$$\Leftrightarrow 2(3abc+2(ab+bc+ca)+a+b+c)\le 3(abc+ab+bc+ca+a+b+c+1)$$
$$\Leftrightarrow 3abc+ab+bc+ca\le a+b+c+3 \quad (2)$$
$$\text{Other: } 8=(a+b)(b+c)(c+a)\ge \frac{8}{9}(a+b+c)(ab+bc+ca)$$
$$\Leftrightarrow (a+b+c)(ab+bc+ca)\le 9$$
$$\Rightarrow 9\ge 3\sqrt[3]{abc}\cdot 3\sqrt[3]{(abc)^2}=9abc \Leftrightarrow abc\le 1 \quad (3)$$
$$\begin{cases}9\ge (a+b+c)(ab+bc+ca)\ge \sqrt{3(ab+bc+ca)}\cdot(ab+bc+ca)\\ \Rightarrow ab+bc+ac\le 3 \quad (4)\end{cases}$$
$$(3),(4)\Rightarrow 3abc+ab+bc+ca\le 6 \quad (5)$$
$$8=(a+b)(b+c)(c+a)\le \frac{\left((a+b)+(b+c)+(c+a)\right)^3}{27}=\frac{8(a+b+c)^3}{27}$$
$$\Rightarrow (a+b+c)^3\ge 27\Rightarrow a+b+c+3\ge 6 \quad (6)$$
$$(5),(6)\Rightarrow 3abc+ab+bc+ca\le a+b+c+3$$
$$\Rightarrow (2)\text{ true}\Rightarrow \frac{a}{a+1}+\sqrt{\frac{2b}{b+1}}+2\cdot\sqrt[4]{\frac{2c}{c+1}}\le\frac{7}{2}$$

PROBLEM 1.098-*Solution by Soumava Chakraborty-Kolkata-India*

$$IA=\frac{r}{\sin\frac{A}{2}}\text{ etc}$$
$$\therefore \sum\frac{1}{IA^2}=\frac{1}{r^2}\sum\sin^2\frac{A}{2} \quad (1)$$
$$\text{Also, } 3\sum\frac{1}{a^2}=\frac{3\sum a^2b^2}{a^2b^2c^2}\overset{Goldstone}{\le}\frac{12R^2s^2}{16R^2r^2s^2}=\frac{3}{4r^2} \quad (2)$$
$$(1),(2)\Rightarrow \text{it suffices to prove: }\sum\sin^2\frac{A}{2}\ge\frac{3}{4}\Leftrightarrow\sum\left(2\sin^2\frac{A}{2}\right)\ge\frac{3}{2}\Leftrightarrow\sum(1-\cos A)\ge\frac{3}{2}$$
$$\Leftrightarrow 3-1-\frac{r}{R}\ge\frac{3}{2}\Leftrightarrow\frac{2R-r}{R}\ge\frac{3}{2}\Leftrightarrow R\ge 2r\to \text{true (Euler) (proved)}$$

PROBLEM 1.099-*Solution by proposer*

$$\text{We have for } x,t,z>0;\ \frac{x}{3x^2+2t^2+z^2}\le\frac{1}{18}\left(\frac{2}{t}+\frac{1}{z}\right)\Leftrightarrow$$
$$\Leftrightarrow 3x^2t+6x^2z+2t^3+2z^3+4t^2z+tx^2\ge 18xtz\Leftrightarrow$$
$$\Leftrightarrow\frac{3x^2t+6x^2z+2t^3+2z^3+4t^2z+tz^2}{18}\ge\sqrt[18]{(x^2t)^3(x^2t)^6(t^3)^2(z^3)^2(t^2z)^4t}=xtz\Rightarrow$$
$$\int_a^b\frac{x\,dt}{3x^2+2t^2+z^2}\le\frac{1}{18}\int_a^b\left(\frac{2}{t}+\frac{1}{z}\right)dt\Rightarrow\int_a^b\frac{x\,dt}{3x^2+2t^2+z^2}\le\frac{1}{9}\ln\frac{b}{a}+\frac{b-a}{18z}\Rightarrow$$
$$\sum_{cyclic}\int_a^b\frac{xdy}{3x^2+2y^2+z^2}\le\sum_{cyclic}\left(\frac{1}{9}\ln\frac{b}{a}+\frac{b-a}{18z}\right)=\frac{1}{3}\ln\frac{b}{a}+\frac{b-a}{18}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

PROBLEM 1.100-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\overset{AM\ge GM}{\ge}\frac{3}{\sqrt[3]{w_aw_bw_c}}\to (1)$$
$$\text{Now, } w_aw_bw_c=\left(\frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}\right)\left(\frac{2\sqrt{ca}}{c+a}\sqrt{s(s-b)}\right)\left(\frac{2\sqrt{ab}}{a+b}\sqrt{s(s-c)}\right)$$

$$=\frac{8abcs\cdot rs}{\prod(a+b)}=\frac{32Rr^2s^3}{\prod(a+b)}\rightarrow (2)$$

$$Again,\ \prod(a+b)=2abc+\sum ab\,(2s-c)=2s(s^2+4Rr+r^2)-4Rrs$$
$$=2s(s^2+2Rr+r^2)\rightarrow\ (3)$$

$$(2),(3)\Rightarrow w_aw_bw_c=\frac{16Rr^2s^2}{s^2+2Rr+r^2}\rightarrow (4)$$

$$(4),(1)\Rightarrow \frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\geq 3\sqrt[3]{\frac{s^2+2Rr+r^2}{16Rr^2s^2}}\geq\frac{3}{R+r}$$
$$\Leftrightarrow (R+r)^3(s^2+2Rr+r^2)\geq 16Rr^2s^2\rightarrow\ (a)$$

$$Now,\ LHS\ of\ (a)\overset{Gerretsen}{\geq}(R+r)^3(18Rr-4r^2)\ \ and$$
$$RHS\overset{Gerretsen}{\leq}16Rr^2(4R^2+4Rr+3r^2)$$

∴ *in order to prove (a), it suffices to prove:*

$$(R+r)^3(18Rr-4r^2)\geq 16Rr^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow 9t^4-7t^3-11t^2-21t-2\geq 0\ \ (where\ t=\frac{R}{r})$$
$$\Leftrightarrow (t-2)(9t^3+11t^2+11t+1)\geq 0\rightarrow true\because t\geq 2\ (Euler)$$

$$\Rightarrow (a)\ is\ true\Rightarrow\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\geq\frac{3}{R+r}\ is\ proved.\ Now,\ \frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\leq\frac{1}{r}\Leftrightarrow\frac{\sum w_aw_b}{w_aw_bw_c}\leq\frac{1}{r}$$

$$\sum w_aw_b=\sum\left(\left(\frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}\right)\left(\frac{2\sqrt{ca}}{c+a}\sqrt{s(s-b)}\right)\right)$$
$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sum\left[\left((a+b)\sqrt{c}\right)\left(\sqrt{(s-a)(s-b)}\right)\right]$$
$$\overset{C-B-S}{\leq}\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{\sum c(a+b)^2}\sqrt{\sum(s-a)(s-b)}$$
$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{\sum c\,(a^2+2ab+b^2)}\sqrt{\sum(s^2-s(a+b)+ab)}$$
$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{\sum ab\,(2s-c)+6abc}\sqrt{3s^2-4s^2+s^2+4Rr+r^2}$$
$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{2s(s^2+4Rr+r^2)+12Rrs}\sqrt{4Rr+r^2}$$
$$=\frac{4s\sqrt{abc}}{\prod(a+b)}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}$$
$$=\frac{4s\sqrt{4Rrs}}{2s(s^2+2Rr+r^2)}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}\ \ (by\ (3))$$
$$\therefore\sum w_aw_b\leq\frac{4s\sqrt{4Rrs}}{2s(s^2+2Rr+r^2)}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}\rightarrow\ (5)$$
$$\therefore\frac{\sum w_aw_b}{w_aw_bw_c}\overset{by\ (5),(4)}{\leq}\frac{4s\sqrt{4Rrs}}{2s(s^2+2Rr+r^2)}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}\cdot\frac{s^2+2Rr+r^2}{16Rr^2s^2}$$
$$=\frac{\sqrt{4Rrs}}{8Rr^2s^2}\sqrt{2s(s^2+10Rr+r^2)}\sqrt{4Rr+r^2}$$
$$=\frac{\sqrt{R(4R+r)(s^2+10Rr+r^2)}}{2\sqrt{2}Rrs}\overset{?}{\leq}\frac{1}{r}\Leftrightarrow 8R^2s^2\overset{?}{\geq}R(4R+r)(s^2+10Rr+r^2)$$

$$\Leftrightarrow (4R-r)s^2 \overset{?}{\geq} (4R+r)(10Rr+r^2) \rightarrow \textit{(b)}$$

$$\Leftrightarrow 8R^2s^2 \overset{?}{\geq} R(4R+r)(s^2+10Rr+r^2)$$

$$\Leftrightarrow (4R-r)s^2 \overset{?}{\geq} (4R+r)(10Rr+r^2) \rightarrow \textit{(b)}$$

$$\textit{Now, LHS of (b)} \geq (4R-r)(16Rr-5r^2) \overset{?}{\geq} (4R+r)(10Rr+r^2)$$

$$\Leftrightarrow 12R^2-25Rr+2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(12R-r) \overset{?}{\geq} 0$$

$$\rightarrow \textit{true} \because R \geq 2r \ \textit{(Euler)} \Rightarrow \textit{(b) is true} \Rightarrow \frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c} \leq \frac{1}{r} \textit{ is proved.}$$

PROBLEM 1.101-*Solution by proposer*

$$\textit{We have } (x-1)^4 \geq 0 \Leftrightarrow x^4-4x^3+6x^2-4x+1 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow 2x^4-4x^3+6x^2-4x+2 \geq x^4+1 \Leftrightarrow x^4-2x^3+3x^2-2x+1 \geq \frac{x^4+1}{2} \Leftrightarrow$$

$$\Leftrightarrow (x^2-x+1)^2 \geq \frac{x^4+1}{2} \Leftrightarrow \frac{\sqrt{x^4+1}}{\sqrt{2}} \leq x^2-x+1. \textit{ Similarly } \frac{\sqrt{x^4+1}}{\sqrt{2}} \leq y^2-y+1, \textit{ and}$$

$$\frac{\sqrt{z^4+1}}{\sqrt{2}} \leq z^2-z+1. \textit{ Adding up these inequalities, we get:}$$

$$\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1} \leq \sqrt{2}(x^2+y^2+z^2)+\sqrt{2}\big(3-(x+y+z)\big) \quad \textit{(1)}$$

By AM-GM inequlity we have $x+y+z \geq 3\sqrt{xyz} = 3$, *so* $3-(x+y+z) \leq 0$. *Now (1) gives*

$$\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1} \leq \sqrt{2}(x^2+y^2+z^2), \textit{ namely}$$

$$\frac{\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1}}{x^2+y^2+z^2} \leq \sqrt{2}. \textit{ Equality holds when } x=y=z=1.$$

PROBLEM 1.102-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\textit{We know, } \left(\sum_{cyc} xy\right)^2 \geq 3xyz(x+y+z)$$

$$\frac{4\sqrt{3xyz(x+y+z)}}{(x+y)(y+z)(z+x)} \leq \frac{4(xy+yz+zx)}{(x+y)(y+z)(z+x)}, \textit{ we need to prove,}$$

$$\sum_{cyc} \frac{1}{x+y} \geq \frac{4(xy+yz+zx)}{(x+y)(y+z)(z+x)} \Leftrightarrow \sum_{cyc} (x+y)(x+z) \geq 4(xy+yz+zx)$$

$$\Leftrightarrow x^2+y^2+z^2 \geq xy+yz+zx, \textit{ which is true.}$$

$$\therefore \sum_{cyc} \frac{1}{x+y} \geq \frac{4\sqrt{3xyz(x+y+z)}}{(x+y)(y+z)(z+x)}$$

PROBLEM 1.103-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\cot\frac{A}{2} = \frac{p(p-a)}{\Delta}, \cot\frac{B}{2} = \frac{p(p-b)}{\Delta} \textit{ and } \cot\frac{C}{2} = \frac{p(p-c)}{\Delta}$$

$$\sum_{cyc} \frac{x}{y+z}\cot^2\frac{A}{2} = (x+y+z)\sum_{cyc} \frac{\cot^2\frac{A}{2}}{y+z} - \sum_{cyc} \cot^2\frac{A}{2}$$

$$\overset{Bergström}{\geq} \frac{1}{2}\left(\sum_{cyc} \cot\frac{A}{2}\right)^2 - \sum_{cyc} \cot^2\frac{A}{2}$$

$$= \frac{1}{2}\left(\sum_{cyc} \frac{p(p-a)}{\Delta}\right)^2 - \sum_{cyc} \frac{p^2(p-a)^2}{\Delta^2} = \frac{p^2}{2r^2} - \frac{p^2\left\{\left(\sum_{cyc}(p-a)\right)^2 - 2\sum_{cyc}(p-a)(p-b)\right\}}{\Delta^2}$$

$$= \frac{p^2}{2r^2} - \frac{p^2 - 2r(r+4R)}{r^2} = \frac{2(r+4R)}{r} - \frac{p^2}{2r^2} \geq \frac{2(r+8r)}{r} - \frac{p^2}{2r^2} = 18 - \frac{p^2}{2r^2}$$

PROBLEM 1.104-*Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia*

$$\frac{ra^2}{h_b m_c} + \frac{r_b^2}{h_c m_a} + \frac{r_c^2}{h_a m_b} \geq \frac{54r^2}{p^2 - r^2 - 4Rr}$$

1) $r_a + r_b + r_c = 4R + r$

2) $h_b m_c + h_c m_a + h_a m_b \overset{\substack{h_a \leq m_a \\ h_b \leq m_b \\ h_c \leq m_c}}{\leq} m_b m_c + m_c m_a + m_a m_b \leq m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

$$LHS: \sum_{\Delta} \frac{r_a^2}{h_b m_c} \overset{Bergström}{\geq} \frac{(\sum r_a)^2}{h_b m_c + h_c m_a + h_a m_b} \overset{(2);(1)}{\geq}$$

$$\geq \frac{(4R+r)^2}{\frac{3}{4}(a^2+b^2+c^2)} \overset{Euler}{\geq} \frac{81r^2}{\frac{3}{4}\cdot 2(p^2 - 4Rr - r^2)} = \frac{54r^2}{p^2 - 4Rr - r^2}$$

PROBLEM 1.105-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS = \sum_{cyc}\left(a^2 \cdot \frac{x}{y+z}\right) (x = a^m, y = b^m, z = c^m)$$

$$\geq 4F\sqrt{\frac{x}{y+z}\cdot\frac{y}{z+x} + \frac{y}{z+x}\cdot\frac{z}{x+y} + \frac{z}{x+y}\cdot\frac{x}{y+z}}$$

$$\left(\begin{matrix} \because a^2 m' + b^2 n' + c^2 p' \geq 4R\sqrt{m'n' + n'p' + p'm'} \\ \forall m', n', p' \in \mathbb{R}^+ \text{ and as}, \frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y} > 0 \\ \because x, y, z > 0 \end{matrix}\right) \overset{?}{\geq} 2\sqrt{3}F$$

$$\Leftrightarrow \frac{xy}{(y+z)(z+x)} + \frac{yz}{(z+x)(x+y)} + \frac{zx}{(x+y)(y+z)} \overset{?}{\geq} \frac{3}{4} \Leftrightarrow \frac{\sum\{xy(x+y)\}}{2xyz + \sum x^2y + \sum xy^2} \overset{?}{\geq} \frac{3}{4}$$

$$\Leftrightarrow 4\sum x^2y + 4\sum xy^2 \overset{?}{\geq} 6xyz + 3\sum x^2y + 3\sum xy^2$$

$$\Leftrightarrow \sum x^2y + \sum xy^2 \overset{?}{\geq} 6xyz \rightarrow \text{true by AM-GM}$$

PROBLEM 1.106-*Solution by Rade Krenkov-Strumica-Macedonia*

Inequality is equivalent with:

$$\frac{8(a+b+c)^2(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)}+5(ab+bc+ca)^2\ge 12(ab+bc+ca)+$$

$+10(a+b+c)(ab+bc+ca)$. *Using equality* $(a+b+c)(ab+bc+ca)=$

$=(a+b)(b+c)(c+a)+abc$ *have*

$$8(a+b+c)^2+\frac{8(a+b+c)^2abc}{(a+b)(b+c)(c+a)}+5(ab+bc+ca)^2\ge 12(ab+bc+ca)+$$

$+10(a+b+c)(ab+bc+ca)$. *From AM-GM we get:*

$5(a+b+c)^2+5(ab+bc+ca)^2\ge 10(a+b+c)(ab+bc+ca)$. *Now, enough to prove that:* $3(a+b+c)^2+\frac{8(a+b+c)^2abc}{(a+b)(b+c)(c+a)}\ge 12(ab+bc+ca)$

$$3(a+b+c)^2+\frac{8(a+b+c)^3abc}{(a+b)(b+c)(c+a)\cdot(a+b+c)}\ge 12(ab+bc+ca)$$

From AM-GM we get:

$$8(a+b+c)^3=[(a+b)+(b+c)+(c+a)]^3\ge\left[3\sqrt[3]{(a+b)(b+c)(c+a)}\right]^3=$$

$=27(a+b)(b+c)(c+a)$. *We, must prove that:*

$$3(a+b+c)^2+\frac{27abc}{a+b+c}\ge 12(ab+bc+ca)\Leftrightarrow$$

$$\Leftrightarrow (a+b+c)^3+9abc\ge 4(a+b+c)(ab+bc+ca)$$

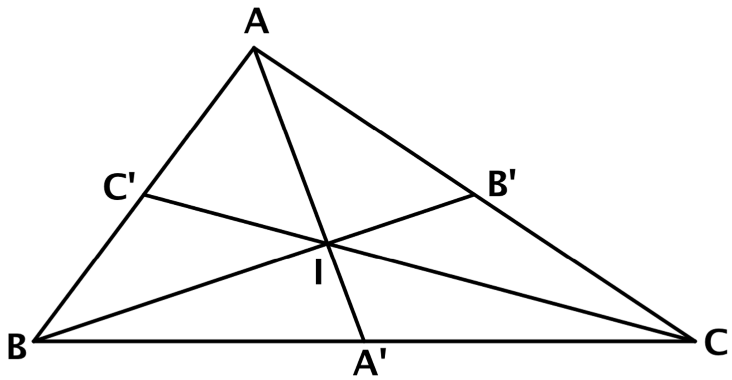
From Schur's inequality we have:

$$(a+b+c)^3=a^2+b^3+c^3+3\sum a^2b+3\sum ab^2+6abc\ge$$

$\ge 4\sum a^2b+4\sum ab^2+3abc$. *Now,* $(a+b+c)^3+9abc\ge$

$$\ge 4\sum a^2b+4\sum ab^2+12abc=4(a+b+c)(ab+bc+ca)$$

PROBLEM 1.107-*Solution by Marian Ursărescu-Romania*



From bisector theorem $\Rightarrow\frac{BA'}{A'C}=\frac{c}{b}\Rightarrow\frac{BA'}{a}=\frac{c}{b+c}\Rightarrow BA'=\frac{ac}{b+c}$

$$\frac{AI}{IA'}=\frac{c}{BA'}=\frac{c}{\frac{ac}{b+c}}=\frac{b+c}{a}\Rightarrow\frac{AI}{AA'}=\frac{b+c}{a+b+c}\Rightarrow\frac{AI}{w_a}=\frac{b+c}{2p}$$

Inequalities become: $\frac{1}{2p}\sum a^2(b+c) \le 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)}$ *(1)*

But $\sum a^2(b+c) = 2p(p^2+r^2-2Rr)$ *(2)*

From (1)+(2) we show: $p^2+r^2-2Rr \le 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)}$ *(3)*

But $R \ge 2r \Rightarrow 3\sqrt{2}\frac{R^2}{r}\sqrt{R(R-r)} \ge 6R^2$ *(4)*

From (3)+(4) we must show: $p^2+r^2-2Rr \le 6R^2$ *(5)*

But (5) its true from Blundon – Gerretsen inequality: $p^2 \le 4R^2+4Rr+3r^2$.

PROBLEM 1.108-*Solution by Ravi Prakash-New Delhi-India*

$$For\ 0 < x < \frac{\pi}{2}, a, b > 0$$

$$4\sqrt{ab}\frac{\sin x}{x} + b\left(\frac{\tan x}{x}\right)^2 + a \ge 4\sqrt{ab}\frac{\sin x}{x} + 2\sqrt{b\left(\frac{\tan x}{x}\right)^2 a}$$

$$= 2\sqrt{ab}\left[\frac{2\sin x}{x} + \frac{\tan x}{x}\right] \quad (1)$$

$$Let\ g(x) = 2\sin x + \tan x - 3x, 0 \le x < \frac{\pi}{2}$$

$$g'(x) = 2\cos x + \sec^2 x - 3, 0 < x < \frac{\pi}{2} :> 3[\cos^2 x \sec^2 x]^{\frac{1}{3}} - 3, 0 < x < \frac{\pi}{2}$$

$$\Rightarrow g'(x) > 0\ \ for\ 0 < x < \frac{\pi}{2} \Rightarrow g(x) > g(0)\ for\ 0 < x < \frac{\pi}{2}$$

$$\Rightarrow 2\frac{\sin x}{x} + \frac{\tan x}{x} > 3\ \ for\ 0 < x < \frac{\pi}{2}\ \ (2)$$

$$From\ (1), (2), we\ get:\ 4\sqrt{ab}\frac{\sin x}{x} + b\left(\frac{\tan x}{x}\right)^2 + a > 6\sqrt{ab}\ for\ 0 < x < \frac{\pi}{2}$$

PROBLEM 1.109-*Solution by Soumava Chakraborty-Kolkata-India*

$$(a+b)\cdot\frac{\sin x}{x} + \frac{2ab}{a+b}\cdot\frac{\tan x}{x} \overset{(1)}{>} \frac{6ab}{a+b}$$

$$(1) \Leftrightarrow (a+b)^2\sin x + 2ab\tan x \overset{(2)}{>} 6abx$$

$$\because (a+b)^2 \ge 4ab \therefore LHS\ of\ (2) \ge 4ab\sin x + 2ab\tan x \overset{?}{>} 6abx \Leftrightarrow 2\sin x + \tan x - 3x \underset{(3)}{\overset{?}{>}} 0$$

$$Let\ f(x) = 2\sin x + \tan x - 3x\ \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$f'(x) = \sec^2 x + 2\cos x - 3\ and\ f''(x) = 2(\sec^2 x\tan x - \sin x) =$$

$$= 2(\tan x(1+\tan^2 x) - \sin x) = 2(\tan x - \sin x + \tan^3 x) \ge 2(\sin x - \sin x + \tan^3 x)$$

$$\left(\because \forall x \in \left[0, \frac{\pi}{2}\right), \tan x \ge x \ge \sin x\right) = 2\tan^3 x \ge 0 \therefore f''(x) \ge 0\ \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$\Rightarrow f'(x) \uparrow on \left[0, \frac{\pi}{2}\right) \Rightarrow f'(x) \ge f'(0) = 0, \forall x \in \left[0, \frac{\pi}{2}\right) \Rightarrow f(x) \uparrow on \left[0, \frac{\pi}{2}\right)$$

$$\Rightarrow f(x) \ge f(0) = 0 \Rightarrow \forall x \in \left[0, \frac{\pi}{2}\right), 2\sin x + \tan x - 3x \ge 0,\ equality\ at\ x = 0$$

$$\therefore \forall a, b > 0, x \in \left(0, \frac{\pi}{2}\right), 2\sin x + \tan x - 3x > 0 \Rightarrow (3)\ is\ true\ (Proved)$$

PROBLEM 1.110-*Solution by Ravi Prakash-New Delhi-India*

$$Let\ f(x) = x(1-x^3), 0 < x < 1; f'(x) = 1 - 4x^3; f'(x) = 0 \Rightarrow x = \left(\frac{1}{4}\right)^{\frac{1}{3}}$$

$$f''(x) = -12x^2 \Rightarrow f''\left(\left(\frac{1}{4}\right)^{\frac{1}{3}}\right) < 0 \Rightarrow f(x) \text{ has a maximum for } x = \left(\frac{1}{4}\right)^3$$

$$\therefore \max_{0<x<1} f(x) = \frac{1}{4^{\frac{1}{3}}}\left(1-\frac{1}{4}\right) = \frac{3}{4^{\frac{4}{3}}} \Rightarrow \frac{1}{x(1-x^3)} \geq \frac{4^{\frac{4}{3}}}{3} \text{ for } 0 < x < 1$$

$$\therefore \frac{\sin^2\left(\frac{A}{2}\right)}{x(1-x^3)} + \frac{\sin^2\left(\frac{B}{2}\right)}{y(1-y^3)} + \frac{\sin^2\left(\frac{C}{2}\right)}{z(1-z^3)} \geq \frac{4^{\frac{4}{3}}}{3}\left[\sin^2\left(\frac{A}{2}\right) + \sin^2\left(\frac{B}{2}\right) + \sin^2\left(\frac{C}{2}\right)\right]$$

$$= \frac{4^{\frac{4}{3}}}{3}\left[1 - \frac{r}{2R}\right] = \frac{2\left(3^{\frac{1}{4}}\right)(2R-r)}{3R}$$

$$\left[\begin{array}{c} 1 - \sin^2\frac{A}{2} - \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = \cos^2\frac{A}{2} - \sin^2\frac{B}{2} - \sin^2\frac{C}{2} \\ = \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \sin^2\frac{C}{2} = \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\frac{A+B}{2}\right] \\ = 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2\frac{(s-a)(s-b)(s-c)}{abc} = \frac{2}{3}\cdot\frac{\Delta^2}{abc} \\ = \frac{1}{2}\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right) = \frac{1}{2}\cdot\frac{r}{R} \end{array}\right]$$

PROBLEM 1.111-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$4\left(\sum_{cyc} a^2 + \sum_{cyc} x^2\right) + 8\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} x^2\right)}$$

$$= 4\left(\sqrt{\sum_{cyc} a^2} + \sqrt{\sum_{cyc} x^2}\right)^2 \overset{\substack{ROOT\ MEAN \\ SQUARE}}{\geq} 4\left(\frac{\sum_{cyc} a}{\sqrt{4}} + \frac{\sum_{cyc} x}{\sqrt{4}}\right)^2$$

$$= (a+b+c+d+x+y+z+t)^2 \text{ (proved)}$$

$$5\left(\sum_{cyc} a^3 + \sum_{cyc} m^3 + \sum_{cyc} x^3\right) + 3\sqrt[3]{\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} m^3\right)\left(\sum_{cyc} x^3\right)}$$

$$\overset{AM\geq GM}{\geq} 18\sqrt[3]{\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} m^3\right)\left(\sum_{cyc} x^3\right)} \geq 18\sqrt[3]{\frac{(a+b+c)^3}{9}\cdot\frac{(m+n+p)^3}{9}\cdot\frac{(x+y+z)^3}{9}}$$

$$= 2(a+b+c)(m+n+p)(x+y+z)$$

PROBLEM 1.112-*Solution by Henry Ricardo-New York-USA*

Recalling that $\alpha + \frac{1}{\alpha} \geq 2$ *for* $\alpha > 0$ *by the AM-GM inequality, we see that*

$$\left(a\cdot\frac{x}{y} + b\cdot\frac{u}{v}\right)^2 + \left(a\cdot\frac{y}{x} + b\cdot\frac{v}{u}\right)^2 =$$

$$= \left(a \cdot \frac{x}{y}\right)^2 + 2ab\frac{xu}{yv} + \left(b \cdot \frac{u}{v}\right)^2 + \left(a \cdot \frac{y}{x}\right)^2 + 2ab\frac{yv}{xu} + \left(b \cdot \frac{v}{u}\right)^2$$
$$= a^2\left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2\right] + 2ab\left(\frac{xu}{yv} + \frac{yv}{xu}\right) + b^2\left[\left(\frac{u}{v}\right)^2 + \left(\frac{v}{u}\right)^2\right]$$
$$\geq 2a^2 + 2b^2 + 4ab = 2(a+b)^2$$

PROBLEM 1.113-*Solution by proposer*

We will prove that $P \geq 15$. *Since* $y^2 + 4yz + 5z^2 = (y+2z)^2 + z^2$. *And*
$$(x^2 + 4xy + 5y^2)(z^2 + 4zx + 5x^2) = [(x+2y)^2 + y^2][x^2 + (z+2x)^2] =$$
$$= [x(x+2y) + y(z+2x)]^2 + [(x+2y)(z+2x) - xy]^2 =$$
$$= (x^2 + 4xy + yz)^2 + (2x^2 + 3xy + 2y + zx)^2$$
By the Cauchy-Schwarz inequality, we have:
$$P \geq [(y+2z)(x^2 + 4xy + yz) + z(2x^2 + 3xy + 2yz + zx)]^2 =$$
$$= [x^2y + y^2z + z^2x + 4(xy^2 + yz^2 + zx^2) + 11xyz]^2 = (5 - 2xyz)^2$$
Thus, it suffices to show that: $(5-2xyz)^2 + 6x^2y^2z^2 \geq 15$.
This inequality is equivalent to $(xyz-1)^2 \geq 0$

The equality holds for $\begin{cases} x^2y + y^2z + z^2x = \frac{16}{5} \\ xy^2 + yz^2 + zx^2 = \frac{-14}{5} \\ xyz = 1 \end{cases}$

PROBLEM 1.114-*Solution by Ravi Prakash-New Delhi-India*

Put $x = \ln a, y = \ln b, z = \ln c$. *With this (a) can be written as*
$(x+y-z)^3 + (y+z-x)^3 + (x+z-y)^3 + 24xyz = (x+y+z)^3$ *and (b) becomes*
$$(x+y)(y+z)(z+x) = \frac{1}{3}[(x+y+z)^3 - x^3 - y^3 - z^3]$$
(a) Consider
$$(x+y+z)^3 + (x+y-z)^3 + \left(z-(x-y)\right)^3 + \left(z+x(x-y)\right)^3$$
$$= 2(x+y)^3 + 6(x+y)z^2 + 2z^3 + 6(x-y)^2z$$
$$= 2(x+y+z)^3 - 6(x+y)z(x+y+z) + 6(x+y)z^2 + 6(x-y)^2z$$
$$= 2(x+y+z)^3 - 6(x+y)^2z - 6(x+y)z^2 + 6(x+y)z^2 + 6(x-y)^2z$$
$$= 2(x+y+z)^3 - 6z[(x+y)^2 - (x-y)^2] = 2(x+y+z)^3 - 6z(4xy)$$
$$\Rightarrow (x+y-z)^3 + (z+x-y)^3 + (z+y-x)^3 + 24xyz = (x+y+z)^3$$
as desired.

(b) Consider
$$(x+y+z)^3 - x^3 - (y^3 + z^3) =$$
$$= (x+y+z-x)[(x+y+z)^2 + x(x+y+z) + x^2] - (y+z)(y^2 - yz + z^2)$$
$$= (y+z)[(x+y+z)^2 + x(x+y+z) + x^2 - y^2 + yz - z^2]$$
$$= (y+z)[(x+y+z-y)(x+y+z+y) + (x^2 + xy + yz + zx) + (x-z)(x+z)]$$
$$= (y+z)[(x+z)\{(x+y) + (y+z)\} + (x+y)(x+z) + (x-z)(x+z)]$$
$$= (y+z)(x+z)[x+y+y+z+x+y+x-z] = 3(x+y)(y+z)(z+x)$$

PROBLEM 1.115-*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Given inequality} \Leftrightarrow \frac{(a+b)(b+c)(c+a)}{abc} \overset{(1)}{\geq} 2+\frac{6\sqrt{3\sum a^2}}{\sum a}\quad (\because \sum a^2=3)$$

Let $a+b=x, b+c=y, c+a=z$. *Then* $x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ *are three sides of a triangle with semiperiemeter, circumradius & inradius* $= s,R,r$ *respectively (say):* $\therefore \sum a = s \Rightarrow c=s-x, a=s-y, b=s-z$. *Using above subtitution, (1)* $\Leftrightarrow \frac{xyz}{r^2s}-2 \geq$

$$\frac{6}{s}\sqrt{3\sum(s-y)^2} \Leftrightarrow \frac{4Rrs}{r^2s}-2 \geq \frac{6}{s}\sqrt{3\sum(s^2-2sy+y^2)} \Leftrightarrow$$

$$\Leftrightarrow \frac{2R-r}{r} \geq \frac{3}{s}\sqrt{3\{3s^2-4s^2+2(s^2-4Rr-r^2)\}} \Leftrightarrow$$

$$\Leftrightarrow \frac{(2R-r)^2}{r^2} \geq \frac{27}{s^2}(s^2-8Rr-2r^2) \Leftrightarrow (2R-r)^2s^2 \overset{(2)}{\geq} 27r^2(s^2-8Rr-2r^2) \because s^2 \geq 27r^2 \therefore \textit{it}$$

suffices to prove: $(2R-r)^2 \geq s^2-8Rr-2r^2 \Leftrightarrow s^2 \leq 4R^2+4Rr+3r^2$, *which is true by Gerretsen* $\Rightarrow$ *(2) is true (hence proved)*

PROBLEM 1.116-*Solution by Marian Ursărescu-Romania*

Because $abc=1 \Rightarrow a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}$ *with* $x,y,z>0$. *Inequality becomes:*

$$\sum \frac{\frac{x}{y}}{\frac{x^2}{y^2}+\frac{y}{z}\cdot\frac{z}{x}} \leq \frac{3}{2} \Leftrightarrow \sum \frac{\frac{x}{y}}{\frac{x^3+y^3}{xy^2}} \leq \frac{3}{2} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{x^2y}{x^3+y^3} \leq \frac{3}{2}\quad (1)$$

But $x^3+y^3 \geq xy(x+y)$ *(because* $x^3+y^3-x^2y-xy^2 \geq 0 \Leftrightarrow$
$\Leftrightarrow x^2(x-y)+y^2(x-y) \geq 0 \Leftrightarrow (x-y)(x^2-y^2) \geq 0 \Leftrightarrow (x-y)^2(x+y) \geq 0$ *true)*

$$\Rightarrow \frac{1}{x^3+y^3} \leq \frac{1}{xy(x+y)} \Rightarrow \frac{x^2y}{x^3+y^3} \leq \frac{x}{x+y}\quad (2)$$

From (1) + (2) we must show this: $\frac{x}{x+y}+\frac{y}{y+z}+\frac{z}{z+x} \leq \frac{3}{2} \Leftrightarrow$

$$\Leftrightarrow 2x(y+z)(z+x)+2y(x+y)(z+x)+2z(x+y)(y+z) \leq$$
$$\leq 3(x+y)(y+z)(z+x) \Leftrightarrow$$
$$\Leftrightarrow 6xyz+4x^2y+4xz^2+4y^2z+2xz^2+2y^2z+2yz^2 \leq$$
$$\leq 6xyz+3x^2y+3xz^2+3y^2z+3x^2z+3xy^2+3yz^2 \Leftrightarrow$$
$$\Leftrightarrow z^2y+x^2z+xy^2-x^2y-y^2z-z^2x \geq 0 \Leftrightarrow$$
$$\left.\begin{array}{c}\Leftrightarrow xy(y-x)+xz(x-z)+zy(z-y) \Leftrightarrow \\ \textit{But } z-y=z-x+x-y\end{array}\right\} \Leftrightarrow$$
$$\Leftrightarrow xy(y-x)+xz(x-z)+zy(z-x+x-y) \geq 0 \Leftrightarrow$$
$$\Leftrightarrow (xy-zy)(y-x)+(xz-zy)(x-z) \geq 0 \Leftrightarrow (y-x)y(x-z)+(x-z)z(x-y) \geq 0 \Leftrightarrow$$
$$\Leftrightarrow (y-x)(x-z)(y-z) \geq 0 \textit{ which is true.}$$

PROBLEM 1.117-*Solution by proposer*

First, we prove: $\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$

Proof.

$$\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$$\Leftrightarrow \frac{b(c+a)-bc}{c(c+a)}+\frac{c(a+b)-ca}{a(a+b)}+\frac{a(b+c)-ab}{b(b+c)} \ge \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$\Leftrightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{a+b}{a+c}+\frac{b+c}{b+a}+\frac{c+a}{c+b}$. *Let* $x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a} \Rightarrow xyz=1$. *So,*

$$\frac{a+b}{a+c}=\frac{1+yz}{1+z}=y+\frac{1-y}{1+z};\ \frac{b+c}{b+a}=\frac{1+zx}{1+x}=z+\frac{1-z}{1+x}$$

$\frac{c+a}{c+b}=\frac{1+xy}{1+y}=x+\frac{1-x}{1+y}$. *We need to prove that:* $\frac{x-1}{y+1}+\frac{y-1}{z+1}+\frac{z-1}{x+1} \ge 0$

$$\Leftrightarrow (x^2-1)(z+1)+(y^2-1)(x+1)+(z^2-1)(y+1) \ge 0$$

$$\Leftrightarrow x^2z+z^2y+y^2x+x^2+y^2+z^2 \ge x+y+z+3 \quad (1)$$

On the other hand, we have: $x^2z+z^2y+y^2x \ge 3xyz=3$. *And*

$x^2+y^2+z^2 \ge \frac{1}{3}(x+y+z)^2 \ge x+y+z$. *So (1) right! Or*

$\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc} \ge \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$. *Next, we prove:*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}+\frac{3}{2(a+b+c)^2}\cdot\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

Proof. Assume

$$a \ge b \ge c \Rightarrow (a-c)^2=\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

We have: $\frac{a}{b+c}-\frac{1}{2}+\frac{b}{c+a}-\frac{1}{2}+\frac{c}{a+b}-\frac{1}{2}$

$$=\frac{1}{2}\left[\frac{(a-b)+(a-c)}{b+c}+\frac{(b-c)+(b-a)}{c+a}+\frac{(c-a)+(c-b)}{a+b}\right]$$

$$=\frac{1}{2}\left[\frac{(a-b)^2}{(a+c)(b+c)}+\frac{(b-c)^2}{(b+a)(c+a)}+\frac{(c-a)^2}{(c+b)(a+b)}\right]$$

$$\ge \frac{1}{2}\cdot\frac{(a-b+b-c+a-c)^2}{(a+c)(b+c)+(b+a)(c+a)+(c+b)(a+b)}$$

$$=\frac{1}{2}\cdot\frac{4(a-c)^2}{a^2+b^2+c^2+3(ab+bc+ca)} \ge \frac{1}{2}\cdot\frac{4(a-c)^2}{(a+b+c)^2+\frac{1}{3}(a+b+c)^2}$$

$$=\frac{3(a-c)^2}{2(a+b+c)^2}=\frac{3}{2(a+b+c)^2}\cdot\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$$

So, $\frac{ab}{c^2+ca}+\frac{bc}{a^2+ab}+\frac{ca}{b^2+bc} \ge \frac{3}{2}+\frac{3}{2(a+b+c)^2}\cdot\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$

Equality occurs if and only if $a=b=c$.

