + c^2)^2}{(b+c)^4} + \frac{(c^2 - ca + a^2)^2}{(c+a)^4} \geq \frac{3}{16}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 3.060.

Let a, b, c be positive real numbers with $a + b + c = 1$. Prove that

$$\left(1 + \frac{1}{2a+b}\right)^c \cdot \left(1 + \frac{1}{2b+c}\right)^a \cdot \left(1 + \frac{1}{2c+a}\right)^b \geq 2$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 3.061.

Prove that in all triangle ABC with usual notations holds the following inequalities:

$$a. \frac{\tan^3\frac{A}{2}}{m \cdot \tan\frac{B}{2} + n \cdot \tan\frac{C}{2}} + \frac{\tan^3\frac{B}{2}}{m \cdot \tan\frac{C}{2} + n \cdot \tan\frac{A}{2}} + \frac{\tan^3\frac{C}{2}}{m \cdot \tan\frac{A}{2} + n \cdot \tan\frac{B}{2}} \geq \frac{\left((4R+r)^2 - 2s^2\right)^2}{(m+n)s^4}$$

$$b. \frac{\tan\frac{A}{2}}{m + n \cdot \tan\frac{B}{2} \cdot \tan\frac{C}{2}} + \frac{\tan\frac{B}{2}}{m + n \cdot \tan\frac{C}{2} \cdot \tan\frac{A}{2}} + \frac{\tan\frac{C}{2}}{m + n \cdot \tan\frac{A}{2} \cdot \tan\frac{B}{2}} \geq \frac{(4R+r)^2}{s(m(4R+r) + 3nr)}$$

$$c.\ \frac{\tan^3\frac{A}{2}}{m\cdot\cot\frac{B}{2}+n\cdot\cot\frac{C}{2}}+\frac{\tan^3\frac{B}{2}}{m\cdot\cot\frac{C}{2}+n\cdot\cot\frac{A}{2}}+\frac{\tan^3\frac{C}{2}}{m\cdot\cot\frac{A}{2}+n\cdot\cot\frac{B}{2}}\ge\frac{(4R+r)r}{(m+n)s^2}$$

$$d.\ \frac{\tan\frac{A}{2}}{\left(x+y\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}\right)^m}+\frac{\tan\frac{B}{2}}{\left(x+y\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2}\right)^m}+\frac{\tan\frac{C}{2}}{\left(x+y\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}\right)^m}\ge\frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m}$$

for any positive real numbers m,n,x,y

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.062.

Given the equilateral triangle ABC and let P be any point in its plane. R, R_a, R_b, R_c denote respectively radii of the circumcircles of the triangles ABC, BPC, CPA, APB and x, y, z are respectively distances from P to the sides BC, CA, AB. Prove that:

$$xR_a+yR_b+zR_c\ge\frac{3}{2}R^2.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.063.

If $a,b,c,d>0$ such that :

$$a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2=6.$$

then:

$$\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\ge 4.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.064.

Let a,b,c,d be non-negative real numbers such that $a+b+c+d=4$. Prove that:

$$ab(a+b)+cd(c+d)+4(a+b)(c+d)\le\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}+16$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.065.

Let $SABC$ be a tetahedron and let M be any point inside the triangle ABC. The lines through M parallel with the planes SBC, SCA, SAB intersect SA, SB, SC at X, Y, Z respectively. Prove that:

$$Vol(MXYZ)\le\frac{2}{27}Vol(SABC)$$

Determine position of the point M such that the equality holds.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.066.

Evaluate:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left(\frac{1}{n^3(2k-1)}\right)$$

Proposed by Shivam Sharma – New Delhi – India

PROBLEM 3.067.

Evaluate:

$$S=\sum_{n=1}^{\infty}H_n\left[\zeta(8)-\frac{1}{1^8}-\frac{1}{2^8}-\frac{1}{3^8}\cdot\ldots\cdot\frac{1}{n^8}\right]$$

Proposed by Shivam Sharma – New Delhi – India

PROBLEM 3.068.

Prove that if $a,b,c\in\mathbb{R}^*$ then:

$$(abc-ab-bc-ca)^2\le 4(1+a^2)(1+b^2)(1+c^2)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.069.

Prove that if $n\in\mathbb{N}^*;\ a>1$ then:

$$(n+a-1)(a-1)^{n-1}\le a^n$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.070.

If $a,b\in\mathbb{R}; a<b; f,g:[a,b]\to\mathbb{R}$ are continuous functions such that:
$f(a+b-x)=f(x), g(a+b-x)g(x)=1, (\forall)x\in[a,b]$ then:

$$\int_a^b\frac{f(x)\cdot g(x)}{1+g(x)}dx=\frac{1}{2}\int_a^b f(x)\,dx$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.071.

Evaluate:

$$\int_0^1\ln\left[\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\right]\frac{x\,dx}{1-x^2}$$

Proposed by Shivam Sharma-New Delhi-India

PROBLEM 3.072.

If $x,y,z>0$ and $b\ge a>0$ then:

$$\frac{x}{z}\ln\frac{x^2+bz}{x^2+az}+\frac{y}{x}\ln\frac{y^2+bx}{y^2+ax}+\frac{z}{y}\ln\frac{z^2+by}{z^2+ay}\le\frac{3}{4}\ln\frac{b}{a}+\frac{b-a}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.073.

Let be $a,b,c\in\mathbb{C}$. Solve the following equation:

$$x^3-(a+b+c)x^2+(ab+bc+ca-1)x+b-abc=0$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.074.

Let $(a_n)_{n\ge1}$ be positive real sequences with $\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a\in R_+^*$ and
$f\in R[x], f(x)\in R_+^*, \forall x\in R_+^*;\ u,v\in R$ such that $u+v=1$. Find the following limit

$$\lim_{n\to\infty}\left((n+1)^u\sqrt[n+1]{\left(a_{n+1}f(n+1)\right)^v}-n^u\sqrt[n]{\left(a_nf(n)\right)^v}\right).$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.075.

Let $(a_n)_{n\ge1}$, $(b_n)_{n\ge1}$, be real positive sequences with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a \in R_+^*$. If $\lim_{n\to\infty}(n(a_n - a)) = b \in R$ and $\lim_{n\to\infty}(n(b_n - a)) = c \in R$, evaluate:

$$\lim_{n\to\infty}\left(a_{n+1}\sqrt[n+1]{(n+1)!} - b_n\sqrt[n]{n!}\right).$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.076.

Evaluate:

$$S = \sum_{n=1}^{\infty}\left(\frac{H_{2n}+1}{n^2}\right)$$

Proposed by Shivam Sharma – New Delhi – India

PROBLEM 3.077.

Evaluate:

$$S = \prod_{n=1}^{\infty}\left(e\left(\frac{n}{n+1}\right)^n\sqrt{\frac{n}{n+1}}\right)$$

Proposed by Shivam Sharma-New Delhi-India

PROBLEM 3.078.

Find:

$$\Omega = \lim_{n\to\infty} n\left(\sqrt[2n+2]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}\right)\left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!}\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.079.

If $x, y, z > 0$ and $b \ge a > 0$ then:

$$\ln\frac{(x+b)(y+b)(z+b)}{(x+a)(x+b)(x+c)} \ge \frac{15}{8}\ln\frac{b}{a} + \frac{1}{16}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)(x^2+y^2+z^2)$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.080.

Let be: $f: (0,\infty) \to (0,\infty)$ a function such that: $\lim_{x\to\infty}\frac{f(x)}{x} = a \in (0,\infty)$ and $\lim_{x\to\infty}\left(\frac{f(x+1)}{f(x)}\right)^x = b \in (0,\infty)$. Find:

$$\Omega = \lim_{x\to\infty}\left(f(x+1) - f(x)\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.081.

If $B_n(t) = n^{1-t}\left(\frac{(n+1)^{2t}}{\left(\sqrt[n+1]{(n+1)!}\right)^t} - \frac{n^{2t}}{\left(\sqrt[n]{n!}\right)^t}\right)$, with $t > 0$, then compute

$$\lim_{n\to+\infty} B_n(t).$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.082.

Let $n \in N$. Calculate:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x \left(\cos x \cos^{2n+1}\left(\frac{\pi}{2}\sin x\right) + \sin x \cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right) dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.083.

Prove that in any triangle ABC the following relationship holds:

$$R\sum (b+c-2a)^2 \le 4(R-2r)\sum a^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.084.

Evaluate:

$$I = \int_0^1 \int_0^1 \frac{(\ln(x)\ln(y))^s}{1-xy} dx\, dy$$

Proposed by Shivam Sharma-New Delhi-India

PROBLEM 3.085.

Let k be positive integer. Calculate:

$$\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{k+1}{x+1}} - \left(\Gamma(x+1)\right)^{\frac{k+1}{x}}\right)\left(\Gamma(x+1)\right)^{-\frac{k}{x}},$$

where $\Gamma(x)$ is the Gamma function (or Euler's second integral).

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.086.

Let $a > 0, b, c > 1$ and $f, g: R \to R$ be continuos and odd functions. Prove that:

$$\int_{-a}^{a} f(x) \ln\left(b^{g(x)} + c^{g(x)}\right) dx = (\ln(bc)) \int_0^a f(x)\, g(x) dx$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.087.

Let $a, b \in R, a < b$ and continuos functions , $g, h: R \to R$ such that
$f(a+b-x) = -f(x), g(a+b-x) = g(x), h(a+b-x) = -h(x), \forall x \in R$. Prove that:

$$\int_a^b f(x)\,(\arctan(x)) \ln\left(1+e^{h(x)}\right) dx = \frac{1}{2}\int_a^b f(x)\, h(x) \arctan(x)\, dx$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.088.

Let $f: R \to R$ be a continuous function such that $f(x) = f(1-x), \forall\, x \in R$. Prove that:

$$\int_0^1 \frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}} f(x) dx = \sqrt{2} \cdot \int_0^1 f(x)\, dx$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.089.
Evalute:

$$\int_0^1 [\ln(x)\ln(1-x)+Li_2(x)]\left(\frac{Li_2(x)}{x(1-x)}-\frac{\zeta(2)}{1-x}\right)dx$$

Proposed by Shivam Sharma-New Delhi-India

PROBLEM 3.090.
Evaluate:

$$\int_0^1 (\ln(\Gamma(x)))(\sin(2k\pi x))\,dx,\qquad k\geq 1$$

Proposed by Shivam Sharma – New Delhi – India

PROBLEM 3.091.
Let be $a\in\mathbb{R}_+^*$ and the continuous functions $f,g,h:\mathbb{R}\to\mathbb{R}$ where f and g are odd and h is even. Prove that:

$$\int_{-a}^{a} f(x)\cdot ln\left(1+e^{g(x)}\right)\cdot arctan\left(h(x)\right)dx=\int_0^a f(x)\,g(x)\,arctan\left(h(x)\right)dx$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.092.
Calculate:

$$\lim_{n\to\infty}\sqrt[3]{n^2}\left(\sqrt[3(n+1)]{(n+1)!}-\sqrt[3n]{n!}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.093.
Let $(a_n)_{n\geq1},(b_n)_{n\geq1}$ be positive real sequences such that there exists $lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}$ and $lim_{n\to\infty}(b_n-u\cdot a_n)$. Find:

$$a.\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)\quad b.\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.094.
Let $(s_n)_{n\geq1}, s_n=\sum_{k=1}^{n}\frac{1}{k^2}$. Calculate:

$$\lim_{n\to\infty}\left(s_n\cdot\sqrt[n+1]{(n+1)!}-\frac{\pi^2}{6}\cdot\sqrt[n]{n!}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.095.

Solve the system of equation: $\begin{cases} \frac{1}{\sqrt{a^3}}+\frac{1}{\sqrt{b^3}}+\frac{1}{\sqrt{c^3}}=3 \\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{(a^3+b^3+c^3)^2}{3} \end{cases}$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.096.

Let $(s_n)_{n\geq 1}, s_n=\sum_{k=1}^{n}\frac{1}{k^2}$. **Calculate:**

$$\lim_{n\to\infty}\left(s_n\cdot\sqrt[n+1]{(2n+1)!!}-\frac{\pi^2}{6}\cdot\sqrt[n]{(2n-1)!!}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.097.

If $x,y,z,a,b,c>0$ **then:**

$$\frac{(x+y)(y+z)(z+x)}{4xyz}\geq\left(\frac{x+z}{y+z}+\frac{y+z}{x+z}\right)^{\frac{a}{a+b+c}}\left(\frac{y+x}{z+x}+\frac{z+x}{y+x}\right)^{\frac{b}{a+b+c}}\cdot\left(\frac{z+y}{x+y}+\frac{x+y}{z+y}\right)^{\frac{c}{a+b+c}}\geq 2$$

(A refinement of Cesaro's inequality)

Proposed by Mihály Bencze Romania

PROBLEM 3.098.

Let $a,b\in\mathbb{R}, a<b$ **and** $f,g:\mathbb{R}\to\mathbb{R}$ **continuos functions such that** $f(x)f(a+b-x)=1, g(x)=g(a+b-x), x\in\mathbb{R}$. **Show that:**

$$\int_a^b\frac{g(x)}{1+f(x)}dx=\frac{1}{2}\cdot\int_a^b g(x)\,dx$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.099.

In an arbitrary triangle ABC **denote by** l_a, m_a, h_a **respectively the lengths of the internal angle-bisector, the median and the altitude corresponding to the side** $a=BC$ **of the triangle. Prove that:**

$$a.\ \frac{l_a^2}{h_a^2}+\frac{l_b^2}{h_b^2}+\frac{l_c^2}{h_c^2}\geq 2\frac{l_a}{h_a}\cdot\frac{l_b}{h_b}\cdot\frac{l_c}{h_c}+1$$

$$b.\ \frac{m_a^2}{h_a^2}+\frac{m_b^2}{h_b^2}+\frac{m_c^2}{h_c^2}\leq 2\frac{m_a}{h_a}\cdot\frac{m_b}{h_b}\cdot\frac{m_c}{h_c}+1$$

c) explain why each of a) and b) are equivalent to the fundamental inequality of the triangle.

Proposed by Vasile Jiglău – Romania

PROBLEM 3.100.

In ΔABC; m_a, m_b, m_c **– median's length. Prove that:**

$$3(a^2+b^2+c^2)<4(am_c+bm_a+cm_b)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.101.

Prove that *if* $a, b, c \in (1, \infty)$ then:

$$3\sqrt{2} + \int_1^a x \sin\frac{\pi}{3x} dx + \int_1^b x \sin\frac{\pi}{3x} dx + \int_1^c x \sin\frac{\pi}{3x} dx > \sqrt{3 + a^2 + b^2 + c^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3. 102.

Solve for real numbers:

$$n^{n(x_1^2 - x_2)} + n^{n(x_2^2 - x_3)} + \cdots + n^{n(x_{n-1}^2 - x_n)} + n^{n(x_n^2 - x_1)} = \frac{n}{\sqrt[4]{n^n}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.103.

Prove that in any triangle ABC the following relationship holds:

$$|\cos A| + |\cos B| + |\cos C| \le \sum\left(\sqrt{|\cos A \cos B|} + \sqrt{\left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.104.

Prove that *if* $x_i \in (0, \infty); i \in \overline{1, n}; n \in \mathbb{N}; n \ge 3; x_{n+1} = x_1; x_1 x_2 \cdot \ldots \cdot x_n = 1$, then:

$$\sum_{i=1}^{n} \frac{\frac{x_i}{x_{i+1}} + \frac{x_{i+1}}{x_i} + 1}{\sqrt{x_i^2 + x_i x_{i+1} + x_{i+1}^2}} \ge n\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.105.

In ABC; a, b, c - length sides; s - semiperimeter; A, B, C - angled's measures. Prove that:

$$\left(\frac{A^3}{b} + \frac{B^3}{c} + \frac{C^3}{a}\right)\left(\frac{A^3}{c} + \frac{B^3}{a} + \frac{C^3}{b}\right)\left(\frac{A^3}{a} + \frac{B^3}{b} + \frac{C^3}{c}\right) \ge \frac{\pi^9}{216 s^3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.106.

Prove that:

$$\cos\frac{A}{4} + \cos\frac{B}{4} + \cos\frac{C}{4} \le \frac{\sqrt{3} - 1}{4\sqrt{6}} t + \frac{11\sqrt{3} + 37}{8\sqrt{6}}, t = \frac{r}{R}$$

Proposed by Vadim Mitrofanov-Kiev-Ukraine

PROBLEM 3.107.

Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$2\sqrt{3}\left(\frac{r}{R}\right)^2 \le \frac{\sum_{cyc} \sin^4 A}{\sum_{cyc} \sin^3 A} \le \frac{\sqrt{3}}{4}\left(\frac{R}{r}\right)^2\left(1 - \frac{r}{R}\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 3.108.

Let ABC be a triangle with circumradius R and inradius r. Prove that:

$$\frac{3}{16}\le \cos^4 A+\cos^4 B+\cos^4 C\le 6\left(\frac{r}{R}\right)^2-\frac{123}{8}\cdot\frac{r}{R}+\frac{51}{8}$$

Proposed by George Apostolopoulos - Messolonghi – Greece

PROBLEM 3.109.

Let a,b,c and d be positive real numbers. Prove or disprove that:

$$\frac{(a+b+c+d)^3}{abc+bcd+cda+dab}\ge 16$$

Proposed by George Apostolopoulos - Messolonghi – Greece

PROBLEM 3.110.

Let m_a,m_b,m_c be the lengths of the medians of a triangle ABC. Prove that:

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\le\frac{R}{2r^2}$$

Proposed by George Apostolopoulos - Messolonghi – Greece

PROBLEM 3.111.

For an acute triangle ABC and a positive integer n, prove that:

$$\left(\sum(\sin A\sin B\sin C)^{\frac{1}{n}}\right)^n\le\frac{3^{n+1}}{8}$$

where the sum is over all cyclic permutations of (A,B,C).

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 3.112.

Solve for positive real numbers:

$$\begin{cases}\dfrac{x^2}{y}+\dfrac{y^2}{x}=\sqrt[8]{128(x^8+y^8)}\\[2mm] 4x^3-3y=\sqrt{\dfrac{1+\sqrt{1-xy}}{2}}\end{cases}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.113.

Let x,y,z be positive real numbers. Prove that:

$$\sqrt{\frac{3x^2+yz}{y^2+z^2}}+\sqrt{\frac{3y^2+zx}{z^2+x^2}}+\sqrt{\frac{3z^2+xy}{x^2+y^2}}\ge\sqrt{\frac{103}{6}+\frac{20x^2y^2z^2}{3(x^2+y^2)(y^2+z^2)(z^2+x^2)}}$$

Proposed by Do Quoc Chinh – Vinh Phuc – Vietnam

PROBLEM 3.114.

Let a,b,c be the sides and R and r the circumradius and the inradius of a triangle ABC respectively. Prove that:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{9}{4r(4R+r)}$$

Proposed by Martin Lukarevski – Skopje – Macedonia

PROBLEM 3.115.

Evaluate:

$$\int_0^{\infty} \frac{ln(x)\,sin(x)}{(x)^{\frac{1}{2}}}dx$$

Proposed by Arafat Rahman Akib – Dahka – Bangladesh

PROBLEM 3.116.

Prove that:

$$\sum_{n=1}^{\infty}\left((-1)^{n-1}\frac{H_n^4+6H_n^2H_n^{(2)}+3\left(H_n^{(2)}\right)^2+8H_nH_n^{(3)}+6H_n^{(4)}}{n}\right)=24Li_5\left(\frac{1}{2}\right)$$

Proposed by Ali Shather – An Nasiriyah – Iraq, Shivam Sharma – New Delhi – India

PROBLEM 3.117.

Let $a,b,c\in\left[\frac{1}{2};3\right)$ be positive real numbers such that: $a+b+c=3$. Prove that:

$$\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}}{30}+\frac{11}{40}\geq\frac{3(ab+bc+ca-2)}{2\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.118.

Prove that:

$$\int_0^{\infty} x^p\sqrt{\frac{1+\sqrt{1+x^2}}{1+x^2}}dx=2^{p+\frac{1}{2}}B\left(\frac{p+1}{2},-p-\frac{1}{2}\right);\ -1<p<-\frac{1}{2}$$

Proposed by Shivam Sharma - New Delhi – India

PROBLEM 3.119.

Prove that:

$$\sum_{k=1}^{\infty}\left(\frac{H_k^{(2)}}{k^9}\right)=9\zeta(2)\zeta(9)+2\zeta(3)\zeta(8)+6\zeta(4)\zeta(7)+4\zeta(5)\zeta(6)-27\zeta(11)$$

Proposed by Ali Shather and Shivam Sharma

PROBLEM 3.120.

Prove that in any acute-angled triangle ABC the following relationship holds:

$$\sqrt{2}\sum(sin\,A+cos\,A)>\sum sin\,C\left(1+cos\,2\,(A-B)\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.121.

Prove that:

$$\int_0^{\infty} x^{p-a}\left(\frac{2}{1+\sqrt{1+4x}}\right)^n dx = \left(\frac{n}{n-p}\right) B(n-2p,p)$$

Proposed by Shivam Sharma – New Delhi – India

PROBLEM 3.122.

Let be $a_n = \sum_{k=1}^{n} arctan\frac{1}{k^2+k+1}$. **Find:**

$$\lim_{n\to\infty} n\left(a_n^{\frac{\pi}{4}} - \frac{\pi^{a_n}}{4}\right)$$

Proposed by Marian Ursărescu-Romania

PROBLEM 3.123.

Find:

$$\Omega = \lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\sqrt{n^4+k^2}+k}{\sqrt{n^4+k^2}-k}\right)$$

Proposed by Marian Ursărescu – Romania

PROBLEM 3.124.

Find all continuous functions such that:

$$\int_{e^x}^{e^{2x}} f(t)\,dt = \int_1^{e^x} f(t)\,dt$$

Proposed by Marian Ursărescu – Romania

PROBLEM 3.125.

Prove that:

$$2\sum_{k=1}^{n}\ln^2\left(1+\frac{1}{k}\right) < \frac{2n}{n+1} < \sum_{k}^{n}\ln\left(1+\frac{2}{k^2+k-1}\right)$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.126.

Prove that:

$$\int_0^1\int_0^1 \frac{\sqrt{x+y}+\sqrt{xy}}{(x\sqrt{y}+y\sqrt{x})}\,dx\,dy = \ln(17-12\sqrt{2})+\sqrt{2}\ln(17+12\sqrt{2}) -$$

$$-\frac{4}{3}(1-3\sqrt{2}+2\ln(2))$$

Proposed by K. Srinivasa Raghava – AIRMC – India

PROBLEM 3.127.

If $x,y,z,t\in\mathbb{R}$ **then:**

$$|\sin x|+|\sin y|+|\sin z|+|\sin t|+|\cos x|+|\cos y|+|\cos z|+$$

$$+|\cos t|+2|\cos(x+y+z+t)|\geq 2$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.128.

If $\alpha,\beta>1$ and $2\alpha-\beta>1$ then: $\zeta(2\alpha-\beta)\zeta(\beta)\ge\zeta^2(\alpha)$ where ζ is the Riemann zeta function.

Proposed by Mihály Bencze – Romania

PROBLEM 3.129.

Let R_+^* be the set of real positive numbers, let $(a_n)_{n\ge1},(b_n)_{n\ge1}$ be two sequences of real positive numbers with $\lim_{n\to\infty}(a_{n+1}-a_n)=a\in R_+^*,\lim_{n\to\infty}(b_{n+1}-b_n)=b\in R_+^*$

Let $P_n=\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$ and we denote $P_n!=P_1P_2\ldots P_n$, for any positive integer n. If $u,v\in R$ with $u+v=1$. Evaluate:

$$\lim_{n\to\infty}\left(b_{n+1}^u\sqrt[n+1]{(P_{n+1}!)^v}-b_n^u\sqrt[n]{(P_n!)^v}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.130.

If $a,b,c,d\in[1,\infty);x\in\mathbb{R}$ then:

$$(ac)^{\sin x}\cdot(bd)^{\cos x}\le2^{\sqrt{2(\log_2^2a+\log_2^2b+\log_2^2c+\log_2^2d)}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.131.

Prove that in any triangle ABC the following relationship holds:

$$2^{\sin A}+2^{\sin B}+2^{\sin C}+2^{\cos A}+2^{\cos B}+2^{\cos C}>\frac{6}{(\sqrt{2})^{\sqrt{2}}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.132.

If $a,b>0;m,n\ge1$ then:

$$\int_0^b\left(\int_0^b(\sin x\sin y)^{2n}(\cos x\cos y)^{2m}dx\right)dy\le\frac{ab}{4^{m+n}}\left(\frac{m}{n}\right)^{m-n}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.133.

Prove that if: $0\le b\le a\le\frac{\pi}{4}$ then:

$$\int_a^b\left(\int_a^b\left(\frac{sin^2(x+y)+sin^2(x-y)-1}{1+2\,sin\,x\,sin\,y}\right)dx\right)dy\ge(a-b)(sin^2a-sin^2b+b-a)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.134.

In $\Delta\,ABC$ the following relationship holds:

$$\frac{a^2(m_a+m_b)}{h_c}+\frac{b^2(m_b+m_c)}{h_a}+\frac{c^2(m_c+m_a)}{h_b}\ge8\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.135.

If $a,b,c>0$ then:

$$\frac{a+c^2}{xb+yc}+\frac{b+a^2}{zc+ya}+\frac{c+b^2}{xa+yb}\geq\frac{3}{x+y}+\frac{a+b+c}{x+y}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.136.

Prove that:

$$\sum_{k=0}^{n}T_{4k}(x)=\frac{1}{4}\left[\frac{2+U_{4n+2}(x)}{x\sqrt{1-x^2}}\right]$$

where, $T_n(x)$ and $U_n(x)$ denotes the Chebyshev Polynomials of First and Second Kind.

Proposed by Shivam Sharma-New Delhi-India

PROBLEM 3.137.

Let $f,g:\mathbb{R}_+^*\to\mathbb{R}_+^*$ be functions such that:

$$\lim_{x\to\infty}\big(f(x+1)-f(x)\big)=a\in\mathbb{R}_+^*,\lim_{x\to\infty}\frac{g(x+1)}{xg(x)}=b\in\mathbb{R}_+^*$$

and exists $\lim_{x\to\infty}\frac{f(x)}{x}$ and $\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}$. For $t\in\mathbb{R}$ calculate the limit:

$$\lim_{x\to\infty}\big(f(x)\big)^{\cos^2 t}\left(\big(g(x)\big)^{\frac{\sin^2 t}{x+1}}-g\big((x)\big)^{\frac{\sin^2 t}{x}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.138.

Let $f,g:\mathbb{R}_+^*\to\mathbb{R}_+^*$ such that: $\lim_{x\to\infty}\big(f(x+1)-f(x)\big)\,a\in\mathbb{R}_+^*,\lim_{x\to\infty}\frac{g(x+1)}{xg(x)}=b\in\mathbb{R}_+^*$ and there is $\lim_{x\to\infty}\frac{f(x)}{x},\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}$. For $t\in\mathbb{R}$, calculate:

$$\lim_{x\to\infty}\big(f(x)\big)^{\sin^2 t}\left(\big(g(x)\big)^{\frac{\cos^2 t}{x+1}}-\big(g(x)\big)^{\frac{\cos^2 t}{x}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.139.