PROBLEM 1.118-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a} \overset{(1)}{\ge} \frac{3\sum a^3}{\sum a}$$

(1) $\Leftrightarrow \frac{ab^4+bc^4+ca^4}{abc} \ge \frac{3\sum a^3}{\sum a}$

$$\Leftrightarrow \left(\sum a\right)(ab^4+bc^4+ca^4) \overset{(2)}{\ge} 3abc\left(\sum a^3\right)$$

Let $s-a=x, s-b=y, s-c=z\ (x,y,z>0)$

$$\therefore s=x+y+z \Rightarrow a=y+z, b=z+x, c=x+y$$

By this substitution, (2) transforms into:

$$2\big((x+y)(y+z)^4+(y+z)(z+x)^4+(z+x)(x+y)^4\big)(x+y+z) \ge$$

$$\geq 3(x+y)(y+z)(z+x)((x+y)^3+(y+z)^3+(z+x)^3)$$
$$\Leftrightarrow 2\sum x^6+6\sum x^5y+7\sum x^4y^2+2\sum x^3y^3\geq 3\sum x^2y^4+2xyz(\sum x^2y)+6xyz(\sum xy^2+18x^2y^2z^2)\ (a)$$
$$\text{Now, } y^6+x^4y^2\overset{A-G}{\geq}2x^2y^4\to (1a)$$
$$z^6+y^4z^2\overset{A-G}{\geq}2y^2z^4\to (1b)$$
$$x^6+z^4x^2\overset{A-G}{\geq}2z^2x^4\to (1c)$$
$$(1a)+(1b)+(1c)\Rightarrow \sum x^6+\sum x^4y^2\geq 2\sum x^2y^4\ (1)$$
$$\text{Again, } y^6+y^6+x^6\overset{A-G}{\geq}3x^2y^4\to (2a)$$
$$z^6+z^6+y^6\overset{A-G}{\geq}3y^2z^4\to (2b)$$
$$x^6+x^6+z^6\overset{A-G}{\geq}3z^2x^4\to (2c)$$
$$(2a)+(2b)+(2c)\Rightarrow 3\sum x^6\geq 3\sum x^2y^2\Rightarrow \sum x^6\geq \sum x^2y^4\ (2)$$
$$\text{Also, } x^4y^2+y^4z^2\overset{A-G}{\geq}2x^2y^3z\to (3a)$$
$$y^4z^2+z^4x^2\overset{A-G}{\geq}2y^2z^3x\to (3b)$$
$$z^4x^2+x^4y^2\overset{A-G}{\geq}2z^2x^3y\to (3c)$$
$$(3a)+(3b)+(3c)\Rightarrow 2\sum x^4y^2\geq 2xyz(\sum xy^2)\Rightarrow 6\sum x^4y^2\geq 6xyz(\sum xy^2)\to (3)$$
$$\text{Moreover, } x^3y^3+x^3y^3+z^3x^3\overset{A-G}{\geq}3x^3y^2z\to (4a)$$
$$y^3z^3+y^3z^3+x^3y^3\overset{A-G}{\geq}3y^3z^2x\to (4b)$$
$$z^3x^3+z^3x^3+y^3z^3\overset{A-G}{\geq}3z^3x^2y\to (4c)$$
$$(4a)+(4b)+(4c)\Rightarrow 3\sum x^3y^3\geq 3xyz(\sum x^2y)\Rightarrow 2\sum x^3y^3\geq 2xyz(\sum x^2y)\to (4)$$
$$\text{Lastly, } 6\sum x^5y\overset{A-G}{\geq}18x^2y^2z^2\to (5)$$
$$(1)+(2)+(3)+(4)+(5)\Rightarrow (a) \text{ is true (Proved)}$$

PROBLEM 1.119-*Solution by proposer*

The inequality need to prove to be equivalent to:

$$\frac{c^3+a+b-c^3}{c^2(c^3+a+b)}+\frac{a^3+b+c-a^3}{a^2(a^3+b+c)}+\frac{b^3+c+a-b^3}{b^2(b^3+c+a)}\geq \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{\sqrt[3]{abc}}$$
$$\Leftrightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{\sqrt[3]{abc}}+\frac{a}{a^3+b+c}+\frac{b}{b^3+c+a}+\frac{c}{c^3+a+b}$$

Applying the Hölder and AM-GM inequality, we have:

$$\sum\frac{a}{a^3+b+c}=\sum\frac{a(1+b+c)^2}{(a^3+b+c)(1+b+c)^2}\leq\sum\frac{a(1+b+c)^2}{(a+b+c)^3}$$
$$=\frac{\sum a\,(1+b^2+c^2+2bc+2b+2c)}{27}$$
$$=\frac{4(ab+bc+ca)+a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+6abc}{27}$$
$$=\frac{4(ab+bc+ca)+(a+b+c)(ab+bc+ca)+3abc+3}{27}$$

$$=\frac{7(ab+bc+ca)+3abc+3}{27}\le\frac{7\cdot\frac{(a+b+c)^2}{3}+3\cdot\frac{(a+b+c)^3}{27}+3}{27}=1$$

Therefore, we need to prove that: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{\sqrt[3]{abc}}+1$

Applying the Cauchy-Schwarz inequality, we have:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{1}{3}\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\ge\frac{3}{a+b+c}\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

We have: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1=\frac{1}{3}\cdot(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1=\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$

We need to prove that: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\ge\frac{2\sqrt{3(a^2+b^2+c^2)}}{\sqrt[3]{abc}}$

We will prove that: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{\sqrt{3(a^2+b^2+c^2)}}{\sqrt[3]{abc}}$. *Which is equivalent to:*

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\ge\frac{3(a^2+b^2+c^2)}{\sqrt[3]{a^2b^2c^2}}\Leftrightarrow\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\frac{3(a^2+b^2+c^2)}{\sqrt[3]{a^2b^2c^2}}$$

Applying the AM-GM inequality, we have:

$$LHS=\sum\frac{a^2}{b^2}+\frac{a}{c}+\frac{a}{c}\ge3\sum\sqrt[3]{\frac{a^4}{b^2c^2}}=\frac{3(a^2+b^2+c^2)}{\sqrt[3]{a^2b^2c^2}}=RHS$$

Similarly, we have: $\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\ge\frac{\sqrt{3(a^2+b^2+c^2)}}{\sqrt[3]{abc}}$

The proof of the inequality is complete. The equality holds for $a=b=c=1$.

PROBLEM 1.120-*Solution by proposer*

Put $\frac{1}{a}=x,\frac{1}{b}=y,\frac{1}{c}=z$. *The inequality need to prove to be equivalent to:*

$$\frac{1}{a^2+ab+ca+kbc}+\frac{1}{b^2+bc+ab+kca}+\frac{1}{c^2+ca+bc+kab}\le\frac{9}{(k+3)(ab+bc+ca)}$$

$$\Leftrightarrow\frac{1}{\frac{1}{x^1}+\frac{1}{xy}+\frac{1}{zx}+\frac{k}{yz}}+\frac{1}{\frac{1}{y^2}+\frac{1}{yz}+\frac{1}{xy}+\frac{k}{zx}}+\frac{1}{\frac{1}{z^2}+\frac{1}{zx}+\frac{1}{yz}+\frac{k}{xy}}\le\frac{9}{(k+3)\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)}$$

$$\Leftrightarrow\frac{x}{kx^2+xy+yz+zx}+\frac{y}{ky^2+xy+yz+zx}+\frac{z}{kz^2+xy+yz+zx}\le\frac{9}{(k+3)(x+y+z)}$$

$$\Leftrightarrow\frac{x(xy+yz+zx)}{kx^2+xy+yz+zx}+\frac{y(xy+yz+zx)}{ky^2+xy+yz+zx}+\frac{z(xy+yz+zx)}{kz^2+xy+yz+zx}\le\frac{9(xy+yz+zx)}{(k+3)(x+y+z)}$$

$$\Leftrightarrow x-\frac{kx^3}{kx^2+xy+yz+zx}+y-\frac{ky^3}{ky^2+xy+yz+zx}+z-\frac{kz^3}{kz^2+xy+yz+zx}$$
$$\le\frac{9(xy+yz+zx)}{(k+3)(x+y+z)}$$

$$\Leftrightarrow k\left(\frac{x^3}{kx^2+xy+yz+zx}+\frac{y^3}{ky^2+xy+yz+zx}+\frac{z^3}{kz^2+xy+yz+zx}\right)$$
$$+\frac{9(xy+yz+zx)}{(k+3)(x+y+z)}\le x+y+z$$

Applying the Cauchy-Schwarz inequality, we have:

$$\sum \frac{x^3}{kx^2+xy+yz+zx} \ge \frac{(\sum x^2)^2}{\sum x\,(kx^2+xy+yz+zx)}$$
$$= \frac{(\sum x^2)^2}{3kxyz+(\sum x)[k(\sum x^2)-(k-1)\sum xy]}$$

Note that we have: $(xy+yz+zx)^2 \ge 3xyz(x+y+z) \Leftrightarrow 3xyz \le \frac{(xy+yz+zx)^2}{x+y+z}$

Therefore, we have: $\sum \frac{x^3}{kx^2+xy+yz+zx} \ge \frac{(\sum x^2)^2(\sum x)}{k(\sum xy)^2+(\sum x)^2[k\sum x^2-(k-1)\sum xy]}$

Therefore, we need to prove that: $\frac{k(\sum x^2)^2(\sum x)}{k(\sum xy)^2+(\sum x)^2[k\sum x^2-(k-1)\sum xy]} + \frac{9(\sum xy)}{(k+3)(\sum x)} \ge \sum x$

$$\Leftrightarrow \frac{k(\sum x^2)^2}{k(\sum xy)^2+(\sum x)^2[k\sum x^2-(k-1)\sum xy]} + \frac{9(\sum xy)}{(k+3)(\sum x)^2} \ge 1$$

Put $t = \frac{x^2+y^2+z^2}{xy+yz+zx}$ $(t \ge 1)$. *The inequality is equivalent to:* $\frac{kt^2}{k+(t+2)(kt-k+1)} + \frac{9}{(k+3)(t+2)} \ge 1$

$$\Leftrightarrow \frac{kt^2(k+3)}{kt^2+kt+t-k+2} - k + \frac{9}{t+2} - 3 \ge 0 \Leftrightarrow (t-1)\left[\frac{k(3t+2-k)}{kt^2+kt+t-k+2} - \frac{3}{t+2}\right] \ge 0$$
$$\Leftrightarrow k(3t+2-k)(t+2) \ge 3[kt^2+kt+t-k+2]$$
$$\Leftrightarrow k(3t^2-tk-2k+8t+4) \ge 3kt^2+3kt+3t-3k+6$$
$$\Leftrightarrow k^2(t+2) - k(5t+7) + 3(t+2) \le 0 \Leftrightarrow k^2 - k\cdot\frac{5t+7}{t+2} + 3 \le 0$$

We see $\frac{5t+7}{t+2} = 5 - \frac{3}{t+2} \ge 5 - \frac{3}{1+2} = 4$. *Therefore, we have:*

$k^2 - k\cdot\frac{5t+7}{t+2} + 3 \le k^2 - 4k + 3 = (k-1)(k-3) \le 0$. *The equality for* $a=b=c$.

PROBLEM 1.121-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{\sin^3 A + \sin^3 B}{\sin^5 A + \sin^5 B} =$$
$$= \frac{(\sin A+\sin B)(\sin^2 A - \sin A\sin B + \sin^2 B)}{(\sin A+\sin B)(\sin^4 A - \sin^3 A\sin B + \sin^2 A\sin^2 B - \sin A\sin^3 B + \sin^4 B)}$$
$$= \frac{(\sin A+\sin B)(\sin^2 A - \sin A\sin B + \sin^2 B)}{\sin^3 A\,(\sin A - \sin B) - \sin^3 B\,(\sin A-\sin B) + \sin^2 A\sin^2 B}$$
$$= \frac{\sin^2 A - \sin A\sin B + \sin^2 B}{(\sin A-\sin B)(\sin A-\sin B)(\sin^2 A + \sin A\sin B + \sin^2 B) + \sin^2 A\sin^2 B}$$
$$\overset{A-G}{\le} \frac{\sin^2 A - \sin A\sin B + \sin^2 B}{3\sin A\sin B\,(\sin A-\sin B)^2 + \sin^2 A\sin^2 B} =$$
$$= \frac{\sin^2 A - \sin A\sin B + \sin^2 B}{\sin A\sin B\,(3\sin^2 A + 3\sin^2 B - 5\sin A\sin B)}$$
$$= \frac{(3\sin^2 A + 3\sin^2 B - 5\sin A\sin B) + 2\sin A\sin B}{3\sin A\sin B\,(3\sin^2 A + 3\sin^2 B - 5\sin A\sin B)} =$$
$$= \frac{1}{3\sin A\sin B} + \frac{2}{3}\left(\frac{1}{3\sin^2 A + 3\sin^2 B - 5\sin A\sin B}\right)$$
$$\overset{A-G}{\le} \frac{1}{3\sin A\sin B} + \frac{2}{3}\left(\frac{1}{\sin A\sin B}\right) = \frac{1}{\sin A\sin B} \Rightarrow \frac{\sin^3 A+\sin^3 B}{\sin^5 A + \sin^5 B} \le \frac{1}{\sin A\sin B}$$

$$\Rightarrow \sum \frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B} \le \sum \frac{1}{\sin A \sin B} \le \sum \frac{1}{\sin^2 A} = 4R^2 \sum \frac{1}{a^2} \overset{Goldstone}{\le} \frac{4R^2}{4r^2} = \frac{R^2}{r^2} \quad (Done)$$

PROBLEM 1.122-*Solution by Myagmarsuren-Yadamsuren-Darkhan-Mongolia*

$$\left.\begin{array}{c} a \ge b \ge c \\ \frac{1}{s-a} \ge \frac{1}{s-b} \ge \frac{1}{s-c} \end{array}\right\} \Rightarrow \frac{a}{s-a} \ge \frac{b}{s-b} \ge \frac{c}{s-c}$$

$$1)\ LHS: 2\left(\frac{R}{r}-1\right) \ge \frac{c}{s-c};\ a \ge b \ge c \Rightarrow a+b \ge 2c \Rightarrow a+b-c \ge c \Rightarrow 2(s-c) \ge c \Rightarrow$$

$$\Rightarrow 2 \ge \frac{c}{s-c};\ \frac{c}{s-c} \le 2 = 2\cdot 1 = 2(2-1) \overset{Euler}{\le} 2\left(\frac{R}{r}-1\right)\ LHS$$

$$2)\ \left.\begin{array}{c} x = s-a \\ y = s-b \\ z = s-c \end{array}\right\} a \ge b \ge c \Rightarrow z \ge y \ge x \Rightarrow 2z \ge y+x \Rightarrow (x+y-2z)xy \le 0 \Rightarrow$$

$$\Rightarrow (x+y)xy + zx(z+x) - zx(z+x) \le 2xyz \overset{zx(z+x)\le yz(z+y)}{\Rightarrow}$$

$$\Rightarrow 2xyz \ge (x+y)xy + zx(z+x) - zx(z+x) \ge (x+y)xy + zx(z+x) - zy(z+y)$$

$$2xyz \ge (x+y)xy + zx(z+x) - zy(z+y) \Leftrightarrow \sum xy(x+y) - 2xyz \le 2zy(z+y) \Rightarrow$$

$$\Rightarrow \prod (x+y) - 4xyz \le 2zy(z+y) \Rightarrow \frac{\prod(x+y) - 4xyz}{4xyz} \le \frac{y+z}{2x} \Leftrightarrow$$

$$\Leftrightarrow 2\left(\underbrace{\frac{\prod(x+y)}{4xyz}}_{\Downarrow\ \frac{R}{2}} - 1\right) \le \underbrace{\frac{y+z}{x}}_{\Downarrow\ \frac{a}{s-a}}$$

$$2\left(\frac{R}{r}-1\right) \le \frac{a}{s-a} \quad RHS$$

PROBLEM 1.123-*Solution by Amit Dutta-Jamshedpur-India*

Let $f(x) = \log_a(b^x + a - b)$ *and* $g(x) = \log_b(a^x + b - a)$. *Let*

$$y = f(x) = \log_a(b^x + a - b) \Rightarrow a^y = b^x + a - b \Rightarrow b^x = (a^y + b - a) \Rightarrow$$

$$\Rightarrow x \log b = \log(a^y + b - a)\ \{Taking\ log\} \Rightarrow x = \log_b(a^y + b - a)$$

$$f^{-1}(y) = \log_b(a^y + b - a) \Rightarrow f^{-1}(x) = \log_b(a^x + b - a) = g(x)$$

Therefore, $f(x)$ *and* $g(x)$ *are inverse of each other. Also, both* $f(x)$ *and* $g(x)$ *are increasing and continuous functions {since they are log functions}*

Considering the last two arguments/statements we can say, the only possible solution lies on the line $y = x$*; i.e.,* $f(x) = g(x) = x \Rightarrow \log_a(b^x + a - b) = \log_b(a^x + b - a) = x$

Taking, $\log_a(b^x + a - b) = x \Rightarrow b^x + a - b = a^x \Rightarrow b^x - a^x + a - b = 0$

Let $h(x) = b^x - a^x + a - b$; $h'(x) = b^x \ln b - a^x \ln a$; $h'(x) = b^x \ln b - a^x \ln a > 0$;

$h'(x) > 0 \Rightarrow h(x)$ *is an increasing function, so it can have atmost one real root*

$$\left\{\begin{array}{c} \because b > a > 1 \\ b^x > a^x \quad (i) \\ \ln b > \ln a \quad (ii) \\ multiplying\ (i); (ii) \\ b^x \ln b > a^x \ln a \end{array}\right.$$

Clearly $h(x)=0$ when $x=1$, which is the only solution.
$\therefore x=1$ *is the unique solution.*

PROBLEM 1.124-*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Weighted GM} \ge \text{weighted HM} \Rightarrow \sqrt[a+b+c]{\left(b\tan\frac{A}{2}\right)^b\left(c\tan\frac{B}{2}\right)^c\left(a\tan\frac{C}{2}\right)^a} \ge$$

$$\ge \frac{a+b+c}{\frac{b}{b\tan\frac{A}{2}}+\frac{c}{c\tan\frac{B}{2}}+\frac{a}{a\tan\frac{C}{2}}} = \frac{2s}{\sum\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}} = \frac{2s}{\sum\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}} =$$

$$= \frac{2s}{\frac{s(3s-2s)}{rs}} = 2r$$

$\therefore \left(b\tan\frac{A}{2}\right)^b\left(c\tan\frac{B}{2}\right)^c\left(a\tan\frac{C}{2}\right)^a \ge (2r)^{a+b+c} \Rightarrow \left(\tan\frac{A}{2}\right)^b\left(\tan\frac{B}{2}\right)^c\left(\tan\frac{C}{2}\right)^a \ge \frac{(2r)^{a+b+c}}{a^ab^bc^c}$. *Now,*

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{rs} \overset{G-A}{\le} \frac{\left(\frac{s-b+s-c}{2}\right)^2}{rs} = \frac{a^2}{4rs} \Rightarrow \left(\frac{\tan\frac{A}{2}}{a^2}\right)^b \overset{(1)}{\le} \left(\frac{1}{4rs}\right)^b. \textit{ Similarly,}$$

$$\left(\frac{\tan\frac{B}{2}}{b^2}\right)^c \overset{(2)}{\le} \left(\frac{1}{4rs}\right)^c \ \& \ \left(\frac{\tan\frac{C}{2}}{c^2}\right)^a \overset{(3)}{\le} \left(\frac{1}{4rs}\right)^a$$

$$(1)\cdot(2)\cdot(3) \Rightarrow \left(\frac{\tan\frac{A}{2}}{a^2}\right)^b\left(\frac{\tan\frac{B}{2}}{b^2}\right)^c\left(\frac{\tan\frac{C}{2}}{c^2}\right)^a \le \left(\frac{1}{4rs}\right)^{a+b+c} \overset{?}{\le} \left(\frac{\sqrt{3}}{36}\right)^{a+b+c}\left(\frac{1}{r^2}\right)^{a+b+c} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{4rs} \overset{?}{\le} \frac{\sqrt{3}}{36}\cdot\frac{1}{r^2} \Leftrightarrow 4rs \overset{?}{\ge} 12\sqrt{3}r^2 \Leftrightarrow s \overset{?}{\ge} 3\sqrt{3}r \to \textit{true}$$

$$\therefore \left(\frac{\tan\frac{A}{2}}{a^2}\right)^b\left(\frac{\tan\frac{B}{2}}{b^2}\right)^c\left(\frac{\tan\frac{C}{2}}{c^2}\right)^a \le \left(\frac{\sqrt{3}}{36}\right)^{a+b+c}\cdot\frac{1}{r^{2(a+b+c)}} \Leftrightarrow$$

$$\Leftrightarrow \left(\tan\frac{A}{2}\right)^b\left(\tan\frac{B}{2}\right)^c\left(\tan\frac{C}{2}\right)^a \le \left(\frac{\sqrt{3}}{36}\right)^{a+b+c}\cdot\frac{a^{2b}\cdot b^{2c}\cdot c^{2a}}{r^{2(a+b+c)}} \quad \textit{(Done)}$$

PROBLEM 1.125-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{4}{3}\sum r_a^2 \overset{(1)}{\ge} 4\sqrt{3}S + \sum(a-b)^2$$

$$(1) \Leftrightarrow \frac{4}{3}\sum r_a^2 + 2\sum ab - \sum a^2 \overset{(2)}{\ge} 4\sqrt{3}S + \sum a^2$$

$$\textit{Now, Hadwiger – Finsler} \Rightarrow 2\sum ab - \sum a^2 \overset{(i)}{\ge} 4\sqrt{3}S \ \& \ \sum a^2 \underset{(ii)}{\overset{Leibnitz}{\le}} 9R^2$$

(i), (ii), (2) ⇒ it suffices to prove:

$$4\{(4R+r)^2 - 2s^2\} \ge 27R^2 \Leftrightarrow 8s^2 \overset{(3)}{\le} 37R^2 + 32Rr + 4r^2$$

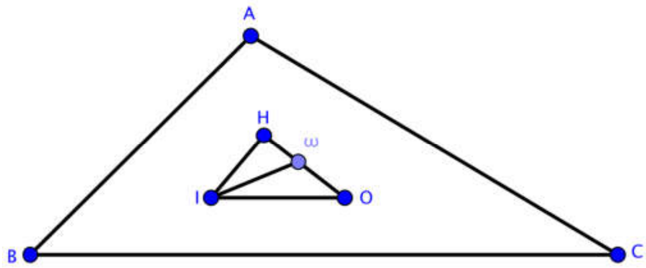
$$\textit{LHS of (3)} \overset{Gerretsen}{\le} 32R^2 + 32Rr + 24r^2 \overset{?}{\le} 37R^2 + 32Rr + 4r^2 \Leftrightarrow 5R^2 \overset{?}{\ge} 20r^2 \Leftrightarrow$$

$$\Leftrightarrow R \overset{?}{\ge} 2r \to \textit{true (Euler) (proved)}$$

PROBLEM 1.126-*Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia*

$$(\sum a^2)^2 \geq (\sum ab)^2 \geq 3abc(a+b+c) \ \ \textit{(True)}$$
$$\left(\sum a^2\right)^2 \geq 3abc(a+b+c) = 24s^2Rr$$
$$\frac{1}{8s^2r}\cdot\left(\sum a^2\right)^2 \geq 3R;\ \frac{1}{8\cdot s^2\cdot r\cdot 2r}\left(\sum a^2\right)^2 \geq \frac{3R}{2r}$$
$$\frac{1}{16s^2r^2}\left(\sum a^2\right)^2 \geq \frac{3R}{2r};\ \frac{1}{4\Delta}\cdot\sum a^2 \geq \sqrt{\frac{3R}{2r}}$$
$$\frac{R}{abc}\cdot\sum a^2 \geq \sqrt{\frac{3R}{2r}} \ \ (*)$$
$$\cot A = \frac{R}{abc}(b^2+c^2-a^2) \ \ (**)$$
$$(*), (**) \to \sum \cot A = \frac{R}{abc}(a^2+b^2+c^2) \geq \sqrt{\frac{3R}{2r}}$$

PROBLEM 1.127-*Solution by Marian Ursărescu – Romania*



$$m_a \leq 2R\cos^2\frac{A}{2} \Rightarrow m_a \leq \frac{a}{\sin A}\cos^2\frac{A}{2} \Rightarrow m_a \leq \frac{a}{2\sin\frac{A}{2}\cos\frac{A}{2}}\cdot\cos^2\frac{A}{2}$$
$$\Rightarrow m_a \leq \frac{a}{2}\cot\frac{A}{2} \Rightarrow m_a \leq \frac{a}{2\tan\frac{A}{2}} \Rightarrow \tan\frac{A}{2} \leq \frac{a}{2m_a} \Rightarrow \tan\left(\frac{\widehat{OIH}}{2}\right) \leq \frac{OH}{2I\omega} =$$
$$= \frac{\sqrt{9R^2-(a^2+b^2+c^2)}}{2\cdot\frac{R-2R}{2}} \Rightarrow \tan\left(\frac{\widehat{OIH}}{2}\right) \leq \frac{\sqrt{9R^2-(a^2+b^2+c^2)}}{R-2r}$$
$$a^2+b^2+c^2 \geq 36r^2 \Rightarrow -(a^2+b^2+c^2) \leq -36r^2$$
$$\tan\left(\frac{\widehat{OIH}}{2}\right) < \frac{3\sqrt{R^2-4r^2}}{R-2r} \Rightarrow \tan\left(\frac{\widehat{OIH}}{2}\right) < 3\sqrt{\frac{R+2r}{R-2r}}$$

PROBLEM 1.128-*Solution by Seyran Ibrahimov-Maasilli-Azerbaijan*

$$LHS = \sum\frac{1}{m_a+m_b} \leq \frac{1}{2r}$$

$$\frac{4}{m_a+m_b}\le\frac{1}{m_a}+\frac{1}{m_b}$$

$$LHS\le\frac{1}{4}\cdot 2\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\le\frac{1}{2}\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{1}{2r}$$

PROBLEM 1.129-*Solution by Marian Ursărescu-Romania*

b) $b^2+c^2\ge 2bc\Rightarrow\frac{1}{\sqrt{b^2+c^2}}\le\frac{1}{\sqrt{2bc}}\Rightarrow\sum\frac{a}{\sqrt{b^2+c^2}}\le\frac{1}{\sqrt{2}}\sum\frac{a}{\sqrt{bc}}\Rightarrow$ *we must show:*

$$\frac{1}{\sqrt{2}}\sum\frac{9}{\sqrt{bc}}\le\sqrt{\frac{6R-3r}{2r}}\Leftrightarrow$$

$$\left(\sum\frac{a}{\sqrt{bc}}\right)^2\le\frac{6R-3r}{r}\quad (1)$$

But from Cauchy's inequality $\Rightarrow\left(\sum\frac{a}{\sqrt{bc}}\right)^2\le 3\sum\frac{a^2}{bc}$ *(2)*

From (1)+(2) we must show: $\sum\frac{a^2}{bc}\le\frac{2R-r}{r}\Leftrightarrow\frac{\sum a^3}{abc}\le\frac{2R-r}{r}$ *(3)*

But $\sum a^3=2s(s^2-3r^2-6Rr)$ *and* $abc=4sRr$ *(4)*

From (3)+(4) we must show: $\frac{s^2-3r^2-6Rr}{2Rr}\le\frac{2R-r}{r}\Leftrightarrow s^2-3r^2-6Rr\le 4R^2-2Rr\Leftrightarrow$

$\Leftrightarrow s^2\le 4R^2+4Rr+3r^2$ *with its Gerretsen's inequality.*

a) Again, $b^2+c^2\ge 2bc\Rightarrow\frac{a^3}{b^2+c^2}\le\frac{a^3}{2bc}\Rightarrow\sum\frac{a^3}{b^2+c^2}\le\frac{1}{2}\sum\frac{a^3}{bc}\Rightarrow$ *we must show this:*

$$\sum\frac{a^3}{bc}\le 3\sqrt{3}\frac{R(R-r)}{r}\quad (5)$$

Now, using sine law $\Rightarrow a=2R\sin A\Rightarrow$ *(5)* $\Leftrightarrow\sum\frac{2R\sin^3 A}{\sin B\sin C}\le 3\sqrt{3}\frac{R(R-r)}{r}\Leftrightarrow$

$$\sum\frac{\sin^3 A}{\sin B\sin C}\le\frac{3\sqrt{3}}{2}\left(\frac{R}{r}-1\right)\quad (6)$$

But (6) its true, proved in IneMath (2/2016) or (5) $\Leftrightarrow\frac{\sum a^4}{abc}\le 3\sqrt{3}\frac{R(R-r)}{r}$ *(6)*

$$\sum a^4=2[s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2]\quad (7)$$

$$abc=4sRr\quad (8)$$

From (6)+(7)+(8)$\Rightarrow\frac{s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2}{sR}\le 3\sqrt{3}R(R-r)$ *and using again Gerretsen's*

$$s^2\le 4R^2+4Rr+3r^2 \text{ and } s\le\frac{3\sqrt{3}}{2}R.$$

PROBLEM 1.130-*Solution by Omran Kouba-Damascus Syria*

Step 1. Consider $f(x)=2(2-3x+2x^2)-1-x^8$ *then* $f(x)\ge 0$ *with equality if and only if* $x=1$*. Indeed, with some algebra we see that* $f(1+t)=t^4h(t)$ *with*

$h(t)=31\left(t^2+\frac{28t}{31}\right)^2+\frac{1820}{31}\left(t+\frac{31}{65}\right)^2+\frac{952}{65}>0$*. This proves the inequality:*

$$\sqrt[4]{\frac{1+x^8}{2}}\le 2x^2-3x+2\quad (1)$$

With equality if and only if $x=1$*.*

Step 2. By the AM-GM inequality, we have for all real x the following:

$$3\sqrt[3]{x^2-x+1}\le x^2-x+1+1+1=x^2-x+3\quad (2)$$

Step 3. For all real x we have:

$$3x^2-4x+5\le 2(x^4-3x+4)\quad (3)$$

with equality if and only if $x=1$. Indeed:

$$2(x^4-3x+4)-(3x^2-4x+5)=2x^4-3x^2-2x+3=(x-1)^2(2(x+1)^2+1)\ge 0$$

Step 4. Combining (1), (2) and (3) we conclude that:

$$3\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{1+x^8}{2}}\le 2(x^4-3x+4)$$

with equality if and only if $x=1$.

PROBLEM 1.131-*Solution by Soumava Chakraborty-Kolkata-India*

Let $\sqrt[3]{x}=a,\sqrt[3]{y}=b,\sqrt[3]{z}=c$. Then, $\sum a^3=3$ & $3\sum a^2+21\overset{(1)}{=}10\sum a^3b^3$

$$\because \sum a^3=3\therefore \left(\sum a^3\right)^2=9\Rightarrow \sum a^6+2\sum a^3b^3=9\Rightarrow 5\sum a^6+10\sum a^3b^3=45$$

$$\Rightarrow 5\sum a^6+3\sum a^2+21=45\Rightarrow 5\sum a^6+3\sum a^2=24\overset{(2)}{=}8\sum a^3\left(\because \sum a^3=3\right)$$

Now, $a^6+a^2+a^2+a^2\overset{A-G}{\ge}4\sqrt[4]{a^{12}}=4a^3\Rightarrow a^6+3a^2\overset{(a)}{\ge}4a^3$.

Similarly, $b^6+3b^2\overset{(b)}{\ge}4b^3$

& $c^6+3c^2\overset{(c)}{\ge}4c^3$. Again, $4\sum a^6\overset{Chebyshev}{\ge}\frac{4}{3}(\sum a^3)^2=\frac{4}{3}\cdot 3\cdot\sum a^3\ (\because \sum a^3=3)$

$$\Rightarrow 4\sum a^6\overset{(d)}{\ge}4\sum a^3$$

(a)+(b)+(c)+(d)$\Rightarrow 5\sum a^6+3\sum a^2\overset{(3)}{\ge}8\sum a^3$, with equality occuring when $a=b=c$

$\therefore$ (2), (3) $\Rightarrow a=b=c$ & $\because \sum a^3=3\therefore 3a^3=3\Rightarrow a=1\Rightarrow a=b=c=1\Rightarrow$

$\Rightarrow x=y=z=1\therefore$ only solution is: $x=y=z=1$ (answer)

PROBLEM 1.132-*Solution by proposer*

First, Milne Inequality is used:

Let $w_j(j=\overline{1,n})>0$, with sum 1; $P_j\in[0,1]\quad (j=\overline{1,n})$, then

$$\left(\sum_{j=1}^{n}\frac{w_j}{1-p_j}\right)\left(\sum_{j=1}^{n}\frac{w_j}{1+p_j}\right)\le\left(\sum_{j=1}^{n}\frac{w_j}{1-p_j^2}\right)^{2-\min\limits_{1\le j\le n}w_j}$$

$$\text{Let's take}\begin{cases}w_1=\sin^2 x,\text{ where } x,y\in\left(0,\frac{\pi}{2}\right)\\ w_2=\cos^2 x\\ p_1=\cos y\\ p_2=\sin y\end{cases}$$

$$\Rightarrow\left(\frac{\sin^2 x}{1-\cos y}+\frac{\cos^2 x}{1-\sin y}\right)\left(\frac{\sin^2 x}{1+\cos y}+\frac{\cos^2 x}{1+\sin y}\right)\le\left(\frac{\sin^2 x}{\sin^2 y}+\frac{\cos^2 x}{\cos^2 y}\right)^{2-\min\limits_{1\le j\le n}w_j}$$

Jordan

$$But\ \forall\ t \in \left(0, \frac{\pi}{2}\right) \Rightarrow \frac{2t}{\pi} \le \sin t \le t$$

$$\Rightarrow \left(\frac{\sin^2 x}{1-\cos y} + \frac{\cos^2 x}{1-\sin y}\right)\left(\frac{\sin^2 x}{1+\cos y} + \frac{\cos^2 x}{1+\sin y}\right) \le \left(\frac{\pi^2}{4} + \frac{\cos^2 x}{\cos^2 y}\right)^k$$

$$where\ k = 2 - \min\{\sin^2 x, \cos^2 x\}$$

PROBLEM 1.133-*Solution by proposer*

- *We have:* $a^8 + b^8 = (a^4+b^4)^2 - \left(a^2b^2\sqrt{2}\right)^2 = \left(a^4 - a^2b^2\sqrt{2} + b^4\right)\left(a^4 + a^2b^2\sqrt{2} + b^4\right)$

$$\Leftrightarrow a^8 + b^8 = \left(a^2 - \sqrt{2-\sqrt{2}}ab + b^2\right)\left(a^2 + \sqrt{2-\sqrt{2}}ab + b^2\right)\left(a^2 - \sqrt{2+\sqrt{2}}ab + b^2\right)\left(a^2 + \sqrt{2+\sqrt{2}}ab + b^2\right)$$

- *Therefore, by inequality AM – GM for 8 positive real numbers.*

$$\frac{a^2 + \sqrt{2+\sqrt{2}}ab + b^2}{b\left(2+\sqrt{2+\sqrt{2}}\right)} + \frac{a^2 - \sqrt{2+\sqrt{2}}ab + b^2}{b\left(2-\sqrt{2+\sqrt{2}}\right)} + \frac{a^2 - \sqrt{2-\sqrt{2}}ab + b^2}{b\left(2-\sqrt{2-\sqrt{2}}\right)} + \frac{a^2 + \sqrt{2-\sqrt{2}}ab + b^2}{b\left(2+\sqrt{2-\sqrt{2}}\right)} + b + b + b + b \ge$$

$$\ge 8 \cdot \sqrt[8]{\frac{a^2 + \sqrt{2+\sqrt{2}}ab + b^2}{b\left(2+\sqrt{2+\sqrt{2}}\right)} \cdot \frac{a^2 - \sqrt{2+\sqrt{2}}ab + b^2}{b\left(2-\sqrt{2+\sqrt{2}}\right)} \cdot \frac{a^2 - \sqrt{2-\sqrt{2}}ab + b^2}{b\left(2-\sqrt{2-\sqrt{2}}\right)} \cdot \frac{a^2 + \sqrt{2-\sqrt{2}}ab + b^2}{b\left(2+\sqrt{2-\sqrt{2}}\right)} \cdot b \cdot b \cdot b \cdot b} \Leftrightarrow$$

$\Leftrightarrow \frac{8a^2}{b} - 12a + 12b \ge 8\sqrt[8]{\frac{a^8+b^8}{2}}$. *Similar:* $\frac{8b^2}{c} - 12b + 12c \ge 8 \cdot \sqrt[8]{\frac{b^8+c^8}{2}}$; $\frac{8c^2}{a} - 12c + 12a \ge 8 \cdot \sqrt[8]{\frac{c^8+a^8}{2}}$

$$\Rightarrow \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge \sqrt[8]{\frac{a^8+b^8}{2}} + \sqrt[8]{\frac{b^8+c^8}{2}} + \sqrt[8]{\frac{c^8+a^8}{2}} \quad (1)$$

- *By inequality AM – GM. We have:*

$$(a+b+c)^{10}\left(\frac{1}{9a} + \frac{1}{9b} + \frac{1}{9c}\right)^9 \ge \frac{1}{9^9}(a+b+c)^{10}\left(3 \cdot \sqrt[3]{\frac{1}{abc}}\right)^9 = \frac{(a+b+c)^{10}}{3^9 \cdot (abc)^3} \quad (2)$$

- *Other:* $(a+b+c)^6 = [(a^2+b^2+c^2) + (ab+bc+ca) + (ab+bc+ca)]^3 \ge$
$\ge 27(a^2+b^2+c^2)(ab+bc+ca)^2 \ge 27(a^2+b^2+c^2) \cdot 3abc(a+b+c)$

$$\Rightarrow (a+b+c)^6 \ge 81abc(a+b+c)(a^2+b^2+c^2) \Leftrightarrow \frac{(a+b+c)^{10}}{3^9(abc)^3} \ge \frac{\left(a^2+b^2+c^2\right)^2}{3abc} \quad (3)$$

- *Let (2), (3):* $\Rightarrow (a+b+c)^{10}\left(\frac{1}{9a} + \frac{1}{9b} + \frac{1}{9c}\right)^9 \ge \frac{\left(a^2+b^2+c^2\right)^2}{3abc} \quad (4)$

- *Let (1), (4). We need to prove:*

$$\frac{(a^2+b^2+c^2)^2}{3abc} \ge \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \Leftrightarrow \frac{(a^2+b^2+c^2)^2}{3abc} \ge \frac{ab^3+bc^3+ca^3}{abc} \Leftrightarrow$$

$$\Leftrightarrow (a^2+b^2+c^2)^2 \ge 3(ab^3+bc^3+ca^3) \Leftrightarrow \frac{1}{2}\sum(a^2 - ac + 2ab - b^2 - bc)^2 \ge 0 \quad (True)$$

- *Therefore:* $\sqrt[8]{\frac{a^8+b^8}{2}}+\sqrt[8]{\frac{b^8+c^8}{2}}+\sqrt[8]{\frac{c^8+a^8}{2}}\le (a+b+c)^{10}\left(\frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c}\right)^9$ *and we get the result.*

PROBLEM 1.134-*Solution by proposer*

$$\text{* We have } 2a^6-a^5-3a^3+a^2+1=2a^5(a-1)+a^4(a-1)+a^3(a-1)-$$
$$-2a^2(a-1)-a(a-1)-(a-1)=(a-1)(2a^5+a^4+a^3-2a^2-a-1)$$
$$=(a-1)\big(2a^4(a-1)+3a^3(a-1)+4a^2(a-1)+2a(a-1)+(a-1)\big)$$
$$=(a-1)^2(2a^4+3a^3+4a^2+2a+1)\ge 0\ \ (\text{because } a>0 \text{ and } (a-1)^2\ge 0)$$
$$\Rightarrow 2a^6-a^5-3a^3+a^2+1\ge 0\Leftrightarrow 2a^6-a^5+a^2+1\ge 3a^3\Leftrightarrow$$
$$\Leftrightarrow 2a^6-a^5+b^4+a^2+1\ge 3a^3+b^4$$
$$\Leftrightarrow \frac{1}{2a^6-a^5+b^4+a^2+1}\le\frac{1}{3a^3+b^2}\Leftrightarrow\frac{ab}{2a^6-a^5+b^4+a^2+1}\le\frac{ab}{3a^3+b^4}\quad (1)$$

- *By AM-GM inequality we have:*

$$3a^3+b^4=a^3+a^3+a^3+b^4\ge 4\sqrt[4]{a^3\cdot a^3\cdot a^3\cdot b^4}=4\sqrt[4]{a^9\cdot b^4}=4a^2b\sqrt[4]{a}\Leftrightarrow$$
$$\Leftrightarrow\frac{ab}{3a^3+b^4}\le\frac{ab}{4a^2b\sqrt[4]{a}}=\frac{1}{4a\sqrt[4]{a}}$$

- *Hence (1) and AM-GM inequality:*

$$\Rightarrow\frac{ab}{2a^6-a^5+b^4+a^2+1}\le\frac{1}{4a\sqrt[4]{a}}\le\frac{1}{4a}\cdot\frac{1}{4}\left(\frac{1}{a}+1+1+1\right)=\frac{1}{16a}\left(\frac{1}{a}+3\right)$$
$$+\text{ Similar: }\frac{bc}{2b^6-b^5+c^4+b^2+1}\le\frac{1}{16b}\left(\frac{1}{b}+3\right);\frac{ca}{2c^6-c^5+a^4+c^2+1}\le\frac{1}{16c}\left(\frac{1}{c}+3\right)$$
$$-\text{ Hence: }\Rightarrow P=\frac{ab}{2a^6-a^5+b^4+a^2+1}+\frac{bc}{2b^6-b^5+c^4+b^2+1}+\frac{ca}{2c^6-c^5+a^4+c^2+1}\le$$
$$\le\frac{1}{16a}\left(\frac{1}{a}+3\right)+\frac{1}{16b}\left(\frac{1}{b}+3\right)+\frac{1}{16c}\left(\frac{1}{c}+3\right)$$
$$\Leftrightarrow P\le\frac{1}{16}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{3}{16}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\quad (2)$$

$-$ *Other* $a^2+b^2+c^2=3abc$ *and inequality:* $(x+y+z)\ge\sqrt{3(xy+yz+zx)}$ *that:*

$$x=\frac{a}{bc},y=\frac{b}{ca},z=\frac{c}{ab}$$

We have:

$$3=\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\ge\sqrt{3\left(\frac{a}{bc}\cdot\frac{b}{ca}+\frac{b}{ca}\cdot\frac{c}{ab}+\frac{c}{ab}\cdot\frac{a}{bc}\right)}=\sqrt{3\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)}\Leftrightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le 3\ \ (3)$$

- *Let (3) and AM-GM inequality:*

$$3\ge\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\left(\frac{1}{a^2}+1\right)+\left(\frac{1}{b^2}+1\right)+\left(\frac{1}{c^2}+1\right)-3\ge\frac{2}{a}+\frac{2}{b}+\frac{2}{c}-3\Leftrightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le 3\ \ (4)$$

- *Let (2), (3), (4):* $\Rightarrow P\le\frac{1}{16}\cdot 3+\frac{3}{16}\cdot 3=\frac{12}{16}=\frac{3}{4}\Rightarrow P\le\frac{3}{4}\Rightarrow P_{\max}=\frac{3}{4}$

$$+\text{ Equality occurs if: }\begin{cases}a,b,c>0;a^2+b^2+c^2=3abc\\ a-1=b-1=c-1=0\\ a^3=b^4;b^3=c^4;c^3=a^4\\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1;\frac{a}{bc}=\frac{b}{ca}=\frac{c}{ab}\end{cases}\Leftrightarrow a=b=c=1.$$

Maximum value of P *is* $\frac{3}{4}$ *when* $a=b=c=1$.