Calculate:

$$\lim_{x\to\infty}\left(x^{\cosh^2 t}\left(\big(\Gamma(x+1)\big)^{-\frac{\sinh^2 t}{x}}-\Big(\big(\Gamma(x+2)\big)\Big)^{-\frac{\sinh^2 t}{x+1}}\right)\right)$$

where $t\in\mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.140.

Calculate: $\lim_{x\to\infty}\left(x^{\sin^2 t}\left(\big(\Gamma(x+2)\big)^{\frac{\cos^2 t}{x+1}}-\big(\Gamma(x+1)\big)^{\frac{\cos^2 t}{x}}\right)\right)$, where $t\in\mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.141.

For $\{a_n\}_{n\geq 0}, a_n = \frac{(n+2)^{n+1}}{(n+1)^n}, x \in (-\infty, \infty), \{b_n(x)\}_{n\geq 1}, b_n(x) = n^{\sin^2 x}\left(a_{n+1}^{\cos^2 x} - a_n^{\cos^2 x}\right)$, find

$$\lim_{n\to\infty} b_n(x)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.142.

Let $(x_n)_{n\geq 1}$ be a sequence which satisfy:

$$-\ln(mn + x_n) + \sum_{k=1}^{mn} \frac{1}{k} = \gamma$$

where m is positive integer and γ is Euler – Mascheroni's constant. Compute:

$$\lim_{n\to\infty} x_n$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.143.

Let $a, b \in \mathbb{R}_+, \gamma_n(a,b) = -\ln(n+a) + \sum_{k=1}^{n} \frac{1}{k+b}$ with

$$\lim_{n\to\infty} \gamma_n(a,b) = \gamma(a,b) \in \mathbb{R}$$

Calculate:

$$\lim_{n\to\infty}\left(\ln\frac{e}{n+a} + \sum_{k=1}^{n}\frac{1}{k+b} - \gamma(a,b)\right)^n$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.144.

If $x, y, z \geq 0$ then:

$$\cosh^2 x \cosh^2 y \cosh^2 z \geq 2(1 + \cosh(x-y) + \cosh(y-z) + \cosh(z-x)) \cdot$$
$$\cdot \sinh\frac{x+y}{2}\sinh\frac{y+z}{2}\sin h\frac{z+x}{2}$$

Proposed by Mihály Bencze – Romania

PROBLEM 3.145.

Let be $(x_n)_{n\geq 1}, x_n \in \mathbb{R}_+^*, \forall n \in \mathbb{N}^*$, such that exists $\lim_{n\to\infty}(x_{n+1} - x_n) = x \in \mathbb{R}_+^*$. Find:

$$\lim_{n\to\infty}\left(\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}} - \frac{nx_n}{\sqrt[n]{(2n-1)!!}}\right)$$

Proposed by Bătinețu – Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.146.

Let $f: (0,\infty) \to (0,\infty)$ be a function with: $\lim_{x\to\infty}\frac{f(x)}{x} = a \in (0,\infty)$ and $t \in \mathbb{R}$. Find:

$$\lim_{n\to\infty}\left((n+1)^{\sin^2 t} \cdot \sqrt[n+1]{\left(f(1)f(2)\dots f(n)f(n+1)\right)^{\cos^2 t}} - n^{\sin^2 t} \cdot \sqrt[n]{\left(f(1)f(2)\dots f(n)\right)^{\cos^2 t}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.147.

In an ABC triangle let be a, b, c the lengths of BC, CA, AB, and r_a, r_b, r_c exradii. Prove that:

$$\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{9(a^2+b^2+c^2)}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.148.

Let a, b, c be positive real numbers such that: $a+b+c=3$. Prove that:

$$2(a^2+b^2+c^2)+3\geq 3\sqrt{3abc(a^3b+b^3c+c^3a)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.149.

Prove that:

$$\sum_{k=-l}^{l}\left[(-1)^k\binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k}\right]=\frac{(l+m+n)!\,(2l)!\,(2m)!\,(2n)!}{(l+m)!\,(l+n)!\,(m+n)!\,(l)!\,(m)!\,(n)!}$$

Proposed by Shivam Sharma – New Delhi – India

PROBLEM 3.150.

Determine all continuos functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$f(x+y)=f(x)+f(y)+xy$ and $f(1)=1$ for all $x, y\in\mathbb{R}$.

Proposed by Mihály Bencze – Romania

PROBLEM 3.151.

Given real numbers $a_1, a_2, \dots, a_n \in [0,1]$.Find the maximum and minimum possible value of

$$a_1+a_2+\dots+a_n+(1-a_1)(1-a_2)\dots(1-a_n).$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.152.

Let $a_1, a_2, \dots, a_n$ $(n\geq 3)$ be positive real numbers such that:

$$\sum_{1\leq i<j\leq n} a_i a_j \geq \frac{n(n-1)}{2}$$

Prove that:

$$\sum_{i=1}^{n}\frac{1}{a_i^2+n-1}\leq 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 3.153.

Find:

$$\Omega=\int_0^1\int_0^\infty\left(\frac{x^{100}}{100^x}\right)(y\ln y)^{100}dx\,dy$$

Proposed by Ekpo Samuel – Nigeria

PROBLEM 3.154.

If $(a_n)_{n\geq 1}\subset(0,\infty); n\in\mathbb{N}^*; \lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}=a>0$ then find:

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n}}\sum_{k=1}^{n}\left[\left(\sqrt[2k]{k!}+\sqrt[2k+2]{(k+1)!}\right)^2\right];$$

$[*]$ - is great integer function.

Proposed by D.M. Bătinețu – Giurgiu; Daniel Sitaru – Romania

PROBLEM 3.155.

If $m\in\mathbb{N}; x,y,z>0$ then in ΔABC the following relationship holds:

$$3m+\left(\frac{a^2x^2}{yz}\right)^{m+1}+\left(\frac{b^2y^2}{zx}\right)^{m+1}+\left(\frac{c^2z^2}{xy}\right)^{m+1}\geq 4(m+1)\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

PROBLEM 3.156.

If $a,b,c>0; b<c; b(a+c)\in\mathbb{N}; b(a+c)\geq 1; n\in\mathbb{N}; n\geq 2; x_k\in[b,c];\ t_k>0; k\in\overline{1,n}$ then:

$$\left(\sum_{k=1}^{n}t_kx_k\right)\left(\sum_{k=1}^{n}\frac{t_k}{x_k}\right)^{b(a+c)}\leq\left(\frac{a+b+c}{1+b(a+c)}\right)^{1+ab+bc}\cdot\left(\sum_{k=1}^{n}t_k\right)^{1+ab+bc}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

PROBLEM 3.157.

If $m\geq 0; x,y>0$ then in ΔABC the following relationship holds:

$$\frac{h_a}{h_b^m h_c^m(xh_b+yh_c)^m}+\frac{h_b}{h_c^m h_a^m(xh_c+yh_a)^m}+\frac{h_c}{h_a^m h_b^m(xh_a+yh_b)^m}\geq\frac{9}{(x+y)^m\cdot s^{2m}\cdot r^{m-1}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

PROBLEM 3.158.

Find:

$$\Omega=\int_0^1 x^2\log(x)\cot\left(\frac{\pi x}{2}\right)dx$$

Proposed by Arafat Rahman Akib-Dhaka-Bangladesh

PROBLEM 3.159.

Find:

$$\Omega=\int_{-1}^{+1}(x^2+2x+1)^{\frac{3}{2}}(1-x)^{\frac{1}{2}}dx$$

Proposed by Ekpo Samuel-Nigeria

PROBLEM 3.160.

Let $x, y, z \in (0; +\infty)$ and $a \geq 1$. Prove:

$$\frac{1}{([x]+\{y\}+1)^a}+\frac{1}{([y]+\{z\}+1)^a}+\frac{1}{([z]+\{x\}+1)^2}\geq \frac{3^{a+1}}{(x+y+z+3)^a}$$

$[x]$ is the integer part of the real number x;$\{x\}$ is the fraction of real numbers x.

Proposed by Nguyen Van Nho – Nghe An – Vietnam

PROBLEM 3.161.

Find:

$$\Omega = \int_0^\infty \int_0^\infty \frac{dxdy}{(x^4+2x^2+1)(y^2+25)^4}$$

Proposed by Ekpo Samuel – Nigeria

PROBLEM 3.162.

If $ABCD$ tetrahedron; $AB = a_1; AC = a_2; AD = a_3; BC = a_4;$
$BD = a_5; CD = a_6$ then:

$$\sum_{1\leq i<j\leq 6} (a_i + a_j)^2 \geq 4\sqrt{3}S[ABCD]$$

$S[ABCD]$ – total area of tetrahedron $ABCD$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.163.

Let a, b, c be positive real numbers such that: $(a^2+b^2)(b^2+c^2)(c^2+a^2) = 8$.
Find the minimum value of:

$$P = \frac{a}{b(b+2c+1)(a+3c)^2}+\frac{b}{c(c+2a+1)(b+3a)^2}+\frac{c}{a(a+2b+1)(c+3b)^2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.164.

Solve the equation: $\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4} = 7-3x$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.165.

Solve the system of equation: $\begin{cases} \frac{1}{\sqrt{a^3}}+\frac{1}{\sqrt{b^3}}+\frac{1}{\sqrt{c^3}} = 3 \\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} = \frac{(a^3+b^3+c^3)^2}{3} \end{cases}$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.166.

Solve the equation in $\mathbb{R}$:

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4} = \frac{x^4-3x^3}{2}+7$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.167.

Let a,b,c be positive real numbers such that: $abc=1$. Find the maximum value of:

$$P=\frac{1}{\sqrt[3]{3a^4-4a+2b^2+11}}+\frac{1}{\sqrt[3]{3b^4-4b+2c^2+11}}+\frac{1}{\sqrt[3]{3c^4-4c+2a^2+11}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3. 168.

Let be $a>0$ and $f:(-\infty,-a-1)\cup(-a,+\infty)\to\mathbb{R}$; $f(x)=\frac{1}{x^2+(2a+1)x+a^2+a}$. Find:

$$\lim_{n\to\infty}\sqrt[n^2]{\left|\lim_{p\to\infty}\sum_{p\to\infty}^{p}f^{(n)}(k)\right|}$$

Proposed by Marian Ursărescu – Romania

PROBLEM 3.169.

Let be the sequence $x_1>0$ and $x_1^p+x_2^p+\cdots+x_n^p=\frac{1}{\sqrt[p+1]{x_{n+1}}}$,

$\forall n\in\mathbb{N}, p\in\mathbb{N}^*$. Find:

$$\lim_{n\to\infty}n^{p+1}\cdot x_n^{p^2+p+1}$$

Proposed by Marian Ursărescu – Romania

PROBLEM 3.170.

Find:

$$\lim_{n\to\infty}\int_0^1\frac{arctan(nx)\,ln(1+x)}{1+x^2}dx$$

Proposed by Marian Ursărescu – Romania

PROBLEMS FOR JUNIORS-SOLUTIONS

PROBLEM 1.001-***Solution by Soumitra Mukherjee-Chandar Nagore-India***

$$\text{Let } f(x) = x^2 + 2\sqrt{x} - 3x, \forall\, x \ge 0;\ f'(x) = 2x + \frac{1}{\sqrt{x}} - 3 \ge 0, \forall x \ge 0$$

$f(x)$ *is continuous on* $[0,\infty)$ *and* $f'(x) \ge 0, \forall\, x \in [0,\infty)$; f *is increasing on* $[0,\infty)$.

$$f(x) \ge f(0) \Rightarrow x^2 + 2\sqrt{x} - 3x \ge 0 \Rightarrow x^2 + 2\sqrt{x} \ge 3x, \forall x \ge 0$$

$$\sum_{cyc} a^2 + 2\sum_{cyc}\sqrt{a} \ge 3\sum_{cyc} a \Rightarrow (a+b+c+d)^2 - 2\sum_{cyc} ab + 2\sum_{cyc}\sqrt{a} \ge 3\sum_{cyc} a$$

$$\Rightarrow 16 + 2\sum_{cyc}\sqrt{a} \ge 12 + 2\sum_{cyc} ab \Rightarrow 4 + 2\sum_{cyc}\sqrt{a} \ge 2\sum_{cyc} ab$$

$$\Rightarrow 2 + \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} \ge ab + ac + ad + bc + bd + cd$$

PROBLEM 1.002-***Solution by Mihály Bencze – Romania***

The inequalities are equivalent with: $\frac{f(x+a-1)}{1+|f(x+a-1)|} \le x \le \frac{f(x)}{1+|f(x)|} + a - 1$

Denote $g(x) = \frac{f(x)}{1+|f(x)|} \Rightarrow g(x+a-1) \le x \le g(x) + a - 1$

In $g(x+a-1) \le x$ *we take* $x \to x - a + 1 \Rightarrow g(x) \le x - a + 1$ *(1)*

but from $x \le g(x) + a - 1 \Rightarrow g(x) = x - a + 1$ *(2)*

(1) $\wedge$ *(2)* $\Rightarrow g(x) = x - a + 1 \Rightarrow f(x) = \frac{x-a+1}{1+|x-a+1|}$

(2)

PROBLEM 1.003-***Solution by Mihály Bencze – Romania***

$$(a^2-b^2)^2 \Rightarrow 4a^4 + 4a^2b^2 + 4b^2 \ge 3a^4 + 6a^2b^2 + 3b^4 \Rightarrow \sqrt{a^4 + a^2b^2 + b^4} \ge \frac{\sqrt{3}}{2}(a^2+b^2)$$

$$\text{If } a,b,c > 0 \Rightarrow \left(\sum\sqrt{a^4 + a^2b^2 + b^4}\right)^2 \ge 3\left(\sum a^2\right)^2$$

$$\left(\sum a\sqrt{2a^2+bc}\right)^2 \le \left(\sum a^2\right)^2\left(\sum(2a^2+bc)\right) \le 3\left(\sum a^2\right)^2 \Rightarrow$$

$$\sqrt{a^4 + a^2b^2 + b^4} + \sqrt{b^4 + b^2c^2 + c^4} + \sqrt{c^4 + c^2a^2 + a^2} \ge \quad (1)$$

$$a\sqrt{2a^2+bc} + b\sqrt{2b^2+ca} + c\sqrt{2c^2+ab}.$$

In (1) we take $c = a$ *and* $c = b$ *therefore*

$$\begin{cases} 2\sqrt{a^4 + a^2b^2 + b^4} + a^2\sqrt{3} \ge 2a\sqrt{2a^2+ab} + b\sqrt{a^2+b^2} \\ 2\sqrt{a^4 + a^2b^2 + b^4} + b^2\sqrt{3} \ge 2b\sqrt{2b^2+ab} + a\sqrt{b^2+2a^2} \end{cases}$$

After addition the conclusion follows.

PROBLEM 1.004-***Solution by Abhay Chandra – India***

$$\sqrt{a_k^2 - a_ka_{k+1} + a_{k+1}^2} = \sqrt{\frac{3}{4}(a_k - a_{k+1})^2 + \frac{1}{4}(a_k + a_{k+1})^2} \ge \frac{a_k + a_{k+1}}{2}$$

And the result follows after summation. Equality at $a_1 = a_2 = \cdots = a_{n+1}$.

PROBLEM 1.005-*Solution by Soumitra Mukherjee-Chandar Nagore-India*

$$\frac{yz}{x}(a^2y+b^2z)+\frac{zx}{y}(a^2z+b^2x)+\frac{xy}{z}(a^2x+b^2y)=$$
$$=y^2\left(\frac{a^2z}{x}+\frac{b^2x}{z}\right)+z^2\left(\frac{b^2y}{x}+\frac{a^2x}{y}\right)+x^2\left(\frac{b^2z}{y}+\frac{a^2y}{x}\right)$$
$$\geq 2ab(x^2+y^2+z^2)\ (Applying\ AM \geq GM) \geq \frac{2ab}{3}(x+y+z)^2$$

PROBLEM 1.006-*Solution by proposer*

$$2a^4+10a^2-5a^3-8a+5=$$
$$=2a^4-3a^3+5a^2-2a^3+3a^2-5a+2a^2-3a+5=$$
$$=a^2(2a^2-3a+5)-a(2a^2-3a+5)+(2a^2-3a+5)=$$
$$=(2a^2-3a+5)(a^2-a+1)=\left[2\left(a-\frac{3}{4}\right)^2+\frac{31}{4}\right]\left[\left(a-\frac{1}{2}\right)^2+\frac{3}{4}\right]>0$$
$$2a^4+10a^2-5a^3-8a+5>0$$
$$2b^4+10b^2-5b^3-8b+5>0$$
$$2c^4+10c^2-5c^3-8c+5>0$$
$$2(a^4+b^4+c^4)+10(a^2+b^2+c^2)+1>5(a^3+b^3+c^3)+8(a+b+c)$$
$$2(a^4+b^4+c^4)+10(a^2+b^2+c^2)>5(a^3+b^3+c^3)+16-15$$
$$2(a^4+b^4+c^4)+10(a^2+b^2+c^2)>5(a^3+b^3+c^3)+1$$

PROBLEM 1.007-*Solution by Hung Nguyen Viet – HaNoi – VietNam*

Let a, b, c *be positive real numbers such that* $a+b+c=3$. *Prove that*

$$a\left(\frac{1}{b^3}+\frac{1}{c^3}\right)+b\left(\frac{1}{c^3}+\frac{1}{a^3}\right)+c\left(\frac{1}{a^3}+\frac{1}{b^3}\right)\geq\frac{18}{a^3+b^3+c^3}$$

By Cauchy – Schwarz inequality we obtain:

$$\sum_{cyc} a\left(\frac{1}{b^3}+\frac{1}{c^3}\right)=\frac{b+c}{a^3}+\frac{c+a}{b^3}+\frac{a+b}{c^3}$$
$$=\frac{(b+c)^2}{(b+c)a^3}+\frac{(c+a)^2}{(c+a)b^3}+\frac{(a+b)^3}{(a+b)c^3}\geq\frac{4(a+b+c)^2}{(b+c)a^3+(c+a)b^3+(a+b)c^3}=$$
$$=\frac{36}{(a+b+c)(a^3+b^3+c^3)-(a^4+b^4+c^4)}=\frac{36}{3(a^3+b^3+c^3)-(a^4+b^4+c^4)}$$

It suffices to show that: $a^4+b^4+c^4 \geq a^3+b^3+c^3$

Indeed, this is true by Cauchy – Schwarz inequality as follows:

$$\frac{a^4+b^4+c^4}{a^3+b^3+c^3}\geq\frac{a^3+b^3+c^3}{a^2+b^2+c^2}\geq\frac{a^2+b^2+c^2}{a+b+c}\geq\frac{a+b+c}{3}=1 \text{ and we are done.}$$

PROBLEM 1.008-*Solution by Soumava Chakraborty – Kolkata – India*

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\Rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=2\left(\frac{b}{a}-\frac{a}{b}+\frac{c}{b}-\frac{b}{c}+\frac{a}{c}-\frac{c}{a}\right)$$

$$= 2\left(\frac{b^2-a^2}{ab}+\frac{c^2-b^2}{bc}+\frac{a^2-c^2}{ca}\right)$$

$$= \frac{2}{bc}\left(c(b^2-a^2)+a(c^2-b^2)+b(a^2-c^2)\right) = \frac{2}{abc}(b-a)(b-c)(c-a)\ (1)$$

If any 2 sides are equal, RHS of (1) $= 0$

Also, if all sides are equal, RHS of (1) $= 0$

But LHS of (1) $= \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$ *(AM≥GM)*

all sides can't be equal. Also 2 sides can't be equal.

(1) $\Rightarrow (b-a)(b-c)(c-a) > 0 \Rightarrow a > b > c$ or $b > c > a$ or $c > a > b$

Case 1 $a > b > c$

$$a-b < c, b-c < a, a-c < b$$

$$\Rightarrow (a-b)(b-c)(a-c) < abc \Rightarrow (b-a)(b-c)(c-a) < abc$$

$$\Rightarrow \frac{2}{abc}(b-a)(b-c)(c-a) < 2 \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} < 2, \textit{false}$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$$

Case 2 $b > c > a$

$$b-c < a, c-a < b, b-a < c$$

$$\Rightarrow (b-a)(b-c)(c-a) < abc \Rightarrow \frac{2}{abc}(b-a)(b-c)(c-a) < 2,$$

$$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} < 2 \textit{ which is false, } \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$$

Case 3 $c > a > b$

$$a-b < c, c-b < a, c-a < b$$

$$\Rightarrow (a-b)(c-b)(c-a) < abc \Rightarrow (b-a)(b-c)(c-a) < abc$$

$$\Rightarrow \frac{2}{abc}(b-a)(b-c)(c-a) < 2 \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} < 2, \textit{false, } \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$$

Hence, in any Δ, $3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) = 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$ *is impossible.*

PROBLEM 1.009-*Solution by Ravi Prakash- New Delhi-India*

For $0 < y \leq x$

$$2(x^2+xy+y^2)-(x+2y)(x+y) = 2x^2+2xy+2y^2-(x^2+3xy+2y^2)$$

$$= x^2-xy = x(x-y) \geq 0. \textit{ Also,}$$

$$(2x+y)(2+y)-2(x^2+xy+y^2) = 2x^2+3xy+y^2-2x^2-2xy-2y^2 = (x-y)y \geq 0$$

$$\textit{for } 0 < y \leq 2x, x+2y \leq \frac{2(x^2+xy+y^2)}{x+y} \leq 2x+y.$$

As $0 < c \leq b \leq a$, *the desired inequality follows.*

PROBLEM 1.010-*Solution by Hamza Mahmood- Lahore – Pakistan*

First we show that $4\cos 12° + 4\cos 36° = \frac{2\sin 48°}{\sin 12°}$:

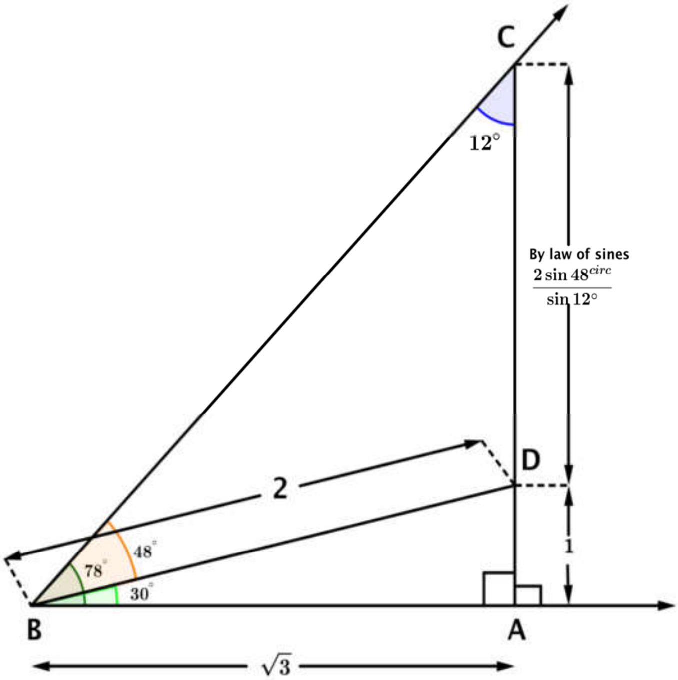
Using identities $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{B-A}{2}\right)$ & $\cos\theta = \frac{\sin 2\theta}{2\sin\theta}$, *we have:*

$$4\cos 12° + 4\cos 36° = 8\cos(24°)\cos(12°) = 8\cdot\frac{\sin 48°}{2\sin 24°}\cdot\frac{\sin 24°}{2\sin 12°} = \frac{2\sin 48°}{\sin 12°}$$

Now consider a right angled triangle ABC with

$$m\angle BAC = 90°, m\angle ABC = 78° \;\&\; m\overline{AB} = \sqrt{3}$$

as shown in the figure below (not drawn to scale):



In right angled triangle BAD: $\tan 30° = \frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AD}}{\sqrt{3}} \Rightarrow \frac{1}{\sqrt{3}} = \frac{m\overline{AD}}{\sqrt{3}} \Rightarrow m\overline{AD} = 1$

and $\sin 30° = \frac{m\overline{AD}}{m\overline{BD}} = \frac{1}{m\overline{BD}} \Rightarrow \frac{1}{2} = \frac{1}{m\overline{BD}} \Rightarrow m\overline{BD} = 2$. *Now in* ΔBCD: *By law of sines:*

$$\frac{m\overline{BD}}{\sin 12°} = \frac{m\overline{CD}}{\sin 48°} \Rightarrow m\overline{CD} = \frac{2\sin 48°}{\sin 12°}$$

Since we have already shown that $\frac{2\sin 48°}{\sin 12°} = 4\cos 12° + 4\cos 36°$,

$$\Rightarrow m\overline{CD} = \frac{2\sin 48°}{\sin 12°} = 4\cos 12° + 4\cos 36°$$

Finally from the figure, $\tan 78° = \frac{m\overline{AC}}{m\overline{AB}} = \frac{m\overline{CD}+m\overline{DA}}{m\overline{AB}}$

$$\Rightarrow \tan 78° = \frac{4\cos 12° + 4\cos 36° + 1}{\sqrt{3}}$$

PROBLEM 1.011-*Solution by Anas Adlany –El Jadida- Morroco*

Let $s = \frac{a+b+c}{2} = 3$ *(the inequality is homogenuous), so we need to prove that* $\sum \frac{a}{\sqrt{3-a}} \geq 3\sqrt{3}$. *Consider* $f(x) = \frac{x}{\sqrt{3-x}} \Rightarrow f''(x) = -\frac{x-12}{4\sqrt{(3-x)^5}} > 0$ *[since* $0 < x < 3$], *so* f *is convex on* $(0,3)$ *hence by Jensen inequality we get*

$$\sum f(a) \geq 3f\left(\sum \frac{a}{3}\right) = 3f(2) = 6 \geq 3\sqrt{3}$$

PROBLEM 1.012-*Solution by Omar Raza – Lahore – Pakistan*

$$\sum \frac{a^4}{a^2 + 2abcd(a)} \geq \frac{(a^2+b^2+c^2+d^2)^2}{a^2+b^2+c^2+d^2+2abcd(a+b+c+d)} =$$

$$= \frac{1}{1+2abcd(a+b+c+d)}$$

From RMS ≥ AM inequality $\left(\frac{1}{4}\right)^{\frac{1}{2}} \ge \frac{a+b+c+d}{4}$; $a+b+c+d \le 2$ *and similarly*

$$\left(\frac{1}{4}\right)^{\frac{1}{2}} \ge (abcd)^{\frac{1}{4}} \text{ so } abcd \le \frac{1}{16}$$

$$\text{thus } \frac{1}{1+2abcd(a+b+c+d)} \ge \frac{1}{(1+2abcd(2))} \ge \frac{1}{1+2\left(\frac{1}{16}\right)\cdot 2} = \frac{1}{1+\frac{1}{4}} \ge \frac{4}{5}$$

PROBLEM 1.013-*Solution by Dana Heuberger – Romania*

Conditions of existence: $x, y \in (0,\infty)$.

a. $\log_a(x+y) = \log_a x + \log_a y \Leftrightarrow x+y = x\cdot y \Leftrightarrow (x-1)(y-1) = 1$ *and any pair* (x,y), *with* $\begin{cases} x > 1 \\ y = \frac{x}{x-1} \end{cases}$ *is solution.*

b. *We choose* $y = a^k$, *with* $k \in \mathbb{N}, k \ge 2$. *We obtain*

$$\log_a(x+a^k) = \log_a(x^k) \Leftrightarrow x + a^k = x^k \text{ and then}$$

$$1 + \frac{1}{x}\cdot a^k = x^{k-1} \qquad (1)$$

Let be $f, g: (0,\infty) \to \mathbb{R}, f(x) = 1 + \frac{1}{x}\cdot a^k, g(x) = x^{k-1}$.

Because f *is strictly decreasing and* g *is strictly increasing, the equation (1) has at least a solution.*

I. $a > 1$. $f(1) > g(1)$ *and* $f(2a) < g(2a) \Rightarrow$ *the equation has a unique solution,* x_k, *which belongs to the interval* $(1,2a)$.

II. $a < 1$. $f(1) > g(1)$ *and* $f\left(\frac{2}{a}\right) < g\left(\frac{2}{a}\right) \Rightarrow$ *the equation has a unique solution,* x_k, *which belongs to the interval* $\left(1,\frac{2}{a}\right)$.

PROBLEM 1.014-*Solution by Soumitra Mukherjee-Chandar Nagore-India*

$$\text{Let } f(x) = 11x + 2\sqrt[4]{x} - 13x^3; \forall x \in (0,1)$$

$$f'(x) = 11 + \frac{1}{2x^{\frac{3}{4}}} - 39x^2; \forall x \in (0,1), f'(x) > 0; \forall x \in (0,1)$$

$f(x)$ *is continuous on* $(0,1)$ *again* $f'(x) > 0; \forall x \in (0,1)$

$f(x)$ *is increasing on* $(0,1)$

$$f(x) \ge f(0) = 0 \Rightarrow 11x + 2\sqrt[4]{x} \ge 13x^3; (\forall)x \in (0,1)$$

for $a, b, c \in (0,1)$ *and* $a+b+c = 3$,

$$11\sum_{cyc} a + 2\sum_{cyc}\sqrt[4]{a} \ge 13\sum_{cyc} a^3 \Rightarrow 33 + 2\sum_{cyc}\sqrt[4]{a} \ge 39abc \Rightarrow 11 + \frac{2}{3}\sum_{cyc}\sqrt[4]{a} \ge 13abc$$

PROBLEM 1.015-*Solution by Soumitra Mukherjee-Chandar Nagore-India*

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = 3$$

$$\sum_{cyc}\frac{a\sqrt{b}+b\sqrt{a}}{a-\sqrt{ab}+b}\le\sum_{cyc}\frac{a\sqrt{b}+b\sqrt{a}}{\sqrt{ab}}=\sum_{cyc}\left(\sqrt{a}+\sqrt{b}\right)=2\sum_{cyc}\sqrt{a}=6$$

Equality at $a=b=c=1$.

PROBLEM 1.016-*Solution by Safal Das Biswas –Kolkata- India*

$m^4=4(p^n-1)$. *Since* $m^4\equiv 0(mod\ 4)$ *then* $m^4=2^4k^4$ *some integer* k.

Then, this leads, $4k^4+1=p^n$, *so* $(2k^2+2k+1)(2k^2-2k+1)=p^n$.

Since p *is prime then we can set* $2k^2+2k+1=p^{n_1}$ *and* $2k^2-2k+1=p^{n_2}$ *for some integer* n_1 *and* n_2 *where* $n_1+n_2=n$.

Now, $p^{n_1}-p^{n_2}=4k$. *Thus,* $k=\frac{p^{n_1}-p^{n_2}}{4}$, *and* $2k^2+1=\frac{p^{n_1}+p^{n_2}}{2}$, *so we have the final set up,*

$2\left(\frac{p^{n_1}-p^{n_2}}{4}\right)^2+1=\left(\frac{p^{n_1}+p^{n_2}}{2}\right)$, *implies* $\left(\frac{p^{n_1}-p^{n_2}}{2}\right)^2=p^{n_1}+p^{n_2}-2$ *so*

we have, or $(p^{n_1}-p^{n_2})^2=4(p^{n_1}+p^{n_2})-8$, *or,*

$p^{2n_1}+p^{2n_2}-4p^{n_1}-4p^{n_2}+8=2p^{n_1}p^{n_2}$, *or,*

$(p^{n_1}-2)^2+(p^{n_2}-2)^2+2(p^{n_1}-2)(p^{n_2}-2)=4(p^{n_1}p^{n_2}-p^{n_1}-p^{n_2}+2)$.

So we have $(p^{n_1}+p^{n_2}-4)^2=4(p^{n_1}p^{n_2}-p^{n_1}-p^{n_2}+2)=$

$=4\left((p^{n_1}-1)(p^{n_2}-1)+1\right)$. *So we have:* $\left(\frac{p^{n_1}+p^{n_2}-4}{2}\right)^2=(p^{n_1}-1)(p^{n_2}+1)+1$.

As, p *is prime, then:* $(p^{n_1}-1)(p^{n_2}-1)\equiv 0(mod\ 4)$

Thus, $(p^{n_1}-1)(p^{n_2}-1)+1=4u+1$. *So,* $\frac{p^{n_1}+p^{n_2}-4}{2}\equiv 1(mod\ 2)$, *which clearly suffices*

$(2a+1)^2=(p^{n_1}-1)(p^{n_2}-1)+1$, *which gives,* $(p^{n_1}-1)(p^{n_2}-1)=4a(a+1)$.

Thus we get:

$\left(\frac{p^{n_1}-1}{2}\right)\left(\frac{p^{n_2}-1}{2}\right)=a(a+1)$, *set,* $\left(\frac{p^{n_1}-1}{2}\right)=x$ *and* $\left(\frac{p^{n_2}-1}{2}\right)=y$ *then we have that,*

$(x+y-1)^2=4xy+1$, *and* $xy=a(a+1)$ *as* $x>y$ *set,* $x=m(mh+1)$

and $y=h$ *and* $a=mh$. *Thus,* $x+y-1=2a+1$ *so,*

$x+h=x+y=2(a+1)=2(mh+1)$, so, $x=2(mh+1)-h$ *again,* $x=m(mh+1)$, *so*

by compairing, $m(mh+1)=2(mh+1)-h$, *so,* $h=(2-m)(mh+1)$.

As $h\ge 0$ *hence* $2-m\ge 0$ *so,* $m\in\{1,2\}$ *if* $m=1$ *then* $h=h+1$ *which is contradiction,*

hence $m=2$ *and* $h=0$, *which gives* $n_2=0$, $n_1=1$, $p=5$, *so* $n=1$ *and* $m=2$ *is the only solution.*

Note 1: Here all the variables that are used $\in\mathbb{Z}$ *and also observed that since* p *is prime we have,* $(p^{n_j}-1)\equiv 0(mod\ 2)\forall\ j\in\{1,2\}$

Note 2: $(x+y-1)^2=4xy+1$ *is true as:* $\left(\frac{p^{n_1}+p^{n_2}-4}{2}\right)^2=(p^{n_1}-1)(p^{n_2}-1)+1$

and we have substituted $\left(\frac{p^{n_1}-1}{2}\right)=x$ *and* $\left(\frac{p^{n_2}-1}{2}\right)=y$.

PROBLEM 1.017-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Para todos los reales no negativos: a,b,c,x,y,z *se cumple la siguiente desigualdad:*

$$(b+c)x+(a+c)y+(a+b)z\ge 2\sqrt{(ab+bc+ac)(xy+yz+zx)}$$

Aplicando: Cauchy – Schwarz:

$$P=(b+c)x+(a+c)y+(a+b)z=(a+b+c)(x+y+z)-(ax+by+cz)$$

$$P=\sqrt{\left((a^2+b^2+c^2)+2(ab+bc+ac)\right)\left((x^2+y^2+z^2)+2(xy+yz+zx)\right)}-$$
$$-(ax+by+cz)$$
$$\geq\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)}+2\sqrt{(ab+bc+ac)(xy+yz+zx)}-$$
$$-(ax+by+cz)\geq 2\sqrt{(ab+bc+ac)(xy+yz+zx)}$$

La igualdad se alcanza cuando: $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$. *Por lo tanto:*

Sea: $a=a^3, b=b^3, c=c^3, x=\frac{1}{b+c}, y=\frac{1}{a+c}, z=\frac{1}{a+b}$

$$\Rightarrow (b^3+c^3)\left(\frac{1}{b+c}\right)+(a^3+c^3)\left(\frac{1}{a+c}\right)+(a^3+b^3)\left(\frac{1}{a+b}\right)\geq$$
$$\geq 2\sqrt{(a^3b^3+b^3c^3+a^3c^3)\left(\frac{2(a+b+c)}{(a+b)(b+c)(a+c)}\right)}$$
$$\Rightarrow 2a^2+2b^2+2c^2\geq ab+bc+ac+2\sqrt{(a^3b^3+b^3c^3+a^3c^3)\left(\frac{2(a+b+c)}{(a+b)(b+c)(a+c)}\right)}$$
$$\Rightarrow a^2+b^2+c^2\geq\frac{ab+bc+ac}{2}+\sqrt{(a^3b^3+b^3c^3+a^3c^3)\left(\frac{2(a+b+c)}{(a+b)(b+c)(a+c)}\right)}$$

PROBLEM 1.018-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Probar en un triángulo ABC: $5r\leq\frac{PA^2}{h_a}+\frac{PB^2}{h_b}+\frac{PC^2}{h_c}\leq\frac{5R}{2}$. *De la siguiente identidad:*

$$aMA^2+bMB^2+cMC^2=(a+b+c)MI^2+abc. \text{ Sea: } M=P$$
$$aPA^2+bPB^2+cPC^2=(2p)PI^2+abc \quad (A)$$

Del gráfico: $PI=r$ *(Caso particular). Dividiendo:* $(\div\, abc)$

$$\frac{PA^2}{bc}+\frac{PB^2}{ac}+\frac{PC^2}{ab}=\frac{2pr^2}{abc}+1,\quad abc=4pRr$$
$$\frac{PA^2}{bc}+\frac{PB^2}{ac}+\frac{PC^2}{ab}=\frac{r}{2R}+1\rightarrow$$
$$\rightarrow\frac{2R}{bc}PA^2+\frac{2R}{ac}PB^2+\frac{2R}{ab}PC^2=2R+r\rightarrow\frac{PA^2}{h_a}+\frac{PB^2}{h_b}+\frac{PC^2}{h_c}=2R+r$$
$$5r\leq 2R+r\leq\frac{5R}{2}\Rightarrow 2R+r\geq 5r\rightarrow R\geq 2r \quad (\text{Desigualdad de Euler})$$
$$\Rightarrow 2R+r\leq\frac{5R}{2}\Rightarrow r\leq\frac{R}{2}\rightarrow R\geq 2r \ (\text{Desigualdad de Euler})$$

PROBLEM 1.019-*Solution by proposer*

We have: $\frac{a^3}{a^2+bc}\geq\frac{4a-b-c}{4}\Leftrightarrow b(a-c)^2+c(a-b)^2\geq 0$ *therefore*

$$\begin{cases}\frac{a^3\ln x}{a^2+bc}\geq\frac{(4a-b-c)\ln x}{4}\\ \frac{b^3\ln y}{b^2+ca}\geq\frac{(4b-c-a)\ln y}{4}\\ \frac{c^3\ln z}{c^2+ab}\geq\frac{(4c-a-b)\ln z}{4}\end{cases}. \text{ After addition we have:}$$

$$\sum \ln x^{\frac{a^3}{a^2+bc}} = \sum \frac{a^3 \ln x}{a^2+bc} \geq \sum \frac{(4a-b-c)\ln x}{4} = \sum \frac{a(4\ln x - \ln y - \ln z)}{4} =$$

$$= \sum \ln\left(\frac{x^4}{yz}\right)^{\frac{a}{4}} \textit{ therefore } \prod x^{\frac{4a^3}{a^2+bc}} \geq \prod \left(\frac{x^4}{yz}\right)^{a}$$

PROBLEM 1.020-*Solution by proposer*

Let $\sum_{k=1}^{n} a_k^2 = n + n(n-1)t^2$, *with* t *be real number,* $0 \leq t < 1$. *We have:*

$$\alpha \sum_{k=1}^{n} \frac{1}{a_k} = \frac{\alpha}{1+(n-1)t} \sum_{k=1}^{n} \frac{1+(n-1)t-a_k}{a_k} + \frac{n\alpha}{1+(n-1)t}$$

Using Cauchy-Schwarz inequality:

$$\sum_{k=1}^{n} \frac{1+(n-1)t-a_k}{a_k} \geq \frac{\left(\sum_{k=1}^{n}[1+(n-1)t-a_k]\right)^2}{\sum_{k=1}^{n} a_k + (n-1)t\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k^2} = \frac{n(n-1)t}{1-t}$$

$$\Rightarrow \alpha \sum_{k=1}^{n} \frac{1}{a_k} \geq \frac{n\alpha(n-1)t}{[1+(n-1)t]} + \frac{n\alpha}{1+(n-1)t} = \frac{n\alpha(nt-2t+1)}{[1+(n-1)t](1-t)}$$

$$\Rightarrow \alpha \sum_{k=1}^{n} \frac{1}{a_k} + \frac{\beta}{\sqrt{\sum_{k=1}^{n} a_k^2}} \geq \frac{n\alpha(nt-2t+1)}{[1+(n-1)t](1-t)} + \frac{\beta}{\sqrt{n+n(n-1)t^2}}$$

We need to prove that: $\frac{n\alpha(nt-2t+1)}{[1+(n-1)t](1-t)} + \frac{\beta}{\sqrt{n+n(n-1)t^2}} \geq n\alpha + \frac{\beta}{\sqrt{n}}$, *(*)*

(*) $\Leftrightarrow \frac{n\alpha(n-1)t^2}{[1+(n-1)t](1-t)} - \frac{\beta(n-1)t^2}{\sqrt{n+n(n-1)t^2}\left(1+\sqrt{1+(n-1)t^2}\right)} \geq 0 \Leftrightarrow$

$$\Leftrightarrow (n-1)\left(\alpha n\sqrt{n}+\beta\right)t^2 - \beta(n-2)t + n\alpha\sqrt{n+n(n-1)t^2} + \alpha n\sqrt{n} - \beta \geq 0$$

Considering the function:

$$f(t) = (n-1)\left(\alpha n\sqrt{n}+\beta\right)t^2 - \beta(n-2)t + n\alpha\sqrt{n+n(n-1)t^2} + \alpha n\sqrt{n} - \beta$$

We have: $f(t) \geq g(t) = (n-1)\left(\alpha n\sqrt{n}+\beta\right)t^2 - \beta(n-2)t + 2\alpha n\sqrt{n} - \beta$

Considering function: $g(t) = (n-1)\left(\alpha n\sqrt{n}+\beta\right)t^2 - \beta(n-2)t + 2\alpha n\sqrt{n} - \beta, \forall t \in [0,1]$

Three quadractic formula $g(t)$ *reaches the minimum value when* $t = \frac{\beta(n-2)}{2(n-1)\left(n\sqrt{n}\alpha+\beta\right)} \Rightarrow$

$$\Rightarrow Min g(t) = 2\alpha n\sqrt{n} - \beta - \frac{\beta^2(n-2)^2}{4(n-1)\left(n\sqrt{n}\alpha+\beta\right)} =$$

$$\frac{4\alpha(n-1)\left(2\alpha n\sqrt{n}+\beta\right) - \beta^2\sqrt{n}}{4(n-1)\left(n\sqrt{n}a+\beta\right)} > 0$$

$\Rightarrow f(t) \geq g(t) \geq Min g(t) > 0$. *Equality occurs when* $a_1 = a_2 = \cdots = a_n = 1$.

PROBLEM 1.021-*Solution by Ngô Minh Ngọc Bảo-Gia Lang Province-VietNam*

Using $AM - GM$ *inequality, we have:*

$$2x\sqrt{y+z} + 2y\sqrt{z+x} + 2z\sqrt{x+y} \leq x^2 + y^2 + z^2 + 2(x+y+z)$$

We need to prove that: $x^3 + y^3 + z^3 \geq x^2 + y^2 + z^2 + 2(x+y+z)(*)$

$$(*) \Leftrightarrow (x^3 - x^2 - 2x) + (y^3 - y^2 - 2y) + (z^2 - z^2 - 2z) \geq 0$$

We have: $x^3 - x^2 - 2x \geq 6x - 12 \Leftrightarrow (x-2)^2(x+3) \geq 0$ *(true)*

$$\text{Similarly, } y^3 - y^2 - 2y \geq 6y - 12 \Leftrightarrow (y-2)^2(y+3) \geq 0 \text{ (true)}$$
$$\text{and } z^3 - z^2 - 2z \geq 6z - 12 \Leftrightarrow (z-2)^2(z+3) \geq 0 \text{ (true)}$$
$$\Rightarrow (x^3 - x^2 - 2x) + (y^3 - y^2 - 2y) + (z^3 - z^2 - 2z) \geq 6(x+y+z) - 36 \geq$$
$$\geq 6 \cdot 3\sqrt[3]{xyz} - 36 = 0$$

PROBLEM 1.022-*Solution by Adil Abdullayev-Baku-Azerbaidian*

$$\text{(VAN AUBEL) } \sum_{cyc} \frac{AH}{h_a} = 2. \text{ Lemma: } \sum_{cyc} AH = 2(R+r)$$
$$\left(\sum_{cyc} \frac{HA}{\sqrt{bc}}\right)^2 \overset{C-B-S}{\leq} \sum_{cyc} AH \cdot \sum_{cyc} \frac{AH}{bc} = 2(R+r) \cdot \sum_{cyc} \frac{AH}{bc} \quad (A)$$
$$\sum_{cyc} \frac{AH}{bc} = \frac{1}{2R} \cdot \sum_{cyc} \frac{AH}{\frac{bc}{2R}} = \frac{1}{2R} \cdot \sum_{cyc} \frac{AH}{h_a} \overset{VAN\ AUBEL}{=} \frac{1}{R} \quad (B)$$
$$(A)(B) \Rightarrow LHS \leq RHS$$

PROBLEM 1.023-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Siendo: a, b, c, d reales positivos. Probar que: $\frac{a}{bc} + \frac{b}{cd} + \frac{c}{da} + \frac{d}{ab} \geq \frac{8}{\sqrt{a^2+b^2+c^2+d^2}}$

$\Rightarrow \frac{a^2d+b^2a+c^2b+d^2c}{abcd}\sqrt{a^2+b^2+c^2+d^2} \geq 8 \Rightarrow$ *Siendo:* $a, b, c, d > 0 \rightarrow$ *Por:* $MA \geq MG$

$$a^2d + b^2a + c^2b + d^2c \geq 4\sqrt[4]{(abcd)^3} \Leftrightarrow \frac{a^2d + b^2a + c^2b + d^2c}{abcd} \geq 4\sqrt[4]{\frac{1}{abcd}} \quad (A)$$
$$\text{Además: } \sqrt{a^2+b^2+c^2+d^2} \geq \sqrt{4\sqrt[4]{(abcd)^2}} = 2\sqrt[4]{abcd} \quad (B)$$

Desde que: $\frac{a^2d+b^2a+c^2b+d^2c}{abcd} \geq 4\sqrt[4]{\frac{1}{abcd}} \wedge \sqrt{a^2+b^2+c^2+d^2} \geq 2\sqrt[4]{abcd}$

Multiplicando las expresiones se tiene que: $\frac{a^2d+b^2a+c^2b+d^2c}{abcd}\sqrt{a^2+b^2+c^2+d^2} \geq 8$

PROBLEM 1.024-*Solution by Kevin Soto Palacios-Huarmey-Peru*

KLAMKIN's INEQUALITY: $\rightarrow x, y, z \in \mathbb{R}, n \in \mathbb{Z}$

$$x^2 + y^2 + z^2 \geq (-1)^{n+1}(2yxz\cos(nA) + 2zx\cos(nB) + 2xy\cos(nC)) \Leftrightarrow$$

$\Leftrightarrow$ *(Demostrado anteriormente). Si:* $n = 2$

$$x^2 + y^2 + z^2 \geq -2yz\cos 2A - 2zx\cos 2B - 2xy\cos 2C$$
$$x^2 + y^2 + z^2 \geq -2yz(1 - 2\operatorname{sen}^2 A) - 2zx(1 - 2\operatorname{sen}^2 B) - 2xy(1 - 2\operatorname{sen}^2 C)$$
$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \geq 4yz\operatorname{sen}^2 A + 4zx\operatorname{sen}^2 B + 4xy\operatorname{sen}^2 C$$
$$(x+y+z)^2 \geq 4yz\operatorname{sen}^2 A + 4zx\operatorname{sen}^2 B + 4xy\operatorname{sen}^2 C$$

Sea: $x = \frac{PA}{h_a} > 0, y = \frac{PB}{h_b} > 0, z = \frac{PC}{h_c} > 0$. *La desigualdad es equivalente:*

$$\frac{1}{4}\left(\frac{PA}{h_a} + \frac{PB}{h_b} + \frac{PC}{h_c}\right)^2 \geq \frac{PB \cdot PC}{h_b h_c}\operatorname{sen}^2 A + \frac{PA \cdot PC}{h_a h_c}\operatorname{sen}^2 B + \frac{PA \cdot PB}{h_a h_b}\operatorname{sen}^2 C$$

$$\frac{1}{4}\left(\frac{PA}{h_a}+\frac{PB}{h_b}+\frac{PC}{h_c}\right)^2\geq\frac{PB\cdot PC}{\frac{ac}{2R}\cdot\frac{ab}{2R}}\cdot\frac{a^2}{4R^2}+\frac{PA\cdot PC}{\frac{bc}{2R}\cdot\frac{ab}{2R}}\cdot\frac{b^2}{4R^2}+\frac{PA\cdot PB}{\frac{bc}{2R}\cdot\frac{ac}{2R}}\cdot\frac{c^2}{4R^2}$$

$$\frac{1}{4}\left(\frac{PA}{h_a}+\frac{PB}{h_b}+\frac{PC}{h_c}\right)^2\geq\frac{PB\cdot PC}{bc}+\frac{PA\cdot PC}{ac}+\frac{PA\cdot PB}{ab}$$

PROBLEM 1.025-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Sea: $n\geq 2$ *un entero y sean "*a,b,c*"*$\mathbb{R}^+$*, tal que:* $ab+bc+ac\leq 1$*. Probar que:*

$\frac{bc}{(2a^2+bc)^n}+\frac{ac}{(2b^2+ca)^n}+\frac{ab}{(2c^2+ab)^n}\geq\frac{1}{3}$*. Realizando la desigualdad ponderada "Jensen" para:*

$$f(x)=\frac{1}{x^n},x>0,n\geq 2\Leftrightarrow\text{(Convexo)}$$

$$\frac{bc}{(2a^2+bc)^n}+\frac{ac}{(2b^2+ca)^n}+\frac{ab}{(2c^2+ab)^n}=bcf(bc+2a^2)+acf(ac+2b^2)+abf(ab+2c^2)$$

$$bcf(bc+2a^2)+acf(ac+2b^2)+abf(ab+2c^2)\geq$$

$$\geq(bc+ac+ab)f\left(\frac{bc(bc+2a^2)+ac(ac+2b^2)+ab(ab+2c^2)}{ab+bc+ac}\right)$$

$$(bc+ac+ab)f\left(\frac{(bc)^2+(ac)^2+(ab)^2+2abc(a+b+c)}{ab+bc+ac}\right)=(bc+ac+ab)f\left(\frac{(ab+bc+ac)^2}{ab+bc+ac}\right)$$

$$(bc+ac+ab)f\left(\frac{(ab+bc+ac)^2}{ab+bc+ac}\right)=(bc+ac+ab)f(ab+bc+ac)=\frac{1}{(ab+bc+ac)^{n-1}}\geq 1$$

PROBLEM 1.026-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Sumando las ecuaciones se tiene:

$$2(ax+by+cz)=x+y+z\rightarrow(ax+by)+cz=\frac{x+y+z}{2}\rightarrow$$

$$\rightarrow cz=\frac{x+y-z}{2}\rightarrow c=\frac{x+y-z}{2z}$$

De forma análoga se tiene que: $a=\frac{y+z-x}{2x}$; $b=\frac{z+x-y}{2y}$*. Por la tanto:*

a) $abc\leq\frac{1}{8}\Leftrightarrow(x+y-z)(z+x-y)(y+z-x)\leq xyz$

$$(x+y-z)(z+x-y)(y+z-x)=(x^2-(y-z)^2)(y+z-x)=$$

$$=(x^2-y^2-z^2+2yz)(y+z-x)$$

$$\rightarrow(x^2-y^2-z^2+2yz)(y+z-x)=x^2y+x^2z-x^3-y^3-y^2z+y^2x-$$

$$-z^2y-z^3+z^2x+2y^2z+2yz^2-2xyz$$

$$\Rightarrow(x+y-z)(z+x-y)(y+z-x)=-x^3-y^3-z^3+xy(x+y)+$$

$$+yz(y+z)+zx(z+x)-2xyz\leq xyz$$

$$\Rightarrow x^3+y^3+z^3+3xyz\geq xy(x+y)+yz(y+z)+zx(z+x)$$

$$\Rightarrow x(x-y)(x-z)+y(y-x)(y-z)+z(z-x)(z-y)\geq 0\rightarrow$$

$\rightarrow$ *(Válido por desigualdad de Schur)*

b) $\frac{1}{2+a+b}+\frac{1}{2+b+c}+\frac{1}{2+a+c}\leq 1$

$$\rightarrow(2+a+c)(2+b+c)+(2+a+b)(2+a+c)+(2+a+b)(2+b+c)\leq$$

$$\leq(2+a+b)(2+b+c)(2+a+c)$$

$$\to 4(3)+\sum(a+c)(b+c)+2\sum(2a+b+c)\le 8+4(2)(a+b+c)+$$
$$+2\sum(a+c)(b+c)+\prod(a+b)$$
$$\to 4+8(a+b+c)\le 8(a+b+c)+\sum(a+c)(b+c)+\sum ab(a+b)+2abc$$
$$\Rightarrow 4\le\sum a^2+3\sum ab+\sum ab(a+b)+2abc$$