PROBLEM 1.135-*Solution by Marian Ursărescu – Romania*

First, we show this: $x^4 - x + 2 \ge x^3 + 1, \forall x \in \mathbb{R}$ *(1)*

(1) $\Leftrightarrow x^4 - x^3 - x + 1 \ge 0 \Leftrightarrow x^3(x-1) - (x-1) \ge 0 \Leftrightarrow (x-1)(x^3-1) \ge 0 \Leftrightarrow$

$\Leftrightarrow (x-1)^2(x^2+x+1) \ge 0$ *(true). From (1)* $\Rightarrow \frac{1}{x^4+y^3-x+2} \le \frac{1}{x^3+y^3+1} \Rightarrow$ *inequality becomes:*

$$\sum \frac{x^3y^3}{x^3+y^3+1} \le \frac{x^4+y^4+z^4+3xyz}{6} \quad (2)$$

From AM-GM $\Rightarrow x^3 + y^3 + 1 \ge 3xy$ *(3)*

From (2)+(3) we must show this: $\sum \frac{x^2y^2}{3} \le \frac{x^4+y^4+z^4+3xyz}{6} \Leftrightarrow \sum x^2y^2 \le \frac{x^4+y^4+z^4+3xyz}{2} \Leftrightarrow$

$\Leftrightarrow x^4+y^4+z^4+3xyz - 2(x^2y^2+x^2z^2+y^2z^2) \ge 0$ *(4). Now, let* $x = \frac{3a}{a+b+c}$,

$y = \frac{3b}{a+b+c}, z = \frac{3c}{a+b+c}$ *with* $a,b,c > 0$.

$$(4) \Leftrightarrow \frac{81(a^4+b^4+c^4)}{(a+b+c)^4} + \frac{81abc}{(a+b+c)^3} - \frac{2\cdot 81(a^2b^2+b^2c^2+c^2a^2)}{(a+b+c)^4} \ge 0$$

$$\Leftrightarrow \frac{a^4+b^4+c^4}{(a+b+c)^4} + \frac{abc}{(a+b+c)^3} - \frac{2(a^2b^2+a^2c^2+b^2c^2)}{(a+b+c)^4} \ge 0 \Leftrightarrow$$

$$\Leftrightarrow a^4+b^4+c^4+abc(a+b+c) - 2(a^2b^2+b^2c^2+c^2a^2) \ge 0 \quad (5)$$

Now, use Cârtoaje theorem: let $f_4(a,b,c)$ *be a symmetric polynomial of degree four. Then:* $f_4(a,b,c) \ge 0, \forall a,b,c \ge 0 \Leftrightarrow f_4(a,1,1) \ge 0, \forall a \ge 0$. *Let* $f_4(a,b,c) = a^4+b^4+c^4+$ $+abc(a+b+c) - 2(a^2b^2+b^2c^2+c^2a^2)$

$f_4(a,1,1) = a^4 + 2 + a(a+2) - 2(2a^2+c) = a^4 + 2 + a^2 + 2a - 4a^2 - 2 =$

$= a^4 - 3a^2 + 2a = a(a^3-3a+2) = a(a-1)^2(a+2) \ge 0, \forall a \ge 0 \Rightarrow f_4(a,b,c) \ge 0$

$\Rightarrow$ *(5) its true.*

PROBLEM 1.136-*Solution by Michael Stergiou-Greece*

$$Q = \sum_{cyc} \frac{1}{\sqrt[3]{2x^5+y^4-x^3+4}} \quad (1)$$

The function $f(x) = \sqrt[3]{x}$ *is concave so (1) becomes:*

$$Q \le 3\sqrt[3]{\frac{1}{3}\sum \frac{1}{2x^5+y^4-x^2+4}} \to \frac{Q^3}{9} \le \sum_{cyc} \frac{1}{2x^5+y^4-x^2+4} \quad (2)$$

As $x^5 + x^5 + x^2 \ge 3x^4$ *(2)* $\to \frac{Q^3}{9} \le \sum_{cyc} \frac{1}{3x^4+y^4-2x^2+4}$ *(3)*

Let $x^2 = a, y^2 = b, z^2 = c, abc = 1$ *(3)* $\to \frac{Q^3}{9} \le \sum \frac{1}{3a^2+b^2-2a+4}$ *(4). But* $a^2+b^2 \ge 2ab$, $a^2+1 \ge 2a$

(4) $\to \frac{Q^3}{9} \le \frac{1}{2}\sum_{cyc} \frac{1}{a+ab+1} \le \cdots \le 1$ *(after calculus the sum reduces to 1 using* $abc = 1$*)*

Therefore $Q \le \sqrt[3]{\frac{9}{2}}$

PROBLEM 1.137-*Solution by Ravi Prakash-New Delhi-India*

Rewrite the inequality: $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} - 2 \ge \frac{4(x-y)^2}{(2x+xy+1)(2y+xy+1)} \Leftrightarrow$

$$\Leftrightarrow \frac{(\sqrt{x}-\sqrt{y})^2}{\sqrt{xy}} \ge \frac{4(\sqrt{x}-\sqrt{y})^2(\sqrt{x}+\sqrt{y})^2}{(2x+xy+1)(2y+xy+1)} \quad (1)$$

If $x = y$*, there is nothing to show. Suppose* $x \ne y$*, then (1) can be written as*

$$\frac{1}{\sqrt{xy}} \geq \frac{4\left(\sqrt{x}+\sqrt{y}\right)^2}{(2x+xy+1)(2y+xy+1)} \Leftrightarrow (xy+1)^2 + 2(x+y)(xy+1) + 4xy \geq$$
$$\geq 4\sqrt{xy}\left(x+y+2\sqrt{xy}\right) \Leftrightarrow x^2y^2 + 1 + 2(x+y)(xy+1) + 6xy \geq 4\sqrt{xy}(x+y) + 8xy$$
$$\Leftrightarrow x^2y^2 - 2xy + 1 + 2(x+y)\left(xy - 2\sqrt{xy} + 1\right) \geq 0 \Leftrightarrow$$
$$\Leftrightarrow (xy-1)^2 + 2(x+y)\left(\sqrt{xy}-1\right)^2 \geq 0 \text{ which is true } \forall x,y > 0.$$

PROBLEM 1.138-*Solution by proposer*

From $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ *we get* $abc = ab+bc+ca \geq 3\sqrt[3]{(abc)^2}$. *Now, we have:* $(abc)^3 \geq 27(abc)^2 |:(abc)^2$ *getting* $abc \geq 27$ *and from that* $\sqrt[3]{abc} \geq 3$. *Note that* $a > 1$ *because* $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ *and we have* $3a-3>0, 3b-3>0$ *and* $3c-3>0$. *Using Hölder's inequality we get:*

$$(4a-3)(4b-3)(4c-3) = [a+(3a-3)][b+(3b-3)][c+(3c-3)] \geq$$
$$\geq \left(\sqrt[3]{abc} + 3\sqrt[3]{(a-1)(b-1)(c-1)}\right)^3$$
$$(4a-3)(4b-3)(4c-3) \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{abc-(ab+bc+ca)+(a+b+c)-1}\right)^3$$
$$(4a-3)(4b-3)(4c-3) \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{abc - abc + 3\sqrt[3]{abc} - 1}\right)^3$$
$$(4a-3)(4b-3)(4c-3) \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{3\cdot \sqrt[27]{27} - 1}\right)^3$$
$$(4a-3)(4b-3)(4c-3) \geq \left(\sqrt[3]{abc} + 3 + 3\right)^3$$
Using AM-GM we have: $(4a-3)(4b-3)(4c-3) \geq \left(3\sqrt[3]{\sqrt[3]{abc}\cdot 3\cdot 3}\right)^3$
$$(4a-3)(4b-3)(4c-3) \geq 3^3 \cdot \sqrt[3]{abc} \cdot 9$$
$$(4a-3)(4b-3)(4c-3) \geq 243\sqrt[3]{abc}$$

PROBLEM 1.139-*Solution by proposer*

- Using Cauchy and Bunhiacopxki inequality. We have:

$$\left(\sqrt{2(y^8+z^8)} + 2y^2z^2\right)^2 \leq 2(2(y^8+z^8)+4y^4z^4) = 4(y^4+z^4)^2 \Leftrightarrow$$
$$\Leftrightarrow \sqrt{2(y^8+z^8)} + 2y^2z^2 \leq 2(y^4+z^4)$$
$$\Leftrightarrow \sqrt{2(y^8+z^8)} \leq 2(y^4 - y^2z^2 + z^4) \Leftrightarrow \sqrt[4]{\frac{y^8+z^8}{2}} \leq \sqrt{y^4 - y^2z^2 + z^4} \quad (1)$$

- Other:

$$\sqrt{y^4 - y^2z^2 + z^4} = \sqrt{(y^2+z^2)^2 - \left(yz\sqrt{3}\right)^2} = \sqrt{\left(y^2 - yz\sqrt{3} + z^2\right)\left(y^2 + yz\sqrt{3} + z^2\right)}$$
$$= \sqrt{\left(2+\sqrt{3}\right)\left(y^2 - yz\sqrt{3}+z^2\right)\cdot\left(2-\sqrt{3}\right)\left(y^2+yz\sqrt{3}+z^2\right)} \leq$$

$$\le \frac{(2+\sqrt{3})(y^2-yz\sqrt{3}+z^2)+(2-\sqrt{3})(y^2+yz\sqrt{3}+z^2)}{2}$$

$$=\frac{2(2y^2-3yz+2z^2)}{2}=2y^2-3yz+2z^2$$

$$\Rightarrow \sqrt{y^4-y^2z^2+z^4}\le 2y^2-3yz+2z^2 \qquad (2)$$

- *Let (1), (2):* $\Rightarrow \sqrt[4]{\frac{y^8+z^8}{2}}\le 2y^2-3yz+2z^2 \Leftrightarrow \sqrt[4]{\frac{y^8+z^8}{2}}+3yz\le 2(y^2+z^2)$

$$\Leftrightarrow \frac{1}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz}\ge\frac{1}{2(y^2+z^2)}\Leftrightarrow\frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz}\ge\frac{x}{2(y^2+z^2)}=\frac{x}{2(3-x^2)}\quad (Let\ x^2+y^2+z^2=3)\quad (3)$$

- *We have:* $\frac{x}{3-x^2}-\frac{x^2}{2}=x\left(\frac{1}{3-x^2}-\frac{x}{2}\right)=\frac{x(x^3-3x+2)}{2(3-x^2)}=\frac{x(x-1)^2(x+2)}{2(3-x^2)}\ge 0$ *(because* $x>0;(x-1)^2\ge 0$*)*

$$\Leftrightarrow \frac{x}{3-x^2}-\frac{x^2}{2}\ge 0\Leftrightarrow\frac{x}{3-x^2}\ge\frac{x^2}{2}.\ Let\ (3):\Rightarrow\frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz}\ge\frac{x^2}{4}\qquad (4)$$

+ *Similar:* $\frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}}+3zx}\ge\frac{y^2}{4};\ \frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}}+3xy}\ge\frac{z^2}{4}\qquad (5)$

- *Let (4), (5) and* $x^2+y^2+z^2=3$*:*

$$\Rightarrow P=\frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz}+\frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}}+3zx}+\frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}}+3xy}\ge\frac{x^2+y^2+z^2}{4}=\frac{3}{4}\Rightarrow$$

$$\Rightarrow Q_{min}=\frac{3}{4}$$

+ *Equality occurs if:*

$$\begin{cases} x,y,z>0;x^2+y^2+z^2=3\\ \sqrt{2(x^8+y^8)}=2x^2y^2;\sqrt{2(y^8+z^8)}=2y^2z^2;\sqrt{2(z^8+x^8)}=2z^2x^2\Leftrightarrow x=y=z=1\\ x-1=y-1=z-1=0\end{cases}$$

PROBLEM 1.140-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS=\frac{bc\sqrt{a+b}+ca\sqrt{b+c}+ab\sqrt{c+a}}{abc}=\frac{\sqrt{bc(a+b)}\sqrt{bc}+\sqrt{ca(b+c)}\sqrt{ca}+\sqrt{ab(c+a)}\sqrt{ab}}{abc}\overset{CBS}{\le}$$

$$\le\frac{\sqrt{\sum ab}\sqrt{3abc+\sum a^2b}}{abc}\overset{?}{\le}\frac{\sum ab}{abc}\sqrt{\sum a-\frac{\sum ab}{\sum a}}\Leftrightarrow 3abc+\sum a^2b\overset{?}{\le}$$

$$\le\sum ab\left(\frac{\sum a^2+\sum ab}{\sum a}\right)\Leftrightarrow$$

$$\Leftrightarrow\left(\sum a^2+\sum ab\right)\left(\sum ab\right)\overset{?}{\ge}\left(\sum a\right)\left(3abc+\sum a^2b\right)\Leftrightarrow ab^2+bc^2+ca^3\underset{(1)}{\overset{?}{\ge}}$$

$$\ge abc\left(\sum a\right).\ But\ ab^3+bc^3+ca^3=abc\left(\frac{b^2}{c}+\frac{c^2}{a}+\frac{a^2}{b}\right)\overset{Bergstrom}{\ge}abc\frac{(\sum a)^2}{\sum a}=$$

$$=abc\left(\sum a\right)\Rightarrow\ (1)\ is\ true\ (proved)$$

PROBLEM 1.141-*Solution by proposer*

$$Let\ be\ g:(0,\infty)\to\mathbb{R};g(x)=\sqrt{ab+bc+ca-\frac{abc}{x}}$$

$$g(x)=\left(ab+bc+ca-\frac{abx}{x}\right)^{\frac{1}{2}};\ g'(x)=\frac{abc}{2x^2}\left(ab+bc+ca-\frac{abc}{x}\right)^{-\frac{1}{2}}$$

$$g''(x)=\frac{abc}{2}\left[\left(-\frac{2}{x^3}(ab+bc+ca-\frac{abc}{x}\right)^{-\frac{1}{2}}-\frac{abc}{2x^4}\left(ab+bc+ca-\frac{abc}{x}\right)\right]$$

$g''(x)<0$; g – *concave. Denote* $q=ab+bc+ca; r=abc$

$$\sum_{cyc}\left(\frac{1}{a}f(a)\right)=\sum_{cyc}\left(\frac{1}{a}\sqrt{ab+bc}\right)=\sum_{cyc}\sqrt{\frac{b+c}{a}}\le$$

$$\overset{JENSEN}{\le}\frac{q}{r}f\left(\frac{3r}{q}\right)=\frac{q}{r}\sqrt{q-\frac{qr}{3r}}=\frac{q}{r}\sqrt{\frac{2q}{3}}=\frac{ab+bc+ca}{abc}\sqrt{\frac{2(ab+bc+ca)}{3}}$$

$$\left(\sum_{cyc}\sqrt{\frac{b+c}{a}}\right)^2\le\frac{2\left(\sum_{cyc}ab\right)^3}{3a^2b^2c^2}$$

PROBLEM 1.142-*Solution by proposer*

$$\text{Let be } f:(0,\infty)\to\mathbb{R}; f(x)=\sqrt{abc-\frac{abc}{x}}$$

$$f(x)=\left(abc-\frac{abc}{x}\right)^{\frac{1}{2}};\ f'(x)=\frac{1}{2}\cdot\frac{abc}{x^2}\left(abc-\frac{abc}{x}\right)^{-\frac{1}{2}}$$

$$f''(x)=\frac{abc}{2}\left[-\frac{2}{x^3}\left(abc-\frac{abc}{x}\right)^{-\frac{1}{2}}-\frac{abc}{2x^4}\left(abc-\frac{abc}{x}\right)^{-\frac{3}{2}}\right]$$

$f''(x)<0$; f *concave. Denote* $p=a+b+c; r=abc$

$$\sum_{cyc}\left(\frac{f(ab)}{ab}\right)=\sum_{cyc}\sqrt{\frac{a-1}{bc}}\overset{JENSEN}{\le}\frac{p}{r}f\left(\frac{3r}{p}\right)=\frac{p}{r}\sqrt{r-\frac{pr}{3r}}=\frac{p}{r}\sqrt{r-\frac{p}{3}}$$

$$abc\sum_{cyc}\sqrt{\frac{a-1}{bc}}\le\left(\sum_{cyc}a\right)\cdot\sqrt{abc-\frac{1}{3}\sum_{cyc}a}$$

$$\sum_{cyc}\sqrt{\frac{a-1}{bc}}\le\left(\sum_{cyc}\frac{1}{ab}\right)\sqrt{abc-\frac{a+b+c}{3}}$$

PROBLEM 1.143-*Solution by Soumava Chakraborty-Kolkata-India*

$$\because w_a^2\le s(s-a), etc\ \therefore\ LHS\le\sum\frac{s(s-a)bc}{4S^2}=\frac{s}{4S^2}\left(s\sum ab-12Rrs\right)$$

$$=\frac{s^2}{4r^2s^2}(s^2-8Rr+r^2)$$

$$\overset{?}{\le}\frac{R^2-r^2}{r^2}\Leftrightarrow s^2\overset{?}{\le}4R^2+8Rr-5r^2.\ Now,\ s^2\overset{Gerretsen}{\le}$$

$$4R^2+4Rr+3r^2\overset{?}{\le}4R^2+8Rr-5r^2\Leftrightarrow 4Rr\overset{?}{\ge}8r^2\Leftrightarrow R\overset{?}{\ge}2r\to true\ (Euler)$$

PROBLEM 1.144-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS=\sum\frac{a^2}{aw_a}\underset{(1)}{\overset{Bergstrom}{\ge}}\frac{4s^2}{\sum aw_a}$$

$$WLOG,\ we\ may\ assume\ a\ge b\ge c\ \therefore w_a\le w_b\le w_c$$

$$(1)\Rightarrow LHS\underset{(2)}{\overset{Chebyshev}{\ge}}\frac{4s^2}{\frac{1}{3}(2s)\sum w_a}=\frac{2s\cdot 3}{\sum w_a}$$

$$Now,\ \sum w_a\underset{(3)}{\overset{w_a\le\sqrt{s(s-a)},etc}{\le}}\sqrt{3s}\sqrt{3s-2s}=\sqrt{3}s$$

$$(2),(3)\Rightarrow LHS\ge\frac{2s\cdot 3}{\sqrt{3}s}=\frac{2\cdot 3}{\sqrt{3}}=\frac{2\cdot 3\sqrt{3}}{3}=\frac{2\cdot 3\sqrt{3}R}{3R}\overset{Mitrinovic}{\ge}\frac{2\cdot 2s}{3R}=\frac{4s}{3R}\ (Done)$$

PROBLEM 1.145-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS=\sum\frac{m_a}{\frac{\Delta}{s}+\frac{\Delta}{s-a}}=\sum\frac{m_a s(s-a)}{rs(b+c)}\overset{m_a<\frac{b+c}{2},etc}{<}\sum\left(\frac{b+c}{2}\right)\cdot\frac{(s-a)}{r(b+c)}=$$

$$=\frac{\sum(s-a)}{2r}=\frac{3s-2s}{2r}=\frac{s}{2r}\ \therefore LHS<\frac{s}{2r}\Rightarrow LHS\le\frac{s}{2r}\ (proved)$$

PROBLEM 1.146-*Solution by proposer*

$$*\ We\ have:\ x^5-x^3-2x+2=x^4(x-1)+x^3(x-1)-2(x-1)=(x-1)(x^4+x^3-2)$$

$$=(x-1)\big(x^3(x-1)+2x^2(x-1)+2x(x-1)+2(x-1)\big)$$

$$=(x-1)^2(x^3+2x^2+2x+2)\ge 0$$

$$\Rightarrow x^5-x^3-2x+2\ge 0\Leftrightarrow x^5-x^3+4\ge 2(x+1)\Leftrightarrow\frac{1}{\sqrt[3]{2(x^5-x^3+4)}}\le\frac{1}{\sqrt[3]{2\cdot 2(x+1)}}$$

- Therefore, by AM-GM inequality for three positive real numbers:

$$\Rightarrow\frac{1}{\sqrt[3]{2(x^5-x^3+4)}}\le\frac{1}{\sqrt[3]{4(x+1)}}\le\frac{1}{3}\left(\frac{1}{\sqrt{2(x+1)}}+\frac{1}{\sqrt{2(x+1)}}+\frac{1}{2}\right)=\frac{1}{3}\left(\sqrt{\frac{2}{x+1}}+\frac{1}{2}\right)$$

$$+\ Similar:\ \frac{1}{\sqrt[3]{2(y^5-y^3+4)}}\le\frac{1}{3}\left(\sqrt{\frac{2}{y+1}}+\frac{1}{2}\right);\ \frac{1}{\sqrt[3]{2(z^5-z^3+4)}}\le\frac{1}{3}\left(\sqrt{\frac{2}{z+1}}+\frac{1}{2}\right)$$

- Therefore:

$$\Rightarrow P=\frac{1}{\sqrt[3]{2(x^5-x^3+4)}}+\frac{1}{\sqrt[3]{2(y^5-y^3+4)}}+\frac{1}{\sqrt[3]{2(z^5-z^3+4)}}\le\frac{1}{3}\left(\sqrt{\frac{2}{x+1}}+\sqrt{\frac{2}{y+1}}+\sqrt{\frac{2}{z+1}}\right)+\frac{1}{2}\quad(1)$$

- Other, because $xyz=1$ then exist positive real numbers a,b,c such that:

$$a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{x}$$

+ Therefore, by inequality Cauchy Schwarz:

$$\left(\sqrt{\frac{2}{x+1}}+\sqrt{\frac{2}{y+1}}+\sqrt{\frac{2}{z+1}}\right)^2=\left(\sqrt{\frac{2}{\frac{a}{b}+1}}+\sqrt{\frac{2}{\frac{b}{c}+1}}+\sqrt{\frac{2}{\frac{c}{a}+1}}\right)^2$$

$$=2\left(\sqrt{\frac{b}{a+b}}+\sqrt{\frac{c}{b+c}}+\sqrt{\frac{a}{c+a}}\right)^2$$

$$=2\left(\sqrt{\frac{b}{(a+b)(b+c)}}\cdot\sqrt{b+c}+\sqrt{\frac{c}{(b+c)(c+a)}}\cdot\sqrt{c+a}+\sqrt{\frac{a}{(c+a)(a+b)}}\cdot\sqrt{a+b}\right)^2\le$$

$$\le 2\big((b+c)+(c+a)+(a+b)\big)\cdot\left(\frac{b}{(a+b)(b+c)}+\frac{c}{(b+c)(c+a)}+\frac{a}{(c+a)(a+b)}\right)$$

$$\le 4(a+b+c)\cdot\frac{b(c+a)+c(a+b)+a(b+c)}{(a+b)(b+c)(c+a)}=\frac{8(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)}\quad (2)$$

- We have: $9(a+b)(b+c)(c+a)-8(a+b+c)(ab+bc+ca)=a(b-c)^2+b(c-a)^2+c(a-b)^2\ge 0$

$$\Rightarrow 9(a+b)(b+c)(c+a)\ge 8(a+b+c)(ab+bc+ca)\Leftrightarrow \frac{8(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)}\le 9$$

By (2): $\Rightarrow \left(\sqrt{\frac{2}{x+1}}+\sqrt{\frac{2}{y+1}}+\sqrt{\frac{2}{z+1}}\right)^2\le 9\Rightarrow \sqrt{\frac{2}{x+1}}+\sqrt{\frac{2}{y+1}}+\sqrt{\frac{2}{z+1}}\le 3\quad$ *(3)*

- Let (1), (3): $\Rightarrow P\le\frac{1}{3}\cdot 3+\frac{1}{2}=\frac{3}{2}\Rightarrow P_{\max}=\frac{3}{2}$.

Equality occurs if: $\begin{cases} xyz=1\\ x=y=z>0\end{cases}\Leftrightarrow x=y=z=1$

PROBLEM 1.147-*Solution by proposer*

- By AM- GM inequality for 4 *positive real numbers we have:*

$$a^3b^3(a^3+b^3)=(a+b)\cdot ab\cdot ab\cdot ab(a^2-ab+b^2)\le(a+b)\left(\frac{ab+ab+ab+a^2-ab+b^2}{4}\right)^4$$

$$\Rightarrow a^3b^3(a^3+b^3)\le(a+b)\left(\frac{(a+b)^2}{4}\right)^4=\frac{(a+b)^9}{4^4}\Leftrightarrow(a^3+b^3)\le\frac{(a+b)^9}{4^4a^3b^3}$$

$$\Leftrightarrow\sqrt[3]{4(a^3+b^3)}\le\frac{(a+b)^3}{4ab}\Leftrightarrow\frac{1}{\sqrt[3]{4(a^3+b^3)}}\ge\frac{4ab}{(a+b)^3}\Leftrightarrow\frac{c^2}{\sqrt[3]{4(a^3+b^3)}}\ge\frac{4abc^2}{(a+b)^3}\quad (1)$$

+ Similar: $\frac{b^2}{\sqrt[3]{4(c^3+a^3)}}\ge\frac{4cab^2}{(c+a)^3};\ \frac{a^2}{\sqrt[3]{4(b^3+c^3)}}\ge\frac{4bca^2}{(b+c)^3}\quad$ *(2)*

- Let (1), (2):

$$\Rightarrow P=\frac{a^2}{\sqrt[3]{4(b^3+c^3)}}+\frac{b^2}{\sqrt[3]{4(c^3+a^3)}}+\frac{c^2}{\sqrt[3]{4(a^3+b^3)}}\ge 4abc\left(\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3}\right)\quad (3)$$

- Other, by inequality Cauchy Schwarz we have:

$$\frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3}=\frac{\left(\frac{a}{b+c}\right)^2}{a(b+c)}+\frac{\left(\frac{b}{c+a}\right)^2}{b(c+a)}+\frac{\left(\frac{c}{a+b}\right)^2}{c(a+b)}\ge\frac{\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2}{2(ab+bc+ca)}$$

+ Because:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a^2}{ab+ac}+\frac{b^2}{bc+ba}+\frac{c^2}{ca+cb}\ge\frac{(a+b+c)^2}{2(ab+bc+ca)}\ge\frac{3(ab+bc+ca)}{2(ab+bc+ca)}=\frac{3}{2}$$

$$\Rightarrow \frac{a}{(b+c)^3}+\frac{b}{(c+a)^3}+\frac{c}{(a+b)^3} \geq \frac{9}{8(ab+bc+ca)} \quad (4)$$

- Let (3), (4) and such that: $a^2+b^2+c^2=3abc, ab+bc+ca \leq a^2+b^2+c^2$*:*

$$\Rightarrow P \geq 4abc \cdot \frac{9}{8(ab+bc+ca)} = \frac{9abc}{2(ab+bc+ca)} \geq \frac{3(a^2+b^2+c^2)}{2(a^2+b^2+c^2)} = \frac{3}{2} \Rightarrow P \geq \frac{3}{2} \Rightarrow P_{\min} = \frac{3}{2}$$

$$+ \textit{Equality occurs if: } \begin{cases} a,b,c>0 \\ a^2+b^2+c^2=3abc \\ ab=a^2-ab+b^2, bc=b^2-bc+c^2, ca=c^2-ca+a^2 \\ a=b=c \end{cases} \Leftrightarrow a=b=c=1$$

PROBLEM 1.148-*Solution by Marian Ursărescu – Romania*

Because $ab+bc+ca=12 \Rightarrow \exists x,y,z>0$ *such that:*

$$a=\frac{2\sqrt{3}x}{\sqrt{xy+yz+zx}}, b=\frac{2\sqrt{3}y}{\sqrt{xy+yz+zx}}, c=\frac{2\sqrt{3}z}{\sqrt{xy+yz+zx}}$$

Inequality becomes:

$$\sum \left(\frac{\frac{24\sqrt{3}(x^3+y^3)}{\left(\sqrt{xy+yz+zx}\right)^3}}{\frac{24y^2-12yz+24z^2}{xy+yz+zx}} \right) \geq 4 \Leftrightarrow 2\sqrt{3}\sum \frac{x^3+y^3}{\sqrt{xy+yz+zx}(2y^2-yz+z^2)} \geq 4 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{x^3+y^3}{2y^3-yz+2z^2} \geq \frac{2}{\sqrt{3}}\sqrt{xy+yz+zx} \quad (1)$$

But $(x+y+z)^2 \geq 3(xy+xz+yz) \Rightarrow \sqrt{xy+xz+yz} \leq \frac{x+y+z}{\sqrt{3}}$ *(2)*

From (1) + (2) we must show: $\sum \frac{x^3+y^3}{2y^2-yz+2z^2} \geq \frac{2}{3}(x+y+z)$ *(3)*

But $2y^2-yz+2z^2 \leq 3(y^2-yz+z^2)$ *(4) (because* $\Leftrightarrow y^2-2yz+z^2 \geq 0$*)*

$$\sum \frac{x^3+y^3}{y^2-yz+z^2} \geq 2(x+y+z) \quad (5)$$

But this inequality its proposed and solved by Vasile Cîrtoaje in 2009, solved by S.O.S method. (Or its solved used $4(x^3+y^3) \geq (x+y)^3$ *and Hölder's inequality)*

Completion: We must (5): $\sum \frac{x^3+y^3}{y^2-yz+z^2} \geq 2(x+y+z)$

We show: (6) $\sum \frac{x^3}{y^2-yz+z^2} \geq x+y+z$ *and* $\sum \frac{y^3}{y^2-yz+z^2} \geq x+y+z$ *(7)*

From (6) + (7) ⇒ (5)

For (6): $\sum \frac{x^3}{y^2-yz+z^2} = \sum \frac{x^4}{x(y^2-yz+z^2)} \geq \frac{(x^2+y^2+z^2)^2}{\sum x(y^2-yz+z^2)}$

(from Cauchy's or Bergström's inequality) ⇒

We must show: $\frac{(x^2+y^2+z^2)^2}{\sum x(y^2-yz+z^2)} \geq x+y+z \Leftrightarrow$

$$\Leftrightarrow (x^2+y^2+z^2)^2 \geq (x+y+z)\cdot \sum x(y^2-yz+z^2) \Leftrightarrow$$

$$\Leftrightarrow (x^2+y^2+z^2)^2-(x+y+z)\sum x(y^2-yz+z^2) \geq 0 \quad (8)$$

Now we use Cîrtoaje's theorem: If $f_n(x,y,z)$ *is a symmetric and homogeneous polynom of degree* 4 *then* $f_4(x,y,z) \geq 0\ \forall x,y,z \in \mathbb{R} \Leftrightarrow f_4(x_1,1,1) \geq 0\ \ \forall x \in \mathbb{R}$ *in our case:* $f_4(x,y,z) = (x^2+y^2+z^2)^2-(x+y+z)\sum x(y^2-yz+z^2)$

$f_4(x_1,1,1) = x^4-2x^3+x^2 = x^2(x-1)^2 \geq 0$ *true ⇒ (6) its true.*

For (7): $\sum\frac{y^3}{y^2-yz+z^2}\geq x+y+z\Leftrightarrow\frac{y^3}{y^2-yz+z^2}+\frac{z^3}{z^2-zx+x^2}+\frac{x^3}{x^2-xy+y^2}\geq x+y+z\Leftrightarrow$

$$\sum x^3\,(y^2-yz+z^2)(z^2-xz+x^2)\geq(x+y+z)\prod(x^2-xy+y^2)\Leftrightarrow$$

$$\Leftrightarrow\sum x^3\,(y^2-yz+z^2)(z^2-xz+x^2)-$$

$$-(x+y+z)(x^2-xy+y^2)(y^2-yz+z^2)(z^2-xz+x^2)\geq 0\quad (9)$$

Now again use Cîrtoaje's theorem: If $f_5(x,y,z)$ it's a symmetric polynomial function of degree 5 then: $f_5(x,y,z)\geq 0\ \forall x,y,z\geq 0\Leftrightarrow f_5(0,4,4)\geq 0$. In our case:

$$f_5(x,y,z)=\sum x^3\,(y^2-yz+z^2)(z^2-xz+x^2)-$$

$$-(x+y+z)(x^2-xy+y^2)(y^2-yz+z^2)(z^2-xz+x^2)$$

$f_5(0,4,4)=2y^7-2y^7\geq 0$ *true* $\Rightarrow$ *(9) its true* $\Rightarrow$ *(7) its true.*

Vasile Cartoaje proof:

Let a,b,c be non-negative real numbers. Prove that:

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2}\geq a+b+c$$

Solution. Applying Cauchy – Schwarz inequality, we have:

$$\sum_{cyc}\frac{a^3}{b^2-bc+c^2}=\sum_{cyc}\frac{a^4}{a(b^2-bc+c^2)}\geq\frac{(a^2+b^2+c^2)^2}{\sum_{cyc}a(b^2-bc+c^2)}$$

It remains to prove that: $\left(\sum_{cyc}a^2\right)^2\geq\left(\sum_{cyc}a(b^2-bc+c^2)\right)\left(\sum_{cyc}a\right)$ *or*

$\sum_{cyc}a^4+2\sum_{cyc}a^2b^2\geq(a+b+c)\sum_{cyc}a^2(b+c)-3abc\sum_{cyc}a$ *or*

$$\sum_{cyc}a^4+abc\sum_{cyc}a\geq\sum_{cyc}a^3(b+c)$$

This is exactly the fourth degree-Schur's inequality, so we are done.

Equality holds for $a=b=c$ or $a=b,c=0$ up to permutation.

PROBLEM 1.149-*Solution by Ravi Prakash-New Delhi-India*

$$\ln(xy)\leq xf(x)+yf(y)\leq xyf(xy),\forall x,y>0$$

Put $x=y=1$, we get: $0\leq f(1)+f(1)\leq f(1)\Rightarrow f(1)=0.$

Put $y=\frac{1}{x}$ to obtain

$0\leq xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)\leq 0\Rightarrow f\left(\frac{1}{x}\right)=-x^2f(x)$. *Taking $y=1$, we get:*

$$\ln(x)\leq xf(x),\forall x>0\quad (1)$$

$$\Rightarrow\ln\left(\frac{1}{x}\right)\leq\frac{1}{x}f\left(\frac{1}{x}\right)=-xf(x)\,,\forall x>0\Rightarrow-\ln(x)\leq-xf(x),\forall x>0$$

$$\Rightarrow xf(x)\leq\ln x\,,\forall x>0\quad (2)$$

From (1), (2): $xf(x)=\ln(x)\,,\forall x>0\Rightarrow f(x)=\frac{1}{x}\ln(x)\,,\forall x>0$

PROBLEM 1.150-*Solution by Ravi Prakash-New Delhi-India*

Let $z=z_1+z_2+z_3$

$(z_1+z_2)(z_2+z_3)(z_3+z_1)+z_1z_2z_3=0\Rightarrow(z-z_3)(z-z_1)(z-z_2)+z_1z_2z_3=0\Rightarrow$

$\Rightarrow z^3-(z_1+z_2+z_3)z^2+(z_2z_3+z_3z_1+z_1z_2)z-z_1z_2z_3+z_1z_2z_3=0\Rightarrow$

$$\Rightarrow z^3 - z(z^2) + z(z_2z_3 + z_3z_1 + z_1z_2) = 0 \Rightarrow z\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)z_1z_2z_3 = 0.$$

As $|z_1| = |z_2| = |z_3| = k > 0, z_1z_2z_3 \neq 0$. *Thus*

$$z\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right) = 0 \quad (1)$$

Also, $k^2 = z_1\overline{z_1} = z_2\overline{z_2} = z_3\overline{z_3}$ *(2)*

From (1), (2): $k^2z(\overline{z_1}+\overline{z_2}+\overline{z_3}) = 0 \Rightarrow k^2z\overline{z} = 0$. *As* $k^2 \neq 0, |z|^2 = 0 \Rightarrow z = 0 \Rightarrow$

$\Rightarrow z_1 + z_2 + z_3 = 0$. *Now,* $|z_2 - z_3|^2 + |z_1|^2 = |z_2 - z_3|^2 + |-z_2 - z_3|^2 =$

$= |z_2 - z_3|^2 + |z_2 + z_3|^2 = 2|z_2|^2 + 2|z_3|^2 = 4k^2 \Rightarrow |z_2 - z_3|^2 + k^2 = 4k^2 \Rightarrow$

$\Rightarrow |z_2 - z_3| = \sqrt{3}k$. *Similarly,* $|z_3 - z_1| = |z_1 - z_2| = \sqrt{3}k$. *Thus,*

$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \Rightarrow$ *triangle with vertices* z_1, z_2, z_3 *is an equilateral triangle.*

PROBLEM 1.151-*Solution by Michael Sterghiou-Greece*

$a, b > 0$. *Find the max of* k *so that the below inequality is true:*

$$\frac{1}{a^4}+\frac{1}{b^4}+\frac{k}{a^4+b^4} \geq \frac{8k+32}{(a+b)^4} \quad (1)$$

(1) homogeneous to $a, b \rightarrow$ *assume* $a + b = 1$.

LHS (1) $\overset{BCS}{\geq} \frac{k+4}{a^4+b^4}$ *which need to be* $\geq \frac{9(k+4)}{1^4}$, *so it suffices that*

$$(k+4)\cdot\left[\frac{1}{x^4+(1-x)^4} - 8\right] \geq 0 \quad \text{(2) where } a = x, b = 1-x$$

$$f(x) = \frac{1}{x^4+(1-x)^4} - 8 \text{ which } f'(x) = -\frac{4[x^3-(1-x)^3]}{[(1-x)^4+x^4]^2} = 0$$

for $x = \frac{1}{2}$ *only and* $\min f(x) \rightarrow -7$ *when* $x \rightarrow 0$ *or* $x \rightarrow 1$ *and* $\max f(x) = 0$ *where*

$x = \frac{1}{2}$. *Therefore* $f(x) \leq 0$. *But* $(k+4)f(x) \geq 0 \rightarrow k \leq -y$ *so that* $k + 4 \leq 0$. *Therefore*

$\max k = -4$ *for which (1) becomes true by BCS. Equality for* $a = b$.

PROBLEM 1.152-*Solution by Soumava Chakraborty-Kolkata-India*

In any $\Delta ABC, \sum\frac{h_ar_a}{w_a^2} \geq 3$

$$\because w_a^2 = \frac{4bcs(s-a)}{(b+c)^2}, \therefore \sum\frac{h_ar_a}{w_a^2} = \sum\frac{2rs}{a}\cdot\frac{rs}{s-a}\cdot\frac{rs}{s-a}\cdot\frac{(b+c)^2}{4bcs(s-a)} =$$

$$= \frac{2r^2s^2}{4\cdot 4Rrs^2}\sum\frac{(b+c)^2}{(s-a)^2} = \frac{r}{8R}\sum\frac{(s+s-a)^2}{(s-a)^2} \overset{(1)}{=} \frac{r}{8R}\sum\frac{s^2+(s-a)^2+2s(s-a)}{(s-a)^2}$$

Now, $\sum(s-b)(s-c) = \sum\{s^2 - s(b+c) + bc\} =$

$$3s^2 - 4s^2 + s^2 + 4Rr + r^2 \overset{(2)}{=} 4Rr + r^2$$

$$(1) \Rightarrow \sum\frac{h_ar_a}{w_a^2} = \frac{rs^2}{8Rr^4s^2}\sum(s-b)^2(s-c)^2 + \frac{3r}{8R} + \frac{2rs}{8R\cdot r^2s}\sum(s-b)(s-c) =$$

$$= \frac{1}{8Rr^3}\left[\left\{\sum(s-b)(s-c)^2 - 2r^2s\{\sum(s-a)\}\right\}\right] + \frac{3r}{8R} + \frac{1}{4Rr}(4Rr + r^2) \text{ (by (2))}$$

$$= \frac{r^{2(4Rr+r)^2} - 2r^2s^2}{8Rr^3} + \frac{3r}{8R} + \frac{4R+r}{4R} \text{ (by (2))} = \frac{(4R+r)^2 - 2s^2 + 3r^2 + 8Rr + 2r^2}{8Rr} \geq 0$$

$$\Leftrightarrow 16R^2 - 8Rr + 6r^2 \geq 2s^2 \Leftrightarrow s^2 \leq 8R^2 - 4Rr + 3r^2. \text{ Now, } s^2 \overset{Gerretsen}{\leq}$$

$$\leq 4R^2 + 4Rr + 3r^2 \overset{?}{\leq} 8R^2 - 4Rr + 3r^2 \Leftrightarrow 4R^2 \overset{?}{\geq} 8Rr \Leftrightarrow R \overset{?}{\geq} 2r \rightarrow \text{true (Euler)}$$

PROBLEM 1.153- ***Solution and generalizations by Marin Chirciu – Romania***

We prove the following lemma: Lemma:

2) In ΔABC:

$$\frac{l_b l_c}{\sin\frac{A}{2}}+\frac{l_c l_a}{\sin\frac{B}{2}}+\frac{l_a l_b}{\sin\frac{C}{2}}=\frac{32s^2Rr}{s^2+r^2+2Rr}$$

Proof: Using $l_a=\frac{2bc}{b+c}\cos\frac{A}{2}, 2\sin\frac{A}{2}\cos\frac{A}{2}=\sin A, \frac{a}{\sin A}=2R$, *we obtain:*

$$\sum\frac{l_b l_c}{\sin\frac{A}{2}}=\sum\frac{\frac{2ac}{a+c}\cos\frac{B}{2}\cdot\frac{2ab}{a+b}\cos\frac{C}{2}}{\sin\frac{A}{2}}=\sum\frac{\frac{8a^2bc}{(a+b)(a+c)}\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=$$

$$=8abc\prod\cos\frac{A}{2}\sum\frac{a}{(a+b)(a+c)\sin A}=32Rrs\cdot\frac{s}{4R}\cdot 2R\sum\frac{1}{(a+b)(a+c)}=$$

$$=16Rrs^2\cdot\frac{2}{s^2+r^2+2Rr}=\frac{32s^2Rr}{s^2+r^2+2Rr}$$

Let's get back to the main problem:

Using the Lemma, the inequality that we have to prove can be written:

$$\frac{32s^2Rr}{s^2+r^2+2Rr}\le\frac{3}{2}\sqrt{3abc(a+b+c)}\Leftrightarrow\left(\frac{32s^2Rr}{s^2+r^2+2Rr}\right)^2\le\frac{9}{4}\cdot 3abc(a+b+c)\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{32s^2Rr}{s^2+r^2+2Rr}\right)^2\le\frac{9}{4}\cdot 12Rrs\cdot 2s\Leftrightarrow\left(\frac{32s^2Rr}{s^2+r^2+2Rr}\right)^2\le 54Rrs^2\Leftrightarrow$$

$$\Leftrightarrow 512s^2Rr\le 27(s^2+r^2+2Rr)^2\Leftrightarrow s^2(27s^2+54r^2-404Rr)+27r^2(2R+r)^2\ge 0$$

We distinguish the following cases:

Case 1). If $(27s^2+54r^2-404Rr)\ge 0$, *the inequality is obvious.*

Case 2). If $(27s^2+54r^2-404Rr)<0$, *we write the inequality:*

$27r^2(2R+r)^2\ge s^2(404Rr-54r^2-27s^2)$, *which follows from Gerretsen's inequality:* $16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$27r^2(2R+r)^2\ge(4R^2+4Rr+3r^2)\big(404Rr-54r^2-27(16Rr-5r^2)\big)\Leftrightarrow$$

$\Leftrightarrow 28R^3-26R^2r-33Rr^2-54r^3\ge 0\Leftrightarrow(R-2r)(28R^2+30Rr+27r^2)\ge 0$, *obviously from Euler's inequality. Equality holds if and only if the triangle is equilateral.*

Remark: We can prove that:

3) In ΔABC:

$$\frac{l_b l_c}{\sin\frac{A}{2}}+\frac{l_c l_a}{\sin\frac{B}{2}}+\frac{l_a l_b}{\sin\frac{C}{2}}\le\frac{27R^2}{2}$$

Proposed by Marin Chirciu – Romania

Solution: Using the Lemma, the inequality can be written:

$$\frac{32s^2Rr}{s^2+r^2+2Rr}\le\frac{27R^2}{2}\Leftrightarrow s^2(27R-64r)+27Rr(2R+r)\ge 0.$$

We distinguish the cases:

Case 1). If $(27R-64r)\ge 0$, *inequality is obvious.*

Case 2). If $(27R-64r)<0$, *inequality can be rewritten:*

$27Rr(2R+r)\ge s^2(64r-27R)$, *which follows from Gerretsen's inequality:*

$s^2 \le 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$27Rr(2R + r) \ge 9(4R^2 + 4Rr + 3r^2)(64r - 27R) \Leftrightarrow 54R^3 - 47R^2r - 74Rr^2 - 96r^3 \ge 0 \Leftrightarrow$

$\Leftrightarrow (R - 2r)(54R^2 + 61Rr + 48r^2) \ge 0$, *obviously Euler's inequality.*

Equality holds if and only if the triangle is equilateral.Remark: Let's highlight an inequality having an opposite sense.

4) In ΔABC*:*

$$\frac{l_b l_c}{\sin\frac{A}{2}} + \frac{l_c l_a}{\sin\frac{B}{2}} + \frac{l_a l_b}{\sin\frac{C}{2}} \ge 27Rr$$

Proposed by Marin Chirciu – Romania

Solution: Using the Lemma, the inequality can be written: $\frac{32s^2Rr}{s^2+r^2+2Rr} \ge 27Rr \Leftrightarrow$

$\Leftrightarrow 5s^2 \ge 27r(2R + r)$, *which follows from Gerretsen's inequality:* $s^2 \ge 16Rr - 5r^2$. *It remains to prove that:*

$5(16Rr - 5r^2) \ge 27r(2R + r) \Leftrightarrow 26Rr \ge 52r^2 \Leftrightarrow R \ge 2r$ *(Euler's inequality)*

Remark: We write the double inequality:

5) In ΔABC*:* $27Rr \le \frac{l_b l_c}{\sin\frac{A}{2}} + \frac{l_c l_a}{\sin\frac{B}{2}} + \frac{l_a l_b}{\sin\frac{C}{2}} \le \frac{27R^2}{2}$

Proposed by Marin Chirciu – Romania

Solution: See inequalities 3) and 4).Equality holds if and only if the triangle is equilateral.

PROBLEM 1.154-*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum \frac{r_a r_b}{\sin\frac{A}{2}\sin\frac{B}{2}} = \sum \frac{s^2 \tan\frac{A}{2}\tan\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = s^2 \sum \frac{\cos\frac{C}{2}}{\left(\prod \cos\frac{A}{2}\right)} = s^2 \sum \frac{\cos\frac{C}{2}}{\left(\frac{s}{4R}\right)} =$$

$$= 2Rs \sum \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\cos\frac{A-B}{2}} \ge 2Rs \sum (\sin A + \sin B)$$

$$\left(\because 0 < \cos\frac{A-B}{2} \le 1 \ as \ -\frac{\pi}{2} < \frac{A-B}{2} < \frac{\pi}{2}\right)$$

$$= 4Rs\left(\sum \sin A\right) = 4Rs\left(\frac{s}{R}\right) = 4s^2 \Rightarrow LHS \overset{(1)}{\ge} 4s^2$$

$$\sum \frac{h_a h_b}{\sin^2\frac{C}{2}} = \sum \frac{4r^2s^2}{ab} \cdot \frac{ab(s-c)}{(s-a)(s-b)(s-c)} = \frac{4r^2s^2}{r^2s}\sum(s-c) = 4s^2 \overset{by\ (1)}{\le} LHS \text{ (Proved)}$$

PROBLEM 1.155-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{a^2}{b^4c\sqrt[3]{4(b^6+1)}} + \frac{b^2}{c^4a\sqrt[3]{4(b^6+1)}} + \frac{c^2}{a^4b\sqrt[3]{4(a^6+1)}} \overset{(1)}{\ge} \frac{3}{2}$$

Firstly, $4(a^6 + 1) \le (3a^2 - 4a + 3)^3 \Leftrightarrow (a-1)^4(23a^2 - 16a + 23) \ge 0 \Leftrightarrow$

$\Leftrightarrow (a-1)^4\{23(a-1)^2 + 30a\} \ge 0 \to$ *true*

$\therefore \sqrt[3]{4(a^6+1)} \overset{(a)}{\le} 3a^2 - 4a + 3$. *Similarly,* $\sqrt[3]{4(b^6+1)} \overset{(b)}{\le} 3b^2 - 4b + 3$ *&*

$\sqrt[3]{4(c^6+1)} \overset{(c)}{\le} 3c^2 - 4c + 3$; *(a), (b), (c)* $\Rightarrow$ *LHS of (1)*

$$\geq \frac{a^2}{b^4c(3b^2-4b+3)}+\frac{b^2}{c^4a(3c^2-4c+3)}+\frac{c^2}{a^4b(3a^2-4a+3)}=$$

$$=\frac{a^3}{ab^4c(3b^2-4b+3)}+\frac{b^2}{bc^4a(3c^2-4c+3)}+\frac{c^2}{a^4bc(3a^2-4a+3)}=$$

$$=\frac{1}{abc}\left\{\frac{\left(\frac{a}{b}\right)^3}{3b^2-4b+3}+\frac{\left(\frac{b}{c}\right)^3}{3c^2-4c+3}+\frac{\left(\frac{c}{a}\right)^3}{3a^2-4a+3}\right\}\geq$$

$$\geq \frac{\left(\frac{a}{b}\right)^3}{3b^2-4b+3}+\frac{\left(\frac{b}{c}\right)^3}{3c^2-4c+3}+\frac{\left(\frac{c}{a}\right)^3}{3a^2-4a+3}\left(\because \sum a=3\overset{A-G}{\geq}3\sqrt[3]{abc}\Rightarrow abc\leq 1\right)$$

$$\underset{(2)}{\overset{Holder}{\geq}}\frac{\left(\sum\frac{a}{b}\right)^3}{3(3\sum a^2-4\sum a+9)}=\frac{\left(\sum\frac{a}{b}\right)^3}{3(3\sum a^2-4\cdot 3+9)}\ \left(\because \sum a=3\right)$$

$$=\frac{\left(\sum\frac{a}{b}\right)^3}{9\left(\sum a^2-\left(\frac{\sum a}{3}\right)^2\right)}\ \left(\because 1=\frac{\sum a}{3}\right)=\frac{\left(\sum\frac{a}{b}\right)^3}{\frac{9}{9}\{9\sum a^2-(\sum a)^2\}}=\frac{\left(\sum\frac{a}{b}\right)^3}{9\sum a^2-(\sum a)^2}$$

Now, $\sum\frac{a}{b}=\sum\frac{a^2}{ab}\overset{Bergstrom}{\geq}\frac{(\sum a)^2}{\sum ab}\Rightarrow\left(\sum\frac{a}{b}\right)^3\overset{(3)}{\geq}\frac{(\sum a)^6}{(\sum ab)^3}$

(2), (3) $\Rightarrow$ *LHS of (1)* $\geq\frac{(\sum a)^6}{(\sum ab)^3\{9\sum a^2-(\sum a)^2\}}\overset{?}{\geq}\frac{3}{2}$

$$\Leftrightarrow 2\left(\sum a\right)^6\overset{?}{\geq}3\left(\sum ab\right)^3\left\{9\sum a^2-\left(\sum a\right)^2\right\}\Leftrightarrow$$

$$\Leftrightarrow\frac{2}{9}\left(\sum a\right)^8\overset{?}{\geq}3\left(\sum ab\right)^3\left\{9\sum a^2-\left(\sum a\right)^2\right\}\left(\because 9=\left(\sum a\right)^2\right)$$

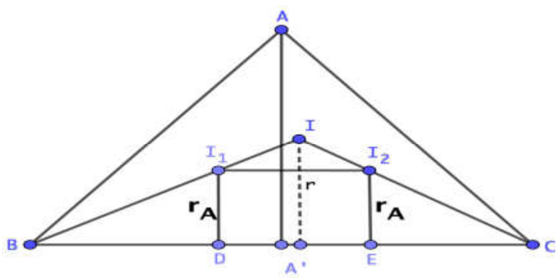
$$\Leftrightarrow 2\left(\sum a^2+2\sum ab\right)^4\overset{?}{\geq}27\left(\sum ab\right)^3\left\{9\sum a^2-\left(\sum a\right)^2\right\}$$

$$\Leftrightarrow 2(x+2y)^4\overset{?}{\geq}27y^3(9x-x-2y)\ \begin{pmatrix}x=\sum a^2,\\ y=\sum ab\end{pmatrix}$$

$$\Leftrightarrow(x+2y)^4\overset{?}{\geq}27y^3(4x-y)\Leftrightarrow t^4+8t^3+24t^2-76t+43\overset{?}{\geq}0\ \left(t=\frac{x}{y}=\frac{\sum a^2}{\sum ab}\right)\Leftrightarrow$$

$$\Leftrightarrow(t-1)^2(t^2+10t+43)\overset{?}{\geq}0\rightarrow true\left(\because t=\frac{\sum a^2}{\sum ab}\geq 1\right)$$

PROBLEM 1.156-*Solution by proposer*



Let be r_A the inradii in $\Delta ABA', ACA'$

$$S = \sigma_{ABA'} + \sigma_{ACA'} = s_{ABA'} \cdot r_A + s_{ACA'} \cdot r_A = r_A(s_{ABA'} + s_{ACA'}) =$$
$$= r_A(s + AA') \Rightarrow r_A(s + AA') = S \quad (1)$$

Let be I_1, I_2 the incenters in $\Delta ABA', ACA'$ and I the incenter in ΔABC

$$\Delta II_1I_2 \sim \Delta IBC \Rightarrow \frac{I_1I_1}{BC} = \frac{r - r_A}{r} \Rightarrow 1 - \frac{r_A}{r} = \frac{I_1I_2}{BC} \Rightarrow \frac{r_A}{r} = 1 - \frac{I_1I_2}{a} \quad (2)$$

Let $D \wedge E$ be the intersection points with side BC of the projections of incenters.

I_1I_2ED is a rectangle $\Rightarrow$

$$I_1I_2 = DE = DA' + A'E = s_{ABA'} - c + s_{ACA'} - b = s - b - c + AA' \quad (3)$$

From (2)+(3)$\Rightarrow \frac{r_A}{r} = 1 - \frac{s - b - c + AA'}{a} = \frac{s - AA'}{a} \Rightarrow r_A = \frac{r}{a}(s - AA') \quad (4)$

From (1)+(4) $\Rightarrow \frac{r}{a}(s - AA')(s + AA') = S \Rightarrow s^2 - AA'^2 = as \Rightarrow$

$$\Rightarrow AA'^2 = s^2 - sa \Rightarrow AA' = \sqrt{s(s-a)} \text{ analogous } BB' = \sqrt{s(s-b)}, CC' = \sqrt{s(s-c)}$$
$$\Rightarrow AA' \cdot BB' \cdot CC'$$

PROBLEM 1.157-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS = \sum\left(4R\sin\frac{A}{2}\cos\frac{A}{2} \cdot \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}} \cdot \frac{b^2c^2}{16R^4}\right) = \frac{1}{4R^3}\sum b^2c^2 \cdot \frac{s(s-a)}{bc} = \frac{s}{4R^3}\sum bc(s-a)$$
$$= \frac{s}{4R^3}\{s(s^2 + 4Rr + r^2) - 12Rrs\} = \frac{s^2}{4R^3}(s^2 - 8Rr + r^2) \overset{Gerretsen}{\geq} \frac{s^2(8Rr - 4r^2)}{4R^3}$$
$$\overset{s \geq 3\sqrt{3}r}{\geq} \frac{27r^2(8Rr - 4r^2)}{4R^3} \overset{?}{\geq} \frac{81r^4}{2R^2(R-r)}$$
$$\Leftrightarrow (4R - 2r)(R - r) \overset{?}{\geq} 3Rr \Leftrightarrow 4R^2 - 9Rr + 2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R - 2r)(4R - r) \overset{?}{\geq} 0$$

true $\because R \overset{Euler}{\geq} 2r$ *(Proved)*

PROBLEM 1.158-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS = \sum\frac{1}{a} + \sum\frac{a^2}{ab^2 + ac^2} \overset{Bergstrom}{\geq} \frac{\sum ab}{abc} + \frac{(\sum a)^2}{\sum a^2b + \sum ab^2} \overset{?}{\geq} 3\sum\frac{1}{b+c}$$
$$\Leftrightarrow \frac{(\sum ab)(\sum a^2b + \sum ab^2) + abc(\sum a)^2}{abc(\sum a^2b + \sum ab^2)} \overset{?}{\geq} \frac{3((\sum a)^2 + \sum ab)}{(a+b)(b+c)(c+a)}$$
$$\Leftrightarrow \sum a^5b^3 + \sum a^3b^5 + abc\left(\sum a^4b + \sum ab^4\right) + 2\sum a^4b^4$$
$$\underset{(1)}{\overset{?}{\geq}} 2abc\left(\sum a^3b^2 + \sum a^2b^3\right) + 2a^2b^2c^2\left(\sum a^2\right)$$

(Simplifying & re-arranging)

$$\because \sum x^2 \geq \sum xy \therefore 2\sum a^4b^4 \overset{(a)}{\geq} 2a^2b^2c^2\left(\sum a^2\right)$$
$$(x = a^2b^2, y = b^2c^2, z = c^2a^2)$$

Again, $abc(\sum a^4b + \sum ab^4) = abc \cdot \sum ab(a^3 + b^3)$

$$\overset{(b)}{\geq} abc\sum ab \cdot ab(a+b) = abc\sum a^2b^2(a+b) = abc\left(\sum a^3b^2 + \sum a^2b^3\right)$$

$$\text{Also, } \sum a^5b^3 + \sum a^3b^5 = \sum c^5(a^3+b^3) \geq \sum c^5 ab(a+b) = abc\{\sum c^4(a+b)\}$$
$$= abc\left(\sum a^4b + \sum ab^4\right) \underset{(c)}{\overset{by\,(b)}{\geq}} abc\left(\sum a^3b^2 + \sum a^2b^3\right)$$
(a)+(b)+(c)⇒ (1) is true

PROBLEM 1.159-*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum a^2h_bh_c = \sum \frac{a^2\cdot ca\cdot ab}{4R^2} = \frac{4Rrs}{4R^2}\left(\sum a^3\right) = \frac{rs}{R}\cdot 2s(s^2-6Rr-3r^2) \leq 4(R+r)^4$$
$$\Leftrightarrow rs^4 - rs^2(6Rr+3r^2) \overset{(1)}{\leq} 2R(R+r)^4$$
$$\text{Now, LHS of (1)} \overset{Rouche}{\leq} rs^2\{2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}-(6Rr+3r^2)\}$$
$$\overset{Rouche}{\leq} r\left\{2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\right\}$$
$$\left(2R^2+4Rr-4r^2+2(R-2r)\sqrt{R^2-2Rr}\right) =$$
$$= r\Big[(2R^2+10Rr-r^2)(2R^2+4Rr-4r^2)$$
$$+4R(R-2r)^3 2(R-2r)\sqrt{R^2-2Rr}(4R^2+14Rr-5r^2)\Big]$$
$$\overset{?}{\leq} 2R(R+r)^4 \Leftrightarrow (R-2r)(R^4+2R^3r+8R^2r^2-19Rr^3+r^4) \underset{(2)}{\overset{?}{\geq}}$$
$$\geq r(R-2r)\sqrt{R^2-2Rr}(4R^2+14Rr-5r^2)$$
$$\because R-2r \overset{Euler}{\geq} 0 \ \&\ R^4+2R^3r+8R^2r^2-19Rr^3+r^4 \overset{Euler}{\geq}$$
$$\geq R^4+R^3r+Rr\cdot 4r^2+8Rr^2\cdot 2r-19Rr^3+r^4 > 0$$
∴ in order to prove (2), it suffices to prove:
$$(R^4+2R^3r+8R^2r^2-19Rr^3+r^4)^2 - R(R-2r)r^2(4R^2+14Rr-5r^2)^2 > 0$$
$$\Leftrightarrow t^8+4t^7+4t^6-86t^5+58t^5+152t^3+72t^2+12t+1 > 0 \quad \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\{(t-2)(t^6+8t^5+32t^2(t^2-4)+98t^2+10t(t^2-4)+32t+160)+684\}$$
$$+729 > 0$$
$$\rightarrow \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow \text{(2) is true} \Rightarrow \text{(1) is true (Proved)}$$

PROBLEM 1.160-*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Given inequality} \Leftrightarrow x\left(\frac{1}{(1+y)^2}+\frac{1}{(1+z)^2}-\frac{1}{1+yz}\right)+y\left(\frac{1}{(1+z)^2}+\frac{1}{(1+x)^2}-\frac{1}{1+zx}\right)+$$
$$+z\left(\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}-\frac{1}{1+xy}\right) \overset{(a)}{\geq} 0. \text{ Now, } \frac{1}{(1+y)^2}+\frac{1}{(1+z)^2}-\frac{1}{1+yz} \geq 0 \Leftrightarrow \frac{(1+z)^2+(1+y)^2}{(1+y)^2(1+z)^2} \geq \frac{1}{1+yz}$$
$$\Leftrightarrow (1+yz)\{(1+z)^2+(1+y)^2\} \geq (1+y)^2(1+z)^2 \ (\because 1+yz \geq 1 > 0)$$
$$\Leftrightarrow y^3z+yz^3-y^2z^2-2yz+1 \geq 0 \Leftrightarrow (y^3z+yz^3-2y^2z^2)+(y^2z^2-2yz+1) \geq 0$$
$$\Leftrightarrow yz(y-z)^2+(yz-1)^2 \geq 0 \rightarrow \text{true} \because yz(y-z)^2 \geq 0 \ (\because yz \geq 0 \text{ as } y,z \geq 0)$$
$$\& \ (yz-1)^2 \geq 0 \Rightarrow yz(y-z)^2+(yz-1)^2 \geq 0 \therefore \frac{1}{(1+y)^2}+\frac{1}{(1+z)^2}-\frac{1}{1+yz} \geq 0$$
$$\Rightarrow x\left(\frac{1}{(1+y)^2}+\frac{1}{(1+z)^2}-\frac{1}{1+yz}\right) \overset{(1)}{\geq} 0 \ (\because x \geq 0)$$

$$\text{Similarly, } y\left(\frac{1}{(1+z)^2}+\frac{1}{(1+x)^2}-\frac{1}{1+zx}\right)\overset{(2)}{\geq} 0 \ \& \ z\left(\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}-\frac{1}{1+xy}\right)\overset{(3)}{\geq} 0$$

(1)+(2)+(3)⇒ (a) is true (Proved)

PROBLEM 1.161-*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Firstly, } \forall x,y,z,w>0, x^3+y^3+z^3+w^3\geq\frac{1}{4}(x^2+y^2+z^2+w^2)(x+y+z+w)$$

$$\Leftrightarrow \{(x^3+y^3)-(x^3y+xy^2)\}+\{(x^3+z^3)-(x^2z+xz^2)\}+$$

$$+\{(x^3+w^3)-(x^2w+xw^2)\}+\{(y^3+z^3)-(y^2z+yz^2)\}+$$

$$+\{(y^3+w^3)-(y^2w+yw^2)\}+\{(z^3+w^3)-(z^2w+zw^2)\}\geq 0\to \text{true}$$

$$\because \forall x,y\geq 0, x^3+y^3\geq xy(x+y) \text{ etc}$$

$$\Rightarrow x^3+y^3+z^3+w^3\overset{(a)}{\geq}\frac{1}{4}(x^2+y^2+z^2+w^2)(x+y+z+w)$$

$$(a)\Rightarrow a^3+b^3+c^3+1\overset{(i)}{\geq}\frac{1}{4}(a^2+b^2+c^2+1)(a+b+c+1),$$

$$b^3+c^3+d^3+1\overset{(ii)}{\geq}\frac{1}{4}(b^2+c^2+d^2+1)(b+c+d+1),$$

$$c^3+d^3+a^3+1\overset{(iii)}{\geq}\frac{1}{4}(c^2+d^2+a^2+1)(c+d+a+1),$$

$$d^3+a^3+b^3+1\overset{(iv)}{\geq}\frac{1}{4}(d^2+a^2+b^2+1)(d+a+b+1)$$

$$\text{(i), (ii), (iii), (iv)}\Rightarrow \text{LHS of (1)}\leq\frac{4}{a+b+c+1}+\frac{4}{b+c+d+1}+\frac{4}{c+d+a+1}+\frac{4}{d+a+b+1}\overset{?}{\leq}4$$

$$\Leftrightarrow (b+c+d+1)(c+d+a+1)(d+a+b+1)+$$

$$+(c+d+a+1)(d+a+b+1)(a+b+c+1)$$

$$+(d+a+b+1)(a+b+c+1)(b+c+d+1)+$$

$$+(a+b+c+1)(b+c+d+1)(c+d+a+1)\leq$$

$$\leq(a+b+c+1)(b+c+d+1)(c+d+a+1)(d+a+b+1)$$

$$\Leftrightarrow a^3(b+c+d)+b^3(a+c+d)+c^3(a+b+d)+d^3(a+b+c)+$$

$$+2(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2)+$$

$$+4(a^2bc+a^2bd+a^2cd+b^2ac+b^2ad+b^2cd+c^2ab+c^2ad+c^2bd+d^2ab+d^2ac+d^2bc)+$$

$$+9abcd\overset{(2)}{\geq}3(a^2+b^2+c^2+d^2)+7(ab+ac+ad+bc+bd+cd)+$$

$$+6(a+b+c+d)+3$$

$$\text{Now, } a^3(b+c+d)+b^3(a+c+d)+c^3(a+b+d)+d^3(a+b+c)\geq$$

$$\overset{A-G}{\geq}3\left(a^3(\sqrt[3]{bcd})+b^3(\sqrt[3]{cad})+c^3(\sqrt[3]{bcd})+d^3(\sqrt[3]{abc})\right)\geq$$

$$\overset{abcd\geq 1}{\geq}3\left(a^3\sqrt[3]{\frac{1}{a}}+b^3\sqrt[3]{\frac{1}{b}}+c^3\sqrt[3]{\frac{1}{c}}+d^3\sqrt[3]{\frac{1}{d}}\right)$$

$$=3\left(a^2\cdot a^{\frac{2}{3}}+b^2\cdot b^{\frac{2}{3}}+c^2\cdot c^{\frac{2}{3}}+d^2\cdot d^{\frac{2}{3}}\right)$$

$$\overset{Chebyshev}{\geq}\frac{3}{4}\left(\sum a^2\right)\left(\sum a^{\frac{2}{3}}\right) \left(\because \text{WLOG, if we assume } a\geq b\geq c\geq d \text{ then, } a^{\frac{2}{3}}\geq b^{\frac{2}{3}}\geq c^{\frac{2}{3}}\geq d^{\frac{2}{3}}\right)$$

$$\overset{A-G}{\geq} \frac{3}{4}\left(\sum a^2\right)\left(4\sqrt[4]{(abcd)^{\frac{2}{3}}}\right) = 3\left(\sum a^2\right)(abcd)^{\frac{1}{6}} \geq 3\sum a^2 \ (\because abcd \geq 1)$$

$$\Rightarrow a^3(b+c+d) + b^3(a+c+d) + c^3(a+b+d) + d^3(a+b+c) \overset{(v)}{\geq} 3\sum a^2$$

Also, $a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 +$

$$+2(a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd + c^2ab + c^2ad + c^2bd + d^2ab + d^2ac + d^2bc) +$$
$$+6abcd = (ab+ac+ad+bc+bd+cd)^2$$

$$\overset{A-G}{\geq} 6\sqrt[6]{a^3b^3c^3d^3}\ (ab+ac+ad+bc+bd+cd)$$

$$= 6(abcd)^{\frac{1}{2}}(ab+ac+ad+bc+bd+cd) \overset{abcd\geq 1}{\geq} 6(ab+ac+ad+bc+bd+cd)$$

$$\Rightarrow a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^3 +$$
$$+2(a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd + c^2ab + c^2ad + c^2bd + d^2ab + d^2ac + d^2bc) + 6abcd$$
$$\overset{(vi)}{\geq} 6(ab+ac+ad+bc+bd+cd)$$

Now, by A-G, $a^2b^2 + a^2c^2 \geq 2a^2bc, a^2b^2 + a^2d^2 \geq 2a^2bd,$
$a^2c^2 + a^2d^2 \geq 2a^2cd, b^2a^2 + b^2c^2 \geq 2b^2ac, b^2a^2 + b^2d^2 \geq 2b^2ad,$
$b^2c^2 + b^2d^2 \geq 2b^2cd, c^2a^2 + c^2b^2 \geq 2c^2ab, c^2a^2 + c^2d^2 \geq 2c^2ad,$
$c^2b^2 + c^2d^2 \geq 2c^2bd, d^2a^2 + d^2b^2 \geq 2d^2ab, d^2a^2 + d^2c^2 \geq 2d^2ac$ &
$d^2b^2 + d^2c^2 \geq 2d^2bc$

Adding the last 12 inequalities, we have

$4P \overset{(b)}{\geq} 2Q$, *where* $P = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2$ &
$Q = a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd + c^2ab + c^2ad + c^2bd +$
$+d^2ab + d^2ac + d^2bc$

Again, $P \overset{A-G}{\geq} 6\sqrt[6]{a^6b^6c^6d^6} = 6abcd \Rightarrow P \overset{(b)}{\geq} 6abcd$

(a)+(b)$\Rightarrow 6P \geq 2Q + 6abcd + P = (ab+ac+ad+bc+bd+cd)^2$

$$\overset{A-G}{\geq} 6\sqrt[6]{a^3b^3c^3d^3}(ab+ac+ad+bc+bd+cd)$$

$$= 6\sqrt{abcd}(ab+ac+ad+bc+bd+cd) \overset{abcd\geq 1}{\geq} 6(ab+ac+ad+bc+bd+cd)$$

$$\Rightarrow a^2b^2 + a^2c^2 + a^2d^3 + b^2c^2 + b^2d^2 + c^2d^2 \overset{(vii)}{\geq} ab+ac+ad+bc+bd+cd$$

Moreover, $2Q = [a^2(bc+bd+cd) + b^2(ac+ad+cd) + c^2(ab+ad+bd) +$
$d^2(ab+ac+bc)]\cdot 2$

$$\overset{A-G}{\geq} 6\left(a^2\left(\sqrt[3]{b^2c^2d^2}\right) + b^2\left(\sqrt[3]{a^2c^2d^2}\right) + c^2\left(\sqrt[3]{a^2b^2d^2}\right) + d^2\left(\sqrt[3]{a^2b^2c^2}\right)\right)$$

$$\overset{abcd\geq 1}{\geq} 6\left(a^2\cdot\sqrt[3]{\frac{1}{a^2}} + b^2\cdot\sqrt[3]{\frac{1}{b^2}} + c^2\cdot\sqrt[3]{\frac{1}{c^2}} + d^2\cdot\sqrt[3]{\frac{1}{d^2}}\right)$$

$$= 6\left(a\cdot a^{\frac{1}{3}} + b\cdot b^{\frac{1}{3}} + c\cdot c^{\frac{1}{3}} + d\cdot d^{\frac{1}{3}}\right) \overset{Chebyshev}{\geq} \frac{6}{4}\left(\sum a\right)\left(\sum a^{\frac{1}{3}}\right)$$

($\because$ if WLOG, we assume $a \geq b \geq c \geq d$, then $a^{\frac{1}{3}} \geq b^{\frac{1}{3}} \geq c^{\frac{1}{3}} \geq d^{\frac{1}{3}}$)

$$\overset{A-G}{\geq} \frac{6}{4}\left(\sum a\right)\left\{4\sqrt[4]{(abcd)^{\frac{1}{3}}}\right\} = 6\left(\sum a\right)(abcd)^{\frac{1}{12}} \overset{abcd\geq 1}{\geq} 6\sum a$$

$$\Rightarrow 2(a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd + c^2ab + c^2ad + c^2bd + d^2ab + d^2ac + d^2bc) \overset{(viii)}{\geq} 6\sum a$$

$$\&\ lastly,\ 3abcd \overset{(ix)}{\geq} 3$$

(v)+(vi)+(vii) +(viii) +(ix)⇒(2) is true ⇒(1) is true (proved)

PROBLEM 1.162-*Solution by Soumava Chakraborty-Kolkata-India*

WLOG, we may assume $x \geq y \geq z$. Then, $\frac{1}{\sqrt{1+yz}} \geq \frac{1}{\sqrt{1+zx}} \geq \frac{1}{\sqrt{1+xy}}$

$$\therefore \sum \frac{x}{\sqrt{1+yz}} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum x\right)\sum \frac{1}{\sqrt{1+yz}} \overset{Bergstrom}{\geq} \frac{\left(\frac{1}{3}\right)\cdot 9}{\sum\sqrt{1+yz}}$$

$$\overset{CBS}{\geq} \frac{3}{\sqrt{3}\sqrt{3+\sum xy}} \overset{(\sum x)^2 \geq 3\sum xy}{\geq} \frac{3}{\sqrt{3}\sqrt{3+\frac{(\sum x)^2}{3}}} = \frac{3}{\sqrt{3}\sqrt{3+\frac{1}{3}}} = \frac{3}{\sqrt{10}}$$

$\therefore \sum \frac{x}{\sqrt{1+yz}} \geq \frac{3}{\sqrt{10}}$, equality at $x = y = z = \frac{1}{3}$

Again, $\sum \frac{x}{\sqrt{1+yz}} \leq 1 \Leftrightarrow \sum \frac{x}{\sqrt{1+yz}} \leq \sum x \Leftrightarrow \sum x\left(1-\frac{1}{\sqrt{1+yz}}\right) \geq 0 \Leftrightarrow \sum x\left(\frac{\sqrt{1+yz}-1}{\sqrt{1+yz}}\right) \overset{(a_1)}{\geq} 0$

$\because 1+yz \geq 1\ (\because yz \geq 0) \therefore \sqrt{1+yz}-1 \geq 0$

Also, $x \geq 0 \Rightarrow \frac{x(\sqrt{1+yz}-1)}{\sqrt{1+yz}} \overset{(i)}{\geq} 0$. Similarly, $\frac{y(\sqrt{1+zx}-1)}{\sqrt{1+zx}}, \frac{z(\sqrt{1+xy}-1)}{\sqrt{1+xy}} \overset{(ii),(iii)}{\geq} 0$

(i)+(ii)+(iii) ⇒(a_1) is true

$\therefore \sum \frac{x}{\sqrt{1+yz}} \leq 1$, equality when $x = 0\ \ (y,z \neq 0)$ or $y = 0 (z,x \neq 0)$ or $z = 0\ (x,y \neq 0)$ or $x = y = 0\ \ (z = 1)$ or $y = z = 0\ \ (x = 1)$ or $z = x = 0\ \ (y = 1)$.