Tener en cuenta lo siguiente: $a=\frac{y+z-x}{2x}\ge 0;\ b=\frac{z+x-y}{2y}\ge 0;\ c=\frac{x+y-z}{2z}\ge 0$

$$\Rightarrow\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2\to\sum(1+b)(1+c)=2\prod(1+a)\to$$
$$\to 3+2\sum a+\sum ab=2+2\sum a+2\sum ab+2abc$$
$$\Rightarrow 1=ab+bc+ac+2abc\to ab+bc+ac=1-2abc\ \wedge abc\le\frac{1}{8}\Leftrightarrow$$
$$\Leftrightarrow ab+bc+ac\ge\frac{3}{4}\Rightarrow a+b+c\ge\sqrt{3(ab+bc+ac)}\ge\sqrt{\frac{9}{4}}=\frac{3}{2}\to a+b+c\ge\frac{3}{2}$$
$$\sum a^2+3\sum ab+\sum ab(a+b)+2abc\ge 4\sum ab+\sum ab\left(\frac{3}{2}-c\right)+2abc=$$
$$=\frac{11}{2}\sum ab-abc\ge\frac{33}{8}-\frac{1}{8}\ge 4$$

c) $a+b+c\ge 2(ab+bc+ca)$

Desde que: $a=\frac{y+z-x}{2x}\ge 0;\ b=\frac{z+x-y}{2y}\ge 0;\ c=\frac{x+y-z}{2z}\ge 0$

$$\Leftrightarrow y=n+m;z=n+p;x=m+p;\ a=\frac{n}{m+p}\ge 0;\ b=\frac{p}{n+m}\ge 0;$$
$$c=\frac{m}{n+p}\ge 0;$$
$$\Rightarrow\frac{n}{m+p}+\frac{p}{m+n}+\frac{m}{n+p}\ge 2\left(\frac{n}{m+p}\cdot\frac{p}{m+n}+\frac{n}{m+p}\cdot\frac{m}{n+p}+\frac{p}{m+n}\cdot\frac{m}{n+p}\right)$$
$$\Rightarrow\frac{n(n+m)(n+p)+p(p+m)(p+n)+m(m+p)(m+n)}{(m+n)(n+p)(m+p)}\ge$$
$$\ge\frac{2np(n+p)+mn(m+n)+mp(m+p)}{(m+n)(n+p)(m+p)}\Rightarrow\sum n^3-\sum mn(m+n)+3mnp\ge 0$$
$$\Rightarrow n(n-m)(n-p)+m(m-n)(m-p)+p(p-m)(p-n)\ge 0\Leftrightarrow$$
$$\Leftrightarrow \text{(Desigualdad de Schur)}$$

PROBLEM 1.027-*Solution by Soumava Chakraborty – Kolkata – India*

$$Let\ \lfloor x\rfloor=I\ and\ \{x\}=f\in[0,1)$$
$$x=I+f$$
$$(I+f+f)^2-(I+f+f)=6If-1\Rightarrow(I+2f)^2-(I+2f)=6If-1$$
$$\Rightarrow I^2+4f^2-2If-I-2f+1=0$$
$$\Rightarrow I^2-I(2f+1)+4f^2-2f+1=0\quad(1)$$
$$\Delta=(2f+1)^2-4(4f^2-2f+1)=-3(4f^2-4f+1)=-3(2f-1)^2\le 0$$

$$But,\ I \in \mathbb{Z} \in \mathbb{R}, \Delta \geq 0. So,\ \Delta \leq 0\ and\ \Delta \geq 0 \Rightarrow \Delta = 0 \Rightarrow 2f - 1 = 0 \Rightarrow f = \frac{1}{2}$$

$$\Delta = 0, from\ (1), we\ get,\ I = \frac{2f+1}{2} = \frac{2\cdot\frac{1}{2}+1}{2} = 1;\ x = I + f = 1 + \frac{1}{2} = \frac{3}{2}$$

PROBLEM 1.028-*Solution by Kevin Soto Palacios – Huarmey – Peru*

$$Probar\ en\ un\ triángulo\ ABC: \frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c} \geq \sqrt{1 + \frac{4R}{r}}. Tener\ en\ cuenta\ lo\ siguiente:$$

$$\frac{4R}{r} = \csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2} = \frac{abc}{(p-a)(p-b)(p-c)} = \frac{8abc}{(b+c-a)(a+c-b)(b+a-c)}$$

$$\frac{r_b}{r_a} = \frac{\frac{S}{p-b}}{\frac{S}{p-a}} = \frac{p-a}{p-b} = \frac{b+c-a}{a+c-b} > 0 \quad (IV), \frac{r_c}{r_a} = \frac{a+c-b}{b+a-c} > 0 \quad (V)\ \frac{r_a}{r_c} = \frac{a+b-c}{b+c-a} > 0 \quad (VI)$$

$$Sean:\ b + c - a = 2x > 0;\ a + c - b = 2z > 0;\ a + b - c = 2y > 0$$

$$Por\ la\ tanto:\ a = y + z, b = x + y, c = x + z$$

$$\Rightarrow \frac{x}{z} + \frac{z}{y} + \frac{y}{x} \geq \sqrt{1 + \frac{(x+y)(y+z)(x+z)}{xyz}}$$

$$Desde\ que:\ x, y, z > 0 \to\ (Elevando\ al\ cuadrado\ la\ expresión\ tenemos)$$

$$\Rightarrow \left(\frac{x}{z} + \frac{z}{y} + \frac{y}{x}\right)^2 \geq 1 + \frac{\sum xy(x+y) + 2xyz}{xyz} \Rightarrow \left(\frac{x}{z}\right)^2 + \left(\frac{z}{x}\right)^2 + \left(\frac{y}{x}\right)^2 + \frac{2x}{y} + \frac{2z}{x} + \frac{2y}{z} \geq$$

$$\geq 3 + \frac{x+y}{z} + \frac{z+x}{y} + \frac{y+z}{x}$$

$$\Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{z}{y}\right)^2 + \left(\frac{y}{x}\right)^2 + \frac{x}{y} + \frac{z}{x} + \frac{y}{z} \geq 3 + \frac{y}{x} + \frac{x}{z} + \frac{z}{y}$$

$$\Rightarrow \left(\left(\frac{x}{z}\right)^2 + 1\right) + \left(\left(\frac{z}{x}\right)^2 + 1\right) + \left(\left(\frac{y}{x}\right)^2 + 1\right) + \frac{x}{y} + \frac{z}{x} + \frac{y}{z} \geq 6 + \frac{y}{x} + \frac{x}{z} + \frac{z}{y}$$

$$Por:\ MA \geq MG$$

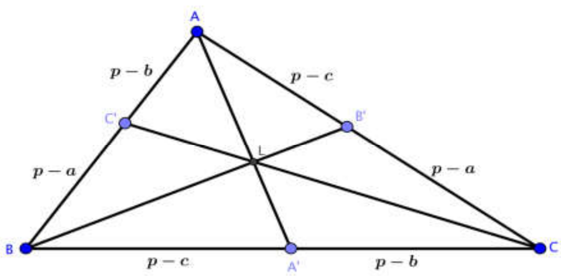
$$\Rightarrow \left(\left(\frac{x}{z}\right)^2 + 1\right) + \left(\left(\frac{z}{y}\right)^2 + 1\right) + \left(\left(\frac{y}{x}\right)^2 + 1\right) + \frac{x}{y} + \frac{z}{x} + \frac{y}{z} \geq \frac{2x}{z} + \frac{2z}{y} + \frac{2y}{x} + \frac{x}{y} + \frac{z}{x} + \frac{y}{z}$$

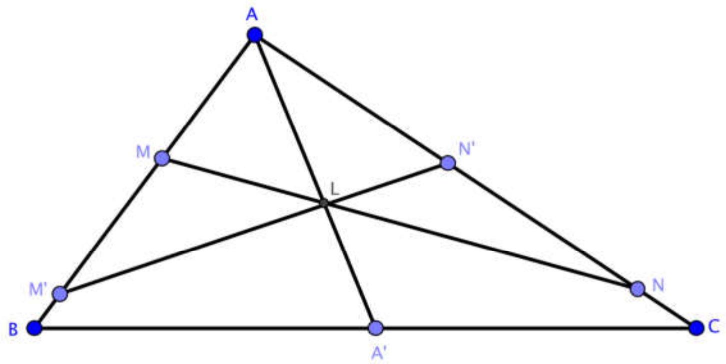
$$\Rightarrow Lo\ cual\ nos\ falta\ probar\ que:$$

$$\frac{2x}{z} + \frac{2z}{y} + \frac{2y}{x} + \frac{x}{y} + \frac{z}{x} + \frac{y}{z} \geq 6 + \frac{y}{x} + \frac{x}{z} + \frac{z}{y} \Leftrightarrow \frac{y}{x} + \frac{x}{z} + \frac{z}{y} + \frac{x}{y} + \frac{z}{x} + \frac{y}{z} \geq 6 \Leftrightarrow (MA \geq MG)$$

$$Por\ la\ tanto; \frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c} \geq \sqrt{1 + \frac{4R}{r}}$$

PROBLEM 1.029-*Solution by Marian Ursărescu – Romania*





From transversal theorem we have:

$$\frac{MB}{MA}\cdot A'C+\frac{NC}{NA}\cdot BA'=\frac{LA'}{LA}\cdot BC\Rightarrow\frac{MB}{MA}\cdot(p-b)+\frac{NC}{NA}\cdot(p-c)=\frac{LA'}{LA}\cdot a\Rightarrow$$

$$\frac{MB}{MA}(a+c-b)+\frac{NC}{NA}(a+b-c)=\frac{LA'}{LA}2a\quad(1)$$

Similarly: $\frac{M'B}{MA}(a+c-b)+\frac{N'C}{N'A}(a+b-c)=\frac{LA'}{LA}2a\quad(2)$

From (1)+(2)$\Rightarrow(a+c-b)\left(\frac{MB}{MA}+\frac{M'B}{M'A}\right)+(a+b-c)\left(\frac{NC}{NA}+\frac{M'C}{M'A}\right)=4a\frac{LA'}{LA}\quad(3)$

From (3) we must show: $4a\frac{LA'}{LA}>b+c-a\Rightarrow\frac{LA'}{LA}>\frac{b+c-a}{4a}\Leftrightarrow\frac{LA'}{LA}>\frac{p-a}{2a}\quad(4)$

From Van Aubel theorem we have: $\frac{LA}{LA'}=\frac{AC'}{C'B}+\frac{AB'}{B'C}=\frac{p-b}{p-a}+\frac{p-c}{p-a}=\frac{a}{p-a}\Rightarrow\frac{LA'}{LA}=\frac{p-a}{a}\quad(5)$

From (4)+(5) we must show: $\frac{p-a}{a}>\frac{p-a}{2a}\Leftrightarrow 1>\frac{1}{2}$ *true.*

PROBLEM 1.030-*Solution by proposer*

First we prove that $x^2+y^2+z^2\le xyz+2$ *for all* $x,y,z\in[0,1]$. *Let be*
$f(x)=x^2+y^2+z^2-xyz-2\Rightarrow f(x)\le\max\{f(0),f(1)\}$ *but*
$f(0)=y^2+z^2-2\le 0$ *with equality for* $y=z=1$ *and* $f(1)=y^2+z^2-yz-1$
Let be $g(y)=y^2+z^2-yz-1\Rightarrow g(y)\le\max\{g(0),g(1)\}=\max\{z^2-1,z^2-z\}\le 0$
Equality holds if and only if $x=y=z\Rightarrow$ *or* $x=y=1;z=0$, *etc.*

$$3\sum_{k=1}^{n}x_k^2=\sum_{cyclic}(x_1^2+x_2^2+x_3^2)\le\sum_{cyclic}(x_1x_2x_3+2)=$$
$$=2n+x_1x_2x_3+x_2x_3x_4+\cdots+x_nx_1x_2$$

PROBLEM 1.031-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Desde que: $a,b,c\ge 0$, *sea:* $a^4+b^4+c^4=3x^4,(ab+bc+ac)^2=9y^4$
$\Rightarrow 9(a+b+c)\ge x+26y$. *Por la desigualdad de Holder:*

$$(x^4+26y^4)(1+26)(1+26)(1+26)\ge(x+26y)^4\to\sqrt[4]{27^3(x^4+26y^4)}\ge x+26y$$

Esto es suficeiente probar: $9(a+b+c)\ge\sqrt[4]{27^3(x^4+26y^4)}\to$

$$\to 3^8(a+b+c)^4\ge 3^9(x^4+26y^4)$$
$$\Rightarrow 3(a+b+c)^4\ge 3(a^4+b^4+c^4)+26(ab+bc+ac)^2$$
$$\Rightarrow 3(a^4+b^4+c^4)+18(a^2b^2+b^2c^2+c^2a^2)+12ab(a^2+b^2)+$$

$$+12bc(b^2+c^2)+12ca(c^2+a^2)+36abc(a+b+c)\geq$$
$$\geq 3(a^4+b^4+c^4)+26(a^2b^2+b^2c^2+c^2a^2)+52abc(a+b+c)$$
$$\Rightarrow 12ab(a^2+b^2)+12bc(b^2+c^2)+12ca(c^2+a^2)\geq$$
$$\geq 8(a^2b^2+b^2c^2+c^2a^2)+16abc(a+b+c)$$
$$\Rightarrow 12ab(a^2+b^2)+12bc(b^2+c^2)+12ca(c^2+a^2)\geq$$
$$\geq 24(a^2b^2+b^2c^2+c^2a^2)\geq 8(a^2b^2+b^2c^2+c^2a^2)+16abc(a+b+c)$$
$$\Rightarrow 16(a^2b^2+b^2c^2+c^2a^2)\geq 16abc(a+b+c)\ \dots\ (LQQD)$$

PROBLEM 1.032-*Solution by Henry Ricardo - New York – USA*

Noting that $\frac{1}{4a+1}=\int_0^1 t^{4a}\,dt$*, we see that the given inequality is equivalent to*

$$\int_0^1 t^{4a}+t^{4b}+6t^{2a+2b}dt\geq\int_0^1 4t^{3a+b}+4t^{3b+a}\,dt,$$

or $t^{4a}+t^{4b}+6t^{2a+2b}\geq 4t^{3a+b}+4t^{3b+a}$*. If we let* $t^a=x$ *and* $t^b=y$*, the inequality is equivalent to* $x^4+y^4+6x^2y^2\geq 4x^3y+4xy^3$*, or* $(x-y)^4\geq 0$*, which is true.*

PROBLEM 1.033-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Siendo: $A+B+C=\pi$*. En un triángulo* ABC*, se cumple:*

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$$

$\Rightarrow$ *Sea:* $a\sqrt{bc}=\cos^2 A,\ b\sqrt{ac}=\cos^2 B,\ c\sqrt{ab}=\cos^2 C,$

$abc=\cos A\cos B\cos C>0$ *(Δ acutángulo). Por lo tanto:*

$a=\frac{\cos^3 A}{\cos B\cos C}>0,\ b=\frac{\cos^3 B}{\cos A\cos C}>0,\ c=\frac{\cos^3 C}{\cos A\cos B}>0$*. La desigualdad es equivalente en … (A):*

$$\Rightarrow\frac{\cos^2 B\cos^2 C}{\cos^4 A}+\frac{\cos^2 A\cos^2 C}{\cos^4 B}+\frac{\cos^2 A\cos^2 B}{\cos^4 C}\geq 2\frac{\cos^3 A}{\cos B\cos C}+2\frac{\cos^3 B}{\cos A\cos C}+2\frac{\cos^3 C}{\cos A\cos B}$$

De la siguiente desigualdad para todos x,y,z *números reales, se cumple en un triángulo* ABC*:*

$x^2+y^2+z^2\geq 2xy\cos A+2yz\cos B+2zx\cos C$*. Siendo:*

$x=\frac{\cos A\cos C}{\cos^2 B}>0,\ y=\frac{\cos B\cos A}{\cos^2 C}>0,\ z=\frac{\cos B\cos C}{\cos^2 A}>0\rightarrow$ *(Δ acutángulo)*

Se obtiene: $\Rightarrow\frac{\cos^2 B\cos^2 C}{\cos^4 A}+\frac{\cos^2 A\cos^2 C}{\cos^4 B}+\frac{\cos^2 A\cos^2 B}{\cos^4 C}\geq$

$\geq 2\frac{\cos^3 A}{\cos B\cos C}+2\frac{\cos^3 B}{\cos A\cos C}+2\frac{\cos^3 C}{\cos A\cos B}$ *… (LQQD)*

PROBLEM 1.034-*Solution by Nguyen Viet Hung – Hanoi – Vietnam*

The equation is equivalent to $(x^2-2x-y)(x^2-yx-1)=2$*.*

There are four possible cases as follows

Case 1: $\begin{cases}x^2-2x-y=1,\\x^2-yx-1=2,\end{cases}\Leftrightarrow\begin{cases}x^2-2x-1=y,\\x(x-y)=3.\end{cases}$

It's easy to find 3 pairs of (x,y) *satisfying this system of equations as*

$(-1,2),(1,-2),(3,2)$.

Case 2: $\begin{cases}x^2-2x-y=2,\\x^2-yx-1=1,\end{cases}\Leftrightarrow\begin{cases}x^2-2x-2=y,\\x(x-y)=2.\end{cases}$

There is only one pair (x,y) *satisfying this system of equations as* $(-1,1)$*.*

$$\text{Case 3: }\begin{cases}x^2-2x-y=-1,\\ x^2-yx-1=-2,\end{cases}\Leftrightarrow\begin{cases}(x-1)^2=y,\\ x(x-y)=-1.\end{cases}$$

There is no pair of (x,y) satisfying this system of equations.

$$\text{Case 4: }\begin{cases}x^2-2x-y=-2,\\ x^2-yx-1=-1,\end{cases}\Leftrightarrow\begin{cases}x^2-2x+2=y,\\ x(x-y)=0.\end{cases}$$

We find 3 pairs (x,y) satisfying this system of equations as $(0,2),(1,1),(2,2)$.

So, there are 7 pairs (x,y) satisfying the requirement as

$(-1,2),(1,-2),(3,2),(-1,1),(0,2),(1,1),(2,2)$.

PROBLEM 1.035-*Solution by Kevin Soto Palacios – Huarmey – Peru*

$$5+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+abc\ge(a+b+c)(ab+bc+ac)$$
$$\Rightarrow 5+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+abc\ge(a+b)(b+c)(c+a)+abc$$
$$\Rightarrow 15+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\ge3(a+b)(b+c)(c+a)$$
$$\Rightarrow a^3+b^3+c^3+15+3\sqrt[3]{a}+3\sqrt[3]{b}+3\sqrt[3]{c}\ge a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$
$$\Rightarrow a^3+b^3+c^3+3\sqrt[3]{a}+3\sqrt[3]{b}+3\sqrt[3]{c}\ge(a+b+c)^3=27$$

$\Rightarrow a^3+b^3+c^3+3\sqrt[3]{a}+3\sqrt[3]{b}+3\sqrt[3]{c}\ge12$. *Desde que:* $a,b,c\ge0$. *Por:* $MA\ge MG$

$\Rightarrow a^3+\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}\ge4\sqrt[4]{a^4}=4a\to a^3+3\sqrt[3]{a}\ge4a$ *… (A)*

Análogamente: $b^3+3\sqrt[3]{b}\ge4b$ *… (B);* $c^3+3\sqrt[3]{c}\ge4c$ *… (C)*

Sumando: (A) + (B) + (C): $(a^3+b^3+c^3)+3\sqrt[3]{a}+3\sqrt[3]{b}+3\sqrt[3]{c}\ge4(a+b+c)=12$

PROBLEM 1.036-*Solution by Rustem Zeynalov – Baku – Azerbaidjian*

$$x+y=a;\ y+z=b;\ z+x=c$$
$$a^4b^4c^4\ge\frac{16^3}{27}\cdot\left(\frac{a+b+c}{2}\right)^3\cdot\left[\frac{a+b-c}{2}\cdot\frac{a+c-b}{2}\cdot\frac{b+c-a}{2}\right]^3$$
$$a^4b^4c^4\ge\frac{1}{27}[(a+b+c)(a+b-c)(a+c-b)(b+c-a)]^3$$
$$(a+b+c)(a+b-c)(a+c-b)(b+c-a)\le\sqrt[3]{27a^4b^4c^4}$$
$$2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4\le\sqrt[3]{27a^4b^4c^4}$$
$$a^4+b^4+c^4+\sqrt[3]{27a^4b^4c^4}\ge2a^2b^2+2a^2c^2+2b^2c^2$$

Schur inequality

PROBLEM 1.037-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Desde que:

$$(4a-3)^2+(4b-3)^2+(4c-3)^2=16(a^2+b^2+c^2)+27-24(a+b+c)=$$
$$=128abc-24(a+b+c)\ge0\Rightarrow\frac{128}{24}\ge\frac{a+b+c}{abc}\to\frac{16}{3}\ge\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

Por: $MA\ge MG$: $a^3+b^3+\frac{27}{64}\ge3\sqrt[3]{\frac{27a^3b^3}{64}}\to\frac{1}{a^3+b^3+\frac{27}{64}}\le\frac{4}{9ab}$

Por la tanto tenemos en … (A): $A=\frac{1}{a^3+b^3+\frac{27}{64}}+\frac{1}{b^3+c^3+\frac{27}{64}}+\frac{1}{c^3+a^3+\frac{64}{27}}\le\frac{4}{9ab}+\frac{4}{9bc}+\frac{4}{9ac}\le$

$$\le \frac{4}{9} \times \frac{16}{3} = \frac{64}{27} \text{... (LQQD)}$$

$A_{Máx} \le \frac{64}{27}$. *La igualdad se alcanza cuando:* $a = b = c = \frac{3}{4}$

PROBLEM 1.038-*Solution by Ngo Minh Ngoc Bao – Vietnam*

We have two lemma: Lemma 1: If $a, b, c > 0$ *then* $\left(\sum \frac{a}{b}\right)\left(\sum a\right) \ge 3\sqrt{3\left(\sum a^2\right)}$

Prove: Use Cauchy – Schwarz

$$\sum \frac{a}{b} \ge \frac{\left(\sum a\right)^2}{\sum ab} \Rightarrow \left(\sum \frac{a}{b}\right)\left(\sum a\right) \ge \frac{\left(\sum a\right)^3}{\sum ab}.$$

We need to prove: $\frac{\left(\sum a\right)^3}{\sum ab} \ge 3\sqrt{3\left(\sum a^2\right)} \Leftrightarrow \left(\sum a\right)^6 \ge 27\left(\sum ab\right)^2\left(\sum a^2\right)$ *(**).*

Use AM – GM inequality: $\left(\sum a\right)^6 = \left(\left(\sum a^2\right) + \left(\sum ab\right) + \left(\sum ab\right)\right)^3 \ge 27\left(\sum a^2\right)\left(\sum ab\right)^2$

$\Rightarrow LHS$ *(**)* $\ge RHS$ *(**).*

Lemma 2: Consider polynomial

$$f(x, y, z) = \sum x^4 + A\sum x^2y^2 + Bxyz\sum x + C\sum x^3y + D\sum xy^3$$

(with A, B, C, D *is the constant)*

$$f(x, y, z) \ge 0 \Leftrightarrow \begin{cases} 1 + A + B + C + D \ge 0 \\ 3(1 + A) < C^2 + CD + D^2 \\ 5 + A + 2C + 2D \ge 0 \\ g(x) = (4 + C + D)(x^3 + 1) + (A + 2C - D - 1)x^2 + (A - C + 2D - 1)x \ge 0, \forall x \ge 0 \end{cases}$$

My solution

(*) $\Leftrightarrow 6\sum a^3b + 6\sum ab^3 + 13abc\sum a \ge 23abc\sqrt{3\left(\sum a^2\right)}$*, we need to prove:*

$$6\sum a^3b + 6\sum ab^3 + 13abc\sum a - \frac{23}{3}abc\left(\sum \frac{a}{b}\right)\left(\sum a\right) \ge 0$$

$$\Leftrightarrow 6\sum a^3b + 6\sum ab^3 + 13abc\sum a - \frac{23}{3}\left(\sum a^2b^2 + \sum ab^3 + abc\sum a\right) \ge 0$$

$$\Leftrightarrow 6\sum a^3b - \frac{5}{3}\sum ab^3 - \frac{23}{3}\sum a^2b^2 + \frac{16}{3}abc\sum a \ge 0 \text{ (***)}$$

Use lemma 2 with $A = -\frac{23}{3}, B = \frac{16}{3}, C = 6, D = -\frac{5}{3}$.

$$\text{We have: } \begin{cases} 1 + A + B + C + D = 1 - \frac{23}{3} + \frac{16}{3} + 6 - \frac{5}{3} = 3 > 0 \\ 5 + A + 2C + 2D = 5 - \frac{23}{3} + 12 - \frac{10}{3} = 6 > 0 \\ 3(1 + A) = -\frac{60}{3} < 6^2 - 6 \cdot \frac{5}{3} + \frac{25}{9} = 26 + \frac{25}{9} = C^2 + CD + D^2 \end{cases}$$

Considering function: $g(x) = \frac{25}{3}x^3 + 5x^2 - 18x + \frac{25}{3} \Rightarrow g'(x) = 25x^2 + 10x - 18$

$$g'(x) = 0 \Leftrightarrow 25x^2 + 10x - 18 = 0 \Leftrightarrow \left[\begin{array}{l} x = \dfrac{-1 + \sqrt{19}}{5} \\ x = \dfrac{-1 - \sqrt{19}}{5} \end{array}\right.$$

x	0		$\frac{-1+\sqrt{19}}{5}$		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$		$\searrow$		$\nearrow$	

$$g(x) \ge g\left(\frac{-1+\sqrt{19}}{5}\right) > 0 \Rightarrow g(x) > 0, \forall x \ge 0$$

PROBLEM 1.039-*Solution by proposer*

If $\lambda \in (0,1); x, y, z \in \mathbb{R}; x+y+z=1$ *then:* $\sum a^x b^y c^z \ge \sum a^{\lambda x+\frac{1-\lambda}{3}} b^{\lambda y+\frac{1-\lambda}{3}} c^{\lambda z+\frac{1-\lambda}{3}}$

For: $x = \frac{a}{a+b+c}; y = \frac{b}{a+b+c}; z = \frac{c}{a+b+c}$. *We have:* $x+y+z=1; a+b+c=2p$

$$\sum a^{\frac{a}{2s}} \cdot b^{\frac{b}{2s}} \cdot c^{\frac{c}{2s}} \ge \sum a^{\frac{\lambda a}{2s}+\frac{1-\lambda}{3}} b^{\frac{\lambda b}{2s}+\frac{1-\lambda}{3}} c^{\frac{\lambda c}{2s}+\frac{1-\lambda}{3}} \quad (1)$$

We take: $\lambda = \frac{2}{3}; \frac{\lambda a}{2s}+\frac{1-\lambda}{3} = \frac{2a}{6s}+\frac{1-\frac{2}{3}}{3} = \frac{a}{3s}+\frac{1}{9} = \frac{3a+s}{9s}$

and analogous: $\frac{\lambda b}{2s}+\frac{1-\lambda}{3} = \frac{3b+s}{9s}; \frac{\lambda c}{2s}+\frac{1-\lambda}{3} = \frac{3c+s}{9s}$. *The relationship (1) can be written:*

$$\sum (a^a \cdot b^b \cdot c^c)^{\frac{1}{2s}} \ge \sum a^{\frac{3a+s}{9s}} b^{\frac{3b+s}{9s}} c^{\frac{3c+s}{9s}} = \sum (a^{3a+s} \cdot b^{3b+s} \cdot c^{3c+s})^{\frac{1}{9s}} =$$

$$= \sum (abc)^{\frac{1}{9}} (a^{3a} b^{3b} c^{3c})^{\frac{1}{9s}} = \sqrt[9]{abc} \sum (a^a b^b c^c)^{\frac{1}{3s}} = \sqrt[9]{4RS} \sum (a^a b^b c^c)^{\frac{1}{3s}}$$

PROBLEM 1.040-*Solution by Anas Adlany - El Jadida- Morroco*

We have $\sqrt[3]{3}(ad-bc) \coloneqq ac+bd \neq 0 \Rightarrow d(a\sqrt{3}-b) \coloneqq c(a+b\sqrt{3})$

Also, we conclude that $ad > bc \Rightarrow \sqrt[4]{abcd} < \sqrt{abcd}$ *and* $a\sqrt{3} > b, 3a > b\sqrt{3}$.