Again, $\because x \geq y \geq z, \therefore \frac{1}{\sqrt{1+y+z}} \geq \frac{1}{\sqrt{1+z+x}} \geq \frac{1}{\sqrt{1+x+y}}$

$$\therefore \sum \frac{x}{\sqrt{1+y+z}} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum x\right)\sum \frac{1}{\sqrt{1+y+z}} \overset{Bergstrom}{\geq} \frac{\left(\frac{1}{3}\right)\cdot 9}{\sum\sqrt{1+y+z}}$$

$$\overset{CBS}{\geq} \frac{3}{\sqrt{3}\sqrt{3+2\sum x}} = \frac{3}{\sqrt{3}\sqrt{5}} = \sqrt{\frac{3}{5}}$$

$\therefore \sum \frac{x}{\sqrt{1+y+z}} \geq \sqrt{\frac{3}{5}}$ equality at $x = y = z = \frac{1}{3}$

Moreover, $\sum \frac{x}{\sqrt{1+y+z}} \leq 1 \Leftrightarrow \sum \frac{x}{\sqrt{1+y+z}} \leq \sum x \Leftrightarrow$

$$\Leftrightarrow \sum x\left(1-\frac{1}{\sqrt{1+y+z}}\right) \geq 0 \Leftrightarrow \sum x\left(\frac{\sqrt{1+y+z}-1}{\sqrt{1+y+z}}\right) \overset{(b_1)}{\geq} 0$$

$\because y+z \geq 0 \therefore 1+y+z \geq 1 \Rightarrow \sqrt{1+y+z}-1 \geq 0$

& $\because x \geq 0 \therefore x\left(\frac{\sqrt{1+y+z}-1}{\sqrt{1+y+z}}\right) \overset{(iv)}{\geq} 0$

Similarly, $y\left(\frac{\sqrt{1+z+x}-1}{\sqrt{1+z+x}}\right) \overset{(v)}{\geq} 0, z\left(\frac{\sqrt{1+x+y}-1}{\sqrt{1+x+y}}\right) \overset{(vi)}{\geq} 0$

(iv)+(v)+(vi)⇒ (b_1) is true

$$\therefore \sum \frac{x}{\sqrt{1+y+z}} \le 1, \text{ equality when } x = y = 0, z = 1 \text{ or } y = z = 0, x = 1 \text{ or}$$
$$z = x = 0, y = 1 \text{ (Hence proved)}$$

PROBLEM 1.163-*Solution by Soumava Chakraborty-Kolkata-India*

$$\because m_a \le R(1 + \cos A) \text{ etc, for acute-angled triangle,}$$
$$\therefore LHS \le \sum \frac{2R\cos^2\frac{A}{2}}{a} = \sum \frac{2Rs(s-a)}{abc} = \frac{2Rs}{4Rrs}\sum(s-a) = \frac{s}{2r} = \frac{2s}{4r} = \frac{a+b+c}{4r} \quad \text{(proved)}$$

PROBLEM 1.164-*Solution by Soumava Chakraborty-Kolkata-India*

$$\forall a, b \ge 0, (a+b)^2 \ge 4ab \ \ (\because (a-b)^2 \ge 0) \Rightarrow |a+b| \ge 2\sqrt{ab} \Rightarrow a + b \overset{(1)}{\ge} 2\sqrt{ab}$$
$$(\because a + b \ge 0 \ as \ a, b \ge 0)$$
$$\text{Also, } \forall a, b \ge 0, a^2 + b^2 \ge 2ab \Rightarrow \sqrt{a^2+b^2} \overset{(2)}{\ge} \sqrt{2ab}$$
$$\text{(1) + (2)} \Rightarrow LHS \ge \left(\left(2+\sqrt{2}\right)\sqrt{ab}\right)^2 = \left(6+4\sqrt{2}\right)ab \overset{?}{\ge} 6\sqrt{3}ab$$
$$\Leftrightarrow \left(6+4\sqrt{2}-6\sqrt{3}\right)ab \overset{?}{\ge} 0 \rightarrow \text{true} \because ab \ge 0 \ \& \ 6+4\sqrt{2}-6\sqrt{3} > 0 \text{ (Proved)}$$

PROBLEM 1.165-*Solution by Soumava Chakraborty-Kolkata-India*

$$\because a, b \ge 0; 2(a^2+b^2) \ge (a+b)^2 \ \ (\because (a-b)^2 \ge 0)$$
$$\Rightarrow \sqrt{a^2+b^2} \overset{(1)}{\ge} \frac{|a+b|}{\sqrt{2}} = \frac{a+b}{\sqrt{2}} \quad (\because a+b \ge 0 \ as \ a, b \ge 0)$$
$$\text{Similarly, } \forall b, c \ge 0, \sqrt{b^2+c^2} \overset{(2)}{\ge} \frac{b+c}{\sqrt{2}} \ \& \ \sqrt{c^2+a^2} \overset{(3)}{\ge} \frac{c+a}{\sqrt{2}}$$
$$\text{(1)+(2)+(3)} \Rightarrow RHS \ge \left(\frac{3\sqrt{3}-2}{\sqrt{2}}\right)\left(2\sum a\right) = \left(3\sqrt{6}-2\sqrt{2}\right)\left(\sum a\right) \overset{?}{\ge} 4\sum a$$
$$\Leftrightarrow \left(3\sqrt{6}-2\sqrt{2}-4\right)\left(\sum a\right) \overset{?}{\ge} 0 \rightarrow \text{true} \because \sum a \ge 0 \ \ (\text{as } a, b, c \ge 0) \ \&$$
$$3\sqrt{6}-2\sqrt{2}-4 > 0 \text{ (Hence proved)}$$

PROBLEMS FROM SENIORS-SOLUTIONS

PROBLEM 2.001-*Solution by Soumava Chakraborty-Kolkata-India*

$$\forall\, a,b,c \in (0,\infty), \sum \frac{2a+3c}{a+2b+5c} \le \frac{273\sum ab + 87\sum a^2}{64\left(\sum\sqrt{ab}\right)^2}$$

$$\because \left(\sum\sqrt{ab}\right)^2 \overset{CBS}{\le} 3\left(\sum ab\right), \therefore \frac{273\sum ab + 87\sum a^2}{64\left(\sum\sqrt{ab}\right)^2} \ge \frac{87\sum a^2 + 273\sum ab}{129\sum ab} \ge$$

$$\overset{?}{\ge} \sum \frac{2a+3c}{a+2b+5c} \Leftrightarrow \left(87\sum a^2 + 273\sum ab\right)(a+2b+5c)(b+2c+5a)(c+2a+5b) -$$

$$-192\left(\sum ab\right)\begin{Bmatrix}(2a+3c)(b+2c+5a)(c+2a+5b)+\\(2b+3a)(c+2a+5b)(a+2b+5c)+\\(2c+3b)(a+2b+5c)(b+2c+5a)\end{Bmatrix} \overset{?}{\ge} 0 \Leftrightarrow$$

$$\Leftrightarrow 870\sum a^5 + 1827\sum a^4b + 2871\sum ab^4 + 3594\sum a^2b^3 + 4272abc\left(\sum a^2\right) \underset{(1)}{\overset{?}{\ge}}$$

$$\ge 366\sum a^3b^3 + 13068abc\left(\sum ab\right). \text{ Now, } \sum ab^4 = abc\left(\frac{b^2}{c}+\frac{c^3}{a}+\frac{a^3}{b}\right) = abc\left(\frac{b^4}{bc}+\frac{c^4}{ca}+\frac{a^4}{ab}\right) \ge$$

$$\overset{Bergstrom}{\ge} abc\cdot\frac{\left(\sum a^2\right)^2}{\left(\sum ab\right)} \ge abc\frac{\left(\sum ab\right)^2}{\sum ab} = abc\left(\sum ab\right) \therefore 2871\sum ab^4 \overset{(a)}{\ge} 2871abc\left(\sum ab\right)$$

$$Also, \sum a^4b = abc\left(\frac{a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\right) = abc\left(\frac{a^4}{ca}+\frac{b^4}{ab}+\frac{c^4}{bc}\right) \overset{Bergstrom}{\ge} abc\frac{\left(\sum a^2\right)^2}{\sum ab} \ge abc\left(\sum ab\right)$$

$$\therefore 1827\sum a^4b \overset{(b)}{\ge} 1827abc\left(\sum ab\right). \text{ Again, } 4272abc\left(\sum a^2\right) \overset{(c)}{\ge} 4272abc\left(\sum ab\right)$$

(a)+(b)+(c) ⇒ LHS of (1)

$$\ge 870\sum a^5 + 3594\sum a^2b^3 + 8970abc\left(\sum ab\right) \overset{?}{\ge} 366\sum a^3b^2 + 13068abc\left(\sum ab\right) \Leftrightarrow$$

$$\Leftrightarrow 870\sum a^5 + 3594\sum a^2b^3 \underset{(2)}{\overset{?}{\ge}} 366\sum a^3b^2 + 4098abc\left(\sum ab\right)$$

Now:

$$\sum(a^5+b^5) \overset{Chebyshev}{\ge} \frac{1}{2}\sum(a^2+b^2)(a^3+b^3) \ge \frac{1}{2}\sum(2ab)\,ab(a+b) = \sum a^2b^2(a+b) =$$

$$= \sum a^3b^2 + \sum a^2b^3 \Rightarrow 2\sum a^5 \ge \sum a^3b^2 + \sum a^2b^3 \Rightarrow 732\sum a^5 \overset{(d)}{\ge} 366\sum a^3b^2 +$$

$$+366\sum a^2b^3$$

$$\text{(d)} \Rightarrow \text{LHS of (2)} \ge 138\sum a^5 + 366\sum a^3b^2 + 366\sum a^2b^3 + 3594\sum a^2b^3 \overset{?}{\ge}$$

$$\ge 366\sum a^3b^2 + 4098abc\left(\sum ab\right) \Leftrightarrow 138\sum a^5 + 3960\sum a^2b^3 \overset{?}{\ge} 4098abc\left(\sum ab\right)$$

$$\Leftrightarrow 23\sum a^5 + 660\sum a^2b^3 \underset{(3)}{\overset{?}{\ge}} 683abc\left(\sum ab\right). \text{ Now, } \sum a^2b^3 = abc\left(\frac{ab^2}{c}+\frac{bc^2}{a}+\frac{ca^2}{b}\right) =$$

$$= abc\left(\frac{a^2b^2}{ca}+\frac{b^2c^2}{ab}+\frac{c^2a^2}{bc}\right) \overset{Bergstrom}{\ge} abc\cdot\frac{\left(\sum ab\right)^2}{\sum ab} = abc\left(\sum ab\right) \Rightarrow$$

$$\Rightarrow 660\sum a^2b^3 \Rightarrow\overset{(e)}{\ge} 660abc\left(\sum ab\right). \text{ Now, } \sum(a^5+b^5) \ge \sum a^3b^2 + \sum a^2b^3 \text{ (proved earlier)}$$

$$= \sum a^3(b^2+c^2) \overset{A-G}{\geq} 2abc\left(\sum a^2\right) \geq 2abc\left(\sum ab\right) \Rightarrow \sum a^5 \geq abc\left(\sum ab\right) \Rightarrow$$

$$\Rightarrow 23\sum a^5 \overset{(f)}{\geq} 23abc\left(\sum ab\right)$$

(e)+(f)⇒ (3) is true (proved)

PROBLEM 2.002-*Solution by proposer*

$$f:(0,\infty)\to\mathbb{R}, f(x) = cosx - x - ln(x+1)$$

$$f'(x) = sinx - 1 + \frac{1}{x+1} = sinx - \frac{x}{x+1} \leq 0, \forall x \in \left[0,\frac{\pi}{2}\right)$$

$$f - decreasing \to f(A) \leq f(0) = 1, f(B) \leq 1, f(C) \leq 1$$

$$By\ adding:\ f(A)+f(B)+f(C) \leq 3$$

$$cosA - A + ln(A+1) + cosB - B + ln(B+1) + cosC - C + ln(C+1) \leq 3$$

$$cosA + cosB + cosC + ln\big((A+1)(B+1)(C+1)\big) \leq 3 + (A+B+C) = 3+\pi$$

PROBLEM 2.003-*Solution by Tin Lu-Binh Son-Quang Ngai-VietNam*

$$\forall\, x_0 \in [0,1]\ and\ f(x)\ is\ a\ differentiable;\ convex,\ we\ have:$$

$$f(x) \geq f'(x)(x-x_0) + f(x_0)$$

$$f(x) \geq f'(a)(x-a) + f(a)$$

$$f(x) \geq f'(b)(x-b) + f(b)$$

$$f(x) \geq f'(c)(x-c) + f(c)$$

$$3f(x) \geq x\sum f'(a) - af(a) + \sum f(a) = x - q + \sum f(a)$$

$$\Rightarrow 3\int_0^1 f(x)\,dx \geq \int_0^1 \left[(x-2) + \sum f(a)\right]dx \Leftrightarrow 3\int_0^1 f(x)\,dx \geq -\frac{3}{2} + \sum f(a)$$

$$\Leftrightarrow \frac{1}{2} + \int_0^1 f(x)\,dx \geq \frac{1}{3}\sum f(a)$$

PROBLEM 2.004-*Solution by Bao Ngo Minh Ngoc – Gia Lai Province– VietNam*

$$Use\ AM-GM\ we\ have:\ x + \frac{1}{x^3} = \frac{x}{3}+\frac{x}{3}+\frac{x}{3}+\frac{1}{x^3} \geq 4\sqrt[4]{\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{1}{x^3}} = \frac{4}{\sqrt{3\sqrt{3}}}$$

$$\Rightarrow x+y+z+\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3} \geq 4\sum\sqrt[4]{\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{1}{x^3}} = \frac{12}{\sqrt{3\sqrt{3}}}$$

PROBLEM 2.005-*Solution by proposer*

First we show that $2(x^3+1)^4 \geq (x^4+1)(x^2+1)^4$ *for all* $x \geq 0$. *But* $(x^2+1)^4 \leq (x+1)^2(x^3+1)^2$ *and we are left with the inequality* $2(x^3+1)^2 \geq (x+1)^2(x^4+1) \Leftrightarrow 2(x^2-x+1)^2 \geq x^4+1 \Leftrightarrow (x-1)^4 \geq 0$ *which follows.*

Therefore $\frac{2}{x^4+1} \geq \left(\frac{x^2+1}{x^3+1}\right)^4$. *If* $x = \sqrt[4]{k^2-1}$ *then* $\left(\frac{1+(k^2-1)^{\frac{1}{2}}}{1+(k^2-1)^{\frac{3}{4}}}\right)^4 \leq \frac{2}{k^2}$

$$therefore\ \sum_{k=1}^{\infty}\left(\frac{1+(k^2-1)^{\frac{1}{2}}}{1+(k^2-1)^{\frac{3}{4}}}\right)^4 \le 2\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{3}$$

PROBLEM 2.006-*Solution by proposer*

$$By\ Jensen's\ inequality:\ \frac{e^x f(e^x)+nx^{n-1}f(x^n)}{e^x+nx^{n-1}} \ge f\left(\frac{e^{2x}+nx^{2n-1}}{e^x+nx^{n-1}}\right)$$

$$\int_0^1 e^x f(e^x)\,dx=\int_1^e f(t)\,dt$$

$$\int_0^1 nx^{n-1}f(x^n)\,dx=\int_0^1 f(t)\,dt$$

$$\int_0^1 f(t)\,dt+\int_1^e f(t)\,dt=\int_0^e f(t)\,dt \Rightarrow \int_0^e f(x)\,dx \ge \int_0^1 (e^x+nx^{n-1})f\left(\frac{e^{2x}+nx^{2n-1}}{e^x+nx^{n-1}}\right)dx$$

$$\int_0^e f(x)\,dx \ge \int_0^1 (e^x+nx^{n-1})f\left(\frac{e^{2x}+nx^{2n}-1}{e^x+nx^{n-1}}\right)dx$$

PROBLEM 2.007-*Solution by proposer*

$$\prod_{k=1}^{n}\log_{x_k}\frac{s-x_k}{n-1} \ge \prod_{k=1}^{n}\log_{x_k}\sqrt[n-1]{x_1\cdot x_{k-1}\cdot x_{k+1}\cdot\ldots\cdot x_n}=$$

$$=\prod_{k=1}^{n}\frac{1}{n-1}\left(\log_{x_k}x_1+\cdots+\log_{x_k}x_{k-1}+\log_{x_k}x_{k+1}+\cdots+\log_{x_k}x_n\right)\ge$$

$$\ge\prod_{k=1}^{n}\sqrt[n-1]{\log_{x_k}x_1\cdot\ldots\cdot\log_{x_k}x_{k-1}\log_{x_k}x_{k+1}\cdot\ldots\cdot\log_{x_k}x_n}=$$

$$=\prod_{cyclic}\log_{x_1}x_2\log_{x_2}x_1=1$$

PROBLEM 2.008-*Solution by Henry Ricardo – New York – USA*

First we note that the AM – GM inequality gives us

$a^2+b^2+9=(a^2+b^2)+9\ge 6\sqrt{a^2+b^2}$ *and* $a^2+b^2\ge 2ab$. *Thus*

$$\frac{c}{a^2+b^2+9}\le\frac{c}{6\sqrt{a^2+b^2}}=\frac{c\sqrt{a^2+b^2}}{6(a^2+b^2)}\le\frac{c\sqrt{a^2+b^2}}{12ab}=\frac{c^2\sqrt{a^2+b^2}}{12abc},$$

Which implies the desired inequality.

PROBLEM 2.009-*Solution by proposer*

$$(g\circ f)(x)\in[0,c]\Rightarrow(g\circ f)(x)\le c;(\forall)x\in[0,a]$$

$$\frac{1}{c}\int_0^a (g\circ f)^2(x)\,dx \le \frac{1}{c}\int_0^a c\cdot(g\circ f)(x)dx = \int_0^a (g\circ f)\,(x)dx$$

$$(f^{-1}\circ g^{-1})(x)\in[0,a] \Rightarrow (f^{-1}\circ g^{-1})(x)\le a;\ (\forall)x\in[0,c]$$

$$\frac{1}{a}\int_0^c (f^{-1}\circ g^{-1})^2(x)\,dx \le \frac{1}{a}\int_0^c a(f^{-1}\circ g^{-1})\,(x)dx = \int_0^c (f^{-1}\circ g^{-1})(x)\,dx$$

$$\frac{1}{c}\int_0^a (g\circ f)^2\,(x)dx \le \int_0^a (g\circ f)\,(x)dx$$

$$\frac{1}{a}\int_0^c (f^{-1}\circ g^{-1})(x)\,dx \le \int_0^c (f^{-1}\circ g^{-1})\,(x)dx$$

$$\frac{1}{c}\int_0^a (g\circ f)^2\,(x)dx + \frac{1}{a}\int_0^c (f^{-1}\circ g^{-1})\,(x)dx \le \int_0^a (g\circ f)(x)\,dx + \int_0^c (f^{-1}\circ g^{-1})\,(x)dx = ac$$

PROBLEM 2.010-*Solution by proposer*

$$If\, x\in\left(0,\frac{\pi}{2}\right) \Rightarrow \cos\frac{x}{2}\le\cosh\frac{x}{2} \Rightarrow \tanh\frac{x}{2}\le\tan\frac{x}{2} \Rightarrow \cosh\frac{x}{2}\le\frac{1}{\sqrt{1-\tan^2\frac{x}{2}}} =$$

$$=\frac{\cos\frac{x}{2}}{\sqrt{\cos x}} \Rightarrow \left(\frac{\cosh\frac{x}{2}}{\cos\frac{x}{2}}\right)^2 \le \frac{1}{\cos x} \Rightarrow \sum\left(\frac{\cosh\frac{A}{2}}{\cos\frac{A}{2}}\right)^2 \le \sum\frac{1}{\cos A} = \frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}$$

PROBLEM 2.011-*Solution by proposer*

$$\sum_{k=n}^{\infty}\frac{1}{k^2} < \sum_{k=n}^{\infty}\frac{1}{k(n-1)} = \sum_{k=n}^{\infty}\left(\frac{1}{k-1}-\frac{1}{k}\right) = \frac{1}{n-1}$$

$$\sum_{k=n}^{\infty}\frac{1}{k^2} > \sum_{n=n}^{\infty}\frac{1}{k(k+1)} = \sum_{k=n}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \frac{1}{n}, so, \frac{1}{n} < \sum_{k=n}^{\infty}\frac{1}{k^2} < \frac{1}{n-1}$$

$$\frac{1}{n}+\sum_{k=1}^{n-1}\frac{1}{k^2} < \sum_{k=1}^{\infty}\frac{1}{k^2} < \sum_{k=1}^{n}\frac{1}{k^2}+\frac{1}{n-1}, or, \frac{1}{n}+\sum_{k=1}^{n-1}\frac{1}{k^2} < \frac{\pi^2}{6} < \sum_{k=1}^{n-1}\frac{1}{k^2}+\frac{1}{n-1}$$

$$Or\,\frac{\pi^2}{6}-\frac{1}{n-1} < \sum_{k=1}^{n-1}\frac{1}{k^2} < \frac{\pi^2}{6}-\frac{1}{n}, or, \frac{\pi^2}{6}-\frac{1}{n} < \sum_{k=1}^{n}\frac{1}{k^2} < \frac{\pi^2}{6}-\frac{1}{n+1}$$

PROBLEM 2.012-*Solution by proposer*

$$Let\, be\, f(t) = \det(A+tB) = t^2\det B + \alpha t + \det A \Rightarrow$$

$$\sum_{k=1}^{n}(\det(A+kB)+\det(A-kB))=\sum_{k=1}^{n}(k^2\det B+\alpha k+\det A+k^2\det B-\alpha k+\det A)$$
$$=2n\det A+\frac{n(n+1)(2n+1)}{3}\det B$$

PROBLEM 2.013-*Solution by Francis Fregeaux-Quebec-Canada*

$$\lim_{n\to\infty}\int_a^b \sin(x)\arctan(nx)\,dx=\alpha$$

$\lim_{n\to\infty}\arctan(n)=\frac{\pi}{2}$. *For any* $x\neq 0$:

$\lim_{n\to\infty}nx=\lim_{n\to\infty}\pm n=\pm\infty$, *depending on the sign of "x".*

$\arctan(-x)=-\arctan(x)$

And since both $\sin(x)$ *and* $\arctan(x)$ *share the same limit when* $x\to 0$

$$\alpha=\int_a^b \sin(x)\cdot\lim_{n\to\infty}\arctan(nx)\,dx\to\frac{\pi}{2}\int_a^b\sin(x)\,dx;\ 0\le a<b$$

$$\alpha=\int_a^b \sin(x)\cdot\lim_{n\to\infty}\arctan(nx)\,dx\to\frac{-\pi}{2}\int_a^b\sin(nx)\,dx;a<b\le 0$$

$$\alpha=\frac{\pi}{2}[\cos(a)-\cos(b)]\ \textit{for: } 0\le a<b$$
$$\alpha=\frac{\pi}{2}[\cos(b)-\cos(a)]\ \textit{for: } a<b\le 0$$

And if $a<0,b>0,a<b$:

$$\alpha=\frac{\pi}{2}[\cos(0)-\cos(a)]+\frac{\pi}{2}[\cos(0)-\cos(b)]=\pi-\frac{\pi}{2}[\cos(a)+\cos(b)]$$

PROBLEM 2.014-*Solution by Ngô Minh Ngọc Bảo –Gia Lai Province-VietNam*

Let $x=e^{bc},y=e^{ac},(x,y>0)$. *We need to prove that:*

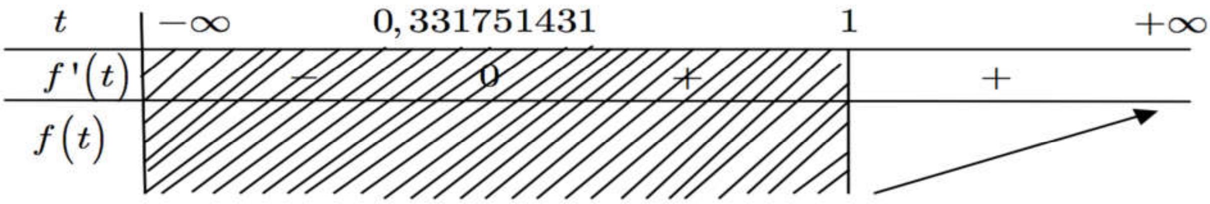
$$2\sqrt{2}(x-y)\le(\ln x-\ln y)\sqrt{x^2+y^2}\Leftrightarrow\sqrt{\left(\frac{x}{y}\right)^2+1}\cdot\ln\frac{x}{y}\ge 2\sqrt{2}\left(\frac{x}{y}-1\right)\ (*)$$

Indeed, let $t=\frac{x}{y}\ge 1$, *we have: (*)* $\Leftrightarrow\sqrt{t^2+1}\cdot\ln t-2\sqrt{2}(t-1)\ge 0$.

Considering function: $f(t)=\sqrt{t^2+1}\cdot\ln t-2\sqrt{2}t+2\sqrt{2},\forall t\ge 1$.

$$f'(t)=\frac{t\ln t}{\sqrt{t^2+1}}+\frac{\sqrt{t^2+1}}{t}-2\sqrt{2},f''(t)=\frac{\ln t}{\left(\sqrt{t^2+1}\right)^3}+\frac{t^2-1}{t^2\sqrt{t^2+1}}>0$$

Therefore, the equation $f'(t)=0$ *has a unique solution.*



$$\Rightarrow f(t) \ge f(1) = \sqrt{1+1} \cdot \ln 1 - 2\sqrt{2} + 2\sqrt{2} = 0, (!)$$

PROBLEM 2.015-*Solution by proposer*

By Young's inequality:

$$px^q + qx^p \ge pqxy; p > 1; \frac{1}{p} + \frac{1}{q} = 1; x, y \ge 0$$

$$\frac{1}{q} = 1 - \frac{1}{p} \Rightarrow \frac{1}{q} = \frac{p-1}{p} \Rightarrow q = \frac{p}{p-1}$$

$$px^{\frac{p}{p-1}} + \frac{p}{p-1} y^p \ge \frac{p^2}{p-1} xy$$

$$p\int_0^x x^{\frac{p}{p-1}}\, dx + \frac{p}{p-1} y^p \int_0^x dx \ge \frac{p^2}{p-1} y \int_0^x x\, dx$$

$$p\frac{x^{\frac{p}{p-1}+1}}{\frac{p}{p-1}+1} + \frac{p}{p-1} y^p x \ge \frac{p^2}{p-1} y \cdot \frac{x^2}{2}$$

$$\frac{x^{\frac{2p-1}{p-1}}}{\frac{2p-1}{p-1}} + \frac{1}{p-1} xy^p \ge \frac{p}{2(p-1)} x^2 y$$

For $p = 6, x = a, y = b$:

$$\frac{x^{\frac{11}{5}}}{\frac{11}{5}} + \frac{1}{5} xy^6 \ge \frac{6}{10} x^2 y \to \frac{5}{11} a^{\frac{11}{5}} + \frac{1}{5} ab^6 \ge \frac{3}{5} a^2 b$$

$$\frac{5}{11} \sum a^{\frac{11}{5}} + \frac{1}{5} \sum ab^6 \ge \frac{3}{5} \sum a^2 b$$

$$25 \sum a^2 \sqrt[5]{a} + 11 \sum ab^6 \ge 33 \sum a^2 b$$

PROBLEM 2.016-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Por desigualdad de Cauchy:

1. $(sa^2 + tb^2 + uc^2)(s + t + u) \ge (sa + tb + uc)^2 \to \frac{sa^2+tb^2+uc^2}{sa+tb+uc} \ge \frac{sa+tb+uc}{s+t+u}$ *(A)*

2. $\frac{sb^2+tc^2+ua^2}{sb+tc+ua} \ge \frac{sb+tc+ua}{s+t+u}$ *(B)*

3. $\frac{sc^2+ta^2+ub^2}{sc+ta+ub} \ge \frac{sc+ta+ub}{s+t+u}$ *(C)*

Sumando: (A) + (B) + (C): $\frac{sa^2+tb^2+uc^2}{sa+tb+uc} + \frac{sb^2+tc^2+ua}{sb+tc+ua} + \frac{sc^2+ta^2+ub^2}{sc+ta+ub} \ge$

$$\ge \frac{s(a+b+c) + t(a+b+c) + u(a+b+c)}{s+t+u} = 1$$

PROBLEM 2.017-*Solution by Ngô Minh Ngọc Bảo-Gia Lang Province-VietNam*

We known: $2 + 4 + 6 + \cdots + 2n = n(n+1)$, *with* $n \in N$. *We have:*

$$\sum_{k=1}^{n}\frac{(k^2-1)a_k+k^2+2k}{a_k^2+a_k+1}=\frac{3}{a_1^2+a_1+1}+\frac{3a_2+8}{a_2^2+a_2+1}+\cdots+\frac{(n^2-1)a_n+n^2+2n}{a_n^2+a_n+1}$$

We prove that: $\frac{(n^2-1)a_n+n^2+2n}{a_n^2+a_n+1}\geq -a_n+2n$, *(*). Indeed,*

$$(*)\Leftrightarrow (n^2-1)a_n+n^2+2n\geq(2n-a_n)(a_n^2+a_n+1)$$
$$\Leftrightarrow (n^2-1)a_n+n^2+2n\geq -a_n^3+(2n-1)a_n^2+(2n-1)a_n+2n$$
$$\Leftrightarrow a_n^3-(2n-1)a_n^2+(n^2-2n)a_n+n^2\geq 0\Leftrightarrow (a_n-n)^2(a_n+1)\geq 0 \text{ (True)}$$
$$\Rightarrow\sum_{k=1}^{n}\frac{(k^2-1)a_k+k^2+2k}{a_k^2+a_k+1}\geq(2+4+\cdots+2n)-\sum_{k=1}^{n}a_k=$$
$$=(2+4+\cdots+2n)-\frac{n(n+1)}{2}=\frac{n(n+1)}{2}$$

Equality occurs when $a_1=1, a_2=2, a_3=3,\dots,a_n=n$.

PROBLEM 2.018-*Solution by proposer*

We have: $\frac{a^3}{a+b}\geq\frac{5a^2-b^2}{8}\Leftrightarrow (a-b)^2(3a+b)\geq 0$ *therefore*

$$\begin{cases}\frac{a^3\ln x}{a+b}\geq\frac{(5a^2-b^2)\ln x}{8}\\ \frac{b^3\ln y}{b+c}\geq\frac{(5b^2-c^2)\ln y}{8}\\ \frac{c^3\ln z}{c+a}\geq\frac{(5c^2-a^2)\ln z}{8}\end{cases}$$

. After addition we obtain:

$$\sum\frac{a^3\ln x}{a+b}=\sum\ln x^{\frac{a^3}{a+b}}\geq\sum\frac{(5a^2-b^2)\ln x}{8}=\sum\frac{a^2(5\ln x-\ln z)}{8}=$$
$$=\sum\ln\left(\frac{x^5}{z}\right)^{\frac{a^2}{8}}\Rightarrow\prod x^{\frac{8a^3}{a+b}}\geq\prod\left(\frac{x^5}{z}\right)^{a^2}$$

PROBLEM 2.019-*Solution by proposer*

$$\sum_{k=1}^{n}(k^2+k)^\alpha\left((k+2)^\alpha-(k-1)^\alpha\right)=\sum_{k=1}^{n}(k+1)^\alpha k^\alpha\left((k+2)^\alpha-(k-1)^\alpha\right)=$$
$$=\sum_{k=1}^{n}\left((k+2)^\alpha(k+1)^\alpha k^\alpha-(k+1)^\alpha k^\alpha(k-1)^\alpha\right)=(n+2)^\alpha(n+1)^\alpha n^\alpha$$

1) If $\alpha=4\Rightarrow (k+2)^4-(k-1)^4=3(2k+1)(2k^2+2k+5)$

2) If $\alpha=6\Rightarrow (k+2)^6-(k-1)^6=9(2k+1)(k^2+k+1)(k^2+k+7)$

PROBLEM 2.020-*Solution by Soumitra Moukherjee-Chandar Nagore-India*

Let $f(x)=\tan x-x$ *for all* $x\in\left(0,\frac{\pi}{2}\right)$. *Now* $f'(x)=\tan^2 x$ *for all* $x\in\left(0,\frac{\pi}{2}\right)$

Now, $f(x)$ *is continuous on* $\left(0,\frac{\pi}{2}\right)$, $f'(x)>0$ *for all* $x\in\left(0,\frac{\pi}{2}\right)$

Hence, $f(x)$ *is increasing on* $\left(0,\frac{\pi}{2}\right)$, $f(x)>f(0)=0$ *for all* $x\in\left(0,\frac{\pi}{2}\right)$

$$\text{So, } \tan x > x \text{ for all } x \in \left(0, \frac{\pi}{2}\right),$$

$$\sum_{cyc} \frac{\tan^2 x}{(y+z)^2} > \sum_{cyc} \left(\frac{x}{y+z}\right)^2 \geq \frac{1}{3}\left(\sum_{cyc} \frac{x}{y+z}\right)^2 \geq \frac{1}{3}\left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

$$\left[\text{since}, \sum_{cyc} \frac{x}{y+z} \geq \frac{3}{2}\right]. \text{Hence}, \sum_{cyc} \frac{\tan^2 x}{(y+z)^2} > \frac{3}{4}$$

PROBLEM 2.021-*Solution by Soumitra Moukherjee– India*

$$2\left(\sum_{cyc} x^3y^3\right)\left\{\sum_{cyc} \frac{1}{(x+y)^5 z}\right\} \geq \left\{\sum_{cyc} z^3(x^3+y^3)\right\}\left\{\sum_{cyc} \frac{1}{(x+y)^5 z}\right\} \geq$$

$$\geq \left\{\sum_{cyc} \frac{z^3}{4}(x+y)^3\right\}\left\{\sum_{cyc} \frac{1}{(x+y)^5 z}\right\} \geq \frac{1}{4}\left(\frac{x}{y+x} + \frac{y}{z+x} + \frac{z}{x+y}\right)^2 \text{[Applying Cauchy – Schwarz]}$$

$$\geq \frac{1}{4}\left(\frac{3}{2}\right)^2 = \frac{9}{16} \text{ [Applying Nesbitt Inequality]} \Rightarrow \left(\sum_{cyc} x^3y^3\right)\left\{\sum_{cyc} \frac{1}{(x+y)^5 z}\right\} \geq \frac{9}{32}$$

PROBLEM 2.022-*Solution by Kunihiko Chikaya – Tokyo – Japan*

$$\frac{b^{n+1}-a^{n+1}}{n+1} + \frac{ab(b^{n-1}-a^{n-1})}{n-1} \leq (b-a)\sqrt{2(a^{2n}+b^{2n})} \qquad (*)$$

$$0 < a \leq b, n \geq 2 \ (n = 2,3,\dots) \ 0 < a \leq b$$

LHS of (*)

$$= \int_a^b (x^n + abx^{n-2})dx$$

$$f(x) = x^n + abx^{n-2} = x^{n-2}(x^2+ab), f'(x) = x^{n-3}\{nx^2 + ab(n-2)\} > 0$$

M.V.T of Integral for $a \leq x \leq b$

$$\leq (b-a)f\left(\frac{a+b}{2}\right)\left(0 < a < \frac{a+b}{2} < b\right) = (b-a)\left(\frac{a+b}{2}\right)^{n-2}\left\{\left(\frac{a+b}{2}\right)^2 + ab\right\}$$

$$\leq (b-a)\left(\frac{a+b}{2}\right)^{n-2} \cdot 2\left(\frac{a+b}{2}\right)^2 = 2(b-a)\left(\frac{a+b}{2}\right)^n \overset{Jensen}{\leq} 2(b-a)\frac{a^n+b^n}{2}$$

$$\leq 2(b-a)\sqrt{\frac{(a^2)^n + (b^2)^n}{2}}$$

PROBLEM 2.023-*Solution by proposer*

$$\left.\begin{aligned}(A+XB^{-1})B = AB + XI_n \\ A(B+XA^{-1}) = AB + xI_n\end{aligned}\right\} \Rightarrow (A+XB^{-1})B = A(B+XA^{-1}) \Rightarrow$$

$$\Rightarrow \det(A+XB^{-1})\det B = \det A \det(B+XA^{-1}) \Rightarrow \det(A+XB^{-1}) = \det(B+XA^{-1})$$

$$\begin{cases}\det(A+XB^{-1}) = \det(B+XA^{-1}) \\ \det(B+yA^{-1}) = \det(A+yB^{-1})\end{cases}. \text{ After multiplication:}$$

$$\det(AB + yI_n + xI_n + xy(AB)^{-1}) = \det(BA + yI_n + xI_n + xy(BA)^{-1}), \text{ finally}$$

$$\det(AB + xy(AB)^{-1} + (x+y)I_n) = \det(BA + xy(BA)^{-1} + (x+y)I_n)$$

PROBLEM 2.024-*Solution by proposer*

First, we will recall without proof two known results below

Lemma 1: For any triangle ABC and all positive real numbers x, y, z then

$$xa^2 + yb^2 + zc^2 \geq 4S_{ABC}\sqrt{xy + yz + zx}$$

Remark 1. We have known that there exists a triangle whose side-lenghts are m_a, m_b, m_c and its area is $S' = \frac{3}{4}S_{ABC}$. Applying lemma 1 for this triangle yields

$$x \cdot m_a^2 + y \cdot m_b^2 + z \cdot m_c^2 \geq 3S_{ABC}\sqrt{xy + yz + zx} \qquad (1)$$

Lemma 2. If ABC is a triangle and P is any point in its plane, then

$$\frac{PB \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \geq 1$$

(Hayashi's inequality)

Back to the main problem

Solution. Applying inequality (1) for $(x, y, z) = \left(\frac{PA}{a}, \frac{PB}{b}, \frac{PC}{c}\right)$ and using lemma 2, we obtain:

$$\frac{PA}{a}m_a^2 + \frac{PB}{b}m_b^2 + \frac{PC}{c}m_c^2 \geq 3S_{ABC}$$

Note that: $m_a = \frac{3}{2}GA, m_b = \frac{3}{2}GB, m_c = \frac{3}{2}GC$. The inequality above may be rewritten as

$$\frac{PA \cdot GA^2}{BC} + \frac{PB \cdot GB^2}{CA} + \frac{PC \cdot GC^2}{AB} \geq \frac{4}{3}S_{ABC}.$$ *The proof is complete.*

PROBLEM 2.025-*Solution by proposer*

$$\sum_{i_1=1,\ldots,i_k=1}^{n} \frac{i_1 \ldots i_k a_{i_1} \ldots a_{i_k}}{i_1 + \cdots + i_k - k + 1} = \sum_{i_1=1,\ldots,i_k=1}^{n} i_1 a_{i_1} \ldots i_k a_{i_k} \int_0^1 t^{i_1+\cdots+i_k-k}\, dt =$$

$$= \int_0^1 \left(\sum_{i_1=1,\ldots,i_k=1}^{n} i_1 a_{i_1} \ldots i_k a_{i_k} t^{i_1-1+\cdots+i_k-1} \right) dt = \int_0^1 \left(\sum_{i=1}^{n} i a_i t^{i-1} \right)^k dt \geq$$

$$\geq \left(\int_0^1 \left(\sum_{i=1}^{n} i a_i t^{i-1} \right) dt \right)^k = \left(\sum_{i=1}^{n} a_i \right)^k$$

PROBLEM 2.026-*Solution by Hamza Mahmood – Lahore – Pakistan*

Let $u = x - n$

$$I = \int_n^{n+1} \frac{\sqrt{n+1-x}}{\sqrt{n+1-x} + \sqrt{x-n}e^{2x-2n-1}} dx = \int_0^1 \frac{\sqrt{1-u}}{\sqrt{1-u} + \sqrt{u}e^{2u-1}} du \qquad (A)$$

Using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, we have:

$$I = \int_0^1 \frac{\sqrt{1-(1-u)}}{\sqrt{1-(1-u)} + \sqrt{1-u}e^{2(1-u)-1}} du = \int_0^1 \frac{\sqrt{u}}{\sqrt{u} + \sqrt{1-u}e^{1-2u}} du \qquad (B)$$

$$\text{Adding } (A)\ \&\ (B)\text{: } 2I=\int_0^1\left(\frac{\sqrt{1-u}}{\sqrt{1-u}+\sqrt{u}e^{2u-1}}+\frac{\sqrt{u}}{\sqrt{u}+\sqrt{1-u}e^{1-2u}}\right)du$$

$$\text{Since } \frac{\sqrt{1-u}}{\sqrt{1-u}+\sqrt{u}e^{2u-1}}=\frac{e\sqrt{1-u}}{e\sqrt{1-u}+u^{2u}\sqrt{u}}.\ \text{And } \frac{\sqrt{u}}{\sqrt{u}+\sqrt{1-u}e^{1-2u}}=\frac{e^{2u}\sqrt{u}}{e^{2u}\sqrt{u}+e\sqrt{1-u}}.\ \text{So}$$

$$2I=\int_0^1\left(\frac{e\sqrt{1-u}}{e\sqrt{1-u}+e^{2u}\sqrt{u}}+\frac{e^{2u}\sqrt{u}}{e^{2u}\sqrt{u}+e\sqrt{1-u}}\right)du=$$

$$=\int_0^1\frac{e\sqrt{1-u}+e^{2u}\sqrt{u}}{e\sqrt{1-u}+e^{2u}\sqrt{u}}du=\int_0^1(1)\,du=1,\ \ I=\int_n^{n+1}\frac{\sqrt{n+1-x}}{\sqrt{n+1-x}+\sqrt{x-n}e^{2x-2n-1}}dx=\frac{1}{2},\ \text{where } n\in\mathbb{N}^*$$

PROBLEM 2.027-*Solution by Ravi Prakash-New Delhi-India*

$$\text{For } x\neq 0,\ \text{let } f(x)=8^x+27^{\frac{1}{x}}+2^{x+1}3^{\frac{x+1}{x}}+2^x3^{\frac{2x+1}{x}}=2^{3x}+3^{\frac{3}{x}}+(15)\left(2^x3^{\frac{1}{x}}\right)$$

$$\text{For } x<0,\ f(x)<1+1+15(1)<125.\ \text{For } 0<x<1$$

$$f'(x)=(2^{3x})(3\ln 2)+3^{\frac{3}{x}}\left(-\frac{3}{x^2}\ln 3\right)+15\left[2^x3^{\frac{1}{x}}\ln 2+2^x3^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln 3\right)\right]$$

$$=(3)2^x\left[2^{2x}\ln 2-\frac{5}{x^2}\left(3^{\frac{1}{x}}\right)\ln 3\right]+\left(3^{\frac{1}{x}}\right)(3)\left[5(2^x)\ln 2-\frac{3^{\frac{2}{x}}}{x^2}\ln 3\right]$$

$$\text{For } 0<x<1,\ 0<2^{2x}<4,\ \ln 2<0.7\Rightarrow 0<2^{2x}\ln 2<2\cdot 8$$

$$\text{For } 0<x<1,\ \frac{1}{x^2}>1,\ \ 3^{\frac{1}{x}}>3,\ \ln 3>1$$

$$2^{2x}\ln 2-\frac{5}{x^2}\left(3^{\frac{1}{x}}\right)\ln 3<0.\ \text{Also, for } 0<x<1,\ 5(2^x)\ln 2<(10)(0.7)=7$$

$$\text{and } \frac{3^{\frac{2}{x}}\ln 3}{x^2}>9\Rightarrow 5(2^x)\ln 2-\frac{3^{\frac{2}{x}}}{x^2}\ln 3<0$$

Thus, $f'(x)<0$ *for* $0<x\le 1$, $f(x)$ *is strictly decreasing* $(0,1]$

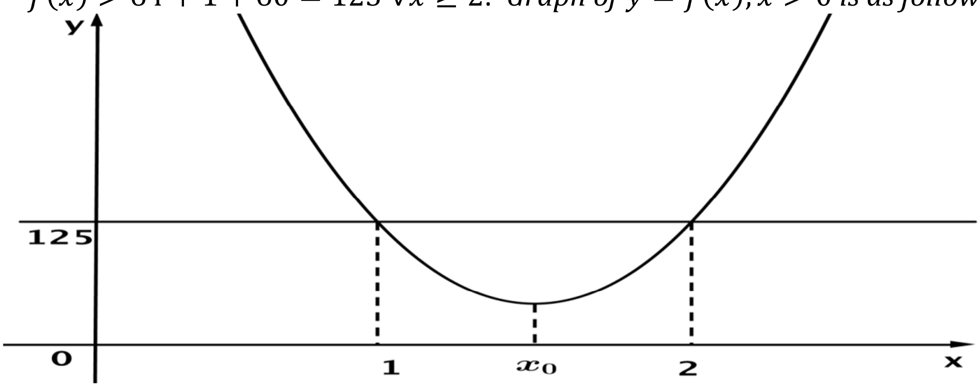
Also, note that $f'(x)$ *is continuous for* $x\ge 1$.

$$f'(1)=24\ln 2-81\ln 3+90\ln 2-90\ln 3<0$$

$$f'(2)=192\ln 2-15\sqrt{3}\ln 3+60\sqrt{3}\ln 2-\frac{9}{4}\sqrt{3}\ln 3>0$$

Thus, $\exists$ *some* $x_0\in(1,2)$ *such that* $f'(x_0)=0$. *For* $x\ge 2,\ 2^{3x}\ge 64,\ 27^{\frac{1}{x}}>1,\ 2^x3^{\frac{1}{x}}>4$

$f(x)>64+1+60=125\ \forall x\ge 2$. *Graph of* $y=f(x),\ x>0$ *is as follow.*



Thus, $f(x)=125$ *has two solutions,* $\alpha=1$ *and* β *where* $1<\beta<2$.