Thus, $d(a+b\sqrt{3}) - c(b-a\sqrt{3}) \coloneqq d(a+b\sqrt{3}) + \frac{c^2}{d}(a+b\sqrt{3}) \coloneqq \left(\frac{c^2+d^2}{d}\right)(a+b\sqrt{3})$

Hence, we have to prove that $\left(\frac{c^2+d^2}{d}\right)(a+b\sqrt{3}) > 4\sqrt[4]{abcd}$. *But,*

$$a > \frac{bc}{d} \Rightarrow a+b\sqrt{3} > b\left(\sqrt{3}+\frac{c}{d}\right) \Rightarrow \left(\frac{c^2+d^2}{d}\right)(a+b\sqrt{3}) > \frac{b}{d}(c^2+d^2)\left(\sqrt{3}+\frac{c}{d}\right)$$

$$\left(\frac{c^2+d^2}{d}\right)(a+b\sqrt{3}) > 2bc\left(\sqrt{3}+\frac{c}{d}\right) > 4bc\sqrt{\frac{bc}{ad}} \coloneqq 4\sqrt{\frac{b^3c^3}{ad}}$$

And note that $\sqrt{\frac{b^3c^3}{ad}} > \sqrt{abcd} \Leftrightarrow (ad)^2 > (bc)^2$

Which is true due to the first observation (see above).

Conclusion: From all those inequalities, we shall obtain the desired inequality.

Comment: this is a great problem for juniors, all thanks to sir DAN SITARU.

PROBLEM 1.041-*Solution by Nguyen Minh Triet - Quang Ngai – Vietnam*

$(\forall)x$*, we have:* $\cos x + \sin x = \sqrt{2}\cos\left(\frac{\pi}{4}-x\right)$. *Hence,* $LHS = \left[\sum_{cyc}(\cos A + \sin A)\right] \cdot \frac{1}{\sqrt{2}}$

Let $x=\cos A+\sin A\,;y=\cos B+\sin B\,,c=\cos C+\sin C$. *Then* $x,y,z\in\left(0,\sqrt{2}\right]$ *and* $LHS=\frac{1}{\sqrt{2}}(x+y+z)$. *So* $\left(\sqrt{2}-x\right)\left(x\sqrt{2}+1\right)\ge 0\Rightarrow x\ge x^2\sqrt{2}-\sqrt{2}$.

Similarly $y\ge y^2\sqrt{2}-\sqrt{2};z\ge z^2\sqrt{2}-\sqrt{2}$

$$\Rightarrow x+y+z\ge\sqrt{2}\cdot(x^2+y^2+z^2-3)=\sqrt{2}\sum_{cyc}[(\sin A+\cos A)^2-1]=$$

$$=\sqrt{2}\cdot\sum_{cyc}\sin 2A=4\sqrt{2}\cdot\sin A\sin B\sin C=\sqrt{2}\cdot\frac{a}{R}\cdot\frac{b}{R}\cdot\sin C=\frac{2S\cdot\sqrt{2}}{R^2}$$

$\Rightarrow\frac{1}{\sqrt{2}}(x+y+z)\ge\frac{2S}{R^2}$ *or* $LHS\ge RHS$. *The equality doesn't hold, so:*

$$\sum_{cyc}\cos\left(\frac{\pi}{4}-A\right)>\frac{2S}{R^2}$$

PROBLEM 1.042-*Solution by Soumava Chakraborty – Kolkata – India*

$$\frac{a^2(b^2+c^2-a^2)^3}{b^2c^2}=\frac{a^2(2bc\cos A)^3}{b^2c^2}=8a^2bc\cos^3 A$$

$$S=\Delta$$

$$=8(4R^2\sin^2 A)(4R^2\sin B\sin C)\cos^3 A=128R^4(\sin A\sin B\sin C)\sin A\cos^3 A$$

$$=64R^2(2R^2\sin A\sin B\sin C)\sin A\cos^3 A=(64R^2\cdot\sin A\cdot\cos^3 A)\Delta$$

Similarly, $\frac{b^2\left(c^2+a^2-b^2\right)^3}{c^2a^2}=(64R^2\sin B\cos^3 B)\Delta$

and $\frac{c^2\left(a^2+b^2-c^2\right)^3}{a^2b^2}=(64R^2\sin C\cos^3 C)\Delta$

given inequality $\Leftrightarrow R^2\sum(\sin A\cos^3 A)\ge\Delta(1-\cos^2 A-\cos^2 B-\cos^2 C)$

$$\sin A\cos^3 A=\frac{1}{4}(2\sin A\cos A)(2\cos^2 A)=\frac{1}{4}(\sin 2A)(1+\cos 2A)$$

$$=\frac{1}{4}(\sin 2A+\sin 2A\cos 2A)=\frac{1}{4}\sin 2A+\frac{1}{8}\sin 4A$$

$$R^2\sum(\sin A\cos^3 A)=\frac{R^2}{4}\sum\sin 2A+\frac{R^2}{8}\sum\sin 4A$$

$$\sum\sin 4A=\sin 4A+\sin 4B+\sin 4C$$

$$=2\sin\big(2(A+b)\big)\cos\big(2(A-B)\big)+2\sin 2C\cos 2C$$

$$=2\sin(2\pi-2C)\cos\big(2(A-B)\big)+2\sin 2C\cos 2C$$

$$=-2\sin 2C\cos\big(2(A-B)\big)+2\sin 2C\cos 2C=2\sin 2C\left\{\cos 2C-\cos\big(2(A-B)\big)\right\}$$

$$=4\sin 2C\sin(C+A-B)\sin(A-B-C)=4\sin 2C\sin(\pi-2B)\sin(2A-\pi)$$

$$=-4\sin 2A\sin 2B\sin 2C$$

$$\frac{R^2}{8}\left(\sum\sin 4A\right)=\frac{R^2}{8}(-32\sin A\sin B\sin C\cos A\cos B\cos C)$$

$$=-2(2R^2\sin A\sin B\sin C)(\cos A\cos B\cos C)=-2\Delta\cos A\cos B\cos C$$

Again, $\frac{R^2}{4}(\sum\sin 2A)=R^2\sin A\sin B\sin C=\frac{\Delta}{2}$ *given inequality* $\Leftrightarrow$

$$\frac{\Delta}{2}-2\Delta\cos A\cos B\cos C\ge\Delta(1-\cos^2 A-\cos^2 B-\cos^2 C)$$

$$\Leftrightarrow 1-4\cos A\cos B\cos C\geq 2-(2\cos^2 A+2\cos^2 B+2\cos^2 C)$$
$$\Leftrightarrow -1-4\cos A\cos B\cos C\geq -(3+\cos 2A+\cos 2B+\cos 2C)$$
$$=-3+1+4\cos A\cos B\cos C\Leftrightarrow 8\cos A\cos B\cos C\leq 1\Leftrightarrow \cos A\cos B\cos C\leq \frac{1}{8}$$

which is true (proved)

PROBLEM 1.043-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Siendo: a,b,c,d *números reales no negativos de tal manera que:* $a+b+c+d=4$. *Probar que:*

a) $ab+bc+cd+da\leq 4$

$$\Rightarrow b(a+c)+d(a+c)=(b+d)(a+c)\leq \frac{[(b+d)+(a+c)]^2}{4}\Rightarrow$$
$$\Rightarrow [(b+d)-(a+c)]^2\geq 0$$

b) $a^2bc+b^2cd+c^2da+d^2ab\leq 4$

Desde que:

$$\Rightarrow a^2bc+b^2cd+c^2da+d^2ab-(ab+cd)(ac+bd)=bd(b-d)(c-a)\ldots \text{(A)}$$
$$\Rightarrow a^2bc+b^2cd+c^2da+d^2ab-(ac+bd)(ad+bc)=ac(a-c)(b-d)\ldots \text{(B)}$$

Multiplicando (A) × (B):

$$\left(\sum a^2bc-(ab+cd)(ac+bd)\right)\left(\sum a^2bc-(ac+bd)(ad+bc)\right)=$$
$$=-(abcd)(b-d)^2(a-c)^2\leq 0$$

Por la tanto se puede afirmar lo siguiente:

$$\Rightarrow a^2bc+b^2cd+c^2da+d^2ab\leq (ab+cd)(ac+bd)\ \text{v}$$
$$a^2bc+b^2cd+c^2da+d^2ab\leq (ac+bd)(ad+bc)$$

Si: $a^2bc+b^2cd+c^2da+d^2ab\leq (ab+cd)(ac+bd)\leq$

$$\leq \frac{[(ab+cd)+(ac+bd)]^2}{4}\leq \frac{[(b+c)(a+d)]^2}{4}\leq \frac{16}{4}=4$$

Si: $a^2bc+b^2cd+c^2da+d^2ab\leq (ac+bd)(ad+bc)\leq$

$$\leq \frac{[(ac+bd)+(ad+bc)]^2}{4}\leq \frac{[(c+d)(a+b)]^2}{4}\leq \frac{16}{4}=4$$

c) $abc+bcd+cda+dab\leq 4$

Solo basta probar lo siguiente: $abc+bcd+cda+dab=ac(b+d)+bd(a+c)\leq$

$$\leq ab+bc+cd+da=(a+c)(b+d)$$
$$\Rightarrow 4ac(b+d)+4bd(a+c)\leq (a+c)(b+d)[(b+d)+(a+c)]$$
$$\Rightarrow (b+d)^2(a+c)+(a+c)^2(b+d)\leq 4(a+c)bd+4ac(b+d)$$
$$\Rightarrow (a+c)(b-d)^2+(b+d)(a-c)^2\geq 0$$

Por la tanto: $\Rightarrow abc+bcd+cda+dab\leq ab+bc+cd+da\leq 4$

d) $ab\sqrt{c}+bc\sqrt{d}+cd\sqrt{a}+da\sqrt{b}\leq 4$. *Desde que:* $a,b,c,d\geq 0$. *Por:* $MA\geq MG$

$$\Rightarrow ab\sqrt{c}+bc\sqrt{d}+cd\sqrt{a}+da\sqrt{b}\leq \frac{ab+abc}{2}+\frac{bc+bcd}{2}+\frac{cd+cda}{2}+\frac{da+dab}{2}\leq \frac{8}{2}=4\ \text{(LQQD)}$$

PROBLEM 1.044-*Solution by Kevin Soto Palacios – Huarmey – Peru*

a) $a\sqrt{bc}+b\sqrt{cd}+c\sqrt{da}+d\sqrt{ab}\leq 4$. *Desde que:* $a,b,c,d\geq 0$. *Por:* $MA\geq MG$

$$\Rightarrow a\sqrt{bc}+b\sqrt{cd}+c\sqrt{da}+d\sqrt{ab}\le \tfrac{a+abc}{2}+\tfrac{b+bcd}{2}+\tfrac{c+cda}{2}+\tfrac{d+dab}{2}\ \dots (A)$$

Anteriormente ya se demostro lo siguiente: $\Rightarrow abc+bcd+cda+dab\le 4$

$\Rightarrow$ *Por lo tanto tenemos en (A):* $\frac{a+abc}{2}+\frac{b+bcd}{2}+\frac{c+cda}{2}+\frac{d+dab}{2}\le 4$

$\Rightarrow$ *Por transitividad:* $a\sqrt{bc}+b\sqrt{cd}+c\sqrt{da}+d\sqrt{ab}\le 4$

b) $\sqrt{abc}+\sqrt{bcd}+\sqrt{cda}+\sqrt{dab}\le 4\Rightarrow$ *Por: MA* $\ge$ *MG*

$$\Rightarrow \sqrt{abc}+\sqrt{bcd}+\sqrt{cda}+\sqrt{dab}\le \tfrac{ab+c}{2}+\tfrac{bc+d}{2}+\tfrac{cd+a}{2}+\tfrac{da+b}{2}\ \dots (B)$$

Asimismo también ya se ha demostrado lo siguiente: $\Rightarrow ab+bc+cd+da\le 4.$

Por lo tanto, por transitividad en (B): $\sqrt{abc}+\sqrt{bcd}+\sqrt{cda}+\sqrt{dab}\le 4$

c) $\sqrt[n]{abc}+\sqrt[n]{bcd}+\sqrt[n]{cda}+\sqrt[n]{dab}\le 4, n\in\mathbb{N}$

Sea: $f(x)=x^{\frac{1}{n}}\ \forall\ x\in<0,+\infty>\wedge$ *considerando para:* $n>1$

Calculamos la primera y segunda derivada: $f'(x)=\frac{x^{\frac{1-n}{n}}}{n}\wedge f''(x)=\frac{x^{\frac{-2n+1}{n}}(-n+1)}{n^2}<0$

Desde que: $f''(x)<0\to$ *entonces* f *es una función concava y se cumple:*

Desigualdad Ponderada de Jensen:

$$f(abc)+f(bcd)+f(cda)+f(dab)\le 4f\left(\frac{abc+bcd+cda+dab}{4}\right)=$$

$$=4\left(\frac{abc+bcd+cda+dab}{4}\right)^{\frac{1}{n}}\le 4(1)^n=4$$

d) $ab\sqrt[n]{c}+bc\sqrt[n]{d}+cd\sqrt[n]{a}+da\sqrt[n]{b}\le 4, n\in\mathbb{N}$

Siendo: $f(x)=x^{\frac{1}{n}}$ *(Concava)* $\forall\ x\in<0,+\infty>\wedge$ *considerando para:* $n>1$

Desigualdad Ponderada de Jensen: $\Rightarrow abf(c)+bcf(d)+cdf(a)+daf(b)\le$

$$\le(ab+bc+cd+da)f\left(\frac{abc+bcd+cda+dab}{ab+bc+cd+da}\right)\le 4(1)^n=4$$

Ya que: $f\left(\frac{abc+bcd+cda+dab}{ab+bc+cd+da}\right)=\left(\frac{abc+bcd+cda+dab}{ab+bc+cd+da}\right)^{\frac{1}{n}}\le(1)^n=1$

PROBLEM 1.045-*Solution by proposer*

In inequality $x^3+y^3+z^3-3xyz\ge 0$ *we take* $x=a-\frac{1}{3}, y=b-\frac{1}{3}, z=c-\frac{1}{3}$

and we obtain $a^3+b^3+c^3-3abc\ge a^2+b^2+c^2-ab-bc-ca$ *or*

$a^3+b^3+c^3+ab+bc+ca-3abc\ge a^2+b^2+c^2$ *or*

$$a^2+\sum a^3+\sum ab-3abc\ge 2a^2+b^2+c^2=(a^2+b^2)+(a^2+c^2)\ge$$

$$\ge\frac{(a+b)^2}{2}+\frac{(a+c)^2}{2}\ge(a+b)(a+c)\ \textit{therefore}$$

$$\prod\left(a^2+\sum a^3+\sum ab-3abc\right)\ge\prod(a+b)(a+c)=\prod(a+b)^2$$

PROBLEM 1.046-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS\underset{(1)}{\overset{A-G}{\ge}}4\sqrt[4]{\frac{1}{(b+c+d)^2(c+d+a)^2(d+a+b)^2(a+b+c)^2}}$$

$$= \frac{4}{\sqrt{(b+c+d)(c+d+a)(d+a+b)(a+b+c)}}$$

$$\sqrt[4]{(b+c+d)(c+d+a)(d+a+b)(a+b+c)} \overset{G-A}{\leq} \frac{3(a+b+c+d)}{4} = 3$$

$$\therefore \sqrt{(b+c+d)(c+d+a)(d+a+b)(a+b+c)} \leq 9$$

$$\Rightarrow \frac{4}{\sqrt{(b+c+d)(c+d+a)(d+a+b)(a+b+c)}} \geq \frac{4}{9} \quad (2)$$

$$(1), (2) \Rightarrow LHS \geq \frac{4}{9} \quad \text{(Proved)}$$

PROBLEM 1.047-***Solution by Kevin Soto Palacios – Huarmey – Peru***

Siendo a, b, c *números* R^+ *de tal manera que* $ab + bc + ca + abc \leq 4$

Probar que $a + b + c + 3 \geq 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$

De la condición $ab + bc + ca + abc \leq 4$ *es equivalente*

$$\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} \geq 1 \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{1}{a+2} - \frac{1}{2}\right) + \left(\frac{1}{b+2} - \frac{1}{2}\right) + \left(\frac{1}{c+2} - \frac{1}{2}\right) \geq 1 - \frac{3}{2} = -\frac{1}{2}$$

$$\Leftrightarrow -\frac{a}{a+2} - \frac{b}{b+2} - \frac{c}{c+2} \leq -1 \Leftrightarrow 1 \geq \frac{a}{a+2} + \frac{b}{b+2} + \frac{c}{c+2}$$

Por la desigualdad de Cauchy $1 \geq \frac{a}{a+2} + \frac{b}{b+2} + \frac{c}{c+2} \geq \frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}{a+2+b+2+c+2}$

$$\Leftrightarrow a + b + c + 6 \geq a + b + c + 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$$

$$\Leftrightarrow 3 \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \quad (A) \wedge a + b + c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \quad (B)$$

Sumando (A) + (B) se obtiene la desigualdad pedida

$$a + b + c + 3 \geq 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) \quad \text{(LQQD)}$$

Siendo a, b, c *números* R^+ *de tal manera que* $ab + bc + ca + abc \leq 4$

Probar que $3 + \frac{5}{3}(a + b + c) \geq (\sqrt[3]{a} + \sqrt[3]{b})(\sqrt[3]{b} + \sqrt[3]{c})(\sqrt[3]{c} + \sqrt[3]{a})$

$$\Leftrightarrow 9 + 5(a + b + c) \geq 3(\sqrt[3]{a} + \sqrt[3]{b})(\sqrt[3]{b} + \sqrt[3]{c})(\sqrt[3]{c} + \sqrt[3]{a})$$

$$\Leftrightarrow 9 + 6(a + b + c) \geq (a + b + c) + 3(\sqrt[3]{a} + \sqrt[3]{b})(\sqrt[3]{b} + \sqrt[3]{c})(\sqrt[3]{c} + \sqrt[3]{a})$$

$$\Leftrightarrow 9 + 6(a + b + c) \geq (\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c})^3.$$ *Por la desigualdad de Holder*

$$\left((2a+1) + (2b+1) + (2c+1)\right)\left(\frac{a}{2a+1} + \frac{b}{2b+1} + \frac{c}{2c+1}\right)(1 + 1 + 1) \geq$$

$$\geq (\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c})^3.$$ *Es necesario demostrar lo siguiente*

$$A = \frac{a}{2a+1} + \frac{b}{2b+1} + \frac{c}{2c+1} \leq 1 \Leftrightarrow \frac{2a}{2a+1} + \frac{2b}{2b+1} + \frac{2c}{2c+1} \leq 2$$

$$\Leftrightarrow \left(\frac{2a}{2a+1} + \frac{1}{2a+1}\right) + \left(\frac{2b}{2b+1} + \frac{1}{2b+1}\right) + \left(\frac{2c}{2c+1} + \frac{1}{2c+1}\right) \geq$$

$$\geq 2 + \frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1}.$$ *Lo cual nos resulta lo siguiente*

$$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} \geq 1$$

$$\Leftrightarrow (2b+1)(2c+1) + (2c+1)(2a+1) + (2a+1)(2b+1) \geq$$

$$\geq (2a+1)(2b+1)(2c+1) \Leftrightarrow 4(ab + bc + ca) + 4(a + b + c) + 3 \geq$$

$$\geq 1 + 2(a+b+c) + 4(ab+bc+ca) + 8abc$$
$$\Leftrightarrow 2(a+b+c) + 2 \geq 8abc \Leftrightarrow a+b+c+1 \geq 4abc \text{ (B)}$$
Por la desigualdad de Cauchy
$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}+\frac{1}{abc} \geq \frac{16}{ab+bc+ca+abc} \geq \frac{16}{4} = 4$$
Por transitividad se tiene lo siguiente
$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}+\frac{1}{abc} \geq 4 \Leftrightarrow a+b+c+1 \geq 4abc \text{ (LQQD)}$$
$$\textit{Por ultimo} \rightarrow 9 + 6(a+b+c) \geq \frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{A} \geq \frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{1} = \left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3$$

PROBLEM 1.048-*Solution by Soumava Chakraborty-Kolkata-India*

$$\textit{Given inequality} \Leftrightarrow \frac{\sum ab^2}{abc} \geq \frac{1}{3} + \frac{2abc+\sum a^2b+\sum ab^2}{\sum a^2b} \Leftrightarrow \frac{\sum ab^2}{abc} \geq \frac{4\sum a^2b+3\sum ab^2+6abc}{3\sum a^2b}$$
$$\Leftrightarrow 3\left(\sum ab^2\right)\left(\sum a^2b\right) \geq 4abc\left(\sum a^2b\right) + 3abc\left(\sum ab^2\right) + 6a^2b^2c^2 \text{ (i)}$$
$$\textit{Now, } \sum ab^2 \overset{A-G}{\geq} 3abc, \textit{ Also } \sum a^2b \overset{A-G}{\geq} 3abc$$
$$\therefore \left(\sum ab^2\right)\left(\sum a^2b\right) \geq 3abc\left(\sum ab^2\right) \text{ (1)}$$
$$\therefore \textit{ it remains to prove: } 2\left(\sum ab^2\right)\left(\sum a^2b\right) \geq 4abc\left(\sum a^2b\right) + 6a^2b^2c^2 \text{ (from (i), (1))}$$
$$\Leftrightarrow \left(\sum ab^2\right)\left(\sum a^2b\right) \geq 2abc\left(\sum a^2b\right) + 3a^2b^2c^2 \text{ (ii)}$$
$$\textit{Now, } \frac{\left(\sum ab^2\right)\left(\sum a^2b\right)}{3} \underset{(2)}{\geq} \frac{3abc\cdot 3abc}{3} = 3a^2b^2c^2$$
$$\therefore \textit{ it remains to prove: } \frac{2}{3}\left(\sum ab^2\right)\left(\sum a^2b\right) \geq 2abc\left(\sum a^2b\right) \text{ (from (ii), (2))}$$
$$\Leftrightarrow \left(\sum ab^2\right)\left(\sum a^2b\right) \geq 3abc\left(\sum a^2b\right) \Leftrightarrow \sum ab^2 \geq 3abc \rightarrow \textit{which is true (Proved)}$$

PROBLEM 1.049-*Solution by Henry Ricardo-New York-USA*

We use the weighted harmonic – geometric – arithmetic means inequality:
$$\left(\sum_{i=1}^{n} \frac{\alpha_i}{x_i}\right)^{-1} \leq \prod_{i=1}^{n} x_i^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i x_i,$$
where $\sum_i \alpha_i = 1$. *In this problem, we let* $\alpha_i = \frac{2i}{n(n+1)}$ *first and then* $\alpha_i = \frac{1}{n}$ *for all i.*
$$\textit{Now we have } 1 = \frac{\frac{n(n+1)}{2}}{\frac{1}{x_1}+\frac{2}{x_2}+\cdots+\frac{n}{x_n}} \leq \left(x_1 \cdot x_2^2 \cdot \ldots \cdot x_n^n\right)^{\frac{2}{n(n+1)}}$$
$$= \left[\left(x_1 \cdot x_2^2 \cdot \ldots \cdot x_n^n\right)^{\frac{1}{n}}\right]^{\frac{2}{n+1}} \leq \left[\frac{x_1 + x_2^2 + \cdots + x_n^n}{n}\right]^{\frac{2}{n+1}}$$
which implies that $x_1 + x_2^2 + \cdots + x_n^n \geq n$.
Equality holds if and only if $x_1 = x_2 = \cdots = x_n = 1$.

PROBLEM 1.050-*Solution by Ngo Minh Ngoc Bao-Vietnam*

$$a^3+b^3+c^3+8(ab+bc+ca) \leq 27 \text{ (*)}$$
$$\textit{The inequality (\&)} \Leftrightarrow a^3+b^3+c^3+\frac{8}{3}(a+b+c)(ab+bc+ca) \leq (a+b+c)^3$$

$$\Leftrightarrow \frac{8}{3}(a+b+c)(ab+bc+ca) \le 3(a+b)(b+c)(c+a)$$
$$\Leftrightarrow \frac{8}{3}(a^2b+abc+ca^2+ab^2+b^2c+abc+abc+bc^2+c^2a)$$
$$\le 9(ab+ac+b^2+bc)(c+a)$$
$$\Leftrightarrow 8\left(\sum a^2b+\sum ab^2+3abc\right) \le 9\left(\sum a^2b+\sum ab^2+2abc\right) \Leftrightarrow$$
$$\Leftrightarrow \sum a^2b+\sum ab^2-6abc \ge 0 \quad (**)$$

Consider the third – order symmetry polynomial

$$P(a,b,c)=\sum a^2b+\sum ab^2-6abc$$
$$P(1,1,1)=3+3-6\ge 0.\ P(a,b,0)=a^2b+ab^2-0\ge 0 \Rightarrow P(a,b,c)\ge 0 \quad (!).$$

Equality when $a=b=c=1$ *or* $a=3, b=c=0$ *or* $a=b=0, c=3$ *or* $a=c=0, b=3$

PROBLEM 1.051-*Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia*

$$0\le n\le \frac{1}{2} \Rightarrow \frac{2R}{R+2r} \le \frac{R}{n\cdot R+(1-n)\cdot 2r} \le \frac{R}{2r}$$
$$\frac{a^2+b^2+c^2}{ab+bc+ca} \le \frac{2R}{R+2r};\ (R+2r)\cdot(a^2+b^2+c^2) \le 2R\cdot(ab+bc+ca)$$
$$2\cdot(p^2-4Rr-r^2)\cdot(R+2r) \le 2R\cdot(p^2+4Rr+r^2);$$
$$2r\cdot p^2 \le 8R^2\cdot r+10R\cdot r^2+2r^3$$
$$p^2 \le 4R^2+5Rr+r^2 = 4R^2+4Rr+R\cdot r+r^2 \overset{Euler}{\ge} 4R^2+4Rr+3r^2$$
$$p^2 \overset{GERRETSEN}{\le} 4R^2+4Rr+3r^2 \le 4R^2+5Rr+r^2$$

PROBLEM 1.052-*Solution by Dang Thanh Tung-– Vietnam*

$$2(c-a)^2+2(c-b)(a-b)$$
$$b=\min\{a,b,c\} \Rightarrow (a-b)(c-b)\ge 0$$
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3=\frac{a-b}{b}+\frac{b-c}{c}+\frac{c-a}{a}=\frac{(c-a)^2}{ca}+\frac{(a-b)(c-b)}{bc}$$
$$a+b+c-3=a+b+c-\sqrt{3(a^2+b^2+c^2)}=-2\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{a+b+c+\sqrt{3(a^2+b^2+c^2)}}$$
$$=-2\cdot\frac{(c-a)^2+(a-b)(c-b)}{a+b+c+3}$$

We have: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+a+b+c-6$

$$=(c-a)^2\left(\frac{1}{ca}-\frac{2}{a+b+c+3}\right)+(a-b)(c-b)\left(\frac{1}{bc}-\frac{2}{a+b+c+3}\right)$$
$$+,\frac{1}{ca}-\frac{2}{a+b+c+3}=\frac{(a-c)^2+b^2+a+b+c}{ca(a+b+c+3)}>0$$
$$+,\frac{1}{bc}-\frac{2}{a+b+c+3}=\frac{(b-c)^2+a^2+a+b+c}{bc(a+b+c+3)}>0 \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+a+b+c\ge 6$$

Equality when $a=b=c=1.$

PROBLEM 1.053-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\sum_{cyc} a\left(\frac{1}{b^n}+\frac{1}{c^n}\right) \ge \frac{18}{a^n+b^n+c^n}$$

Let $a \ge b \ge c$ *then* $\frac{1}{b^n+c^n} \ge \frac{1}{c^n+a^n} \ge \frac{1}{a^n+b^n}$

$$\sum_{cyc} a\left(\frac{1}{b^n}+\frac{1}{c^n}\right) \ge 4\sum_{cyc}\frac{a}{b^n+c^n} \overset{CHEBYSHEV'S\ INEQUALITY}{\ge} \frac{4}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc}\frac{1}{a^n+b^n}\right)$$

$\ge 4\frac{9}{\sum_{cyc}(a^n+b^n)} \ge \frac{18}{a^n+b^n+c^n}$ *(proved) equality* $a=b=c=1$.

PROBLEM 1.054-*Solution by Soumava Chakraborty-Kolkata-India*

In any $\Delta\ ABC, \frac{9}{4R+r} \le \sum\frac{1}{m_a} \le \frac{1}{r}$

$\sum\frac{1}{m_a} \ge \frac{(1+1+1)^2}{\sum m_a}$ *(Bergstrom)* $\ge \frac{9}{4R+r}$ $(\because \sum m_a \le 4R+r)$

Tereshin $\Rightarrow m_a \ge \frac{b^2+c^2}{4R} \overset{A-G}{\ge} \frac{2bc}{4R} = \frac{bc}{2R}$

$$\therefore \sum\frac{1}{m_a} \le 2R\sum\frac{1}{bc} = 2R\frac{(a+b+c)}{abc} = \frac{2R\cdot 2s}{4Rrs} = \frac{1}{r}$$

PROBLEM 1.055-*Solution by Vasile Jiglău – Romania*

a) We will use the following formulas (see [1]):

$a+c=b+d=p$; $\quad ef=ac+bd$ *(first theorem of Ptolemy)*

$abcd=p^2r^2$; $\quad ef=2r\left(\sqrt{4R^2+r^2}+r\right)$;

Let's simplify the expression $\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}$. *We have:* $\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}=\frac{(a+c)^2}{ac}+\frac{(b+d)^2}{bd}-4$

$$=\frac{bd(a+c)^2+ac(b+d)^2}{abcd}-4=\frac{p^2ef}{p^2r^2}-4=$$

$$=\frac{ef}{r^2}-4=\frac{2r\left(\sqrt{4R^2+r^2}+r\right)}{r^2}-4=\frac{2\sqrt{4R^2+r^2}}{r}-2$$

So $2\frac{R^2}{r^2} \ge \frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}$

$$\Leftrightarrow 2\frac{R^2}{r^2} \ge \frac{2\sqrt{4R^2+r^2}}{r}-2 \Leftrightarrow R^2 \ge r\sqrt{4R^2+r^2}-r^2 \Leftrightarrow R^4+r^4+2R^2r^2 \ge 4R^2r^2+r^4$$

$\Leftrightarrow R^2 \ge 2r^2$

We now prove the second inequality

$$\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b} \ge 2\sqrt{2}\frac{R}{r} \Leftrightarrow \frac{2\sqrt{4R^2+r^2}}{r}-2 \ge 2\sqrt{2}\frac{R}{r} \Leftrightarrow$$

$$\sqrt{2}R \le \sqrt{4R^2+r^2}-r \Leftrightarrow \left(R\sqrt{2}+r\right)^2 \le 4R^2+r^2 \Leftrightarrow$$

$$2R^2+r^2+2\sqrt{2}Rr \le 4R^2+r^2 \Leftrightarrow 2\sqrt{2}Rr \le 2R^2 \Leftrightarrow r\sqrt{2} \le R$$

namely Euler's inquality, true in every bicentric quadrilateral. Equality holds when $ABCD$ is a square.

b) We have:

$$\left(\frac{a}{c}+\frac{c}{a}\right)\left(\frac{b}{d}+\frac{d}{b}\right)=\frac{[(a+c)^2-2ac][(b+d)^2-2bd]}{abcd}=$$

$$=\frac{p^4-2p^2(ac+bd)+4abcd}{abcd}=\frac{p^2-4R\sqrt{4R^2+r^2}}{r^2}$$

So the inequality form enunciation becomes equivalent with:

$$4\frac{R^2}{r^2}-4\geq\frac{p^2-4r\sqrt{4R^2+r^2}}{r^2}\Leftrightarrow 4R^2-4r^2\geq p^2-4r\sqrt{4R^2+r^2}\Leftrightarrow$$

$$\Leftrightarrow 4R^2+4r\sqrt{4R^2+r^2}-4r^2\geq p^2$$

Taking into account that $p^2\leq\left(\sqrt{4R^2+r^2}+r\right)^2$, its enough to prove that:

$$4R^2+4r\sqrt{4R^2+r^2}-4r^2\geq\left(\sqrt{4R^2+r^2}+r\right)^2=4R^2+2r^2+2r\sqrt{4R^2+r^2}$$

$$\Leftrightarrow 2r\sqrt{4R^2+r^2}\geq 6r^2\Leftrightarrow 4R^2\geq 8r^2\Leftrightarrow R\geq r\sqrt{2}$$

So the inequality from enunciation is true, the equality is being obtained also, when $ABCD$ is a square.Reference:

[1] OT Pop , N Minculete , M Bencze – *An introduction to quadrilateral geometry* , EDP , 2013

PROBLEM 1.056-*Solution by Soumava Chakraborty-Kolkata-India*

WLOG, we may assume $a\geq b\geq c$; $r_a\geq r_b\Leftrightarrow\frac{\Delta}{s-a}\geq\frac{\Delta}{s-b}\Leftrightarrow -b\geq -a\Leftrightarrow a\geq b\rightarrow$ *true*

$\therefore r_a\geq r_b$ *Similarly,* $r_b\geq r_c$ $\therefore r_a\geq r_b\geq r_c$; $\frac{1}{s_a+r}\geq\frac{1}{s_b+r}\Leftrightarrow s_b\geq s_a\Leftrightarrow s_b^2\geq s_a^2$

$$\Leftrightarrow\frac{c^2a^2}{(c^2+a^2)^2}(2c^2+2a^2-b^2)\geq\frac{b^2c^2}{(b^2+c^2)^2}(2b^2+2c^2-a^2)$$

$$\Leftrightarrow a^6b^2+4a^4b^2c^2+2a^4c^4+2a^2c^6\geq a^2b^6+4a^2b^4c^2+2b^4c^4+2b^2c^6\quad (a)$$

$$\because a^4\geq b^4,\therefore a^4\cdot a^2b^2\geq b^4\cdot a^2b^2\Rightarrow a^6b^2\overset{(1)}{\geq}a^2b^6$$

$$\because a^2\geq b^2,\therefore 4a^2b^2c^2\cdot a^2\geq 4a^2b^2c^2\cdot b^2\Rightarrow 4a^4b^2c^2\overset{(2)}{\geq}4a^2b^4c^2$$

$$\because a^4\geq b^4,\therefore 2a^4c^4\overset{(3)}{\geq}2b^4c^4\quad\because a^2\geq b^2,\therefore 2a^2c^6\overset{(4)}{\geq}2b^2c^6$$

(1) + (2) + (3) + (4) ⇒ (a) is true ⇒ $\frac{1}{s_a+r}\geq\frac{1}{s_b+r}$

Similarly, $\frac{1}{s_b+r}\geq\frac{1}{s_c+r}$ $\therefore\frac{1}{s_a+r}\geq\frac{1}{s_b+r}\geq\frac{1}{s_c+r}$

$\therefore$ *applying Chebyshev, LHS* $\geq\frac{1}{3}\left(\sum r_a\right)\left(\sum\frac{1}{s_a+r}\right)\overset{Bergstrom}{\geq}\frac{1}{3}(4R+r)\frac{9}{\sum s_a+3r}=\frac{3(4R+r)}{4R+4r}$

$$\left(\because\sum s_a\leq\sum m_a\leq\sum r_a\leq 4R+r\right)$$

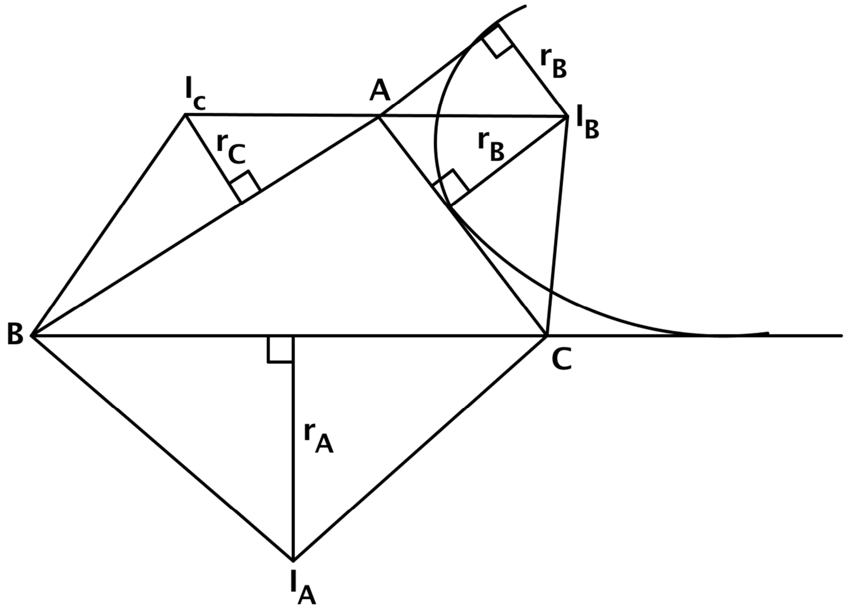
$\therefore$ *it suffices to prove:* $\frac{3(4R+r)}{4(R+r)}\geq\frac{9r^2}{R^2}$

$$\Leftrightarrow 4R^3+R^2r-12Rr^2-12r^3\geq 0\Leftrightarrow 4t^3+t^2-12t-12\geq 0$$

$$\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t^2+9t+6) \geq 0 \to \textit{true} \because t = \frac{R}{r} \geq 2 \textit{ (Euler) (Proved)}$$

PROBLEM 1.057-*Solution by Saptak Bhattacharya-Kolkata-India*



Observe that for any triangle, given h_1, h_2 as altitudes, their ratio is the ratio of corresponding bases. Using this;

$$\prod_{i=1}^{9} h_i = r_a^3 r_b^3 r_c^3 \frac{a^2b^2c^2}{\prod(AI_B \cdot CI_B)}$$

Remains to show that: $\frac{a^2b^2c^2}{\prod(AI_b \cdot CI_B)} = 1$. *Thus; ratio becomes*

$\frac{a^2b^2c^2}{8\prod r_B \cdot \prod R_B} = \frac{16\Delta^2R^2}{8\prod r_B \prod R_B}$. *Now,* $r\prod r_B = \Delta^2$, *so, we have;* $\frac{2R^2r}{\prod R_B}$

Now, in $\Delta\ AI_BC$; *be sine rule, and using* $\sin 2B = 2\sin\frac{B}{2}\sin\frac{B}{2}$, *we have*

$$R_B = 2R\sin\frac{B}{2}. \textit{ Thus, } \frac{2R^2r}{\prod R_B} = \frac{r}{R\left(4\prod\sin\frac{B}{2}\right)} \Leftrightarrow \frac{a^3b^3c^3}{\prod(AI_B \cdot CI_B b)} = 1$$

$$\textit{Now, } \prod(AI_B \cdot CI_B \cdot b) = \prod\left(4 \cdot \frac{1}{2} \cdot r_B \cdot b \cdot R_B\right)$$

[$\because abc = 4\Delta R$ is a triangle R_b = circumradius of $\Delta\ AI_BC$]

$= 8\prod r_B \cdot \prod R_B \cdot abc$. *Now, we know,* $\sum\cos A = 1 + 4\prod\sin\frac{B}{2} = 1 + \frac{r}{R}$;

Thus, $4\prod\sin\frac{B}{2} = \frac{r}{R}$ *and hence;* $\frac{a^2b^2c^2}{\prod(AI_B \cdot CI_B)} = 1$ *(Proved)*

PROBLEM 1.058-*Solution by Ravi Prakash-New Delhi-India*

$|\sin(\cos x)| < \cos(\sin x)$. *For* $x = 0, \sin(\cos 0) = \sin 1 < 1 = \cos(\sin 0)$

For $0 < x \leq \frac{\pi}{2}$, $\sin x < x \Rightarrow \cos x < \cos(\sin x)$. *Also,* $\sin(\cos x) \leq \cos x$

$|\sin(\cos x)| = \sin(\cos x) < \cos(\sin x)$ *for* $0 \leq x \leq \frac{\pi}{2}$. *For* $\frac{\pi}{2} \leq x \leq \pi \Rightarrow 0 \leq \pi - x \leq \frac{\pi}{2}$

$$|\sin(\cos(\pi - x))| < \cos(\sin(\pi - x))$$
$$\Rightarrow |\sin(-\cos x)| < \cos(\sin x) \Rightarrow |\sin(\cos x)| < \cos(\sin x)$$
Thus, $|\sin(\cos x)| < \cos(\sin x)$. *For* $0 \le x \le \pi$. *For* $\pi \le x \le 2\pi, 0 \le x - \pi \le \pi$
$$\Rightarrow |\sin(\cos(x - \pi))| < \cos(\sin(x - \pi)) \Rightarrow |\sin(\cos x)| < \cos(-\sin x) = \cos(\sin x)$$
Hence, $|\sin(\cos x)| < \cos(\sin x), 0 \le x \le 2\pi$. *Now, let* $x \in \mathbb{R}$, *then*
$x = 2k\pi + \theta$ *for some integer* k *and* $0 \le \theta < 2\pi \Rightarrow \theta = 2k\pi - x$
We have $|\sin(\cos\theta)| < \cos(\sin\theta) \Rightarrow |\sin\{\cos(2k\pi - x)\}| < \cos(\sin(2k\pi - x))$
$$\Rightarrow |\sin(\cos x)| < \cos(\sin x)$$

PROBLEM 1.059-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{1}{a^3}\tan\frac{A}{2} = \frac{1}{a^2}\cdot\frac{1}{2R\sin A}\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{1}{a^2}\cdot\frac{\sin\frac{A}{2}}{4R\sin\frac{A}{2}\cos^2\frac{A}{2}} = \frac{1}{4Ra^2}\sec^2\frac{A}{2} \quad (1)$$
WLOG, we may assume $a \ge b \ge c$
$$\therefore \cos\frac{A}{2} \le \cos\frac{B}{2} \le \cos\frac{C}{2} \Rightarrow \sec^2\frac{A}{2} \ge \sec^2\frac{B}{2} \ge \sec^2\frac{C}{2} \text{ and } \frac{1}{a^2} \le \frac{1}{b^2} \le \frac{1}{c^2}$$
$$\therefore LHS = \sum\frac{1}{a^3}\tan\frac{A}{2} = \frac{1}{4R}\sum\left(\frac{1}{a^2}\cdot\sec^2\frac{A}{2}\right) \text{ (from (1))}$$
$$\overset{Chebyshev}{\le} \frac{1}{12R}\sum\frac{1}{a^2}\cdot\sum\sec^2\frac{A}{2} = \frac{1}{12R}\cdot\frac{(\sum a^2b^2)}{a^2b^2c^2}\cdot\sum\sec^2\frac{A}{2}$$
$$\le \frac{1}{12R}\cdot\frac{(4R^2s^2)}{16R^2r^2S^2}\sum\sec^2\frac{A}{2} \text{ (by Goldstone's inequality)}$$
$$= \frac{1}{48Rr^2}\sum\sec^2\frac{A}{2} \therefore \text{it suffices to prove: } \frac{1}{48Rr^2}\sum\sec^2\frac{A}{2} \le \frac{R}{48r^4} \Leftrightarrow \sum\sec^2\frac{A}{2} \le \frac{R^2}{r^2}$$
$$\Leftrightarrow \frac{bc}{s(s-a)} + \frac{ca}{s(s-b)} + \frac{ab}{s(s-c)} \le \frac{R^2}{r^2}$$
$$\Leftrightarrow \frac{1}{s\cdot rs}\left(bc\cdot\frac{\Delta}{s-a} + ca\cdot\frac{\Delta}{s-b} + ab\cdot\frac{\Delta}{s-c}\right) \le \frac{R^2}{r^2}$$
$$\Leftrightarrow \frac{1}{s^2}(bc\cdot r_a + ca\cdot r_b + ab\cdot r_c) \le \frac{R^2}{r} \quad (2)$$
$$\because a \ge b \ge c, \therefore bc \le ca \le ab \text{ and } r_a \ge r_b \ge r_c$$
$$\therefore LHS \text{ of } (2) \overset{Chebyshev}{\le} \frac{1}{3s^2}\left(\sum ab\right)\left(\sum r_a\right) \le \frac{1}{3s^2}\cdot\sum a^2\cdot(4R + r)$$
$$\overset{Leibnitz}{\le} \frac{1}{3s^2}\cdot 9R^2(4R + r) = \frac{3R^2(4R+r)}{s^2} \therefore \text{it suffices to prove: } \frac{3R^2(4R+r)}{s^2} \le \frac{R^2}{r}$$
$$\Leftrightarrow s^2 \ge 12Rr + 3r^2 \text{ Now, Gerretsen} \Rightarrow s^2 \ge 16Rr - 5r^2$$
$$\therefore \text{it suffices to prove: } 16Rr - 5r^2 \ge 12Rr + 3r^2$$
$$\Leftrightarrow 4Rr \ge 8r^2 \Leftrightarrow R \ge 2r \to \text{true (Euler) (Proved)}$$

PROBLEM 1.060-*Solution by Evgenidis Nikolaos-Larisa-Greece*

By AM-HM inequality we deduce that:
$$\frac{2ab}{a+b} \le \frac{a+b}{2}, \frac{2bc}{b+c} \le \frac{b+c}{2}, \frac{2ca}{c+a} \le \frac{c+a}{2}$$
Therefore, it suffices to prove that $a + b + c \le 3\sqrt{3}R$ *or, if we denote the semiperimeter of the triangle by* s, *it suffices to show that* $2s \le 3\sqrt{3}R$. *Blundon's inequality states that* $s \le (3\sqrt{3} - 4)r + 2R$. *Then, it suffices to prove that*

$$(3\sqrt{3}-4)2r+4R \leq 3\sqrt{3}R \Leftrightarrow (3\sqrt{3}-4)2r \leq (3\sqrt{3}-4)R,$$

which obviously holds by Euler's inequality $R \geq 2r$. Equality holds if and only if the given triangle is equilateral, i.e. $a=b=c$.

PROBLEM 1.061-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Realizamos los siguientes cambios de variables

$x=a^3+b^3+c^3>0, y=ab+bc+ca>0$. *La desigualdad propuesta es equivalente* $\frac{1}{x}+\frac{8}{y}\geq 3$. *Aplicando la desigualdad de Cacuchy*

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}+\frac{1}{y}\right)(x+y+y+y+y+y+y+y+y)\geq 9^2=81$$

Es suficiente probar

$$x+8y\leq 27 \Leftrightarrow a^3+b^3+c^3+8(ab+bc+ca)\leq (a+b+c)^3 \Leftrightarrow$$

$\Leftrightarrow 8(ab+bc+ca)\leq 3(a+b)(b+c)(c+a)$. *Lo cual es cierto ya que*

$$3(a+b)(b+c)(c+a)\geq 3\cdot\frac{8}{9}(a+b+c)(ab+bc+ca)=8(ab+bc+ca)$$

Por lo tanto $\rightarrow \frac{1}{x}+\frac{8}{y}\geq\frac{81}{x+8y}\geq 3 \Leftrightarrow \frac{1}{a^3+b^3+c^3}+\frac{8}{ab+bc+ca}\geq 3$

PROBLEM 1.062-*Solution by proposer*

We prove by induction by n that $x_1, x_2, \dots, x_n \in [1,\infty)$; $n\in\mathbb{N}^*$ implies:

$x_1+x_2+\dots+x_n\leq n-1+x_1x_2\dots x_n$. *Checking:*

$$n=1;\quad x_1\leq 1-1+x_1 \Leftrightarrow x_1\leq x_1$$

$n=2;\quad x_1+x_2\leq 1+x_1x_2 \Leftrightarrow (x_1-1)(x_2-1)\geq 0$ *which it's true.*

$P(k): x_1+x_2+\dots+x_k\leq k-1+x_1x_2\dots x_k$. *Suppose that it's true.*

$P(k+1): x_1+x_2+\dots+x_k+x_{k+1}\leq k+x_1x_2\dots+x_kx_{k+1}$. *To proved.*

$$x_1+x_2+\dots+x_k+x_{k+1}\leq k-1+x_1x_2\dots x_k+x_{k+1}$$

Remains to prove that: $k-1+x_1x_2\dots x_k+x_{k+1}\leq k+x_1x_2\dots x_kx_{k+1}$

$$0\leq x_1x_2\dots x_k(x_{k+1}-1)-(x_{k+1}-1)$$

$$(x_1x_2\dots x_k-1)(x_{k+1}-1)\geq 0$$

$$P(k)\rightarrow P(k+1)$$

For $x_1=a; x_2=b; x_3=c; x_4=d$

$$a+b\leq 1+ab$$

$$a+b+c\leq 2+abc$$

$$a+b+c+d\leq 3+abcd$$

By adding: $3a+3b+2c+d\leq 6+ab(1+c+cd)$

PROBLEM 1.063-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Como $x,y,z\geq 0$. Aplicando la desigualdad de Holder

$$\sqrt[3]{\frac{(x+y+y+z+z+z+z+z)(1+1+1+1+1+1+1+1)(1+1+1+1+1+1+1+1)}{64}}\geq\frac{\sqrt[3]{\left(\sqrt[3]{x}+2\sqrt[3]{y}+5\sqrt[3]{z}\right)^3}}{4}$$

$$\Rightarrow \sqrt[3]{x+2y+5z}\geq\frac{\sqrt[3]{x}+2\sqrt[3]{y}+5\sqrt[3]{z}}{4}\quad (A)$$

Análogamente para los siguientes términos se cumplirá

$$\sqrt[3]{y+2z+5x} \geq \frac{\sqrt[3]{y}+2\sqrt[3]{z}+5\sqrt[3]{x}}{4} \quad (B),$$

$$\sqrt[3]{z+2x+5y} \geq \frac{\sqrt[3]{z}+2\sqrt[3]{x}+5\sqrt[3]{y}}{4} \quad (C)$$

Sumando (A) + (B) + (C)

$$E = \sqrt[3]{x+2y+5z} + \sqrt[3]{y+2z+5x} + \sqrt[3]{z+2x+5y} \geq \frac{8(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z})}{4} = \frac{8\cdot 3}{4} = 6$$

La igualdad se alcanza cuando $x = y = z = 1$

PROBLEM 1.064-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Como $a, b, c \geq 0$. Aplicando la desigualdad de MA ≥ MG y Holder

$$\sqrt[3]{a^3+1} + \sqrt[3]{1+b^3} \geq 2\sqrt[6]{\frac{(a^3+1)(1+b^3)(1+1)}{2}} = 2\sqrt[6]{\frac{(a+b)^3}{2}} = \frac{2\sqrt{a+b}}{\sqrt[6]{2}} \quad (A)$$

Análogamente para los siguientes términos se cumplirá

$$\sqrt[3]{b^3+1} + \sqrt[3]{1+c^3} \geq \frac{2\sqrt{b+c}}{\sqrt[6]{2}} \quad (B),$$

$$\sqrt[3]{c^3+1} + \sqrt[3]{1+a^3} \geq \frac{2\sqrt{c+a}}{\sqrt[6]{2}} \quad (C)$$

Sumando (A) + (B) + (C)

$$2\sqrt[3]{1+a^3} + 2\sqrt[3]{1+b^3} + 2\sqrt[3]{1+c^3} \geq \frac{2\sqrt{a+b}+2\sqrt{b+c}+2\sqrt{c+a}}{\sqrt[6]{2}}$$

$$\Rightarrow \sqrt[3]{1+a^3} + \sqrt[3]{1+b^3} + \sqrt[3]{1+c^3} \geq \frac{\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}}{\sqrt[6]{2}}$$

PROBLEM 1.065-*Solution by Ravi Prakash-New Delhi-India*

Let's take circle to be $x^2 + y^2 = R^2$ (1)

If $A(R\cos\theta, R\sin\theta)$ and $B(R\cos\phi, R\sin\phi)$, are two points on (1), then

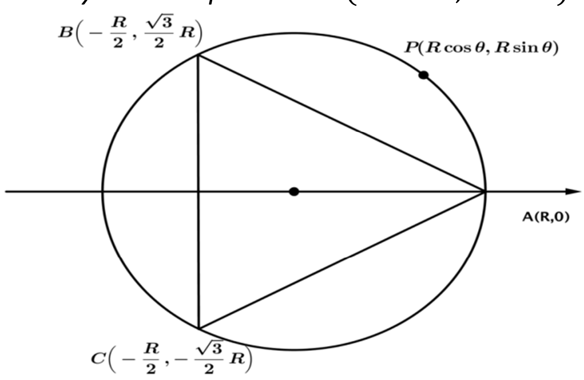
$$AB^2 = R^2(\cos\theta - \cos\phi)^2 + R^2(\sin\theta - \sin\phi)^2$$

$$= R^2[\cos^2\theta + \cos^2\phi - 2\cos\theta\cos\phi + \sin^2\theta + \sin^2\phi - 2\sin\theta\sin\phi]$$

$$= R^2[2 - 2\cos(\theta-\phi)] = 4R^2\sin^2\left(\frac{\theta-\phi}{2}\right) \Rightarrow AB = 2R\left|\sin\left(\frac{\theta-\phi}{2}\right)\right|$$

Let's take $\Delta\, ABC$ with vertices as $A(R,0), B\left(-\frac{R}{2}, \frac{\sqrt{3}}{2}R\right), C\left(-\frac{R}{2}, -\frac{\sqrt{3}}{2}R\right)$

In view of the symmetry we take point P as $(R\cos\theta, R\sin\theta)$ where $0 \leq \theta \leq \pi$



Now, $PA = 2R\left|\sin\frac{\theta}{2}\right|$ *[For* $A, \phi = 0$*]*$= 2R\sin\frac{\theta}{2}$ $[\because 0 \le \phi \le \pi]$

$$PB = 2R\left|\sin\left(\frac{\theta - \frac{2\pi}{3}}{2}\right)\right| [\text{For } B, \phi = \tfrac{2\pi}{3}] = 2R\left|\sin\left(\frac{\theta}{2} - \frac{\pi}{3}\right)\right|$$