PROBLEM 2.028-*Solution by Ravi Prakash-New Delhi-India*

$$I=\int_1^n \frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx=\int_1^n \frac{x(x^3+4x^2+12x+9)}{15x^4+90x^3+270x^2+405x}dx$$

$$=\frac{1}{15}\int_1^n \frac{(x^3+4x^2+12x+9)x}{x^4+6x^3+18x^2+27x}dx=\frac{1}{15}\int_1^n \frac{x(x^3+4x^2+12x+9)}{x(x+3)(x^2+3x^2+9)}dx$$

$$=\frac{1}{15}\int_1^n \frac{x(x+1)(x^2+3x^2+9)}{x(x+3)(x^2+3x+9)}dx=\frac{1}{15}\int_1^n \frac{x+3-1}{x+3}dx$$

$$=\frac{1}{15}\int_1^n \left[1-\frac{1}{x+3}\right]dx=\frac{1}{15}[x-\ln|x+3|]_1^n=\frac{1}{15}[n-\ln(n+3)-1+\ln 4]$$

$$\lim_{n\to\infty}\frac{1}{n}(I)=\frac{1}{15}\lim_{n\to\infty}\left[1-\frac{1}{n}\ln(n+3)-\frac{1}{n}+\frac{\ln 4}{n}\right]=\frac{1}{15}$$

PROBLEM 2.029-*Solution by Hamza Mahmood-Lahore-Pakistan*

We shall use the following theorem: If $(c_n)_{n\ge 1}$ *is a convergent sequence with* $\lim_{n\to\infty} c_n=L$ *then* $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n c_k=L$

Consider a sequence $(a_n)_{n\ge1}$ *defined as:* $a_n=\int_0^1 \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx$,

we now find $\lim_{n\to\infty} a_n$

$$a_n=\int_0^{\frac{1}{2}} \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx+\int_{\frac{1}{2}}^1 \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx$$

For $0<x<\frac{1}{2}\Rightarrow 0>-2x>-1\Rightarrow 1>1-2x>0\Rightarrow \text{as } n\to\infty, n^{1-2x>0}\to\infty$

$$\Rightarrow \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}\to 0;\ \lim_{n\to\infty}\int_0^{\frac{1}{2}} \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx=0$$

For $\frac{1}{2}<x<1\Rightarrow -1>-2x>-2\Rightarrow 0>1-2x>-1\Rightarrow as\ n\to\infty, n^{1-2x<0}\to$

$$\to 0\Rightarrow \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}\to\frac{x\sin(\pi x)}{x}=\sin(\pi x)$$

$$\lim_{n\to\infty}\int_{\frac{1}{2}}^1 \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx=\int_{\frac{1}{2}}^1 \sin(\pi x)=-\frac{1}{\pi}\left(\cos\pi-\cos\frac{\pi}{2}\right)=\frac{1}{\pi}$$

$\lim_{n\to\infty} a_n=\frac{1}{\pi}\Rightarrow (a_n)_{n\ge1}$ *is a convergent sequence with* $\lim_{n\to\infty} a_n=\frac{1}{\pi}$

so from the above theorem: $\Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n a_k=\frac{1}{\pi}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_0^1\frac{x\sin(\pi x)}{x+(1-x)k^{1-2x}}dx=\frac{1}{\pi}$$

PROBLEM 2.030-*Solution by Naren Bhandari-Bajura-Nepal*

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\cos^2 2015\,x-\cos^2 2016\,x}{\sin x}dx>0.0001$$

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\left[2\cos\left[\frac{4031x}{2}\right]\cdot\cos\left(\frac{x}{2}\right)\right]\left[2\omega\sin\left(\frac{40131x}{2}\right)\cdot\sin\left(\frac{x}{2}\right)\right]}{\sin x}dx$$

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\left[d\sin\left(\frac{4031x}{2}\right)\cdot\cos\left(\frac{4031x}{2}\right)\right]\left[2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)\right]}{\sin x}dx$$

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sin(4031x)\,dx;\ I=-\frac{-\cos(4031x)}{4031}\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}}>0{,}001$$

$$I=+\frac{1}{4031}\left[\cos\left(4031\frac{\pi}{3}\right)-\cos\left(4031\frac{\pi}{2}\right)\right]>0{,}0001$$

$$I=\frac{1}{4031}\left[\cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{2}\right)\right]>0.0001$$

$$I=\frac{1}{4031\cdot 2}>\frac{1}{10000}\Rightarrow 10000>4031\cdot 2\ \therefore I=\frac{1}{4031\cdot 2}>\frac{1}{10000}\quad \textit{(proved)}$$

PROBLEM 2.031-*Solution by Soumitra Mandal – Kolkata – India*

Theorem: Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. If $(a_n)_{n\ge1}\in S(\mathbb{R}_+^*)$ is a $B-(t+1,a)$ sequence, then $\left(\sqrt[n]{a_n}\right)_{n\ge1}$ is a $L-\left(t,a(t+1),e^{-(t+1)}\right)$ sequence.

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a\in(0,\infty)$$

$$\Omega=\lim_{n\to\infty}\left(\sqrt[2n+2]{a_{n+1}}-\sqrt[2n]{a_n}\right)\sqrt{n}=\left\{\lim_{n\to\infty}\left(\sqrt[2n+2]{a_{n+1}}-\sqrt[2n]{a_n}\right)\right\}\left(\lim_{n\to\infty}\sqrt{n}\right)$$

$$=\left\{\lim_{n\to\infty}\left(\sqrt[n+1]{c_{n+1}}-\sqrt[n]{c_n}\right)\right\}\left(\lim_{n\to\infty}\sqrt{n}\right)\ \textit{where } c_n=\sqrt{a_n}\ \textit{for all } n\ge1$$

$$\lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n}=\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right)\left(\lim_{n\to\infty}\sqrt{\frac{a_{n+1}}{n\cdot a_n}}\right)=\sqrt{a}\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right)$$

Hence, $(c_n)_{n\ge1}$ is $B-\left(1,\sqrt{a}\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right)\right)$ sequence, so by the above theorem $\left(\sqrt[n]{c_n}\right)_{n\ge1}$ is a $L-\left(0,\sqrt{a}\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right)\cdot 1\cdot e^{-1}\right)$ sequence.

$$\Omega=\frac{\sqrt{a}}{e}\left(\lim_{n\to\infty}\sqrt{n}\right)\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right)=\frac{\sqrt{a}}{e}\lim_{n\to\infty}\left(\sqrt{n}\times\frac{1}{\sqrt{n}}\right)=\frac{\sqrt{a}}{e}\ \textit{(Ans:)}$$

PROBLEM 2.032-*Solution by Soumitra Mandal – Kolkata – India*

Theorem: Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. *If* $(a_n)_{n\geq1}\in S(\mathbb{R}_+^*)$ *is a* $B-(t+1,a)$ *sequence, then* $\left(\sqrt[n]{a_n}\right)_{n\geq1}$ *is a* $L-\left(t,a(t+1),e^{-(t+1)}\right)$ *sequence.*

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=b;\ \Omega=\lim_{n\to\infty}\left(\sqrt[2n+2]{a_{n+1}b_{n+1}}-\sqrt[2n]{a_nb_n}\right)$$

$$=\lim_{n\to\infty}\left(\sqrt[n+1]{c_{n+1}}-\sqrt[n]{c_n}\right),\text{ where } c_n=\sqrt{a_nb_n} \text{ for all } n\geq1$$

$$\lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n}=\lim_{n\to\infty}\frac{\sqrt{a_{n+1}b_{n+1}}}{n\cdot\sqrt{a_nb_n}}=\left(\lim_{n\to\infty}\sqrt{\frac{a_{n+1}}{n\cdot a_n}}\right)\left(\lim_{n\to\infty}\sqrt{\frac{b_{n+1}}{n\cdot b_n}}\right)=\sqrt{ab}$$

Hence $(c_n)_{n\geq1}$ *is a* $B-\left(1,\sqrt{ab}\right)$ *sequence, so by the above theorem* $\left(\sqrt[n]{c_n}\right)_{n\geq1}$ *is a* $L-\left(0,\sqrt{ab}\cdot1\cdot e^{-1}\right)$ *sequence i.e.* $L-\left(0,\frac{\sqrt{ab}}{e}\right)$ *sequence. So ,*$\Omega=\frac{\sqrt{ab}}{3}$

PROBLEM 2.033-*Solution by Soumitra Mandal – Kolkata – India*

Theorem: Let $(t,a)\in\mathbb{R}_+^*\times\mathbb{R}_+^*$. *If* $<a_n>_{n\geq1}\in S(\mathbb{R}_+^*)$ *is a* $B-(t+1,a)$ *sequence then* $<\sqrt[n]{a_n}>_{n\geq1}$ *is a* $L-\left(t,a(t+1),e^{-(t+1)}\right)$ *sequence.*

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot a_n}=a\in(0,\infty) \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{n^{s+1}\cdot b_n}=b\in(0,\infty)$$

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{a_{n+1}b_{n+1}}-\sqrt[n]{a_nb_n}\right)e^{-(r+s)x_n}\right)$$

$$=\left\{\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}b_{n+1}}-\sqrt[n]{a_nb_n}\right)\right\}\left(\lim_{n\to\infty}e^{-(r+s)x_n}\right)$$

Let $c_n=a_nb_n$ *for all* $n\geq1$

$$\lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n}=\left(\lim_{n\to\infty}n^{r+s}\right)\left(\lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot a_n}\right)\left(\lim_{n\to\infty}\frac{b_{n+1}}{n^{s+1}\cdot b_n}\right)=ab\left(\lim_{n\to\infty}n^{r+s}\right)$$

Hence $<c_n>_{n\geq1}$ *is a* $B-\left(1,ab(\lim_{n\to\infty}n^{r+s})\right)$ *is a sequence. Hence the above theorem yields* $<\sqrt[n]{c_n}>_{n\geq1}$ *a* $L-(0,ab(\text{li}_{n\to\infty}n^{r+s})\cdot1\cdot e^{-1})$ *sequence.*

$$\Omega=\frac{ab}{e}\left(\lim_{n\to\infty}n^{r+s}\right)\left(\lim_{n\to\infty}e^{-(r+s)x_n}\right)=\frac{ab}{e}\left(\lim_{n\to\infty}n^{r+s}\,e^{-(r+s)(\gamma_n+\ln n)}\right)$$

Where $\gamma_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n$ *is Euler's Constant* $=\frac{ab}{e^{(r+s)\gamma_n+1}}$ *(Ans :)*

PROBLEM 2.034-*Solution by Marian Ursărescu – Romania*

Let $a_n=f(n)\Rightarrow\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a\wedge\exists\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}$. *We must calculate:*

$$\Omega=\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[2n+2]{a_{n+1}}}-\frac{n}{\sqrt[2n]{a_n}}\right)\sqrt{n}\quad(1)$$

$$\Omega=\lim_{n\to\infty}\left(e^{\ln\frac{n+1}{\sqrt[2n+2]{a_{n+1}}}}-e^{\ln\frac{n}{\sqrt[2n]{a_n}}}\right)\sqrt{n}=\lim_{n\to\infty}e^{\frac{n}{\sqrt[2n]{a_n}}}\left(e^{\ln\frac{n+1}{\sqrt[2n+2]{a_{n+1}}}-\ln\frac{n}{\sqrt[2n]{a_n}}}-1\right)\sqrt{n}=$$

$$=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[2n]{a_n}}\cdot n\cdot\left(e^{\ln\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{a_n}}{\sqrt[2n+2]{a_{n+1}}}\right)}-1\right)\ \text{(2)}$$

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[2n]{a_n}}=\lim_{n\to\infty}\frac{\sqrt[2n]{n^n}}{\sqrt[2n]{a_n}}=\sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}}}\overset{C.D.}{=}\sqrt{\lim_{n\to\infty}\frac{(n+1)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{n^n}}=$$

$$=\sqrt{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{a_n(n+1)}{a_{n+1}}}=\sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{a_n\cdot n}{a_{n+1}}\cdot\frac{n+1}{n}}=\sqrt{\frac{e}{a}}\ \text{(3)}$$

$$\lim_{n\to\infty}n\left(e^{\ln\left(\frac{n+1}{n}\cdot\frac{\sqrt[2n]{a_n}}{\sqrt[2n+2]{a_{n+1}}}\right)}-1\right)=\lim_{n\to\infty}\frac{\left(e^{\ln\frac{\sqrt[2n]{a_n}}{\sqrt[2n+1]{a_{n+1}}}-1}\right)}{\ln\frac{\sqrt[2n]{a_n}}{\sqrt[2n+2]{a_{n+1}}}}\cdot\ln\frac{\sqrt[2n]{a_n}}{\sqrt[2n+2]{a_{n+1}}}=$$

$$=\lim_{n\to\infty}n\ln\sqrt{\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}}=\lim_{n\to\infty}\frac{n}{2}\ln\left(\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}\right)=$$

$$=\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}\right)^n=\frac{1}{2}\ln\left(\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}}\cdot\sqrt[n+1]{a_{n+1}}\right)\right)=\frac{1}{2}\ln\left(\lim_{n\to\infty}\frac{na_n}{a_{n+1}}\cdot\frac{1}{n}\sqrt[n+1]{a_{n+1}}\right)=$$

$$=\frac{1}{2}\ln\left(\lim_{n\to\infty}\frac{na_n}{a_{n+1}}\cdot\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\right)=\frac{1}{2}\ln\left(\frac{1}{a}\cdot1\cdot\frac{a}{e}\right)=\frac{-1}{2}\ \text{(4)}$$

For (1)+(2)+(3)+(4)$\Rightarrow\Omega=-\frac{1}{2}\sqrt{\frac{e}{a}}$

PROBLEM 2.035-*Solution by Henry Ricardo - New York – USA*

To simplify things typographically, we introduce the notation $(mm\ldots mm)_k$ *to denote the* k *- digit number each of whose digits is* m*.*

First we see that for any positive integer k*:* $(44\ldots44)_{2k}=(44\ldots44)_k\cdot10^k+(44\ldots44)_k$

$$=(44\ldots44)_k\cdot(10^k+1)=4(11\ldots11)_k\cdot(9(11\ldots11)_k+2)$$

$$=36\cdot(11\ldots11)_k^2+8(11\ldots11)_k=(66\ldots66)_k^2+8(11\ldots11)_k$$

$$<(66\ldots66)_k^2+8(11\ldots11)_k+\frac{4}{9}=\left((66\ldots66)_k+\frac{2}{3}\right)^2$$

Thus $(66\ldots66)_k^2<(44\ldots44)_{2k}<\left((66\ldots66)_k+\frac{2}{3}\right)^2$*, implying that* $(66\ldots66)_k<\sqrt{(44\ldots44)_{2k}}<(66\ldots66)_k+\frac{2}{3}$ *and so* $\left\lfloor\sqrt{(44\ldots44)_{2k}}\right\rfloor=(66\ldots66)_k$*. Now we have* $\frac{\sum_{k=1}^n\left\lfloor\sqrt{(44\ldots44)_{2k}}\right\rfloor}{10^n}=\frac{6\cdot\sum_{k=1}^n(11\ldots11)_k}{10^n}=\frac{6\cdot\sum_{k=1}^n\left(\frac{10^k-1}{9}\right)}{10^n}$

$$=\frac{2}{3}\left(\frac{\sum_{k=1}^n10^k-n}{10^n}\right)=\frac{2}{3}\frac{\left(\frac{10^{n+1}-1}{9}-n\right)}{10^n}=\frac{2}{27}\left(\frac{10^{n+1}-1-9n}{10^n}\right)=\frac{2}{27}\left(10-\frac{1}{10^n}-\frac{9n}{10^n}\right)\to\frac{20}{27}\ \textit{as } n\to\infty.$$

PROBLEM 2.036-*Solution by Soumitra Mandal – Kolkata – India*

$$3(a+b)(b+c)(c+a) \geq \frac{8}{\sqrt[8]{a^3+b^3+c^3}}$$

$$\Rightarrow \sum_{cyc} a^3 + 3\prod_{cyc}(a+b) \geq \frac{8}{\sqrt[8]{a^3+b^3+c^3}} + (a^3+b^3+c^3)$$

$$\geq (8+1)\sqrt[9]{\left\{\frac{1}{\sqrt[8]{a^3+b^3+c^3}}\right\}^8 (a^3+b^3+c^3)} = 9$$

$\Rightarrow (a+b+c)^3 \geq 9 \Rightarrow a+b+c \geq \sqrt[3]{9}$ *(proved). Equality at* $a=b=c=\frac{1}{\sqrt[3]{3}}$

PROBLEM 2.037-*Solution by Mirza Uzair Baig-Lahore-Pakistan*

It is easy to prove the following asymptotic expansions

$$n\ln\left(1+\frac{a}{n}\right)^b = \left(\frac{a}{n}\right)^b\left(\frac{a^2b(3b+5)}{24n} - \frac{ab}{2} + n + O(n^{-2})\right)$$

$$= \frac{a^{2+b}b(3b+5)}{24n^{1+b}} - \frac{a^{1+b}b}{2n^b} + a^b n^{1-b} + O(n^{-2-b})$$

$$\left(1+\frac{a}{n}\right)^b = 1 + \frac{ab}{n} + O(n^{-2}).$$

Now now that

$$n\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta} = n\left(\frac{\tan(x)+\sec(x)}{n}\right)^{\cos(x)} + O\left(n^{-\delta}\right)$$

$$\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta} = 1 + \frac{\cos\theta\cot\theta}{n} + O(n^{-2})$$

$$\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta} = 1 + \frac{\sin\theta\sec^2\theta\cot\theta}{n} + O(n^{-2})$$

For $x \in \left(\frac{\pi}{4},\frac{\pi}{3}\right)$ *we have,* $n^{1-\cos(x)} \to \infty, n \to \infty$. *Thus limit is* $+\infty$.

PROBLEM 2.038-*Solution by proposer*

First we show that if $a,b,c,x,y,z \in \mathbb{R}$ *then:*

$$ax+by+cz+\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} \geq \frac{2}{3}(a+b+c)(x+y+z) \quad (1)$$

Let us $t=\sqrt{\frac{x^2+y^2+z^2}{a^2+b^2+c^2}}$, $x=pt, y=qt, z=rt \Rightarrow a^2+b^2+c^2 = p^2+q^2+r^2$ *and (1) becomes*

$$ap+bq+cr+a^2+b^2+c^2 \geq \frac{2}{3}(c+b+c)(p+q+r) \text{ or}$$

$4(a+b+c)(p+q+r) \leq \left((a+b+c)+(p+q+r)\right)^2$ *it suffices to prove that:*

$(a+b)^2+(b+q)^2+(c+r)^2 \geq \frac{1}{3}\left((a+p)+(b+q)+(c+r)\right)^2$. *This is clearly true.*

In (1) we take $a=x^k, b=y^k, c=z^k \Rightarrow$

$$\sum_{k=0}^{n}\left(x^{k+1}+y^{k+1}+z^{k+1}+\sqrt{(x^2+y^2+z^2)(x^{2k}+y^{2k}+z^{2k})}\right)\ge$$

$$\ge \frac{2}{3}\sum_{k}^{n}(x+y+z)(x^k+y^k+z^k) \text{ or}$$

$$\frac{1}{3}\sum_{cyclic}(x-2y-2z)\left(\frac{x^{n+1}-1}{x-1}\right)+\sqrt{x^2+y^2+z^2}\sum_{k=0}^{n}\sqrt{x^{2k}+y^{2k}+z^{2k}}\ge 0$$

PROBLEM 2.039-***Solution by Nguyen Phuc Tang – Hanoi – Vietnam***

$$\text{We have } \left|\ln\frac{ab}{c}\right|+\left|\ln\frac{ac}{b}\right|+\left|\ln\frac{bc}{a}\right|\ge\left|\ln\frac{ab}{c}+\ln\frac{ac}{b}+\ln\frac{bc}{a}\right|=\ln(abc)$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{3}{\sqrt[3]{abc}} \quad \text{(AM-GM)}$$

$$LHS=e^{\left|\ln\frac{ab}{c}\right|+\left|\ln\frac{ac}{b}\right|+\left|\ln\frac{bc}{a}\right|}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3\ge e^{\ln(abc)}\cdot\frac{27}{abc}\ge abc\cdot\frac{27}{abc}=27$$

Equality holds if $a=b=c$.

PROBLEM 2.040-***Solution by Nguyen Phuc Tang – Hanoi – Vietnam***

We have $\ln a\ge 1, \ln b\ge 1, \ln c\ge 1$. *The given inequality is equivalent to*

$$\frac{\ln b+\ln c}{1+2\ln a}+\frac{\ln a+\ln c}{1+2\ln b}+\frac{\ln b+\ln a}{1+2\ln c}\ge\sum\frac{\ln c}{1+2\sqrt{\ln a\ln b}}$$

$\Leftrightarrow \sum(\ln c)\left(\frac{1}{1+2\ln a}+\frac{1}{1+2\ln b}-\frac{2}{1+2\sqrt{\ln a\ln b}}\right)\ge 0$ *(*)- (*) is true, by the well – known inequality:*

$\frac{1}{1+x^2}+\frac{1}{1+y^2}\ge\frac{2}{1+xy}$ *for all* $x,y>0$ & $xy\ge 1$. *Equality holds if* $a=b=c$.

PROBLEM 2.041-***Solution by Ravi Prakash – New Delhi – India***

$$\sum_{k=1}^{n}(n-k+1)^2\sum_{j=1}^{k}f\left(\frac{j}{n}\right)=$$

$$=n^2f\left(\frac{1}{n}\right)+(n-1)^2\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)\right]+(n-2)^2\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)\right]+\cdots$$

$$\ldots+1^2\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right]=$$

$$=\sum_{k=1}^{n}f\left(\frac{k}{n}\right)[1^1+2^2+\cdots+(n-k+1)^2] \quad (1)$$

$$\text{We know } \frac{1}{3}k^3<1^2+2^2+\cdots+k^2<\frac{1}{3}(k+1)^3$$

$$\frac{1}{3}(n-k+1)^3<\sum_{j=1}^{n-k+1}j^2<\frac{1}{3}(n-k+2)^3$$

Using (1), we get $\sum_{k=1}^{n}\frac{1}{3}\frac{(n-k+1)^3f\left(\frac{k}{n}\right)}{n^2(n+1)(2n+1)}<J<\sum_{k=1}^{n}\frac{1}{3}\frac{(n-k+2)^3f\left(\frac{k}{n}\right)}{n^2(n+1)(2n+1)}$ *(2)*

When $J=\frac{\sum_{k=1}^{n}\left((n-k+1)^2\sum_{j=1}^{k}f\left(\frac{j}{n}\right)\right)}{n^2(n+1)(2n+1)}$. *Now,* $\sum_{k=1}^{n}\frac{1}{3}\cdot\frac{(n-k+1)^3f\left(\frac{k}{n}\right)}{n^2(n+1)(2n+1)}=$

$$=\frac{1}{6}\sum_{k=1}^{n}\frac{1}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)}\left(1+\frac{1}{n}-\frac{k}{n}\right)^3 f\left(\frac{k}{n}\right)$$

$$=\frac{1}{6}\cdot\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)}\cdot\frac{1}{n}\cdot\sum_{k=1}^{n}\left\{\left(1-\frac{k}{n}\right)^3 f\left(\frac{k}{n}\right)+\frac{3}{n}\left(1-\frac{k}{n}\right)^2 f\left(\frac{k}{n}\right)+\frac{3}{n^2}\left(1-\frac{k}{n}\right)f\left(\frac{k}{n}\right)+\frac{1}{n^3}f\left(\frac{k}{n}\right)\right\}$$

$$\to\frac{1}{6}\left[\int_0^1(1-x)^3 f(x)dx+(0)\int_0^1(1-x)^2 f(x)\,dx+(0)\int_0^1(1-x)f(x)\,dx+(0)\int_0^1 f(x)\,dx\right]=$$

$$=\frac{1}{6}\int_0^1(1-x)^3 f(x)dx$$

Similarly, expression on RHS of (2) approaches:

$$\frac{1}{6}\int_0^1(1-x)^3 f(x)dx;\ J\to\frac{1}{6}\int_0^1(1-x)^3 f(x)dx\ \text{ as } n\to\infty$$

PROBLEM 2.042-*Solution by proposer*

With elementary calculus holds:

$$\det(xI_2+yAB+zBA)=x^2+x(y+z)Tr(AB)+(y^2+z^2)\det(AB)+$$

$$+yz\left((Tr(AB))^2-Tr(A^2B^2)\right)\text{ and using the inequality } x^2+y^2+z^2\geq xy+yz+zx$$

holds $\det(xI_2+yAB+zBA)+\det(yI_2+zAB+xBA)+\det(zI_2+xAB+yBA)=$

$$=(x^2+y^2+z^2)+2(xy+yz+zx)Tr(AB)+2(x^2+y^2+z^2)\det(AB)+$$

$$+(xy+yz+zx)\left((Tr(AB))^2-Tr(A^2B^2)\right)\geq(xy+yz+zx)\left(1+2Tr(AB)+Tr(AB)\right)^2+$$

$$+2\det(AB)-Tr(A^2B^2)=(xy+yz+zx)\left((1+Tr(AB))^2+2\det(AB)-Tr(A^2B^2)\right)$$

PROBLEM 2.043-*Solution by proposer*

We have: $ax^3y+by^3z+cz^3x\geq(a+b+c)((x^3y)^a(y^3z)(z^3x)^c)^{\frac{1}{a+b+c}}=$

$$=(a+b+c)(x^{3a+c}y^{3b+a}z^{3c+b})^{\frac{1}{a+b+c}}\Rightarrow(a+b+c)\sum x^3y=$$

$$=\sum(ax^3y+by^3z+cz^3x)\geq(a+b+c)\sum(x^{3a+c}y^{3b+a}z^{3c+b})^{\frac{1}{a+b+c}}$$

PROBLEM 2.044-*Solution by proposer*

$$\frac{x_1^\alpha+x_2^\alpha+x_3^\alpha}{3}\geq\left(\frac{x_1+x_2+x_3}{3}\right)^\alpha\ \textit{for all } x_1,x_2,x_3>0$$

If $x_1=-a+b+c+d,x_2=a-b+c+d,x_3=a+b-c+d,x_4=a+b+c-d$ *and*

$x_1,x_2,x_3,x_4>0$ *then* $x_1^\alpha+x_2^\alpha+x_3^\alpha+x_4^\alpha=\sum_{cyclic}\frac{x_1^\alpha+x_2^\alpha+x_3^\alpha}{3}\geq\sum\left(\frac{x_1+x_2+x_3}{3}\right)^\alpha$

or $\sum(-a+b+c+d)^\alpha\geq\sum\left(\frac{a+b+c}{3}+d\right)^\alpha$ *etc.*

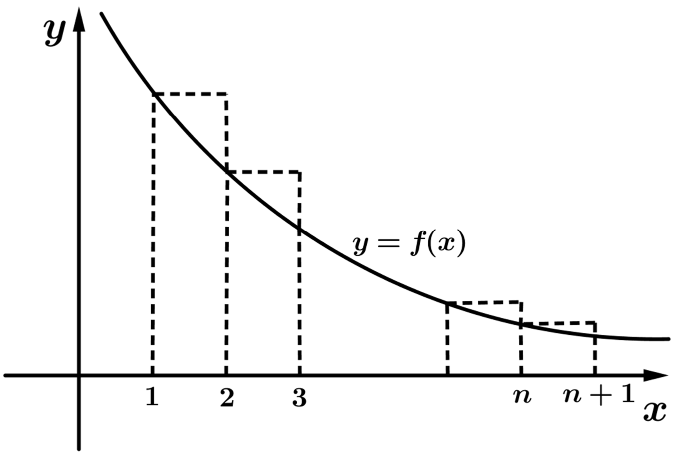
PROBLEM 2.045-*Solution by proposer*

$$\text{We have: } \frac{1}{1-a^4} \geq \frac{5\sqrt[5]{5}}{4}a \Leftrightarrow a\sqrt[5]{4} = x, \frac{1}{5-x^4} \geq \frac{5x}{4} \Leftrightarrow$$

$$\Leftrightarrow (x-1)^2(x^3+2x^2+3x+4) \geq 0 \Rightarrow \frac{1}{a(1-a^4)} \geq \frac{5\sqrt[5]{5}}{4} \Rightarrow$$

$$\sum \frac{1}{\left(a(1-a^4)\right)^{4n}} \geq \sum \left(\frac{5\sqrt[5]{5}}{4}\right)^{4n} = 3\left(\frac{3125}{256}\right)^n$$

PROBLEM 2.046-*Solution by Shahlar Maharrahmov-Jebrail-Azerbaidian*



Let us use figure. Take $f(x) = \frac{1}{x}$ *and partition* $a_k = \frac{1}{k}$*, then we obtain*

$$\int_1^{n+1} \frac{1}{x}dx \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \int_1^n f(x)\,dx \Rightarrow \ln(n+1) \leq$$

$$\leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \ln n \quad (*)$$

since $\ln\frac{n+1}{2} < \ln(n+1)$ *and* $\log_2\frac{n+1}{2} > 1 + \ln n$

then from (*) $\Rightarrow \ln\frac{n+1}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log_2\frac{n+1}{2}$

PROBLEM 2.047-*Solution by Ravi Prakash-New Delhi-India*

$$\sum_{k=1}^{17} \cos^4\left(\frac{k\pi}{36}\right)$$

$$= \sum_{k=1}^{8}\left[\cos^4\left(\frac{k\pi}{36}\right) + \left(\cos\left(\frac{\pi}{2} - \frac{k\pi}{36}\right)\right)^4\right] + \cos^4\left(\frac{\pi}{4}\right) = \sum_{k=1}^{8}\left[\cos^3\left(\frac{k\pi}{36}\right) + \sin^3\left(\frac{k\pi}{36}\right)\right] + \frac{1}{4}$$

$$= \frac{1}{2}\sum_{k=1}^{8}\left[\left(\cos^2\frac{k\pi}{36} + \sin^2\frac{k\pi}{36}\right)^2 + \left(\cos^2\frac{k\pi}{36} - \sin^2\frac{k\pi}{36}\right)^2\right]$$

$$= \frac{1}{2}\sum_{k=1}^{8}\left[1 + \cos^2\left(\frac{k\pi}{18}\right)\right] + \frac{1}{4} = \frac{1}{2}\sum_{k=1}^{8}\left[1 + \frac{1 + \cos\left(\frac{k\pi}{a}\right)}{2}\right] + \frac{1}{4}$$

$$= \frac{1}{4}\sum_{k=1}^{8}\left[3 + \cos\left(\frac{k\pi}{9}\right)\right] + \frac{1}{4} = \frac{25}{4} + \frac{1}{2}S_1$$

where $S_1 = \sum_{k=1}^{8}\cos\left(\frac{k\pi}{9}\right)$

$$= \sum_{k=1}^{4}\left[\cos\left(\frac{k\pi}{9}\right) + \cos\left(\pi - \frac{k\pi}{9}\right)\right] = \sum_{k=1}^{4}\left[\cos\left(\frac{k\pi}{9}\right) - \cos\left(\frac{k\pi}{9}\right)\right] = 0$$

$$\sum_{k=1}^{17}\cos^4\left(\frac{k\pi}{36}\right) = \frac{25}{4}$$

PROBLEM 2.048-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Probar para todos los reales no negativos: a, b, c *la siguiente desigualdad:*

$$(a^4 + b^4 + c^4)(ab^3 + bc^3 + c^3a) \geq (a^3b + b^3c + a^3c)(a^2b^2 + b^2c^2 + c^2a^2)$$

Siendo: $a, b, c \geq 0$. *Por la desigualdad de Cauchy:*

$$(a^4 + b^4 + c^4)(a^2b^2 + b^2c^2 + c^2a^2) \geq (a^3b + b^3c + a^3c)^2 \dots \text{(A)}$$

$$(ab^3 + bc^3 + c^3a)(a^3b + b^3c + a^3) \geq (a^2b^2 + b^2c^2 + c^2a^2)^2 \dots \text{(B)}$$

Multiplicando, se obtiene: (A) $\times$ *(B):*

$$(a^4 + b^4 + c^4)(ab^3 + bc^3 + c^3a) \geq (a^3b + b^3c + a^3c)(a^2b^2 + b^2c^2 + c^2a^2)$$

PROBLEM 2.049-*Solution by Soumitra Mandal-Chandar Nagore-India*

Knonw, results $\frac{x+y}{2} \geq \frac{2xy}{x+y}$ *and* $x^xy^y \geq \left(\frac{x+y}{2}\right)^{x+y}$

Now, $x^yy^x \overset{WEIGHTED\ AM \geq GM}{\lesssim} \left(\frac{xy+yx}{x+y}\right)^{x+y} = \left(\frac{2xy}{x+y}\right)^{x+y} \leq \left(\frac{x+y}{2}\right)^{x+y} \leq x^xy^y$

$\therefore x^{y-x}y^{x-y} \leq 1$ *(proved) equality at* $x = y$

PROBLEM 2.050-*Solution by SK Rejuan -West Bengal-India*

Given $a \geq b \geq c > 0$. *We have to prove*

$$a^{a-b}b^{b-c}c^{c-a} \geq 1 \Leftrightarrow a^ab^bc^c \geq a^bb^cc^a \quad (1)$$

Let us take $a, b, c \in \mathbb{R}^+$ *with the associated weight* a, b, c *respectively, by applying AM* $\geq$ *HM*

we get, $(a^ab^bc^c)^{\frac{1}{a+b+c}} \geq \frac{a+b+c}{\frac{a}{a}+\frac{b}{b}+\frac{c}{c}}$ $\quad [\because a, b, c \neq 0] \Rightarrow a^ab^bc^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}$ *(2)*

Now let us take $a, b, c > 0$ with the associated weight b, c, a respectively by applying AM $\geq$ GM we get,

$$\left(\frac{a\cdot b+b\cdot c+c\cdot a}{b+c+a}\right) \geq (a^b \cdot b^c \cdot c^a)^{\frac{1}{a+b+c}} \Rightarrow \left(\frac{ab+bc+ca}{a+b+c}\right)^{a+b+c} \geq a^b b^c c^a \quad (3)$$

Now, $(a+b+c)^2 - 3(ab+bc+ca) = \sum a^2 - \sum ab$

$$= \frac{1}{2}\left\{\sum(a-b)^2\right\} \geq 0 \Rightarrow (a+b+c)^2 \geq 3(ab+bc+ca)$$

$$\Rightarrow \left(\frac{a+b+c}{3}\right) \geq \left(\frac{ab+bc+ca}{a+b+c}\right) \Rightarrow \left(\frac{\sum a}{3}\right)^{a+b+c} \geq \left(\frac{\sum ab}{\sum a}\right)^{a+b+c} \quad (4)$$

Combining (3) & (4) we get,

$$\left(\frac{\sum a}{3}\right)^{a+b+c} \geq \left(\frac{\sum ab}{\sum a}\right)^{a+b+c} \geq a^b b^c c^a \Rightarrow \left(\frac{\sum a}{3}\right)^{a+b+c} \geq a^b b^c c^a \quad (5)$$

Combining (2) & (5) we get $a^a b^b c^c \geq \left(\frac{\sum a}{3}\right)^{a+b+c} \geq a^b b^c c^a \Rightarrow a^a b^b c^c \geq a^b b^c c^a$

$$\Rightarrow a^{a-b} b^{b-c} c^{c-a} \geq 1 \quad \textit{[Proved]}$$

PROBLEM 2.051-*Solution by Marian Ursărescu-Romania*

We must show:

$$\frac{x^{m+2}}{(axy+bxz)^{m+1}} + \frac{y^{m+2}}{(ayz+bxy)^{m+1}} + \frac{z^{m+2}}{(axz+byz)^{m+1}} \geq \frac{3^{m+1}}{(a+b)^{m+1}(x+y+z)^m} \quad (1)$$

From Hölder's inequality we have: $\frac{x^{m+2}}{(axy+bxz)^{m+1}} + \frac{y^{m+2}}{(ayz+bxy)^{m+1}} + \frac{z^{m+2}}{(axz+byz)^{m+1}} \geq$

$$\geq \frac{(x+y+z)^{m+2}}{(axy+bxz+ayz+bxy+axz+byz)^{m+1}} = \frac{(x+y+z)^{m+2}}{(a+b)^{m+1}(xy+xz+yz)^{m+1}} \quad (2)$$

From (1)+(2) we must show: $\frac{(x+y+z)^{m+2}}{(a+b)^{m+1}(xy+xz+yz)^{m+1}} \geq \frac{3^{m+1}}{(a+b)^{m+1}(x+y+z)^m} \Leftrightarrow$

$$\Leftrightarrow ((x+y+z)^2)^{m+1} \geq 3^{m+1}(xy+xz+yz)^{m+1} \Leftrightarrow (x+y+z)^2 \geq 3(xy+xz+yz) \Leftrightarrow$$

$$\Leftrightarrow x^2+y^2+z^2 \geq xy+xz+yz \quad \textit{(true)}$$

PROBLEM 2.052-*Solution by George Apostolopoulos-Messolonghi-Greece*

Using the AM-GM inequality, we have $\frac{1}{(\cos A+\cos B)^2} + \frac{1}{(\cos B+\cos C)^2} + \frac{1}{(\cos C+\cos A)^2} \geq$

$$\geq \frac{3}{\sqrt[3]{\left((\cos A+\cos B)(\cos B+\cos C)(\cos C+\cos A)\right)^2}}$$

It is well – known that in any triangle ABC holds:

$$(\cos A+\cos B)(\cos B+\cos C)(\cos C+\cos A) = \frac{2R^2r^2+r^3+rs^2}{4R^3}$$

Also, we know that $R \geq 2r$ (Euler) and $s = \frac{a+b+c}{2} \leq \frac{3\sqrt{3}}{2}R$. So

$$\frac{1}{(\cos A+\cos B)^2} + \frac{1}{(\cos B+\cos C)^2} + \frac{1}{(\cos C+\cos A)^2} \geq$$

$$\frac{3}{\sqrt[3]{\left(\frac{2Rr^2+r^3+rs^2}{4R^3}\right)^2}} = \frac{3\sqrt[3]{4R^3}^2}{\sqrt[3]{(2Rr^2+r^3+rs^2)^2}} =$$

$$\frac{3R^2\sqrt[3]{16}}{\sqrt[3]{(2Rr^2+r^3+rs^2)^2}} \geq \frac{3R^2\sqrt[3]{16}}{\left(\sqrt[3]{2R\left(\frac{R}{2}\right)^2+\left(\frac{R}{2}\right)^3+\frac{R}{2}\cdot\frac{27R^2}{4}}\right)^2} =$$

$$\frac{3R^2\sqrt[3]{16}}{\sqrt[3]{\left(\frac{R^3}{2}+\frac{R^3}{8}+\frac{27R^3}{8}\right)^2}} = \frac{3R^2\sqrt[3]{16}}{\sqrt[3]{(4R^3)^2}} = \frac{3R^2\sqrt[3]{16}}{\sqrt[3]{16}\cdot R^2} = 3$$

Equality holds when the triangle ABC is equilateral.

PROBLEM 2.053-*Solution by George Apostolopoulos-Messolonghi-Greece*

Let $x+y+z=k>0$. Consider the function $f(t)=\frac{t}{(k-t)^3}, t>0$. Then $f''(t)>0$. So the function f is convex on $(0,+\infty)$. By Jensen's Inequality, we have $f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)$ namely

$$\frac{x}{(y+z)^3}+\frac{y}{(z+x)^3}+\frac{z}{(x+y)^3}\geq 3f\left(\frac{k}{3}\right)=3\cdot\frac{\frac{k}{3}}{\left(k-\frac{k}{3}\right)^3}=\frac{27}{8k^2}=\frac{27}{8(x+y+z)^2}$$

Equality holds when $x=y=z$.

PROBLEM 2.054-*Solution by Yen Thung Chung-Taichung-Taiwan*

$$\int_{-a}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx$$

$$=\int_0^a\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx+\underbrace{\int_{-a}^{0}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx}_{let\ x=-t\Rightarrow dx=-dt}$$

$$=\int_0^a\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx+\int_a^0\frac{f(-t)+g(-t)}{(b-\cos(-t))^m h(-t)+k(-t)\sin^{2n}(-t)}\cdot(-dt)$$

$$=\int_0^a\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx+\int_0^a\frac{-f(t)-g(t)}{(b-\cos t)^m h(t)+k(t)\sin^{2n}t}dt$$

$$=\int_0^a\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx-\int_0^a\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx=0$$

PROBLEM 2.055-*Solution by Soumava Chakraborty-Kolkata-India*

Using Tereshin's Inequality, $m_a\geq\frac{b^2+c^2}{4R}, m_b\geq\frac{c^2+a^2}{4R}, m_c\geq\frac{a^2+b^2}{4R}$

$$\therefore \sum m_a \geq \frac{2\sum a^2}{4R} \quad \text{Again, } \sum \sin^2 A = \frac{\sum a^2}{4R^2}$$
$$\therefore \frac{\sum a}{\sum \sin^2 A} \geq \frac{2\sum a^2}{4R} \cdot \frac{4R^2}{\sum a^2} = 2R \geq 4r \text{ (Euler) (Proved)}$$

PROBLEM 2.056-*Solution by Ravi Prakash-New Delhi-India*

$$(-\sin A + \sin B + \sin C)\left(\frac{1}{\sin B} + \frac{1}{\sin C}\right) = 2 \quad (1)$$
$$\Rightarrow \frac{(\sin B + \sin C)^2}{\sin B \sin C} - \frac{\sin A(\sin B + \sin C)}{\sin B \sin C} = 2 \Rightarrow \sin A = \frac{\sin^2 B + \sin^2 C}{\sin B + \sin C} \quad (1)$$
Also, from (2): $\sin A(\sin B + \sin C) = \sin^2 B + \sin^2 C - \sin^2 A + \sin^2 A$
$$= \sin^2 B + \sin(C - A)\sin(C + A) + \sin^2 A = \sin^2 B + \sin(C - A)\sin B + \sin^2 A$$
$$= \sin B[\sin(C + A) + \sin(C - A)] + \sin^2 A$$
$$\Rightarrow \sin A(-\sin A + \sin B + \sin C) = 2\sin B \cos A \sin C$$
$$\Rightarrow \sin A\left(\frac{2\sin B \sin C}{\sin B + \sin C}\right) = 2\sin B \sin C \cos A \quad [\text{using (1)}]$$
$$\Rightarrow \frac{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\text{cs}\left(\frac{B-C}{2}\right)} = \cos A \Rightarrow \cos A = \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}$$
$$\Rightarrow \cos A \geq \sin\left(\frac{A}{2}\right) > 0 \Rightarrow \cos^2 A \geq \sin^2\left(\frac{A}{2}\right) = \frac{1 - \cos A}{2}$$
$$\Rightarrow 2\cos^2 A + \cos A - 1 \geq 0 \Rightarrow (2\cos A - 1)(\cos A + 1) \geq 0$$
$$\Rightarrow 2\cos A \geq 1 \Rightarrow \cos A \geq \frac{1}{2} \Rightarrow A \leq \frac{\pi}{3}$$

PROBLEM 2.057-*Solution by Shivam Sharma-New Delhi-India*

As we know the following lemma,
If $f(x)$ is a continuous function defined on $[a, b]$, then,
$$\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$$

Using the above lemma, we get,
$$I = \int_a^b \frac{f(a + b - x - a)\big(c + df(b - a - b + x)\big)}{c\big(f(a + b - x - a) + f(b - a - b + x)\big) + 2df(a + b - x - a)f(b - a - b + x)}dx$$
$$\Rightarrow \int_a^b \frac{f(b - x)\big(c + df(x - a)\big)}{c\big(f(b - x) + f(x - a)\big) + 2df(b - x)f(x - a)}dx$$
$$2I = \int_a^b \frac{c\big(f(b - x) + f(x - a)\big) + 2df(b - x)f(x - a)}{c\big(f(b - x) + f(x - a)\big) + 2df(b - x)f(x - a)}dx$$
$$2I = \int_a^b (1)\,dx \Rightarrow [x]_a^b;\ 2I = b - a. \text{ Hence, } I = \frac{b - a}{2} \text{ (Q.E.D)}$$

PROBLEM 2.058-*Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan*

I. The aproximations for $n \to +\infty, \tan\frac{1}{n+k} \approx \frac{1}{n+k}, \cos\frac{1}{n+k} \approx 1$

$$S_n = \sum_{k=1}^{n}\left(\tan\frac{1}{n+k} - \tan\frac{1}{n+k+1}\right)\left(\cos\frac{1}{n+1} + \cos\frac{1}{n+2} + \cdots + \cos\frac{1}{n+k}\right) \approx$$

$$\approx \sum_{k=1}^{n}\left(\frac{1}{n+k} - \frac{1}{n+k+1}\right)\cdot k = \sum_{k=1}^{n}\frac{k}{n+k} - \sum_{k=2}^{n}\frac{k-1}{n+k} = \sum_{k=1}^{n}\frac{1}{n+k} - \frac{n}{2n+1} = E_n$$

Using the fact that the sequence $\sum_{k=1}^{n}\frac{1}{k} - \ln n$ *is convergent with the limit* γ *(the Euler-Mascheroni constant), results*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} = \lim_{n\to\infty}\left(\sum_{k=1}^{2n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k}\right) = \lim_{n\to\infty}[(\gamma + \ln 2n) - (\gamma + \ln n)] = \ln 2$$

$$\lim_{n\to\infty} S_n \approx \lim_{n\to+\infty} E_n = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{n+k}\right) - \lim_{n\to\infty}\frac{n}{2n+1} = \ln 2 - \frac{1}{2}$$

II. The evaluation of the errors. For $0 < x < \frac{\pi}{4}, x < \tan x < x + x^3$*.*

$$\frac{1}{n+k} - \frac{1}{n+k+1} - \frac{1}{(n+k+1)^3} < \tan\frac{1}{n+k} - \tan\frac{1}{n+k+1} < \frac{1}{n+k} - \frac{1}{n+k+1} + \frac{1}{(n+k)^3}.$$

$$k - \left(\cos\frac{1}{n+1} + \cos\frac{1}{n+2} + \cdots + \cos\frac{1}{n+k}\right) = \sum_{i=1}^{n}\left(1 - \cos\frac{1}{n+i}\right) = \sum_{i=1}^{n} 2\sin^2\frac{1}{2(n+i)}$$

$$0 < \sum_{i=1}^{n} 2\sin^2\frac{1}{2(n+i)} < \sum_{i=1}^{n}\frac{1}{2(n+i)^2} < \frac{k}{2(n+1)^2} < \frac{1}{2n} \Rightarrow$$

$$\Rightarrow k - \frac{1}{2n} < \cos\frac{1}{n+1} + \cos\frac{1}{n+2} + \cdots + \cos\frac{1}{n+k} < k$$

$$S_n > \sum_{i=1}^{n}\left(\frac{1}{n+k} - \frac{1}{n+k+1} - \frac{1}{(n+k+1)^3}\right)\left(k - \frac{1}{2n}\right) >$$

$$> E_n - \frac{n^2}{(n+2)^3} - \frac{1}{2n}\cdot\frac{n}{(n+1)(2n+1)} + \frac{1}{2}\cdot\frac{1}{(2n+1)^3} \Rightarrow \lim_{n\to\infty} S_n \geq \lim_{n\to\infty} E_n$$

$$S_n < \sum_{k=1}^{n}\left(\frac{1}{n+k} - \frac{1}{n+k+1} + \frac{1}{(n+k)^3}\right)\cdot k < E_n + \frac{n^2}{(n+1)^3} \Rightarrow \lim_{n\to\infty} S_n \leq \lim_{n\to\infty} E_n$$

Results: $\lim_{n\to\infty} S_n = \lim_{n\to\infty} E_n = \ln 2 - \frac{1}{2}$

PROBLEM 2.059-*Solution by proposer*

Let be $x_n = \frac{1}{n^{p+1}}(1^p + 2^p + \cdots + n^p)$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{1}{n^{p+1}}\sum_{k=1}^{n} k^p = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^p = \int_0^1 f(x)\,dx = \int_0^1 x^p\,dx = \frac{x^{p+1}}{p+1}\Big|_0^1 = \frac{1}{p+1}$$

$$a_{nk} = \frac{1^p + 2^p + \cdots + k^p}{1^p + 2^p + \cdots + n^p}; \lim_{n\to\infty} a_{nk} = 0$$

By Toeplitz's theorem:

$$\lim_{n\to\infty}\sum_{k=1}^{n} a_{nk}x_k = \lim_{n\to\infty} x_n = \frac{1}{p+1}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} a_{nk}x_k = \lim_{n\to\infty}\frac{1^p+2^p+\cdots+k^p}{1^p+2^p+\cdots+n^p}\cdot\frac{1}{n^{p+1}}(1^p+2^p+\cdots+k^p) =$$

$$= \lim_{n\to\infty}\frac{(1^p+2^p+\cdots+k^p)^2}{n^{p+1}(1^p+2^p+\cdots+n^p)} = \frac{1}{p+1}$$

PROBLEM 2.060-*Solution by Soumava Chakraborty-Kolkata-India*

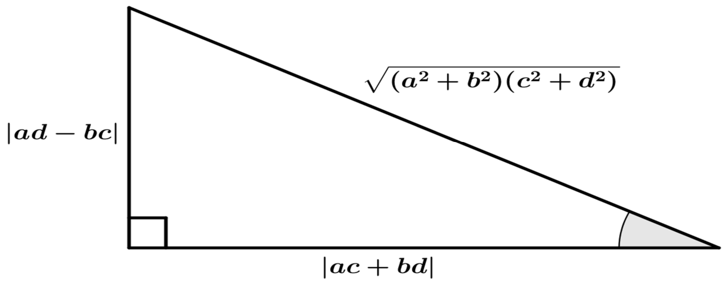
$$(ac+bd)^2+(ad-bc)^2 \overset{(a)}{=} (a^2+b^2)(c^2+d^2)$$

$$\therefore 3(a^2+b^2)(c^2+d^2)-4(ad-bc)^2$$

$$= 3\{(ac+bd)^2+(ad-bc)^2\}-4(ad-bc)^2$$

$$\overset{(1)}{=} 3(ac+bd)^2-(ad-bc)^2 = 3|ac+bd|^2-|ad-bc|^2$$

Case 1: $ac+bd\neq 0, a-bc\neq 0$



$\therefore |ac+bd| = p\cos\theta$ *(2)*

$|ad-bc| = p\sin\theta$ *(3), where,* $p=\sqrt{(a^2+b^2)(c^2+d^2)}$

$\therefore LHS = \frac{(ad-bc)(3p^2\cos^2\theta-p^2\sin^2\theta)}{p^3}$ *(using (1), (2), (3))* $\overset{(4)}{=} \frac{(ad-bc)(3\cos^2\theta-\sin^2\theta)}{p}$

Now, according as $ad-bc\geq 0$ *or* $ad-bc<0$

$ad-bc=\pm|ad-bc|$ *(6) Again* $\frac{|ad-bc|}{p}=\sin\theta$

$\therefore LHS = \pm\sin\theta(3\cos^2\theta-\sin^2\theta)$ *(using (4), (5), (6))*

$= \pm\sin\theta(3(1-\sin^2\theta)-\sin^2\theta) = \pm(3\sin\theta-4\sin^3\theta) = \pm\sin 3\theta$

When $LHS=\sin 3\theta$, *then* $LHS\leq 1$, $\because \sin 3\theta\leq 1$

$LHS=-\sin 3\theta$, *then, also,* $LHS\leq 1$, $\because \sin 3\theta\geq -1$

$\therefore LHS=\pm\sin 3\theta\leq 1$ *(proved under case (1))*

Case 2: $ad-bc=0$ *(a)* $\Rightarrow ac+bd\neq 0$ *Then,* $LHS=0\leq 1$

Case 3: $ac+bd=0$ *(a)* $\Rightarrow ad-bc\neq 0$ $(\because (a^2+b^2)(c^2+d^2)\neq 0)$

$\therefore LHS=\frac{-(ad-bc)}{|ad-bc|}=\pm 1\leq 1$. *Hence, in all 3 cases,* $LHS\leq 1$. *(Done)*

PROBLEM 2.061-*Solution by proposer*

From the given condition and by the AM-GM inequality, we obtain

$$\frac{1}{1+x_1}=\frac{2x_1}{1+x_2}+\frac{3x_3}{1+x_3}+\cdots+\frac{nx_n}{1+x_n}$$
$$\geq(2+3+\cdots+n)\sqrt[2+3+\cdots+n]{\left(\frac{x_2}{1+x_2}\right)^2\left(\frac{x_3}{1+x_3}\right)^3\ldots\left(\frac{x_n}{1+x_n}\right)^n}$$
$$\frac{1}{1+x_2}=\frac{x_1}{1+x_1}+\frac{x_2}{1+x_2}+\frac{3x_3}{1+x_3}+\cdots+\frac{nx_n}{1+x_n}$$
$$\geq(2+3+\cdots+n)\sqrt[2+3+\cdots+n]{\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)\left(\frac{x_3}{1+x_3}\right)^3\ldots\left(\frac{x_n}{1+x_n}\right)^n}$$
$$\frac{1}{+x_3}=\frac{x_1}{1+x_1}+\frac{2x_2}{1+x_2}+\frac{2x_3}{1+x_3}+\cdots+\frac{nx_n}{1+x_n}$$
$$\geq(2+3+\cdots+n)\sqrt[2+3+\cdots n]{\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)^2\left(\frac{x_3}{1+x_3}\right)^2\ldots\left(\frac{x_n}{1+x_n}\right)^n}$$
$$\ldots\quad\ldots\quad\ldots$$
$$\frac{1}{1+x_n}=\frac{x_1}{1+x_1}+\frac{2x_2}{1+x_2}+\cdots+\frac{(n-1)x_{n-1}}{1+x_{n-1}}+\frac{(n-1)x_n}{1+x_n}$$
$$\geq(2+3+\cdots+n)\sqrt[2+3+\cdots n]{\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)^2\ldots\left(\frac{x_{n-1}}{1+x_{n-1}}\right)^{n-1}\left(\frac{x_n}{1+x_n}\right)^{n-1}}$$

From these relations above, we infer that

$$\frac{1}{1+x_1}\cdot\frac{1}{(1+x_2)^2}\cdot\frac{1}{(1+x_3)^3}\cdots\frac{1}{(1+x_n)^n}\geq$$
$$(2+3+\cdots+n)^{1+2+\cdots+n}\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)^2\left(\frac{x_3}{1+x_3}\right)^3\ldots\left(\frac{x_n}{1+x_n}\right)^n$$

Which implies that $x_1x_2^2\ldots x_n^n\leq\frac{1}{(2+3+\cdots+n)^{1+2+\cdots+n}}$

The equality holds if and only if: $x_1=x_2=\cdots=x_n=\frac{1}{2+3+\cdots+n}$

Thus $\max P=\frac{1}{(2+3+\cdots+n)^{1+2+\cdots+n}}$

PROBLEM 2.062-*Solution by proposer*

$$|a^3+b^3+c^3-3abc|=|(a+b+c)(a+b\varepsilon+c\varepsilon^2)(a+b\varepsilon+c\varepsilon)|\leq$$
$$\leq|a+b+c|(|a|+|b\varepsilon|+|c\varepsilon^2|)(|a|+|b\varepsilon|+|c\varepsilon|)=$$
$$=|a+b+c|(|a|+|b|+|c|)(|a|+|b|+|c|)=|a+b+c|(|a|+|b|+|c|)^2\text{ when:}$$
$$\varepsilon=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

PROBLEM 2.063-*Solution by Kevin Soto Palacios – Huarmey – Peru*

$$\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\geq$$
$$\geq\sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\frac{1}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)$$

En un $\Delta\ ABC\rightarrow\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}>0$

Dividiendo $(\div)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ *a la desigualdad propuesta*

$$\Leftrightarrow \sec\frac{C}{2}+\sec\frac{A}{2}+\sec\frac{B}{2}\geq\sqrt{3}+\frac{1}{2}\left(\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}\right)$$

$$\Leftrightarrow \sec\frac{C}{2}+\sec\frac{A}{2}+\sec\frac{B}{2}\geq\sqrt{3}+\frac{1}{2}\left(\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\left(\frac{C+A}{2}\right)}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\left(\frac{A+B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}\right)$$

$$\Leftrightarrow \sec\frac{C}{2}+\sec\frac{A}{2}+\sec\frac{B}{2}\geq\sqrt{3}+\frac{1}{2}\left(\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\right)$$

$$\Leftrightarrow \sec\frac{A}{2}-\tan\frac{A}{2}+\sec\frac{B}{2}-\tan\frac{B}{2}+\sec\frac{C}{2}-\tan\frac{C}{2}\geq\sqrt{3}$$

Calculamos la primera y segunda derivada

Sea $f(x)=\sec x-\tan x\,,x\in<0,\frac{\pi}{2}>,f'(x)=\sec x\tan x-\sec^2 x,$

$$f''(x)=\sec x\,(\sec x-\tan x)^2>0$$

Como $f''(x)>0$, *entonces* $f(x)$ *es estrictamente convexo en* $<0,\frac{\pi}{2}>$

Dado que $\frac{A}{2},\frac{B}{2},\frac{C}{2}\in<0,\frac{\pi}{2}>$ *de tal manera que* $A+B+C=\pi$.

Aplicamos la desigualdad de Jensen

$$\sec\frac{A}{2}-\tan\frac{A}{2}+\sec\frac{B}{2}-\tan\frac{B}{2}+\sec\frac{C}{2}-\tan\frac{C}{2}=f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)\geq$$

$$\geq 3f\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)=3f\left(\frac{\pi}{6}\right)$$

$$\Leftrightarrow f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)\geq 33f\left(\frac{\pi}{6}\right)=3\left(\sec\frac{\pi}{6}-\tan\frac{\pi}{6}\right)=3\left(\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)=\sqrt{3}$$

$$\Rightarrow \sec\frac{A}{2}-\tan\frac{A}{2}+\sec\frac{B}{2}-\tan\frac{B}{2}+\sec\frac{C}{2}-\tan\frac{C}{2}\geq\sqrt{3}$$

PROBLEM 2.064-*Solution by Soumitra Mandal-Chandar Nagore-India*

Let $f(x)=x-x^3$ *for all* $x\in(0,1), f'(x)=1-3x^2, f''(x)=-6x$

$\because f'(x_0)=0$ *where* $x_0\in(0,1)\Rightarrow x_0=\pm\frac{1}{\sqrt{3}}$ *choosing* $x_0=\frac{1}{\sqrt{3}}$

$\therefore f''\left(\frac{1}{\sqrt{3}}\right)<0$ *hence* f *attains maximum at* $x=\frac{1}{\sqrt{3}}\Rightarrow f(x)\leq f\left(\frac{1}{\sqrt{3}}\right)$

$$\therefore \sum_{cyc}\left(\frac{yz}{1-x^2}\right)^{2n}=(xyz)^{2n}\sum_{cyc}\frac{1}{(x-x^3)^{2n}}\geq(xyz)^{2n}\frac{3}{\left(\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}}\right)^{2n}}=\frac{3^{3n+1}}{4^n}(xyz)^{2n}\quad (proved)$$

PROBLEM 2.065-*Solution by Sanong Huayrerai-Nakon Pathom-Thailand*

Prove that $x^n+y^n+z^n\geq x^{n-1}z+z^{n-1}y+y^{n-1}x;\ x,y,z>0,n\in\mathbb{N}$

Since $x^n+x^n+x^n+\cdots+x^n(n-1)term+2^n\geq nx^{n-1}z$

$$z^n+z^n+z^n+\cdots+z^n(n-1)term+y^n\geq nz^{n-1}y$$

$$y^n+y^n+y^n+\cdots+y^n(n-1)term+x^n\geq ny^{n-1}x$$

Hence $n(x^n+y^n+z^n)\geq n(x^{n-1}z+z^{n-1}y+y^{n-1}x)$

$$x^n+y^n+z^n\geq x^{n-1}+z^{n-1}y+y^{n-1}x$$

Solution

For $a,b,c>0$ *and* $n\in\mathbb{N}$, *we have*

$$\frac{a^n+b^n}{2}\ge\left(\frac{a+b}{2}\right)^n,\frac{b^n+c^n}{2}\ge\left(\frac{b+c}{2}\right)^n,\frac{c^n+a^n}{2}\ge\left(\frac{c+a}{2}\right)^n$$

$$\text{Hence } a^n+b^n+c^n\ge\left(\tfrac{a+b}{2}\right)^n+\left(\tfrac{b+c}{2}\right)^n+\left(\tfrac{c+a}{2}\right)^n\Rightarrow$$

$$\Rightarrow 2^n(a^n+b^n+c^n)\ge(a+b)^n+(b+c)^n+(c+a)^n\ge$$

$$\ge(a+b)^{n-1}(c+a)+(b+c)^{n-1}(b+a)+(c+a)^{n-1}(c+b)$$

Therefore it is to be true.

PROBLEM 2.066-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x^t}=\lim_{\substack{x\to\infty\\ n\in\mathbb{N}}}\frac{\sqrt[n]{f(n)}}{n^t}=\lim_{x\to\infty}\sqrt[n]{\frac{f(n)}{n^{nt}}}\overset{CESARO-STOLZ}{\cong}\lim_{x\to\infty}\frac{\frac{f(n+1)}{(n+1)^{(n+1)t}}}{\frac{f(n)}{n^{nt}}}$$

$$=\lim_{x\to\infty}\left(\frac{f(n+1)}{n^tf(n)}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{nt}}\cdot\left(\frac{n}{n+1}\right)^t\right)=\frac{1}{e^t}\cdot\lim_{x\to\infty}\frac{f(n+1)}{n^tf(n)}=\frac{1}{e^t}\cdot\lim_{x\to\infty}\frac{f(x+1)}{x^tf(x)}$$

PROBLEM 2.067-*Solution by proposer*

$$\text{For all } x,y,z>0 \text{ we have: }\frac{x}{3x^2+2y^2+z^2}\ge\frac{1}{18}\left(\frac{2}{y}+\frac{1}{z}\right)\Leftrightarrow$$

$$\Leftrightarrow\frac{3x^2y+6x^2z+2y^3+2z^3+4y^2z+yz^2}{18}\ge\sqrt[18]{(x^2y)^3(x^2z)(y^3)^2(z^3)^2(y^2z)^4yz^2}=$$

$$=xyz \text{ and }\frac{x}{x^2+yz}\le\frac{1}{4}\left(\frac{1}{y}+\frac{1}{z}\right)\Leftrightarrow y(x-z)^2+z(x-y)^2\ge0\Rightarrow$$

$$\frac{x^2}{(x^2+yz)(3x^2+2y^2+z^2)}\le\frac{1}{72}\left(\frac{2}{y}+\frac{1}{z}\right)\left(\frac{1}{y}+\frac{1}{z}\right)=\frac{(y+z)(y+2z)}{72y^2z^2}\Rightarrow$$

$$\sum_{cyclic}\frac{1}{(x^2+yz)(3x^2+2y^2+z^2)}\le\sum_{cyclic}\frac{(y+z)(y+2z)}{72x^2y^2z^2}=\frac{3\sum x^2+3\sum xy}{72x^2y^2z^2}=\frac{\sum x^2+\sum xy}{24x^2y^2z^2}$$

PROBLEM 2.068-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\Omega=\lim_{n\to\infty}\left(\sqrt[m(n+1)]{(2n+1)!!}-\sqrt[mn]{(2n-1)!!}\right)n^{\frac{m-1}{m}}$$

$$\lim_{n\to\infty}\left(\frac{\sqrt[mn]{(2n-1)!!}}{\sqrt[m]{(n-1)}}\cdot\sqrt[m]{\left(1-\frac{1}{n}\right)}\frac{u_m-1}{\ln u_n}\cdot\ln u_n^n\right)\text{ where } u_n=\frac{\sqrt[m(n+1)]{(2n+1)!!}}{\sqrt[mn]{(2n-1)!!}}\text{ for all } n\in\mathbb{N}$$

$$\text{Now, }\lim_{n\to\infty}\sqrt[m]{\sqrt[n]{\frac{(2n-1)!!}{(n-1)^m}}}\overset{D'ALEMBERT}{\cong}\lim_{n\to\infty}\sqrt[m]{\frac{(2n+1)!!}{n^{n+1}}\cdot\frac{(n-1)^n}{(2n-1)!!}}$$

$$= \lim_{n\to\infty} \sqrt[m]{\frac{\frac{(2n+1)!}{2^m n!}}{\frac{(2n-1)!}{2^{m-1}(n-1)!}} \cdot \left(1-\frac{1}{n}\right)^n \cdot \frac{1}{n}} = \lim_{n\to\infty} \sqrt[m]{\frac{2n(2n-1)}{2n(n-1)} \cdot \left(1-\frac{1}{n}\right)^n} = \sqrt[m]{\frac{2}{e}}$$

$$\lim_{n\to\infty} u_m = \lim_{n\to\infty} \frac{\frac{\sqrt[m(n+1)]{(2n+1)!!}}{\sqrt[m]{n}}}{\frac{\sqrt[mn]{(2n-1)!!}}{\sqrt[m]{n-1}}} \cdot \sqrt[m]{\frac{n}{n-1}} = 1\text{ . Hence } \lim_{n\to\infty} \frac{u_m - 1}{\ln u_n} = 1$$

$$\text{Now, } \lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \sqrt[m]{\frac{(2n+1)!!}{(2n-1)!!}} \cdot \frac{1}{\sqrt[m(n+1)]{(2n+1)!!}}$$

$$= \lim_{n\to\infty} \sqrt[m]{\frac{\frac{(2n+1)!!}{n^n}}{\frac{(2n-1)!!}{(n-1)^{n-1}}} \cdot \frac{1}{\left(1-\frac{1}{n}\right)^m}} \cdot \sqrt[m]{1-\frac{1}{n}} \cdot \frac{1}{\frac{\sqrt[m(n+1)]{(2n+1)!!}}{\sqrt[m]{n}}} = \sqrt[m]{e}$$

$$\therefore \Omega = \sqrt[m]{\frac{2}{e}} \cdot 1 \cdot \ln \sqrt[m]{e} = \frac{1}{m} \sqrt[m]{\frac{2}{e}} \textit{ (proved)}$$

PROBLEM 2.069-*Solution by proposer*

$$\text{For all } y,z,t>0 \text{ we have: } \frac{t}{t^2+yz} \le \frac{1}{y}\left(\frac{1}{y}+\frac{1}{z}\right) \Leftrightarrow y(t-z)^2 + z(y-t)^2 \ge 0$$

$$\text{therefore } \int_a^b \frac{tdt}{t^2+y} \le \int_a^b \frac{1}{4}\left(\frac{1}{y}+\frac{1}{z}\right) dt \Rightarrow \frac{1}{2}\ln\frac{b^2+yz}{a^2+yz} \le \frac{1}{4}\left(\frac{1}{y}+\frac{1}{z}\right)(b-a) \Rightarrow$$

$$\sum_{cyclic} \ln\frac{b^2+yz}{a^2+yz} \le \frac{1}{2}\sum_{cyclic}\left(\frac{1}{y}+\frac{1}{z}\right)(b-a) = (b-a)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

PROBLEM 2.070-*Solution by Nguyen Phuc Tang-Hanoi-Vietnam*

$$(a^2+2)(b^2+2) = (a+b+1)^2 + (a-1)^2 + (b-1)^2 + (ab-1)^2 \ge$$
$$\ge (1-a^2) + (1-b)^2 + (ab-1)^2. \textit{ By Cauchy}-\textit{Schwarz}$$
$$(a^2+2)(b^2+2)(c^2+2) \ge [(1-a)^2 + (1-b)^2 + (ab-1)^2](1+1+c^2) \ge$$
$$\ge (2-a-b-c+abc)^2$$
$$\text{Equality holds if} \begin{cases} a+b+1=0 \\ 1-a=1-b=\frac{ab-1}{c} \end{cases} \Leftrightarrow a=b=c=-\frac{1}{2}$$

PROBLEM 2.071-*Solution by Ravi Prakash-New Delhi-India*

$$3abc - a^3 - b^3 - c^3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \Rightarrow (3abc - a^3 - b^3 - c^3)^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$
$$= \begin{vmatrix} x & y & y \\ y & x & y \\ y & y & x \end{vmatrix} = x^3 + 2y^3 - 3xy^2 \text{ where } x = a^2+b^2+c^2$$
$$y = bc + ca + ab = x^3 - y^2(2x-2y) - xy^2$$
$$= (a^2+b^2+c^2)^3 - (ab+bc+ca)^2(a^2+b^2+c^2) -$$
$$-(ab+bc+ca)^2\{(a-b)^2 + (b-c)^2 + (c-a)^2\} \le (a^2+b^2+c^2)^3$$

PROBLEM 2.072-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\sum_{cyc}\left(\frac{2na}{b+(2n-1)c}\right)^{\frac{2}{3}}=\sum_{cyc}\frac{2na}{\sqrt[3]{2na(b+(2n-1)c)^2}}\overset{AM\geq GM}{\geq}\sum_{cyc}\frac{6na}{2na+2b+2(2n-1)c}$$
$$=3n\sum_{cyc}\frac{a}{na+b+(2n-1)c}=3n\sum_{cyc}\frac{a^2}{na^2+ab+(2n-1)ca}\geq$$
$$\geq 3n\frac{(a+b+c)^2}{n\sum_{cyc}a^2+\sum_{cyc}ab+(2n-1)\sum_{cyc}ab}=3\ (proved)$$

PROBLEM 2.073-*Solution by Tran Hong-Vietnam*

$$Let\ f(t)=\log\left(1+\frac{1}{t}\right) for\ t>0$$
$$\Rightarrow f'(t)=\frac{\left(1+\frac{1}{t}\right)'}{\left(1+\frac{1}{t}\right)\ln 10}=-\frac{1}{t(t+1)\ln 10}\Rightarrow f''(t)=\frac{1}{\ln 10}\cdot\frac{2t+1}{[t(t+1)]^2}>0\ \forall t>0$$
$$\Rightarrow using\ Jensen's\ inequality\ we\ have$$
$$LHS=f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)=3\log\left(1+\frac{3}{x+y+z}\right)$$
$$Proved.\ Equality\ y\Leftrightarrow x=y=z.$$

PROBLEM 2.074-*Solution by proposer*

** By AM-GM inequality we have:*

$$\frac{x^4}{y^4\cdot\sqrt[3]{4z(x^5+1)}}=\frac{x^4}{y^4\cdot\sqrt[3]{4z(x+1)(x^4-x^3+x^2-x+1)}}=$$
$$=\frac{x^4}{y^4\cdot\sqrt[3]{2(zx+z)(2x^4-2x^3+2x^2-2x+2)}}\geq$$
$$\geq\frac{x^4}{y^4\left(\frac{2+zx+z+2x^4-2x^3+2x^2-2x+2}{3}\right)}=$$
$$=\frac{3x^4}{y^4(2x^4-2x^3+2x^2+zx-2x+z+4)}\Rightarrow$$
$$\Rightarrow\frac{x^4}{y^4\cdot\sqrt[3]{4z(x^5+1)}}\geq\frac{3x^4}{y^4(2x^4-2x^3+2x^2+zx-2x+z+4)}$$
$$+\ Similar:\frac{y^4}{z^4\cdot\sqrt[3]{4x(y^5+1)}}\geq\frac{3y^4}{z^4(2y^4-2y^3+2y^2+xy-2y+x+4)}$$
$$\frac{z^4}{x^4\cdot\sqrt[3]{4y(z^5+1)}}\geq\frac{3z^4}{x^4(2z^4-2z^3+2z^2+yz-2z+y+4)}$$
$$-\ Hence:\Rightarrow\frac{P}{3}=\frac{1}{3}\left(\frac{x^4}{y^4\cdot\sqrt[3]{4z(x^5+1)}}+\frac{y^4}{z^4\cdot\sqrt[3]{4x(y^5+1)}}+\frac{z^4}{x^4\cdot\sqrt[3]{4y(z^5+1)}}\right)\geq$$
$$\geq\frac{x^4}{y^4(2x^4-2x^3+2x^2+zx-2x+z+4)}+\frac{y^4}{z^4(2y^4-2y^3+2y^2+xy-2y+x+4)}+\frac{z^4}{x^4(2z^4-2z^3+2z^2+yz-2z+y+4)}\quad (1)$$

- Other, by Cauchy Schwarz inequality we have:

$$\frac{x^4}{y^4(2x^4-2x^3+2x^2+zx-2x+z+4)}+\frac{y^4}{z^4(2y^4-2y^3+xy-2y+x+4)}+$$
$$+\frac{z^4}{x^4(2z^4-2z^3+2z^2+yz-2z+y+4)}=$$
$$=\frac{\left(\frac{x^2}{y^2}\right)^2}{2x^4-2x^3+2x^2+zx-2x+z+4}+\frac{\left(\frac{y^2}{z^2}\right)^2}{2y^4-2y^3+2y^2+xy-2y+x+4}+$$
$$+\frac{\left(\frac{z^2}{x^2}\right)^2}{2z^4-2z^3+2z^2+yz-2z+y+4}\geq$$
$$\geq\frac{\left(\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)^2}{(2x^4-2x^3+2x^2+zx-2x+z+4)+(2y^4-2y^3+2y^2+xy-2y+x+4)+(2z^4-2z^3+2z^2+yz-2z+y+4)}\quad (2)$$

- Let (1), (2):

$$\Rightarrow\frac{P}{3}\geq\frac{\left(\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)^2}{(2x^4-2x^3+2x^2+zx-2x+z+4)+(2y^4-2y^3+2y^2+xy-2y+x+4)+(2z^4-2z^3+2z^2+yz-2z+y+4)}\quad (3)$$

- By AM-GM inequality and $x+y+z=3$. We have:

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}=\frac{\frac{x^2}{y^2}+\frac{x^2}{y^2}+\frac{y^2}{z^2}}{3}+\frac{\frac{y^2}{z^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}}{3}+\frac{\frac{z^2}{x^2}+\frac{z^2}{x^2}+\frac{x^2}{y^2}}{3}\geq$$
$$\geq\frac{3\cdot\sqrt[3]{\frac{x^2}{y^2}\cdot\frac{x^2}{y^2}\cdot\frac{y^2}{z^2}}}{3}+\frac{3\cdot\sqrt[3]{\frac{y^2}{z^2}\cdot\frac{y^2}{z^2}\cdot\frac{z^2}{x^2}}}{3}+\frac{3\cdot\sqrt[3]{\frac{z^2}{x^2}\cdot\frac{z^2}{x^2}\cdot\frac{x^2}{y^2}}}{3}=\sqrt[3]{\frac{x^4}{y^2z^2}}+\sqrt[3]{\frac{y^4}{z^2x^2}}+\sqrt[3]{\frac{z^4}{x^2y^2}}$$
$$=\frac{x^2+y^2+z^2}{\sqrt[3]{x^2y^2z^2}}$$
$$\Rightarrow\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq\frac{x^2+y^2+z^2}{\sqrt[3]{x^2y^2z^2}}\geq\frac{x^2+y^2+z^2}{\left(\frac{x+y+z}{3}\right)^2}=\frac{x^2+y^2+z^2}{\left(\frac{3}{3}\right)^2}=x^2+y^2+z^2\quad (4)$$

- Let (3), (4): $\Rightarrow\frac{P}{3}\geq\frac{(x^2+y^2+z^2)^2}{2(x^4+y^4+z^4)-2(x^3+y^3+z^3)+2(x^2+y^2+z^2)+(xy+yz+zx)-(x+y+z)+12}\quad (5)$

We will prove:

$$\frac{(x^2+y^2+z^2)^2}{2(x^4+y^4+z^4)-2(x^3+y^3+z^3)+2(x^2+y^2+z^2)+(xy+yz+zx)-(x+y+z)+12}\geq\frac{1}{2}\quad (6)$$
$$\Leftrightarrow 2(x^2+y^2+z^2)^2\geq 2(x^4+y^4+z^4)-2(x^3+y^3+z^3)+$$
$$+2(x^2+y^2+z^2)+2(x^2+y^2+z^2)+(xy+yz+zx)-(x+y+z)+12$$
$$\Leftrightarrow 2(x^3+y^3+z^3)+4(x^2y^2+y^2z^2+z^2x^2)\geq 2(x^2+y^2+z^2)+xy+yz+zx-3+12$$
$$(x+y+z=3)$$
$$\Leftrightarrow 2(x^3+y^3+z^3)+4(x^2y^2+y^2z^2+z^2x^2)\geq 2(x^2+y^2+z^2)+xy+yz+zx+9$$
$$\Leftrightarrow 18(x^3+y^3+z^3)+36(x^2y^2+y^2z^2+z^2x^2)\geq$$
$$18(x^2+y^2+z^2)+9(xy+yz+zx)+81$$
$$\Leftrightarrow 6(x+y+z)(x^3+y^3+z^3)+36(x^2y^2+y^2z^2+z^2x^2)\geq$$
$$\geq 2(x+y+z)^2(x^2+y^2+z^2)+(x+y+z)^2(xy+yz+zx)+(x+y+z)^4$$

(because $x+y+z=3$ then: $18=6(x+y+z)$; $18=2(x+y+z)^2$; $81=(x+y+z)^4$*)*

$$\Leftrightarrow 6(x^4+y^4+z^4)+6\big(xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\big)$$
$$+36(x^2y^2+y^2z^2+z^2x^2)\geq 3(x+y+z)^2(x^2+y^2+z^2+xy+yz+zx)$$
$$\Leftrightarrow 2(x^4+y^4+z^4)+2\big(xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\big)$$
$$+12(x^2y^2+y^2z^2+z^2x^2)\geq$$
$$\geq (x^2+y^2+z^2+2xy+2yz+2zx)(x^2+y^2+z^2+xy+yz+zx)$$
$$\Leftrightarrow 2(x^4+y^4+z^4)+2\big(xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\big)$$
$$+12(x^2y^2+y^2z^2+z^2x^2)\geq$$
$$\geq x^4+y^4+z^4+3\big(xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\big)+4\big(x^2y^2+y^2z^2+z\ \ x^2\big)$$
$$+7xyz(x+y+z)$$
$$\Leftrightarrow x^4+y^4+z^4+8(x^2y^2+y^2z^2+z^2x^2)\geq xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)+ 7xyz(x+y+z)\quad (7)$$

- We have:

$$(x^2+y^2+z^2-xy-yz-zx)^2\geq 0\Leftrightarrow (x^2+y^2+z^2)+(xy+yz+zx)^2$$
$$\geq 2(x^2+y^2+z^2)(xy+yz+zx)$$
$$\Leftrightarrow x^4+y^4+z^4+3(x^2y^2+y^2z^2+z^2x^2)\geq 2\big(y(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\big)\quad (8)$$

- By AM-GM inequality:

$$xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\geq xy\cdot 2xy+yz\cdot 2yz+zx\cdot 2zx=2(x^2y^2+ y^2z^2+z^2x^2)\quad (9)$$

- Let (8), (9): $\Rightarrow x^4+y^4+z^4+3(x^2y^2+y^2z^2+z^2x^2)\geq xy(x^2+y^2)+yz(y^2+z^2)+ zx(z^2+x^2)+2(x^2y^2+y^2z^2+z^2x^2)$

$$\Leftrightarrow x^4+y^4+z^4+(x^2y^2+y^2z^2+z^2x^2)\geq xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\quad (10)$$

+ Other: $x^2y^2+y^2z^2+z^2x^2=\frac{x^2(y^2+z^2)}{2}+\frac{y^2(z^2+x^2)}{2}+\frac{z^2(x^2+y^2)}{2}\geq\frac{x^2\cdot 2yz}{2}+\frac{y^2\cdot 2zx}{2}+\frac{z^2\cdot 2xy}{2}$

$$\Rightarrow x^2y^2+y^2z^2+z^2x^2\geq xyz(x+y+z)\Leftrightarrow xyz(x+y+z)\Leftrightarrow 7(x^2y^2+y^2z^2+z^2x^2)\geq 7xyz(x+y+z)\quad (11)$$

− Hence (10), (11):

$$\Rightarrow x^4+y^4+z^4+8(x^2y^2+y^2z^2+z^2x^2)$$
$$\geq xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)+7xyz(x+y+z)$$

$\Rightarrow$ *Inequality (7) true* $\Rightarrow$ *(6) true.*

− Let (5), (6): $\Rightarrow \frac{P}{3}\geq\frac{1}{2}\Rightarrow P\geq\frac{3}{2}\Rightarrow P_{Min}=\frac{3}{2}$*. Equality occurs if:*

$$\begin{cases}x,y,z>0;x+y+z=3\\ x=y=z\end{cases}\Leftrightarrow x=y=z=1$$

PROBLEM 2.075-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Aplicando la desigualdad de Cauchy

$$\frac{x^2}{x(y^3+z^3)+x}+\frac{y^2}{y(z^3+x^3)+y}+\frac{z^2}{z(x^3+y^3)+z}\geq$$
$$\geq\frac{(\sum x)^2}{\sum xy(x^2+y^2)+\sum x}=\frac{(\sum x)^2}{\sum xy(x^2+y^2)+(\sum x)xyz}=\frac{(\sum x)^2}{(\sum x^2)(\sum xy)}$$