Similarly, $PC = 2R\left|\sin\left(\frac{\theta}{2} + \frac{\pi}{3}\right)\right| = 2R\sin\left(\frac{\theta}{2} + \frac{\pi}{3}\right)$

$$E(\theta) = 6\sqrt{3}\cos\left(\frac{\theta}{2}\right) + 8\sin^3\left(\frac{\theta}{2}\right)$$

$$0 \le \theta \le \frac{2\pi}{3}$$

For $\frac{2\pi}{3} < \theta < \pi$

$$E(\theta) = 2\begin{bmatrix} 3\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right) - 3\sin\left(\frac{\pi}{3} - \frac{\theta}{2}\right) + \sin\left(\pi - \frac{3\theta}{2}\right) + \\ +3\sin\left(\frac{\pi}{3} + \frac{\theta}{2}\right) - \sin\left(\pi + \frac{3\theta}{2}\right) \end{bmatrix}$$

$$= 2\left[3\sin\left(\frac{\theta}{2}\right) + 3(2)\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{3\theta}{2}\right)\right]$$

$$= 2\sin\left(\frac{3\theta}{2}\right) + 12\sin\left(\frac{\theta}{2}\right) = 18\sin\left(\frac{\theta}{2}\right) - 8\sin^3\frac{\theta}{2}$$

Not that $E(\theta)$ *is continuous at* $\theta = \frac{2\pi}{3}$ *and hence on* $[0, \pi]$

Using $4\sin^3 A = 3\sin A - \sin(3A)$*, we get*

For $0 \le \theta \le \frac{2\pi}{3}$

$$E(\theta) = 2\begin{bmatrix} 3\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right) + 3\sin\left(\frac{\pi}{3} - \frac{\theta}{2}\right) - \sin\left(\pi - \frac{3\theta}{2}\right) + \\ +3\sin\left(\frac{\pi}{3} + \frac{\theta}{2}\right) - \sin\left(\pi - \frac{3\theta}{2}\right) \end{bmatrix}$$

if $0 \le \theta \le \frac{2\pi}{3}$

$$= 6\left[\sin\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\theta}{2}\right)\right] - 2\sin\left(\frac{3\theta}{2}\right)$$

$$= 6\sin\left(\frac{\theta}{2}\right) + 6\sqrt{3}\cos\left(\frac{\theta}{2}\right) - 6\sin\left(\frac{\theta}{2}\right) + 8\sin^3\left(\frac{\theta}{2}\right)$$

$$E'(\theta) = \begin{cases} -3\sqrt{3}\sin\left(\frac{\theta}{2}\right) + 12\sin^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right); 0 < \theta < \frac{2\pi}{3} \\ 9\cos\left(\frac{\theta}{2}\right) - 12\sin^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right); \frac{2\pi}{3} < \theta < \pi \end{cases}$$

$$= \begin{cases} 6\sin\left(\frac{\theta}{2}\right)\left[\sin\theta - \frac{\sqrt{3}}{2}\right]; 0 < \theta < \frac{2\pi}{3} \\ 12\cos\left(\frac{\theta}{2}\right)\left(\frac{\sqrt{3}}{2} - \sin\theta\right)\left(\frac{\sqrt{3}}{2} + \sin\theta\right); \frac{2\pi}{3} < \theta < \pi \end{cases}$$

Note that $E'(\theta) = 0$ *for* $\frac{\pi}{3}$*. Also, at* $\frac{2\pi}{3}, E'(\theta) = 0$*. Now,* $E(0) = 8\left[2\sin^3\left(\frac{\pi}{3}\right)\right] = 6\sqrt{3}$

$$E\left(\frac{\pi}{3}\right) = 8\left[\sin^3\left(\frac{\pi}{6}\right) + \sin^3\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{2}\right)\right] = 8\left(\frac{1}{8} + \frac{1}{8} + 1\right) = 10$$

$$E\left(\frac{2\pi}{3}\right)=8\left[\frac{3\sqrt{3}}{8}+0+\frac{3\sqrt{3}}{8}\right]=6\sqrt{3}.\ Also,\ E(\pi)=10.\ As\ 10<6\sqrt{3},$$
$$\min E(\theta)=10\sqrt{100}>\sqrt{72}=6\sqrt{2}\ and\ \max E(\theta)=6\sqrt{3}=3\left(2\sqrt{3}\right)=3(2^4\cdot 3^2)^{\frac{1}{4}}$$
$$=3(144)^{\frac{1}{4}}<3\left(\frac{2}{6}\right)^{\frac{1}{4}}$$

PROBLEM 1.066-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Tener en cuenta lo siguiente

$$abc=4RS, a+b+c=2s=\frac{2S}{r}\Leftrightarrow (a+b+c)abc=8S^2\cdot\frac{R}{r}$$

Ahora bien

$$\frac{R}{r}-\frac{2s}{3\sqrt{3}r}=\frac{r(3\sqrt{3}R-2s)}{3\sqrt{3}r}\geq 0\Leftrightarrow \text{Lo cual es cierto ya que en un}$$
$$\Delta\ ABC\rightarrow 3\sqrt{3}R\geq 2s.\ \text{Como } n\leq\frac{3}{2}$$
$$\rightarrow n\cdot\frac{R}{r}+(3-2n)\cdot\frac{s}{3\sqrt{3}r}=n\cdot\left(\frac{R}{r}-\frac{2s}{3\sqrt{3}r}\right)+\frac{3s}{3\sqrt{3}r}\leq\frac{3}{2}\left(\frac{R}{r}-\frac{2s}{3\sqrt{3}r}\right)+\frac{s}{\sqrt{3}r}=\frac{3R}{2r}$$
$$\text{Es suficiente probar}\sum\frac{4m_a^2}{4h_bh_c}\geq\frac{3R}{2r}$$
$$\Leftrightarrow\frac{(2b^2+2c^2-a^2)bc}{16S^2}+\frac{(2c^2+2a^2-b^2)ca}{16S^2}+\frac{(2a^2+2b^2-c^2)ab}{16S^2}\geq\frac{3R}{2r}$$
$$\Leftrightarrow 2bc\sum(b^2+c^2)-abc(a+b+c)\geq 16S^2\cdot\frac{3R}{2r}=3abc(a+b+c)$$

Aplicando MA ≥ MG

$$2bc\sum(b^2+c^2)-abc(a+b+c)\geq 4\sum b^2c^2-abc(a+b+c)\geq$$
$$\geq 4abc(a+b+c)-abc(a+b+c)=3abc(a+b+c)$$

PROBLEM 1.067-*Solution by Kevin Soto Palacios Peru – Huarmey – Peru*

Tener en cuenta lo siguiente

1) *Siendo a,b,c los lados de un triángulo se cumple la siguiente desigualdad*

$$(ab+bc+ca)\left(\frac{1}{ba}+\frac{1}{bc}+\frac{1}{ca}\right)\geq 9\Leftrightarrow\frac{(s^2+r^2+4Rr)}{2Rr}\geq 9\Leftrightarrow$$
$$\Leftrightarrow s^2+r^2-14Rr\geq 0$$

2) *Para todo $k\geq 2$ it checks* $\left(\frac{2r}{R}\right)^{k-1}\leq 1\Leftrightarrow\left(\frac{2r}{R}\right)^k\leq\frac{2r}{R}\Leftrightarrow\frac{2r}{R}\leq\sqrt[k]{\frac{2r}{R}}$

La desigualdad propuesta s equivalente

$$n\cdot\left(\frac{s^2+r^2}{Rr}-14\right)+\sqrt[k]{\frac{2r}{R}}\geq 1\Leftrightarrow n\left(\frac{s^2+r^2-14Rr}{Rr}\right)+\sqrt[k]{\frac{2r}{R}}\geq 1$$
$$\text{Ahora bien } n\left(\frac{s^2+r^2-14Rr}{Rr}\right)+\sqrt[k]{\frac{2r}{R}}\geq\frac{1}{2}\left(\frac{s^2+r^2-14Rr}{Rr}\right)+\frac{2r}{R}$$

Es suficiente probar

$$\frac{1}{2}\left(\frac{s^2+r^2-14Rr}{Rr}\right)+\frac{2r}{R}\geq 1\Leftrightarrow\frac{s^2+r^2}{2Rr}+\frac{4r^2}{2Rr}\geq 8\Leftrightarrow s^2\geq 16Rr-5r^2$$

(Válido por desigualdad de Gerretsen)

PROBLEM 1.068-*Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia*

1) $\sum\frac{a}{b+c}\leq 2-\frac{r}{R}$ *(ASSURE)*

$$\sum\frac{a}{b+c}=\frac{2\cdot(p^2-r^2-Rr)}{p^2+r^2+2Rr}\leq 2-\frac{r}{R}\Leftrightarrow$$

$$\Leftrightarrow\frac{2\cdot(p^2-r^2-Rr)}{p^2+r^2+2Rr}+\frac{r}{R}\leq 2\Leftrightarrow\frac{2R\cdot(p^2-r^2-Rr)+r(p^2+r^2+2Rr)}{R\cdot(p^2+r^2+2Rr)}\leq 2$$

$$\Leftrightarrow p^2\leq 6R^2+2Rr-r^2\overset{Gerretsen}{\Rightarrow}4R^2+4Rr+3r^2\leq 6R^2+2Rr-r^2$$

$$\Leftrightarrow 2R^2-2Rr-4r^2\geq 0\Leftrightarrow(R-2r)\cdot(R+r)\geq 0\ \textit{Euler}$$

True

2) $$-\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)\overset{Schwarz}{\leq}-\frac{(a+b+c)^2}{2\cdot(a+b+c)}=-\frac{a+b+c}{2}=-p$$

3) $$\sum\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{a+b}=\sum\frac{\sqrt{\frac{p-c}{p}}\cdot\left(\sqrt{\frac{p-b}{p-a}}+\sqrt{\frac{p-a}{p-b}}\right)}{a+b}=$$

$$=\sum\frac{\sqrt{\frac{p-c}{p}}\cdot\left(\frac{p-b+p-a}{\sqrt{(p-a)\cdot(p-b)}}\right)}{a+b}=\sum\frac{\sqrt{\frac{p-c}{p}}\cdot\left(\frac{c}{\sqrt{(p-a)(p-b)}}\right)}{a+b}=$$

$$=\sum\frac{\frac{(p-c)\cdot c}{S}}{a+b}=\frac{1}{S}\cdot\sum\frac{(p-c)\cdot c}{a+b}=\frac{1}{S}\cdot\left(p\cdot\sum\frac{c}{a+b}-\frac{c^2}{a+b}\right)\overset{(1)\,(2)}{\leq}$$

$$\leq\frac{1}{S}\cdot\left(p\cdot\left(2-\frac{r}{R}\right)-p\right)=\frac{1}{r}\cdot\left(2-\frac{r}{R}-1\right)=\frac{1}{r}-\frac{1}{R}$$

PROBLEM 1.069-*Solution by SK Rejuan-West Bengal-India*

$a,b\in\mathbb{R}^+$ *and* $a^2+ab+b^2=9$ *(i)*

By AM ≥ GM we get, $a^2+ab+b^2\geq 3ab\Rightarrow 9\geq 3ab$ *[from (i)]*$\Rightarrow ab\leq 3$ *(ii)*

Again, $a^2+ab+b^2=9\Rightarrow(a+b)^2-ab=9$

$\Rightarrow(a+b)^2=9+ab\leq 9+3$ *[from (ii)]* $\Rightarrow(a+b)^2\leq 12$ *(iii)*

$\Rightarrow LHS=(a+b)^6+(ab)^5+2(ab)^3+(ab)^2-17$

$\leq(12)^3+(3)^5+2\cdot(3)^3+(3)^2-17$ *[from (iii) & (ii)]*

$=1728+243+54+9-17=2017\Rightarrow(a+b)^6+(ab)^5+2(ab)^3+(ab)^2-17\leq 2017$

∴ *Maximum value is* 2017

PROBLEM 1.070-*Solution by SK Rejuan-West Bengal-India*

$a,b\in\mathbb{R}^+$ *and* $a^2+ab+b^2=k^2$ *(i)*

By AM ≥ GM we get, $a^2+ab+b^2\geq 3ab\Rightarrow k^2\geq 3ab$ *[as* $k^2=a^2+ab+b^2$*]*

$\Rightarrow ab\leq\frac{k^2}{3}$ *(ii)*$\Rightarrow\sqrt[4]{ab}\leq\frac{\sqrt{k}}{\sqrt[4]{3}}$ *(iii)*

Again, $a^2+ab+b^2=k^2 \Rightarrow (a+b)^2-2ab+ab=k^2$

$\Rightarrow (a+b)^2=k^2+ab \le k^2+\frac{k^2}{3}$ *[from (ii)]* $\Rightarrow (a+b)^2 \le \frac{4}{3}k^2$

$\Rightarrow \sqrt{a+b} \le \frac{\sqrt{2}}{\sqrt[4]{3}}\sqrt{k}$ *(iv)*

Adding (iv) & (iii) we get $\sqrt{a+b}+\sqrt[4]{ab} \le \frac{\sqrt{2}}{\sqrt[4]{3}}\sqrt{k}+\frac{\sqrt{k}}{\sqrt[4]{3}} \Rightarrow \sqrt{a+b}+\sqrt[4]{ab} \le \frac{\sqrt{2}+1}{\sqrt[4]{3}}\sqrt{k}$

PROBLEM 1.071-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS \le \sum\left[\frac{s(s-a)}{a^2}\tan\frac{A}{2}\right] \quad \left(\because w_a \le \sqrt{s(s-a)}\right) = \sum\left(s\tan\frac{A}{2}\cdot\frac{s-a}{a^2}\right)$$

$$=\sum\left(r_a\cdot\frac{s-a}{a^2}\right)=\sum\left(\frac{\Delta}{s-a}\cdot\frac{s-a}{a^2}\right)$$

$$=\Delta\left(\sum\frac{1}{a^2}\right)=\Delta\cdot\frac{\sum a^2b^2}{16R^2r^2s^2}\overset{Goldstone}{\le}\frac{\Delta\cdot 4R^2S^2}{16R^2r^2s^2}$$

$$=\frac{rs\cdot 4R^2s^2}{16R^2r^2s^2}=\frac{s}{4r}\overset{Mitrinovic}{\le}\frac{3\sqrt{3}R}{2\cdot 4r}=\frac{3\sqrt{3}}{8}\cdot\frac{R}{r}$$

PROBLEM 1.072-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Teniendo en cuenta las siguientes identidades y desigualdades en triángulo ABC

$$r_a=p\tan\frac{A}{2}, r_b=p\tan\frac{B}{2}, r_c=p\tan\frac{C}{2}, S=pr, \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{\sqrt{3}}{2r}$$

$$\tan\frac{A}{2}=\frac{(s-b)(s-c)}{S}=\frac{(a+c-b)(a+b-c)}{4S}=\frac{a^2-(b-c)^2}{4S}\le\frac{a^2}{4S},$$

$\tan\frac{B}{2}\le\frac{b^2}{4S}, \tan\frac{C}{2}\le\frac{c^2}{4S}$. *Utilizando las desigualdades previas en (A)*

$$\frac{r_a}{a^3}+\frac{r_b}{b^3}+\frac{r_c}{c^3}=\frac{p\tan\frac{A}{2}}{a^3}+\frac{p\tan\frac{B}{2}}{b^3}+\frac{p\tan\frac{C}{2}}{c^3}\le\frac{p}{4Sa}+\frac{p}{4Sb}+\frac{p}{4Sc}=$$

$=\frac{1}{4r}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le\frac{1}{4r}\cdot\frac{\sqrt{3}}{2r}=\frac{\sqrt{3}}{8r^2}$ *(LQQD)*

Probar en un triángulo ABC: $\left(\frac{3R}{2r}\right)^2\ge\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6$

1) *Teniendo en cuenta las siguientes identidades en un triángulo ABC*

$$r_a=\frac{S}{s-a}, r_b=\frac{S}{s-b}, r_c=\frac{S}{s-c}, \frac{R}{2r}=\frac{abc}{8(s-a)(s-b)(s-c)}$$

Realizando los siguientes cambios de variables

$x=s-a, y=s-b, z=s-c, x+y=b, y+z=a, z+x=c$

2) *Siendo* $x,y,z>0$*, se cumple la siguiente desigualdad*

$9(x+y)(y+z)(z+x)\ge 8(x+y+z)(xy+yzx+zx)$

Solo es necesario demostrar lo siguiente

$$\frac{9R}{2r}\ge\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6\Leftrightarrow\frac{9(x+y)(y+z)(z+x)}{8xyz}\ge\frac{y}{x}+\frac{z}{y}+\frac{x}{z}+6$$

Es suficiente probar

$$\frac{(x+y+z)(xy+yz+zx)}{xyz}=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq\frac{y}{x}+\frac{z}{y}+\frac{x}{z}+6\Leftrightarrow$$
$$\Leftrightarrow 3+\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\geq\frac{y}{x}+\frac{z}{y}+\frac{x}{z}+6\ (MA\geq MG)$$
Se concluye que $\rightarrow\left(\frac{3R}{2r}\right)^2\geq\frac{9R}{2r}\geq\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6.$ *(LQQD)*

PROBLEM 1.073-*Solution by Nguyen Minh Tri-Vietnam*

$$\frac{3n(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq n+1$$
$$\Leftrightarrow 3n(a^4+b^4+c^4)+(a^2+b^2+c^2)(ab+bc+ac)\geq(n+1)(a^2+b^2+c^2)^2$$
$$\Leftrightarrow n[3(a^4+b^4+c^4)-(a^2+b^2+c^2)^2]+\sum ab\,(a^2+b^2)+$$
$$+abc(a+b+c)\geq(a^2+b^2+c^2)^2\quad (**)$$
We have $n\geq 1; 3(a^4+b^4+c^4)\geq(a^2+b^2+c^2)^2$
$$\Rightarrow(n-1)[3(a^4+b^4+c^4)-(a^2+b^2+c^2)]\geq 0$$
$$\Leftrightarrow n[3(a^4+b^4+c^4)-(a^2+b^2+c^2)^2]\geq 3(a^4+b^4+c^4)-(a^2+b^2+c^2)^2$$
So we have to prove that: $3(a^4+b^4+c^4)-(a^2+b^2+c^2)^2+\sum ab\,(a^2+b^2)+$
$$+abc(a+b+c)\geq(a^2+b^2+c^2)\quad (*)$$
$$\Leftrightarrow 3(a^4+b^4+c^4)+\sum ab\,(a^2+b^2)+abc(a+b+c)\geq 2(a^2+b^2+c^2)^2$$
$$\Leftrightarrow a^4+b^4+c^4+abc(a+b+c)+\sum ab\,(a^2+b^2)\geq 4(a^2b^2+b^2c^2+a^2c^2)$$
Use Schur inequality exponent two and we have:
$a^4+b^4+c^4+abc(a+b+c)\geq\sum ab\,(a^2+b^2)$. *So we need to prove:*
$$\sum ab\,(a^2+b^2)\geq 2\left(\sum a^2b^2\right)\quad (1)$$
Have: $a^2+b^2\geq 2ab\Leftrightarrow ab(a^2+b^2)\geq 2a^2b^2$. *Similarly:* $bc(b^2+c^2)\geq 2b^2c^2$
$$ac(a^2+c^2)\geq 2a^2c^2\Rightarrow\sum ab\,(a^2+b^2)\geq 2\left(\sum a^2b^2\right)$$
$\Rightarrow$ *(1) true* $\Rightarrow$ *(*) true* $\Rightarrow$ *(**) true* $\Rightarrow$ *Q.E.D.*

PROBLEM 1.074-*Solution by Ravi Prakash-New Delhi-India*

$$n(ab+bc+ca)+2abc=n^3$$
$$\Leftrightarrow\left(\frac{a}{n}\right)\left(\frac{b}{n}\right)+\left(\frac{b}{n}\right)\left(\frac{c}{n}\right)+\left(\frac{c}{n}\right)\left(\frac{a}{n}\right)+2\left(\frac{a}{n}\right)\left(\frac{b}{n}\right)\left(\frac{c}{n}\right)=1\quad (1)$$
Let $\frac{a}{n}+1=x,\frac{b}{n}+1=y,\frac{c}{n}+1=z$
(1) now becomes
$$(x-1)(y-1)+(y-1)(z-1)+(z-1)(x-1)+$$
$$+2(x-1)(y-1)(z-1)=1$$
$$\Leftrightarrow xy-x-y+1+yz-y-z+1+zx-x-z+1+$$
$$+2(xyz-xy-yz-zx+x+y+z-1)=1$$
$\Leftrightarrow 2xyz=xy+yz+zx\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2.$ *Let* $\frac{1}{x}=\alpha,\frac{1}{y}=\beta,\frac{1}{z}=\gamma$
So that $\alpha+\beta+\gamma=2$. *Now, to prove* $\frac{1}{a+b+2n}+\frac{1}{b+c+2n}+\frac{1}{c+a+2n}\leq\frac{1}{n}$

$$\Leftrightarrow \frac{1}{\left(\frac{a}{n}+1\right)+\left(\frac{b}{n}+1\right)}+\frac{1}{\left(\frac{b}{n}+1\right)+\left(\frac{c}{n}+1\right)}+\frac{1}{\left(\frac{c}{n}+1\right)+\left(\frac{a}{n}+1\right)} \le 1$$

$$\Leftrightarrow \frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \le 1 \Leftrightarrow \frac{\alpha\beta}{\alpha+\beta}+\frac{\beta\gamma}{\beta+\gamma}+\frac{\gamma\alpha}{\alpha+\gamma} \le 1 \quad (2)$$

LHS of (2)

$$\frac{1}{2}\left\{\frac{2\alpha\beta}{\alpha+\beta}+\frac{2\beta\gamma}{\beta+\gamma}+\frac{2\gamma\alpha}{\gamma+\alpha}\right\} \le \frac{1}{2}\left\{\frac{\alpha+\beta}{2}+\frac{\beta+\gamma}{2}+\frac{\gamma+\alpha}{2}\right\} = \frac{\alpha+\beta+\gamma}{2} = \frac{2}{2} = 1$$

$\therefore$ *(2) is true*

PROBLEM 1.075-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Tener en cuenta las siguientes identidades y desigualdades en un $\Delta\, ABC$

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} = \frac{1}{2Rr}, R \ge 2r \quad (Euler)$$

La desigualdad propuesta es equivalente $2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge \frac{3\sqrt{3}R}{R+r} \Leftrightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \frac{3\sqrt{3}}{2(R+r)}$

Luego $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \sqrt{3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)} = \sqrt{\frac{3}{2Rr}} \ge \frac{3\sqrt{3}}{2(R+r)}$

Es necesario demostrar lo siguiente

$$\sqrt{\frac{3}{2Rr}} \ge \frac{3\sqrt{3}}{2(R+r)} \Leftrightarrow \sqrt{2}(R+r) \ge 3\sqrt{Rr} \Leftrightarrow 2(R+r)^2 \ge 9Rr \Leftrightarrow$$

$$\Leftrightarrow 2(R+r)^2 - 9Rr = (R-2r)(2R-r) \ge 0 \quad (R \ge 2r)$$

PROBLEM 1.076-*Solution by Ravi Prakash-New Delhi-India*

$$(a\cot A)^a(b\cot B)^b(c\cot C)^c \le \left[\frac{a(a\cot A)+b(b\cot B)+c(\cot C)}{a+b+c}\right]^{a+b+c}$$

But $a(a\cot A)+b(b\cot B)+c(\cot C)$

$$= 2R^2(2\sin A\cos A+2\sin B\cos B+2\sin C\cos C)$$

$$= 2R^2(\sin 2A+\sin 2B+\sin 2C) = 2R^2(4\sin A\sin B\sin C)$$

$$= 8R^2\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right) = \frac{abc}{R} = 4S \quad [S = area\ of\ \Delta\, ABC]$$

$$\therefore (a\cot A)^a(b\cot B)^b(c\cot C)^c \le \left(\frac{4S}{2s}\right)^{a+b+c} = (2r)^{a+b+c}$$

PROBLEM 1.077-*Solution by Ravi Prakash-New Delhi-India*

$$\sum_{i=1}^{a} a_i = 1, a_i \ge 0 \Rightarrow \sum_{i=1}^{a} a_i^2 + 2\sum_{i<j} a_i a_j = 1$$

Put

$$\sum_{i=1}^{a} a_i^2 = \sin^2\theta, 2\sum_{i<j} a_i a_j = \cos^2\theta$$

$\left(0 \le \theta \le \frac{\pi}{2}\right)$. *Also,*

$$\frac{1}{9}\sum_{i=1}^{a} a_i^2 \ge \left(\frac{1}{9}\sum_{i=1}^{a} a_i\right)^2 = \frac{1}{81} \Rightarrow \sum_{i=1}^{a} a_i^2 \ge \frac{1}{9}$$

$\Rightarrow \frac{1}{9} \le \sin^2\theta \le 1 \Rightarrow \frac{1}{3} \le \sin\theta \le 1$. *Now,*

$$\sqrt{\sum_{i=1}^{a} a_i^2} + \lambda\sqrt{\sum_{i<j} a_i a_j} = \sin\theta + \frac{\lambda}{\sqrt{2}}\cos\theta = f(\theta) \ \ \textit{(say)}$$

For $\lambda = u$, $f(\theta) \le \sqrt{1+8} = 3 = \frac{2\lambda+1}{3}$. *So assume* $\lambda > 4$

$$f'(\theta) = \cos\theta - \frac{\lambda}{\sqrt{2}}\sin\theta$$

$$f'(\theta) \ge 0 \Leftrightarrow \cos\theta \ge \frac{\lambda}{\sqrt{2}}\sin\theta \Leftrightarrow 1 - \sin^2\theta \ge \frac{\lambda^2}{2}\sin^2\theta \Leftrightarrow 1 \ge \left(\frac{\lambda^2}{2}+1\right)\sin^2\theta$$

$\Leftrightarrow \sin^2\theta \le \frac{2}{\lambda^2+2} < \frac{1}{9}$. *Not possible as* $\sin^2\theta \ge \frac{1}{9} \therefore f'(\theta) < 0$

$\Rightarrow f(\theta)$ *decreases in* $\left[\sin^{-1}\left(\frac{1}{3}\right), \frac{\pi}{2}\right] \Rightarrow \max f(\theta) = f\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

$$= \frac{1}{3} + \frac{\lambda}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{3} \Rightarrow f(\theta) \le \frac{2\lambda+1}{3}$$

PROBLEM 1.078-*Solution by Ravi Prakash-New Delhi-India*

Let $b = a\cos\theta$, $c = a\sin\theta$

$$\left(0 < \theta < \frac{\pi}{2}\right)$$

Now, $ab + bc + ca + (\sqrt{2}-1)\frac{abc}{a+b+c} \le 2a^2$

$$\Leftrightarrow \cos\theta + \sin\theta + \cos\theta\sin\theta + \frac{(\sqrt{2}-1)\cos\theta\sin\theta}{1+\cos\theta+\sin\theta} \le 2$$

LHS

$$E = \cos\theta + \sin\theta + \cos\theta\sin\theta + \frac{(\sqrt{2}-1)(\cos\theta\sin\theta)}{1+\cos\theta+\sin\theta}$$

$$= (\cos\theta+\sin\theta) + \frac{(\cos\theta+\sin\theta)^2-1}{2} + \frac{(\sqrt{2}-1)}{2}\frac{\{(\cos\theta+\sin\theta)^2-1\}}{\cos\theta+\sin\theta+1}$$

$= t + \frac{1}{2}(t^2-1) + \frac{1}{2}(\sqrt{2}-1)(t-1)$ *where* $t = \cos\theta + \sin\theta \le \sqrt{2}$

$$\therefore E \le \sqrt{2} + \frac{1}{2}(2-1) + \frac{1}{2}(\sqrt{2}-1)(\sqrt{2}-1) = 2$$

PROBLEM 1.079-*Solution by Ravi Prakash-New Delhi-India*

$$\frac{1}{a+b} + \frac{2}{b+c} + \frac{1}{c+a} \ge \frac{16}{a+b+2b+2c+c+a}$$

$\Rightarrow \frac{1}{a+b} + \frac{2}{b+c} + \frac{1}{c+a} \ge \frac{16}{2a+3b+3c}$. *Similarly,* $\frac{2}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{16}{3a+3b+2c}$

and $\frac{1}{a+b} + \frac{1}{b+c} + \frac{2}{c+a} \ge \frac{16}{3a+2b+3c}$. *Adding, we get* $4\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \ge$

$$\geq 16\left(\frac{1}{2a+3b+3c}+\frac{1}{3a+3b+2c}+\frac{1}{3a+2b+3c}\right)$$
$$\Rightarrow \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq \frac{4}{2a+3b+3c}+\frac{4}{3a+3b+2c}+\frac{4}{3a+2b+3c}$$

PROBLEM 1.080-*Solution by Kevin Soto Palacios-Huarmey-Peru*

Tener presente en un triángulo ABC, lo siguiente:

$m_a^2=\frac{2b^2+2c^2-a^2}{4}\Leftrightarrow m_a^2+\frac{3a^2}{4}=\frac{2b^2+2c^2+2a^2}{4}$. *Por MA≥ MG*

$$\frac{2b^2+2c^2+2a^2}{4}=m_a^2+\frac{3a^2}{4}\geq \frac{2\sqrt{3}m_a a}{2}\Leftrightarrow a^2+b^2+c^2\geq 2\sqrt{3}m_a a$$

Invirtiendo tenemos: $\frac{2\sqrt{3}a}{a^2+b^2+c^2}\leq \frac{1}{m_a}$ *... (A)*

por lo tanto: $\frac{2\sqrt{3}b}{a^2+b^2+c^2}\leq \frac{1}{m_b}$ *... (B) y* $\frac{2\sqrt{3}c}{a^2+b^2+c^2}\leq \frac{1}{m_c}$ *... (C)*

Sumando: (A)+(B)+(C)

$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\geq \frac{2\sqrt{3}(a+b+c)}{a^2+b^2+c^2}\Leftrightarrow \left(\frac{a^2+b^2+c^2}{a+b+c}\right)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\geq 2\sqrt{3}$ *(LQQD)*

PROBLEM 1.081-*Solution by Anas Adlany-El Zemamra-Morocco*

More generaly, we prove that $\sum\sqrt{\frac{x}{mx+ny}}\leq \frac{3}{\sqrt{m+n}}$, *where* $(m,n)\in(\mathbb{N}^*)^2$.

$$\sum\sqrt{\frac{x}{mx+ny}}=\frac{\sum\sqrt{x(my+nz)(mz+nx)}}{\sqrt{\prod(mx+ny)}}\leq\sqrt{\frac{(\sum x(my+nz))(\sum mz+nx)}{\prod(mx+ny)}}$$
$$=(m+n)\cdot\sqrt{\frac{(\sum xy)(\sum x)}{\prod(mx+ny)}}=(m+n)\cdot\sqrt{\frac{xyz+\prod(x+y)}{\prod(mx+ny)}}$$
$$=\frac{1}{\sqrt{m+n}}\cdot\sqrt{8\left(1+\frac{xyz}{\prod(x+y)}\right)}\leq\frac{1}{\sqrt{m+n}}\cdot\sqrt{8\left(1+\frac{1}{8}\right)}=\frac{3}{\sqrt{m+n}}$$

where the last step come from Chebyshev's inequality, and we are done!

PROBLEM 1.082-*Solution by Nguyen Minh Tri-Ho Chi Minh-Vietnam*

If $a,b,c>0, a+b+c=3$ *then*

$$\frac{a}{1+3b^4}+\frac{b}{1+3c^4}+\frac{c}{1+3a^4}\geq\frac{3}{4}\Rightarrow ab+bc+ca\leq 3$$
$$\Leftrightarrow \frac{a(1+3b^4)-3ab^4}{1+3b^4}+\frac{b(1+3c^4)-3bc^4}{1+3c^4}+\frac{c(1+3a^4)-3ca^4}{1+3a^4}\geq\frac{3}{4}$$
$$\Leftrightarrow a-\frac{3ab^4}{1+3b^4}+b-\frac{3bc^4}{1+3c^4}+c-\frac{3ca^4}{1+3ab^4}\geq\frac{3}{4}$$
$$\Leftrightarrow (a+b+c)-\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\geq\frac{3}{4}$$

$$\Leftrightarrow 3-\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\geq\frac{3}{4}\ (1)$$

Use Cauchy for 4 numbers, we have: $3b^4+1=b^4+b^4+b^4+1\geq 4\sqrt[4]{b^{12}}=4b^3$

$$\Leftrightarrow -\frac{3ab^4}{1+3b^4}\geq-\frac{3ab^4}{4b^3}=\frac{-3ab}{4}.\ \text{Similarly } -\frac{3bc^4}{1+3c^4}\geq-\frac{3bc}{4};\ \frac{-3ca^4}{1+3a^4}\geq-\frac{3ac}{4}$$

$$\Rightarrow -\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\geq-\frac{3}{4}(ab+bc+ca)\ \text{but } ab+bc+ca\leq 3$$

$$\Rightarrow -\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\geq-\frac{3}{4}\cdot 3=-\frac{9}{4}$$

$$\Rightarrow 3-\frac{3ab^4}{1+3b^4}-\frac{3bc^4}{1+3c^4}-\frac{3ca^4}{1+3a^4}\geq\frac{3}{4}\Rightarrow (1)\ \text{true}\Rightarrow Q.E.D.$$

PROBLEM 1.083-*Solution by SK Rejuan-West Bengal-India*

For $\Delta\ ABC$: With the help & m-th power theorem we get,

$$\frac{\sum a^{2m}}{3}\geq\left(\frac{\sum a^2}{3}\right)^m\Rightarrow\sum a^{2m}\geq 3^{1-m}\left(\sum a^2\right)^m\quad (1)$$

By Cauchy inequality we get, $(a+b)^2\leq 2(a^2+b^2)\Rightarrow (a+b)^{2n}\leq 2^n(a^2+b^2)^n$

$$\Rightarrow\sum\frac{1}{(a+b)^{2n}}\geq\frac{1}{2^n}\sum\frac{1}{(a^2+b^2)^n}\quad (2)$$

Again by mth power theorem we get, $\frac{\sum\left(\frac{1}{a^2+b^2}\right)^n}{3}\geq\left\{\frac{\sum\left(\frac{1}{a^2+b^2}\right)}{3}\right\}^n$

$$\Rightarrow\sum\left(\frac{1}{a^2+b^2}\right)^n\geq 3^{1-n}\left(\sum\frac{1}{a^2+b^2}\right)^n\Rightarrow\frac{1}{2^n}\sum\frac{1}{(a^2+b^2)^n}\geq\frac{3^{1-n}}{2^n}\left(\sum\frac{1}{a^2+b^2}\right)^n\quad (3)$$

Again by AM≥HM we get,

$$\sum\frac{1}{a^2+b^2}\geq\frac{9}{\sum(a^2+b^2)}=\frac{9}{2\sum a^2}\Rightarrow\frac{3^{1-n}}{2^n}\times\left(\sum\frac{1}{a^2+b^2}\right)^n\geq\frac{3^{1-n}}{2^n}\left(\frac{9}{2\sum a^2}\right)^n\quad (4)$$

Combining (2), (3), (4) we get

$$\sum\frac{1}{(a+b)^{2n}}\geq\frac{3^{1-n}}{2^n}\left(\frac{9}{2\sum a^2}\right)^n\quad (5)$$

Multiplying (1) & (5) we get, $\sum a^{2m}\cdot\sum\frac{1}{(a+b)^{2n}}\geq 3^{1-m}\left(\sum a^2\right)^m\cdot\frac{3^{1-n}}{2^n}\left(\frac{9}{2\sum a^2}\right)^n$

$$=3^{2-m-n+2n}\,2^{-2n}\left(\sum a^2\right)^{m-n}=3^{n-m+2}\cdot 4^{-n}\cdot\left(\sum a^2\right)^{(m-n)}\quad (6)$$

Also we know that $\sum a^2\geq 36r^2\Rightarrow\left(\sum a^2\right)^{m-n}\geq(36)^{m-n}\cdot r^{2(m-n)}$

$$\Rightarrow 3^{n-m+2}\cdot 4^{-n}\cdot\left(\sum a^2\right)^{m-n}\geq 3^{n-m+2}\cdot 4^{-n}\cdot 36^{(m-n)}\cdot r^{2(m-n)}$$

$$=3^{(n-m+2+2m-2n)}\cdot 4^{(-n+m-n)}\cdot r^{2(m-n)}$$

$$=3^{(m-n+2)}\cdot 4^{(m-2n)}\cdot r^{2(m-n)}\quad (7)$$

Combining (6) & (7) we get, $\sum a^{2m}\cdot\sum\frac{1}{(a+b)^{2n}}\geq 3^{(m-n+2)}\cdot 4^{(m-2n)}\cdot r^{2(m-n)}$

PROBLEM 1.084-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Siendo $m\geq n\geq 1$. Probar en un triángulo la siguiente desigualdad

$$\left((a+b)\tan\frac{C}{2}+(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{C}{2}\right)^m\cdot$$

$$\left(\frac{1}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{2n}}+\frac{1}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)^{2n}}+\frac{1}{\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)^{2n}}\right)\geq 3^{n+2}\cdot 4^{m-n}\cdot r^m$$

Recordar las siguientes identidades y desigualdades en un Δ ABC

$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{r}{p}, R\geq 2r, abc=4Rpr$. *Ahora bien, utilizando MA $\geq$ MG*

$$(a+b)\tan\frac{C}{2}+(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{C}{2}\geq 3\sqrt[3]{(a+b)(b+c)(c+a)\cdot\frac{r}{p}}\geq 3\sqrt[3]{8abc\cdot\frac{r}{p}}$$

$$=3\sqrt[3]{32Rr^2}\geq 3\sqrt[3]{64r^3}=12r$$

IRAN INEQUALITY

Siendo $x,y,z\geq 0$ se cumple la siguiente desigualdad

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}\quad (A)$$

Realizamos los siguientes cambios de variables

$$x=\tan\frac{A}{2}>0, y=\tan\frac{B}{2}>0, z=\tan\frac{C}{2}>0\Leftrightarrow xy+yz+zx=1$$

Por lo tanto tenemes en (A)

$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\geq\frac{9}{4}$. *De la desigualdad ponderada de Cauchy*

$\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)\cdot 3^{n-1}\geq\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^n$, *donde $a,b,c>0, n\geq 1$*

Siendo $a=(x+y)^2, b=(y+z)^2, c=(z+x)^2\Leftrightarrow x,y,z>0$

$$\Rightarrow\frac{1}{(x+y)^{2n}}+\frac{1}{(y+z)^{2n}}+\frac{1}{(z+x)^{2n}}\geq\frac{3}{3^n}\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^n\geq\frac{3}{3^n}\cdot\frac{9^n}{4^n}=\frac{3^{n+1}}{4^n}$$

Utilizando $\rightarrow m-n\geq 0\wedge m\geq 1$. Se concluye que

$$LHS\geq(12r)^m\cdot\frac{3^{n+1}}{4^n}=4^m\cdot 3^m\cdot r^m\cdot\frac{3^{n+1}}{4^n}=4^{m-n}\cdot 3^{n+m+1}\cdot r^m\geq 4^{m-n}\cdot 3^{n+2}\cdot r^m$$

PROBLEM 1.085-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Lemma

Siendo I - Incentro. En un triángulo ABC, se cumple los siguiente

$$IA+IB+IC\leq 2(R+r)$$

Tener en cuenta las siguientes identidades y desigualdades en un ΔABC

$$IA=\frac{bc}{p}\cos\frac{A}{2}=\frac{bc}{p}\sqrt{\frac{p(p-a)}{bc}}=\sqrt{\frac{bc(p-a)}{p}},$$

$$IB=\sqrt{\frac{ca(p-b)}{p}}, IC=\sqrt{\frac{ab(p-c)}{p}}$$

$$a+b+c=2p, ab+bc+ca=p^2+r^2+4Rr,$$

$p^2\leq 4R^2+4Rr+3r^2$ *(Gerretsen Inequality)*

Aplicando la desigualdad de Cauchy

$$IA+IB+IC=\sqrt{\frac{bc(p-a)}{p}}+\sqrt{\frac{ca(p-b)}{p}}+\sqrt{\frac{ab(p-c)}{p}}\le$$
$$\le\sqrt{\left(\frac{p-a}{p}+\frac{p-b}{p}+\frac{p-c}{p}\right)(bc+ca+ab)}=\sqrt{ab+bc+ca}=$$
$$=\sqrt{p^2+r^2+4Rr}\le\sqrt{4R^2+4Rr+3r^2+r^2+4Rr}=$$
$$=\sqrt{4(R^2+2Rr+r^2)}=\sqrt{4(R+r)^2}=2(R+r)$$

(LQQD)

Siendo ABC un triángulo, con I - Incentro, R – circumradio, r - inradio, además x,y,z son los inradios de triángulos IBC,ICA,IAB. Probar que

$$\frac{\sin A}{x}+\frac{\sin B}{y}+\frac{\sin C}{z}\le\frac{4+3\sqrt{3}}{2r}+\frac{2}{r}$$

Recordar las siguientes desigualdades conocidas en un triángulo ABC

$\frac{p}{R}\le\frac{3\sqrt{3}}{2}$, $IA+IB+IC\le 2(R+r)$. *De las condiciones, se verifica lo siguiente*

$$\frac{a}{x}=\frac{pa+ab\cos\frac{C}{2}+ca\cos\frac{B}{2}}{S}=\frac{pa+pIC+pIB}{S}=\frac{p(a+IC+IB)}{pr}=\frac{a+IC+IB}{r}$$

$$\frac{a}{x}=\frac{pa+ab\cos\frac{C}{2}+ca\cos\frac{B}{2}}{S}=\frac{pa+pIC+pIB}{S}=\frac{p(a+IC+IB)}{pr}=\frac{a+IC+IB}{r}$$
$$\frac{b}{y}=\frac{pb+bc\cos\frac{A}{2}+ab\cos\frac{C}{2}}{S}=\frac{pb+pIA+pIC}{S}=\frac{p(b+IA+IC)}{pr}=\frac{b+IA+IC}{r}$$
$$\frac{c}{z}=\frac{pc+ca\cos\frac{B}{2}+bc\cos\frac{A}{2}}{S}=\frac{pc+pIB+pIA}{S}=\frac{p(c+IB+IA)}{pr}=\frac{c+IB+IA}{r}$$

Lo cual implica $\Rightarrow\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{2p}{r}+\frac{2(IA+IB+IC)}{r}\Leftrightarrow$

$$\Leftrightarrow\frac{\sin A}{x}+\frac{\sin B}{y}+\frac{\sin C}{z}=\frac{p}{Rr}+\frac{IA+IB+IC}{Rr}\le\frac{3\sqrt{3}}{2r}+2\left(\frac{1}{R}+\frac{1}{r}\right)=\frac{4+3\sqrt{3}}{2r}+\frac{2}{R}$$

PROBLEM 1.086-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS\overset{C-B-S}{\le}\sqrt{\sum AI}\cdot\sqrt{\sum\frac{1}{a^2}}=\frac{1}{4Rrs}\cdot\sqrt{\sum AI}\sqrt{\sum a^2b^2}$$
$$\overset{Goldstone}{\le}\frac{1}{4Rrs}\sqrt{\sum AI}\sqrt{4R^2s^2}=\frac{2Rs}{4Rrs}\sqrt{\sum AI}=\frac{1}{2r}\sqrt{\sum AI}\overset{?}{\le}\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{R+r}}{r}$$
$$\Leftrightarrow\sum AI\le 2(R+r)\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\le\frac{2(R+r)}{r}$$
$$\Leftrightarrow\frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}}+\frac{\sqrt{ca}}{\sqrt{(s-c)(s-a)}}+\frac{\sqrt{ab}}{\sqrt{(s-a)(s-b)}}\le\frac{2(R+r)}{r}$$
$$\Leftrightarrow\frac{\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}}{\sqrt{(s-a)(s-b)(s-c)}}\le\frac{2(R+r)}{r}$$

$$\Leftrightarrow \frac{\sqrt{s}}{rs}\left\{\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}\right\} \le \frac{2(R+r)}{r}$$

$$\Leftrightarrow \sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)} \le 2\sqrt{s}(R+r) \quad (1)$$

$$\text{LHS of (1)} \overset{C-B-S}{\le} \sqrt{\sum ab}\sqrt{3s-2s} = \sqrt{\sum ab}\sqrt{s} \overset{?}{\le} 2\sqrt{s}(R+r)$$

$$\Leftrightarrow \sum ab \le 4(R+r)^2 \Leftrightarrow s^2+4Rr+r^2 \le 4R^2+8Rr+4r^2$$

$$\Leftrightarrow s^2 \le 4R^2+4Rr+3r^2 \to \text{true, by Gerretsen (proved)}$$

PROBLEM 1.087-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\sum_{cyc}\sqrt{\sin A\cdot\sin B\cdot\cos C} = \sum_{cyc}\sqrt{\frac{a}{2R}\cdot\frac{b}{2R}\cdot\left(\frac{b^2+c^2-a^2}{2ab}\right)}$$

$$= \frac{1}{2\sqrt{2}R}\sum_{cyc}\sqrt{b^2+c^2-a^2} \le \frac{1}{2\sqrt{2}R}\sqrt{3\sum_{cyc}(b^2+c^2-a^2)}$$

[$\because \sqrt{x}$ is concave hence by Jensen's Inequality]

$$= \frac{1}{2\sqrt{2}R}\sqrt{3\textstyle\sum_{cyc}a^2} \le \frac{1}{2\sqrt{2}R}\sqrt{3\cdot 9R^2} = \frac{3}{2}\sqrt{\frac{3}{2}} \text{ (Proved)}$$

PROBLEM 1.088-*Solution by Dinh Tien Dung-Hanoi-Vietnam*

By Holder inequality we have: $\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca} \ge \frac{\left(\sqrt[3]{a^3+b^3}+\sqrt[3]{b^3+c^3}+\sqrt[3]{c^3+a^3}\right)^3}{3(a^2+b^2+c^2+ab+bc+ca)}$

Since $4(a^3+b^3) \ge (a+b)^3$, *we have:* $\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca} \ge \frac{2(a+b+c)^3}{3(a^2+b^2+c^2+ab+bc+ca)}$

We need to show that: $\frac{2(a+b+c)^3}{3(a^2+b^2+c^2+ab+bc+ca)} \ge \frac{9abc}{ab+bc+ca}$. *The inequality is equivalent to*

$$2(a+b+c)^3(ab+bc+ca) \ge 27abc(a^2+b^2+c^2+ab+bc+ca)$$

We assume that $a = \min\{a,b,c\}$ *and let* $a = b-q, a = c-q \ (p,q \ge 0)$

The inequality is equivalente to

$$27a^3(p^2-pq+q^2)+3a^2(5p^3+6p^2q+6pq^2+5q^3)+$$
$$+a(4p^4+7p^3q+33p^2q^2+7pq^3+4q^4)+2pq(p^3+3p^2q+3pq^2+q^3) \ge 0$$

Equality if and only if $a=b=c$. *Q.E.D.*

PROBLEM 1.089-*Solution by Richdad Phuc-Hanoi-Vietnam*

We have $X = \frac{a}{b}+\frac{b}{a} \ge 2$ *(AM-GM). Similar* $Y \ge 2, Z \ge 2$

$$(X-2)(Y-2) \ge 0 \Rightarrow XY \ge 2(X+Y-2)$$

similar $YZ \ge 2(Y+Z-2)$ *and* $ZX \ge 2(Z+X-2)$

Let $t = X+Y+Z$ $(t \ge 6)$. *We get*

$$X^2+Y^2+Z^2 = (X+Y+Z)^2-2(XY+YZ+ZX) \le t^2-8t+24$$

we need to prove that

$$t^4 \ge 16(t^2-8t+21)(t+3) \Leftrightarrow (t-6)(t^3-10t^2+20t+168) \ge 0$$

$$(t-6)[t(t-6)^2+2(t-4)^2+136]\geq 0 \text{ (true with } t\geq 6)$$

equality holds if $t=6\Leftrightarrow a=b=c$

PROBLEM 1.090-*Solution by Kevin Soto Palacios-Huarmey-Peru*

Recordar las siguientes identidades en un triángulo ABC

$$\sin^2 A+\sin^2 B+\sin^2 C=2(1+\cos A\cos B\cos C),\ S=2R^2\sin A\sin B\sin C$$

$$\Rightarrow \cot A+\cot B+\cot C=\frac{\sin^2 A+\sin^2 B+\sin^2 C}{2\sin A\sin B\sin C}=\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}=$$

$=\frac{a^2+b^2+c^2}{4S}=\frac{a^2+b^2+c^2}{4sr}$. *La desigualdad propuesta es equivalente*

$$\frac{a^2+b^2+c^2}{4sr}\geq\frac{s}{3r}\Leftrightarrow 3(a^2+b^2+c^2)\geq 4s^2=(a+b+c)^2$$

(Válido por desigualdad de Cauchy)

PROBLEM 1.091-*Solution by Vadim Mitrofanov-Kiev-Ukraine*

We have $C-S\sum_{cyc}\frac{1}{a(b+c)}=\frac{(a+b+c)^2}{\sum_{cyc}a^3(b+c)}\geq\frac{3}{2}\cdot\frac{(a+b+c)}{a^3+b^3+c^3}\Leftrightarrow 2(a^4+b^4+c^4)\geq\sum_{cyc}a^3(b+c)$

We have $C-S\sum_{cyc}\frac{a^3}{ab+c^2}\geq\frac{(a^2+b^2+c^2)^2}{\sum_{cyc}a(b^2+c^2)}\geq\frac{3}{2}\cdot\frac{(a^2+b^2+c^2)}{a+b+c}\Leftrightarrow$

$$\Leftrightarrow 2(a^3+b^3+c^3)\geq\sum_{cyc}a(b^2+c^2)$$

PROBLEM 1.092-*Solution by Nguyen Ngoc Tu-Ha Giang-Vietnam*

a. *We have* $\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\geq\frac{3(a+b+c)}{a^2+b^2+c^2}\Leftrightarrow\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\right)\geq 3$

Use Cauchy – Schwarz and AM-GM inequality we have

$a^2+b^2+c^2\geq\frac{1}{3}(a+b+c)^2\Rightarrow\frac{a^2+b^2+c^2}{a+b+c}\geq\frac{a+b+c}{3}\geq\sqrt[3]{abc}$ *and*

$\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\geq\frac{3}{\sqrt[3]{abc}}$. *Hence* $\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\right)\geq 3.$

b. *Use Lemma* $(a+b+c)(a^2+b^2+c^2)\geq 3(a^2b+b^2c+c^2a)$ *and Cauchy – Schwarz inequality we have* $(a+b+c)(a^2+b^2+c^2)\left(\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\right)\geq$

$$\geq 3(a^2b+b^2c+c^2a)\left(\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\right)\geq 3(a^2+b^2+c^2)^2$$

$$\Rightarrow\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\geq\frac{3(a^2+b^2+c^2)}{a+b+c}$$

PROBLEM 1.093-*Solution by Vadim Mitrofanov-Kiev-Ukraine*

$$\sum_{cyc}\frac{1}{a+bc}=\sum_{cyc}\frac{1}{(a+b)(a+c)}=\frac{2}{(a+b)(b+c)(a+c)}\leq\frac{1}{4abc}$$

$$\sum_{cyc}\frac{\sqrt{a}}{a+\sqrt{bc}}\le\sum_{cyc}\frac{\sqrt{a}}{2\sqrt{a\sqrt{bc}}}=\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}}{2\sqrt[4]{abc}}\le\frac{1}{2\sqrt{abc}}\Leftrightarrow\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)^4abc\le 1$$

$$\text{we have }\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\right)^4\le\left(3\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\right)^2\le 27\Rightarrow 27abc\le(a+b+c)^3=1$$

PROBLEM 1.094-*Solution by proposer*

By Hölder's inequality we obtain:

$$\left(\sum_{cyc}bc\sqrt{a^2+2bc}\right)^2\left(\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab}\right)\ge(bc+ca+ab)^3=1$$

The proof will be completed if we show that $\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab}\le 1$. *Indeed, we will use Cauchy – Schwarz inequality by the following way*

$$\sum_{cyc}\frac{bc}{a^2+2bc}=\sum_{cyc}\frac{(a^2+2bc)-a^2}{2(a^2+2bc)}=$$

$$=\frac{3}{2}-\sum_{cyc}\frac{a^2}{2(a^2+2bc)}\le\frac{3}{2}-\frac{(a+b+c)^2}{2(a^2+2bc+b^2+2ca+c^2+2ab)}=1\text{ and we are done.}$$

PROBLEM 1.095-*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum\frac{a(b^2+c^2)}{2a^2+bc}=\sum\frac{abc(b^2+c^2)}{bc(2a^2+bc)}=abc\sum\frac{b^2+c^2}{bc(2a^2+bc)}\ge$$

$$\overset{BERGSTROM}{\ge}abc\cdot\frac{2(\sum a)^2}{\sum b^2c^2+2abc\sum a}=abc\cdot\frac{2(\sum a)^2}{(\sum ab)^2}\ge abc\cdot\frac{2\cdot 3\sum ab}{(\sum ab)^2}=\frac{6abc}{ab+bc+ca}$$

PROBLEM 1.096-*Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam*

$$\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\overset{AM-GM}{\ge}3\sqrt[3]{\frac{1}{a^2b^2c^2}}\cdot 3\sqrt[3]{\frac{1}{a^2b^2c^2}}=$$

$$=\frac{9}{\sqrt[3]{a^4b^4c^4}}\overset{AM-GM}{\ge}\frac{9}{\frac{a^4+b^4+c^4}{3}}=9$$

PROBLEM 1.097-*Solution by Hoang Le Nhat Tung-Hanoi-Vietnam*

By AM-GM:

$$\frac{a}{a+1}+\sqrt{\frac{2b}{b+1}\cdot 1}+2\cdot\sqrt[4]{\frac{2c}{c+1}\cdot 1\cdot 1\cdot 1}\le\frac{a}{a+1}+\frac{\frac{2b}{b+1}+1}{2}+\frac{2\left(\frac{2c}{c+1}+1+1+1\right)}{4}$$

$$=\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+2\quad(1)$$

We prove that: $\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}\le\frac{3}{2}$

$$\Leftrightarrow\frac{a(b+1)(c+1)+b(c+1)(a+1)+c(a+1)(b+1)}{(a+1)(b+1)(c+1)}\le\frac{3}{2}$$