Como $x,y,z>0$*. Aplicando MA ≥ MG*

$$\frac{(\sum x)^2(\sum xy)}{(\sum x^2)(\sum xy)(\sum xy)} \geq \frac{3(\sum x)^2}{\left(\frac{\sum x^2+\sum xy+\sum xy}{3}\right)^3} = \frac{81(\sum x)^2}{(\sum x)^6} = \frac{81}{(x+y+z)^4}$$

$$P = 2(x+y+z) + \frac{x}{y^3+z^3+1} + \frac{y}{z^3+x^3+1} + \frac{z}{x^3+y^3+1} \geq$$

$$\geq 2(x+y+z) + \frac{81}{(x+y+z)^4}$$

Nuevamente por MA ≥ MG

$$2(x+y+z) + \frac{81}{(x+y+z)^4} = \left(\frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{81}{(x+y+z)^4}\right) +$$

$$+ \frac{2(x+y+z)}{3} \geq 5+2 = 7$$

Por transitividad $\rightarrow P = 2(x+y+z) + \frac{x}{y^3+z^3+1} + \frac{y}{z^3+x^2+1} + \frac{z}{x^3+y^3+1} \geq 7$

La igualdad se alcanza cuando $x = y = z = 1$

PROBLEM 2.076-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Como es un triángulo acutángulo $\rightarrow \cos A, \cos B, \cos C > 0$. *Recordar las siguientes identidades en un triángulo* ABC

$$b^2+c^2-a^2 = 2bc\cos A,\ c^2+a^2-b^2 = 2ca\cos B,\ a^2+b^2-c^2 = 2ab\cos C$$

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C,\ abc = 4RS,$$

$$16S^2 = (a+b+c)(b+c-a)(c+a-b)(b+a-c)$$

$\tan A + \tan B + \tan C = \tan A\tan B\tan C$. *Lo cual es equivalente*

$$\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \frac{\sin A\sin B\sin C}{\cos A\cos B\cos C} \Leftrightarrow$$

$$\Leftrightarrow \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B = \sin A\sin B\sin C$$

Probaremos lo siguiente $(b+c-a)(a+c-b)(b+a-c) \geq$

$$\geq (b^2+c^2-a^2)^a \cdot a^a \cdot (c^2+a^2-b^2)^b b^b (a^2+b^2-c^2)^c c^c$$

$$\Leftrightarrow (b+c-a)(a+c-b)(b+a-c) \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

Recordar $\rightarrow 2p = a+b+c = 1$. *Aplicando la desigualdad ponderada MA ≥ MG*

$$\frac{(2abc\cos A)a + (2abc\cos B)b + (2abc)\cos C}{a+b+c} \geq$$

$$\geq \sqrt[a+b+c]{(2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c}$$

$$\Leftrightarrow 2abc(a\cos A + b\cos B + c\cos C) \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$\Leftrightarrow 2abcR(\sin 2A + \sin 2B + 2\sin 2C) \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$\Leftrightarrow 2abc\ R(4\sin A\sin B\sin C) \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$\Leftrightarrow 2abcR\left(\frac{abc}{2R^3}\right) \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$\Leftrightarrow \frac{a^2b^2c^2}{R^2} \geq (2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c$$

$$(2abc\cos A)^a(2abc\cos B)^b(2abc\cos C)^c \leq 16S^2 =$$

$$= (a+b+c)(b+c-a)(a+c-b)(b+a-c) = (b+c-a)(a+c-b)(b+a-c)$$

(LQQD). Por último, probaremos

$$a^ab^bc^c(b+c-a)(a+c-b)(b+a-c) \geq$$

$$\geq (b^2+c^2-a^2)^{b+c}(c^2+a^2-b^2)^{c+a}(a^2+b^2-c^2)^{a+b}$$
$$\Leftrightarrow a^a b^b c^c (b+c-a)(a+c-b)(b+a-c) \geq$$
$$\geq (2bc\cos A)^{b+c}(2ca\cos B)^{c+a}(2ab\cos C)^{a+b}$$
$$\Leftrightarrow (b+c-a)(a+c-b)(b+a-c) \geq$$
$$\left[\frac{(2ab\cos C)(2ca\cos B)}{a}\right]^a \left[\frac{(2bc\cos A)(2ab\cos C)}{b}\right]^b \left[\frac{(2ca\cos B)(2bc\cos A)}{c}\right]^c$$
$$\Leftrightarrow (b+c-a)(a+c-b)(b+a-c) \geq$$
$$\geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$

Utilizando la desigualdad ponderada $MA \geq MG$

$$\frac{(4abc\cos B\cos C)a+(4abc\cos C\cos A)b+(4abc\cos A\cos B)c}{a+b+c} \geq$$
$$\geq \sqrt[a+b+c]{(4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c}$$
$$\Leftrightarrow 4abc(a\cos B\cos C+b\cos C\cos A+c\cos A\cos B) \geq$$
$$\geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$
$$\Leftrightarrow 8Rabc\,(\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B) \geq$$
$$\geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$
$$\Leftrightarrow Rabc\,(8\sin A\sin B\sin C) \geq$$
$$\geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$
$$\Leftrightarrow \frac{a^2b^2c^2}{R^2} \geq (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c$$
$$\Leftrightarrow (4abc\cos B\cos C)^a(4abc\cos C\cos A)^b(4abc\cos A\cos B)^c \leq 16S^2 =$$
$$=(a+b+c)(b+c-a)(a+c-b)(b+a-c)=(b+c-a)(a+c-b)(b+a-c)$$

PROBLEM 2.077-*Solution by Kevin Soto Palacios-Huarmey-Peru*

Como es un triángulo acutángulo $\cos A, \cos B, \cos C > 0$. Recordar la siguientes identidades y desigualdades en un $\Delta\ ABC$ $h_a=\frac{bc}{2R}, h_b=\frac{ca}{2R}, h_c=\frac{ab}{2R}$; $m_a \geq \frac{b^2+c^2}{4R}, m_b \geq \frac{c^2+a^2}{4R}, m_c \geq \frac{a^2+b^2}{4R}$

Lo cual implica $\Rightarrow \frac{m_a}{h_a}\cos A+\frac{m_b}{h_b}\cos B+\frac{m_c}{h_c}\cos C \geq a\left(\frac{b^2+c^2}{2abc}\right)\cos A+b\left(\frac{c^2+a^2}{2abc}\right)\cos B+$
$+c\left(\frac{a^2+b^2}{2abc}\right)\cos C=\frac{3}{2}$. *Lo cual es cierto ya que*

$$\Rightarrow a(b^2+c^2)\cos A+b\,(c^2+a^2)\cos B+c(a^2+b^2)=3abc$$
$$\Leftrightarrow a^2(b^2+c^2)(2bc\cos A)+b^2(c^2+a^2)(2ca\cos B)+c^2(a^2+b^2)(2ab\cos C)=6a^2b^2c^2$$
$$\Leftrightarrow a^2(b^2+c^2)(b^2+c^2-a^2)+b^2(c^2+a^2)(c^2+a^2-b^2)+$$
$$+c^2(a^2+b^2)(a^2+b^2-c^2)=a^2(b^2+c^2)^2-a^4(b^2+c^2)+$$
$$+b^2(c^2+a^2)^2-b^4(c^2+a^2)+c^2(a^2+b)^2-c^4(a^2+b^2)=$$
$$=a^2(b^4+c^4)-a^4(b^2+c^2)+b^2(c^4+a^4)-b^4(c^2+a^2)$$
$$+c^2(a^4+b^4)-c^4(a^2+b^2)+6a^2b^2c^2=6a^2b^2c^2$$

PROBLEM 2.078-*Solution by proposer*

Applying the Weighted AM-GM inequality we obtain

$$a^{-a}b^{-b}c^{-c}=\left(\frac{1}{a}\right)^a\left(\frac{1}{b}\right)^b\left(\frac{1}{c}\right)^c \leq a\cdot\frac{1}{a}+b\cdot\frac{1}{b}+c\cdot\frac{1}{c},$$

$$a^{-b}b^{-c}c^{-a}=\left(\frac{1}{a}\right)^b\left(\frac{1}{b}\right)^c\left(\frac{1}{c}\right)^a\le b\cdot\frac{1}{a}+c\cdot\frac{1}{b}+a\cdot\frac{1}{c},$$
$$a^{-c}b^{-a}c^{-b}=\left(\frac{1}{a}\right)^c\left(\frac{1}{b}\right)^a\left(\frac{1}{c}\right)^b\le c\cdot\frac{1}{a}+a\cdot\frac{1}{b}+b\cdot\frac{1}{c}.$$

Adding up these relations yields

$$a^{-a}b^{-b}c^{-c}+a^{-b}b^{-c}c^{-a}+a^{-c}b^{-a}c^{-b}\le(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=a^{-1}+b^{-1}+c^{-1}$$

as desired.

PROBLEM 2.079-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\frac{a^n+b^n+c^n}{9}\left(\sum_{cyc}\frac{1}{a^n}\right)=\frac{1}{9}\sum_{cyc}\left(\frac{a^n+b^n}{2}\right)\cdot\left(\sum_{cyc}\frac{1}{a^n}\right)$$
$$=\frac{1}{9}\sum_{cyc}\left(\frac{a+b}{2}\right)^n\left(\sum_{cyc}\frac{1}{a^n}\right)=\sum_{cyc}\left(\frac{a+b}{6}\right)^n\cdot\left(\sum_{cyc}\frac{1}{a^n}\right)\cdot 3^{n-2}$$
$$=\sum_{cyc}\left(\frac{a+b}{6}\right)^n\cdot\left(\sum_{cyc}\frac{1}{a^n}\right)\cdot(1+1+1)\cdot(1+1+1)\dots((n-2)times)$$
$$\overset{HOLDER}{\ge}\left(\sum_{cyc}\sqrt[n]{\left(\frac{a+b}{6}\right)^n\cdot\frac{1}{c^n}\cdot 1}\right)^n=\left(\sum_{cyc}\frac{a+b}{6c}\right)^n$$

PROBLEM 2.080-*Solution by proposer*

To solve this problem we must need the following results

Lemma 1. For any positive real numbers u,v,w,x,y,z then $\frac{u^3}{x}+\frac{v^3}{y}+\frac{w^3}{z}\ge\frac{(u+v+w)^3}{3(x+y+z)}$

Proof. By Hölder's inequality we have

$\left(\frac{u^3}{x}+\frac{v^3}{y}+\frac{w^3}{z}\right)(x+y+z)(1+1+1)\ge(u+v+w)^3$ *and the conclusion follows.*

Lemma 2. For all non-negative real numbers a,b,c then

$$a^2b+b^2c+c^2a+abc\le\frac{4}{27}(a+b+c)^3.$$

Proof. Because the variables a,b,c are cyclic, without loss of generality we can suppose that b is between a and c. Then

$$a^2b+b^2c+c^2a+abc-(a^2b+c^2b+2abc)=c(b-a)(b-c)\le 0$$

Consequently $a^2b+b^2c+c^2a+abc\le(a^2b+c^2b+2abc)=b(a+c)^2$

On the other hand, applying the AM-GM inequality we get

$$b(a+c)^2=\frac{1}{2}(2b)(a+c)(a+c)\le\frac{1}{2}\left(\frac{2b+(a+c)+(a+c)}{3}\right)^3=\frac{4}{27}(a+c+c)^3$$

Hence $a^2b+b^2c+c^2a+abc\le\frac{4}{27}(a+b+c)^3$. *Come back to the main problem*

We use lemma 1 and lemma 2, respectively, to obtain $LHS=\frac{(b+c)^3}{b^3+c^3}+\frac{(c+a)^3}{c^3+a^3}+\frac{(a+b)^3}{a^3+b^3}\geq$
$\geq\frac{(2a+2b+2c)^3}{3(2a^3+2b^3+2c^2)}=\frac{4(a+b+c)^3}{3(a^3+b^3+c^3)}\geq\frac{9(a^2b+b^2c+c^2a+abc)}{a^3+b^3+c^3}$ *and we are done.*

PROBLEM 2.081-*Solution by proposer*

Using the AM-GM inequality we obtain:

$$\sum_{cyc}\sqrt{\frac{bc}{(b+ka)(c+ka)}}=\sum_{cyc}\frac{bc}{\sqrt{(bc+kca)(bc+kb)}}\geq\sum_{cyc}\frac{2bc}{2bc+k(ca+ab)}$$

After setting $bc=x,ca=y,ab=z$, *the required inequality reduces to:*

$$\frac{2x}{2x+k(y+z)}+\frac{2y}{2y+k(z+x)}+\frac{2z}{2z+k(x+y)}\geq\frac{3}{k+1}$$

Applying the Cauchy-Schwarz inequality yields

$$\sum_{cyc}\frac{2x}{2x+k(y+z)}=\sum_{cyc}\frac{2x^2}{2x^2+kx(y+z)}\geq\frac{2(x+y+z)^2}{2(x^2+y^2+z^2)+2k(xy+yz+zx)}=$$

$$=\frac{(x+y+z)^2}{(x+y+z)^2+(k-2)(xy+yz+zx)}\geq\frac{(x+y+z)^2}{(x+y+z)^2+\frac{k-2}{3}(x+y+z)^2}=$$

$=\frac{1}{1+\frac{k-2}{3}}=\frac{3}{k+1}$. *This completes the proof. The equality holds when* $a=b=c$.

PROBLEM 2.082-*Solution by proposer*

We first see easily that: $x\overrightarrow{MA}+y\overrightarrow{MB}+z\overrightarrow{MC}=\vec{0}$ *(1)*

Next, we have:

$$x\cdot PA\cdot MA+y\cdot PB\cdot MB+z\cdot PC\cdot MC\geq x\cdot\overrightarrow{PA}\cdot\overrightarrow{MA}+y\cdot\overrightarrow{PB}\cdot\overrightarrow{MB}+z\cdot\overrightarrow{PC}\cdot\overrightarrow{MC}=$$
$$=x\left(\overrightarrow{PM}+\overrightarrow{MA}\right)\overrightarrow{MA}+y\left(\overrightarrow{PM}+\overrightarrow{MB}\right)\overrightarrow{MB}+z\left(\overrightarrow{PM}+\overrightarrow{MC}\right)\overrightarrow{MC}=$$
$$=\overrightarrow{PM}\left(x\overrightarrow{MA}+y\overrightarrow{MB}+z\overrightarrow{MC}\right)+xMA^2+yMB^2+zMC^2=xMA^2+yMB^2+zMC^2.$$

Now we square both sides of (1) to obtain:

$$x^2MA^2+y^2MB^2+z^2MC^2+2xy\overrightarrow{MA}\cdot\overrightarrow{MB}+2yz\overrightarrow{MB}\cdot\overrightarrow{MC}+2zx\overrightarrow{MC}\cdot\overrightarrow{MA}=0$$
or $x^2MA^2+y^2MB^2+z^2MC^2+xy(MA^2+MB^2-AB^2)+yz(MB^2+MC^2-BC^2)+$
$+zx(MC^2+MA^2-CA^2)=0$ *or*
$(x+y+z)(xMA^2+yMB^2+zMC^2)=yza^2+zxb^2+xyc^2$ *or*
$xMA^2+yMB^2+zMC^2=\frac{(xy+yz+zx)a^2}{x+y+z}$ *(since* $a=b=c$*)*

Furthermore, it's not difficult to observe that: $x+y+z=h=\frac{a\sqrt{3}}{2}$. *Hence*
$xMA^2+yMB^2+zMC^2=\frac{2a}{\sqrt{3}}(xy+yz+zx)$. *Thus we have proved*
$x\cdot PA\cdot MA+y\cdot PB\cdot MB+z\cdot PC\cdot MC\geq\frac{2a}{\sqrt{3}}(xy+yz+zx)$ *(2)*

Also, using the AM-GM inequality we get:

$$xPA^2+xMA^2\geq 2xPA\cdot MA,$$
$$yPB^2+yMB^2\geq 2yPB\cdot MB,$$
$$zPC^2+zMC^2\geq 2zPC\cdot MC,$$

It follows that:

$(xPA^2 + yPB^2 + zPC^2) + (xMA^2 + yMB^2 + zMC^2) \geq 2x \cdot PA \cdot MA + 2y \cdot PB \cdot MB +$
$+2z \cdot PC \cdot MC$. *On the other hand, according to the above proof, we have*

$$x \cdot PA \cdot MA + y \cdot PB \cdot MB + zc \cdot PC \cdot MC \geq xMA^2 + yMB^2 + zMC^2$$

Adding up two last results we obtain

$$xPA^2 + yPB^2 + zPC^2 \geq x \cdot PA \cdot MA + y \cdot PB \cdot MB + z \cdot PC \cdot MC \quad (3)$$

Combining (2) and (3) gives us

$$xPA^2 + yPB^2 + zPC^2 \geq x \cdot PA \cdot MA + y \cdot PB \cdot MB + z \cdot PC \cdot MC \geq \frac{2a}{\sqrt{3}}(xy + yz + zx)$$

The equalities occur if and only if $P \equiv M$. *From here we take again* $P \equiv O$ *which is the center of the equilateral triangle* ABC *then to get*

$$(x + y + z)R \geq xMA + yMB + zMC \geq \frac{2a}{R\sqrt{3}}(xy + yz + zx)$$

Also, $x + y + z = \frac{a\sqrt{3}}{2}$ *and* $R = \frac{a}{\sqrt{3}}$. *Therefore we find the desired result.*

PROBLEM 2.083-*Solution by proposer*

Applying the Cauchy – Schwarz Inequality, we have:

$$\frac{1}{m_a + x} + \frac{1}{m_b + x} + \frac{1}{m_c + x} \geq \frac{(1 + 1 + 1)^2}{m_a + m_b + m_c + 3x} = \frac{9}{m_a + m_b + m_c + 3x}, x \geq 0$$

It is well – known that $m_a + m_b + m_c \leq \frac{9R}{2}$. *So,* $\frac{1}{m_a+x} + \frac{1}{m_b+x} + \frac{1}{m_c+x} \geq \frac{9}{\frac{9R}{2}+3x} = \frac{3}{\frac{3R}{2}+x}$. *Now,*

$$\int_0^1 \left(\frac{1}{m_a+x} + \frac{1}{m_b+x} + \frac{1}{m_c+x}\right) dx \geq \int_0^1 \frac{3}{\frac{3R}{2}+x} dx. \text{ So,}$$

$$[\ln(m_a + x) + \ln(m_b + x) + \ln(m_c + x)]_0^1 \geq 3\left[\ln\left(\frac{3R}{2} + x\right)\right]_0^1 \Leftrightarrow$$

$$\Leftrightarrow \ln(m_a + 1) + \ln(m_b + 1) + \ln(m_c + 1) - \ln m_a - \ln m_b - \ln m_c \geq$$

$$\geq 3\left(\ln\left(\frac{3R}{2} + 1\right) - \ln\frac{3R}{2}\right) \Leftrightarrow \ln\frac{m_a + 1}{m_a} + \ln\frac{m_b + 1}{m_b} + \ln\frac{m_c + 1}{m_c} \geq 3\ln\left(1 + \frac{2}{3R}\right) \Leftrightarrow$$

$$\Leftrightarrow \ln\left(\frac{m_{a+1}}{m_a} \cdot \frac{m_b+1}{m_b} \cdot \frac{m_c+1}{m_c}\right) \geq \ln\left(1 + \frac{2}{4R}\right)^3. \text{ Namely}$$

$$\left(1 + \frac{1}{m_a}\right)\left(1 + \frac{1}{m_b}\right)\left(1 + \frac{1}{m_c}\right) \geq \left(1 + \frac{2}{3R}\right)^3.$$ *Equality holds when triangle ABC is equilateral.*

PROBLEM 2.084-*Solution by proposer*

$$\int_0^1 (x^{n-1} - \arctan(x^n))^2 dx \geq 0$$

$$\int_0^1 x^{2n-2}\, dx - 2\int_0^1 x^{n-1} \arctan(x^n)\, dx + \int_0^1 \arctan^2(x^n)\, dx \geq 0$$

$$x \in [0,1] \Rightarrow x^{n-1} \in [0,1]$$

$$\tan x \geq x; \arctan x \leq x \Rightarrow \arctan(x^{n-1}) \leq x^{n-1}$$

$$\frac{1}{2n-1}+\int_0^1 \arctan^2(x^n)\,dx \geq 2\int_0^1 x^{n-1}\arctan(x^n)\,dx \geq 2\int_0^1 \arctan(x^{n-1})\arctan(x^n)\,dx$$

PROBLEM 2.085-*Solution by proposer*

From means inequality:

$$\frac{a}{b^n}+\frac{b}{a^n}\geq 2\sqrt{\frac{ab}{a^nb^n}} \quad (1)$$

$$\frac{a^n}{b^n}+\frac{b}{a}\geq 2\sqrt{\frac{a^nb}{b^na}} \quad (2)$$

$$\frac{b^n}{a^n}+\frac{a}{b}\geq 2\sqrt{\frac{b^na}{a^nb}} \quad (3)$$

By multiplying the relationships (1); (2); (3):

$$\left(\frac{a}{b^n}+\frac{b}{a^n}\right)\left(\frac{a^n}{b^n}+\frac{b}{a}\right)\left(\frac{b^n}{a^n}+\frac{a}{b}\right)\geq \frac{8}{\sqrt{a^{n-1}\cdot b^{n-1}}} \quad (4)$$

We prove that: $\frac{a^n}{b}+\frac{b^n}{a}\geq a^{n-1}+b^{n-1}$ *(5)*

$$a^{n+1}+b^{n+1}\geq a^nb+ab^n,\ a^n(a-b)-b^n(a-b)\geq 0$$

$$(a-b)(a^n-b^n)\geq 0,\ \ (a-b)^2(a^{n-1}+a^{n-2}b+\cdots+b^{n-1})\geq 0 \quad \textit{(true)}$$

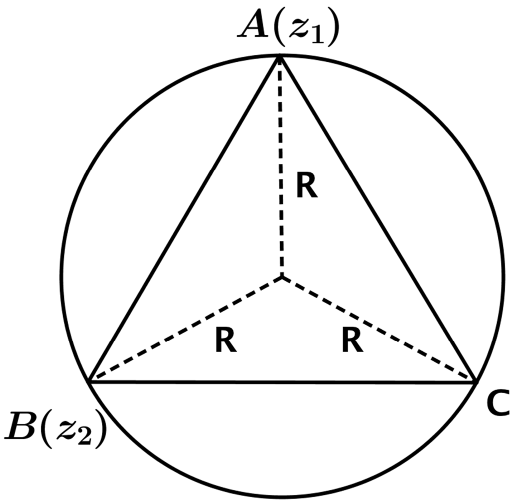
We multiply the relationships (4); (5):

$$\left(\frac{a^n}{b}+\frac{b^n}{a}\right)\left(\frac{a}{b^n}+\frac{b}{a^n}\right)\left(\frac{a^n}{b^n}+\frac{b}{a}\right)\left(\frac{b^n}{a^n}+\frac{a}{b}\right)\geq \frac{8(a^{n-1}+b^{n-1})}{\sqrt{a^{n-1}\cdot b^{n-1}}}=8\left(\sqrt{\left(\frac{a}{b}\right)^{n-1}}+\sqrt{\left(\frac{b}{a}\right)^{n-1}}\right)$$

PROBLEM 2.086-*Solution by proposer*

$$\sin^2 a+\sin^2 b+\sin^2 c=1-\cos^2 a-\cos^2 b-\cos^2 c+2\geq$$
$$\geq 1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c=$$
$$=1-\sum\frac{1+\cos 2a}{2}+[\cos(a+b)+\cos(a-b)]\cos c=$$
$$=\frac{-1-\sum\cos 2a}{2}+\frac{\cos(a+b+c)+\sum\cos(a+b-c)}{2}=$$
$$=\frac{\cos(a+b+c)}{2}+\frac{1}{2}\sum(\cos(b+c-a)-\cos 2a)=$$
$$=-\sin^2\frac{a+b+c}{2}-\sum\sin\frac{a+b+c}{2}\sin\frac{b+c-3a}{2}=$$
$$=-\sin\frac{a+b+c}{2}\left[\left(\sin\frac{a+b+c}{2}+\sin\frac{b+c-3a}{2}\right)+\left(\sin\frac{a+b-3c}{2}+\frac{a-3b+c}{2}\right)\right]=$$
$$=4\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}\sin\frac{a+c-b}{2}\sin\frac{a+b-c}{2}=$$
$$=4\sin\frac{2s}{2}\sin\frac{2s-2a}{2}\sin\frac{2s-2b}{2}\sin\frac{2s-2c}{2}=4\sin s\sin(s-a)\sin(s-b)\sin(s-c)$$

PROBLEM 2.087-*Solution by Ravi Prakash-New Delhi-India*



Let's take O the circumcentre of $\Delta\ ABC$.

Let's take $z_1 = R(\cos\alpha + i\sin\alpha);\ z_2 = R(\cos\beta + i\sin\beta);\ z_3 = R(\cos\gamma + i\sin\gamma)$

Now, $\frac{z_2-z_3}{z_2+z_3} = \frac{1-\frac{z_3}{z_2}}{1+\frac{z_3}{z_2}} = \frac{1-\cos(\gamma-\beta)-i\sin(\gamma-\beta)}{1+\cos(\gamma-\beta)+i\sin(\gamma-\beta)} \Rightarrow \left|\frac{z_2-z_3}{z_2+z_3}\right|^2 = \frac{(1-\cos(\gamma-\beta))^2+\sin^2(\gamma-\beta)}{(1+\cos(\gamma-\beta))^2+\sin^2(\gamma-\beta)}$

$= \frac{2[1-\cos(\gamma-\beta)]}{2[1+\cos(\gamma-\beta)]} = \frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A \therefore \left|\frac{z_2-z_3}{z_2+z_3}\right| = \tan A$. *Similalry for other expressions*

$$\therefore LHS = \prod(\tan A + \tan B) = \prod\frac{\sin(A+B)}{\cos A + \cos B} = \prod\frac{\sin C}{\cos A\cos B}$$

$$= \frac{\sin A\sin B\sin C}{\cos^2 A\cos^2 B\cos^2 C} = \frac{16R^4\sin A\sin B\sin C}{(4R^2\cos A\cos B\cos C)^2} \quad (1)$$

We now show that $4R^2\cos A\cos B\cos C = s^2 - (2R+r)^2 = s^2 - (4R^2 + 4Rr + r^2)$

Now, $4Rr + r^2 = \frac{abc}{\Delta}\cdot\frac{\Delta}{s} + \frac{\Delta^2}{s^2}\ [\Delta = area\ of\ \Delta ABC] = \frac{abc}{s} + \frac{1}{s}(s-a)(s-b)(s-c)$

$$= \frac{1}{s}[abc + (s-a)(s-b)(s-c)]$$

$$= \frac{1}{s}[abc + s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc] = -s^2 + ab + bc + ca$$

$$\therefore s^2 - (4R^2 + 4Rr + r^2) = s^2 - 4R^2 + s^2 - (ab+bc+ca)$$

$$= \frac{1}{2}(a+b+c)^2 - (ab+bc+ca) - (2R)^2 = \frac{1}{2}[a^2+b^2+c^2 - 2(2R)^2]$$

$$= \frac{1}{2}(2R)^2[\sin^2 A + \sin^2 B + \sin^2 C - 2] = 2R^2[-\cos^2 A - (\cos^2 B - \sin^2 C)]$$

$$= 2R^2[-\cos^2 A - \cos(B+C)\cos(B-C)] = 2R^2[-\cos^2 A + \cos A\cos(B-C)]$$

$$= 2R^2(\cos A)[\cos(B-C) + \cos(B+C)] = 4R^2\cos A\cos B\cos C \quad (2)$$

Also, $16R^4\sin A\sin B\sin C = 4R^2(2R\sin A)(2R\sin B)\sin C$

$$= 4R^2 ab\sin C = 4R^2(2\Delta) = 8R^2\Delta = 8R^2(sr) \geq 8(2r)^2(sr)$$

$$\therefore 16R^4\sin A\sin B\sin C \geq 32sr^3 \quad (3)$$

From (1), (2), (3), we get $\prod\left(\frac{|z_2-z_3|}{|z_2+z_3|} + \frac{|z_1-z_3|}{|z_1+z_3|}\right) \geq \frac{32sr^3}{[s^2-(2R+r)^2]^2}$

PROBLEM 2.088-*Solution by Kevin Soto Palacios-Huarmey-Peru*

Siendo $a, b, c > 0$ *de tal manera que* $ab + bc + ca + abc = 4$. *Probar que*

$$(a+1)\sqrt{(b+1)(c+1)} + (b+1)\sqrt{(c+1)(a+1)} + (c+1)\sqrt{(a+1)(b+1)} \geq$$
$$\geq a + b + c + 9$$

De la condición, realizamos las siguientes sustituciones

$$a = \frac{2x}{y+z} > 0, b = \frac{2y}{z+x} > 0, c = \frac{2z}{x+y} > 0, \text{ donde } x, y, z > 0$$

Aplicando la desigualdad de Cauchy y MA ≥ MG

$$(a+1)\sqrt{(b+1)(c+1)} = \left(\frac{(x+y)+(x+z)}{y+z}\right)\sqrt{\frac{(y+z)+(y+x)}{z+x} \cdot \frac{(y+z)+(z+x)}{x+y}} \geq$$
$$\geq \left(\frac{(x+y)+(x+z)}{y+z}\right)\left(\frac{y+z}{\sqrt{(z+x)(x+y)}} + 1\right) = \frac{(x+y)+(x+z)}{y+z} \cdot \frac{y+z}{\sqrt{(z+x)(x+y)}} + \frac{2x}{y+z} + 1 \geq \frac{2x}{y+z} + 3 \quad (A)$$

Analogamente para los siguientes términos

$$(b+1)\sqrt{(c+1)(a+1)} \geq \frac{2y}{z+x} + 3 \quad (B)$$
$$(c+1)\sqrt{(a+1)(b+1)} \geq \frac{2z}{x+y} + 3 \quad (C)$$

Sumando (A)+(B)+(C)

$$(a+1)\sqrt{(b+1)(c+1)} + (b+1)\sqrt{(c+1)(a+1)} + (c+1)\sqrt{(a+1)(b+1)} \geq$$
$$\geq \frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y} + 9 = a + b + c + 9$$

PROBLEM 2.089-*Solution by Soumava Chakraborty-Kolkata-India*

WLOG, we may assume $a \geq b \geq c$. *Then,* $r_a^2 \geq r_b^2 \geq r_c^2$ *and* $\frac{1}{h_a} \geq \frac{1}{h_b} \geq \frac{1}{h_c}$

$$\therefore LHS \overset{Chebyshev}{\geqq} \frac{1}{3}\left(\sum r_a^2\right)\left(\sum \frac{1}{h_a}\right) = \frac{1}{3r}\{(4R+r)^2 - 2s^2\} \overset{?}{\geqq} \frac{2s^2(R-r)}{3Rr}$$

$$\Leftrightarrow R(4R+r)^2 - 2Rs^2 \overset{?}{\geqq} 2Rs^2 - 2rs^2 \Leftrightarrow (4R-2r)s^2 \overset{?}{\leqq} R(4R+r)^2$$

Now, Rouche $\Rightarrow (4R-2r)s^2 \leq (2R^2 + 10Rr - r^2)(4R-2r) +$

$$+2(R-2r)(4R-2r)\sqrt{R^2-2Rr} \overset{?}{\leqq} R(4R+r)^2$$

$$\Leftrightarrow (R-2r)(8R^2 - 12Rr + r^2) \overset{?}{\geqq} 2(R-2r)(4R-2r)\sqrt{R^2-2Rr}$$

$\because R - 2r \geq 0$ *by Euler* $\therefore$ *it suffices to prove:* $8R^2 - 12Rr + r^2 > 4(2R-r)\sqrt{R^2-2Rr}$

$$[8R^2 - 12Rr + r^2 = (R-2r)(8r+4r) + 9r^2 > 0]$$
$$\Leftrightarrow (8R^2 - 12Rr + r^2)^2 - 16(2R-r)^2(R^2 - 2Rr) > 0$$
$$\Leftrightarrow 16R^2r^2 + 8Rr^3 + r^4 > 0 \rightarrow \text{true (Proved)}$$

PROBLEM 2.090-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\sum_{cyc} \frac{\sin\alpha}{u\sin\beta + v\sqrt{\sin\alpha\sin\beta}} \overset{AM \geq GM}{\geqq} 2\sum_{cyc} \frac{\sin\alpha}{2u\sin\beta + v(\sin\alpha + \sin\beta)}$$

$$\geq 2\sum_{cyc}\frac{\sin^2 a}{v\sin^2\alpha+(2u+v)\sin\alpha\sin\beta}\overset{Bergstrom}{\geq}\frac{2(\sin\alpha+\sin\beta+\sin\gamma)^2}{v\ \sum_{cyc}\alpha+(2u+v)\sum_{cyc}\sin\alpha\sin\beta}$$

$$=\frac{2\left(\sum_{cyc}\sin\alpha\right)^2}{v\left(\sum_{cyc}\sin\alpha\right)^2+(2u-v)\sum_{cyc}\sin\alpha\sin\beta}$$

$$\geq 2\frac{\left(\sum_{cyc}\sin\alpha\right)^2}{v\left(\sum_{cyc}\sin\alpha\right)^2+\frac{2u-v}{3}\left(\sum_{cyc}\sin\alpha\right)^2}[\because 2u-v>0]=\frac{3}{u+v}$$

PROBLEM 2.091-*Solution by proposer*

We have: $\frac{a^2}{a+b+c}=\frac{5a-b-c}{9}+\frac{(b+c-2a)^2}{9(a+b+c)},\frac{b^2}{b+c+d}=\frac{5b-c-d}{9}+\frac{(c+d-2b)^2}{9(b+c+d)},$

$$\frac{c^2}{c+d+a}=\frac{5c-d-a}{9}+\frac{(d+a-2c)^2}{9(c+d+a)},\frac{d^2}{d+a+b}=\frac{5d-a-b}{9}+\frac{(a+b-2d)^2}{9(d+a+b)}$$

Adding up these relations we obtain: $\sum_{cyc}\frac{a^2}{a+b+c}=\frac{a+b+c+d}{3}+\sum_{cyc}\frac{(b+c-2a)^2}{9(a+b+c)}.$

Now we use Cauchy – Schwarz inequaity to get

$$\sum_{cyc}\frac{(b+c-2a)^2}{9(a+b+c)}=\frac{(b+c-2a)^2}{9(a+b+c)}+\frac{(c+d-2b)^2}{9(b+c+d)}+\frac{(-d-a+2c)^2}{9(c+d+a)}+$$

$$+\frac{(-a-b+2d)^2}{9(d+a+b)}\geq\frac{4(2a+b-2c-d)^2}{27(a+b+c+d)}$$

Therefore $\frac{a^2}{a+b+c}+\frac{b^2}{b+c+d}+\frac{c^2}{c+d+a}+\frac{d^2}{d+a+b}\geq\frac{a+b+c+d}{3}+\frac{4(2a+b-2c-d)^2}{27(a+b+c+d)}$ *as desired.*

PROBLEM 2.092-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}+\frac{(b-c)^2}{2(a+b+c)}$$

Given inequality $\Leftrightarrow\frac{\sum a^2\left(\sum ab+a^2\right)}{(a+b)(b+c)(c+a)}\geq\frac{(a+b+c)^2+(a+b-2c)^2}{2(a+b+c)}$

$$\Leftrightarrow 2\left(\sum a\right)\left\{\sum a^4+\left(\sum ab\right)\left(\sum a^2\right)\right\}\geq$$

$$\geq(a+b)(b+c)(c+a)\{(a+b+c)^2+(a+b-2c)^2\}$$

$$\Leftrightarrow 2(a^5+b^5+c^5)+2a^4b+2a^4c+2a^3c^2+2ab^4+2b^4c+2b^3c^2\geq$$

$$\geq 4a^3b^2+4a^2b^3+4a^2b^2c+a^2bc^2+a^2c^3+ab^2c^2+ac^4+b^2c^3+bc^4\ (1)$$

Now, $2(a^5+ab^4)\overset{A-G}{\geq}4a^3b^2\quad(a)$

$$2(b^5+a^4b)\overset{A-G}{\geq}4a^2b^3\quad(b)$$

$$2(a^4c+b^4c)\overset{A-G}{\geq}4a^2b^2c\quad(c)$$

$$c^2(a^3+b^3)\geq c^2ab(a+b)=a^2bc^2+ab^2c^2\quad(d)$$

$$c^2(a^3+c^3)\geq c^2ac(a+c)=a^2c^3+ac^4\quad(e)$$

$$c^2(b^3+c^3)\geq c^2bc(b+c)=b^2c^3+bc^4\quad(f)$$

(a)+(b)+(c)+(d)+(e)+(f)⇒ (1)

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge\frac{a+b+c}{2}+\frac{(a+b-2c)^2}{2(a+b+c)}$$

$$\textit{Given inequality} \Leftrightarrow \frac{\sum a^2(\sum ab+a^2)}{(a+b)(b+c)(c+a)}\ge\frac{(a+b+c)^2+(a+b-2c)^2}{2(a+b+c)}$$

$$\Leftrightarrow 2\left(\sum a\right)\left\{\sum a^4+\left(\sum ab\right)\left(\sum a^2\right)\right\}\ge$$

$$\ge (a+b)(b+c)(c+a)\{(a+b+c)^2+(a+b-2c)^2\}$$

$$\Leftrightarrow 2(a^5+b^5+c^5)+2a^4b+2a^4c+2a^3c^2+2ab^4+2b^4c+2b^3c^2\ge$$

$$\ge 4a^3b^2+4a^2b^3+4a^2b^2c+a^2bc^2+a^2c^3+ab^2c^2+ac^4+b^2c^3+bc^4$$

$$\text{Now, } 2(a^5+ab^4)\overset{A-G}{\ge}4a^3b^2 \quad (a)$$

$$2(b^5+a^4b)\overset{A-G}{\ge}4a^2b^3 \quad (b)$$

$$2(a^4c+b^4c)\overset{A-G}{\ge}4a^2b^2c \quad (c)$$

$$c^2(a^3+b^3)\ge c^2ab(a+b)=a^2bc^2+ab^2c^2 \quad (d)$$

$$c^2(a^3+c^3)\ge c^2ac(a+c)=a^2c^3+ac^4 \quad (e)$$

$$c^2(b^3+c^3)\ge c^2bc(b+c)=b^2c^3+bc^4 \quad (f)$$

(a)+(b)+(c)+(d)+(e)+(f)⇒(1) is true (Proved)

PROBLEM 2.093-*Solution by Soumava Chakraborty-Kolkata-India*

Let $s-a=x, s-b=y, s-c=z$. *Then* $x,y,z>0$ *and* $s=x+y+z$

$\therefore a=y+z, b=z+x, c=x+y$. *Now, given inequality* $\Leftrightarrow$

$$\Leftrightarrow \frac{(b+c)a}{2b^2+2c^2-a^2}+\frac{(c+a)b}{2c^2+2a^2-b^2}+\frac{(a+b)c}{2a^2+2b^2-c^2}\overset{(1)}{\ge}2$$

$$\text{Now, } 2b^2+2c^2-a^2=2(z+x)^2+2(x+y)^2-(y+z)^2$$

$$=2z^2+2x^2+4zx+2x^2+2y^2+4xy-y^2-z^2-2yz$$

$$=z^2+y^2+4x^2+2yz+4xy+4zx-4yz\overset{(a)}{=}(y+z+2z)^2-4yz$$

$$(a)\Rightarrow\frac{(b+c)a}{2b^2+2c^2-a^2}=\frac{(y+z)(y+z+2x)}{(y+z+2x)^2-4yz} \quad (i)$$

$$\textit{Similalry, } \frac{(c+a)b}{2c^2+2a^2-b^2}\overset{(ii)}{=}\frac{(z+x)(z+x+2y)}{(z+x+2y)^2-4zx} \;\&\; \frac{(a+b)c}{2a^2+2b^2-c^2}\overset{(iii)}{=}\frac{(x+y)(x+y+2z)}{(x+y+2z)^2-4xy}$$

(i)+(ii)+(iii)⇒ given inequality ⇔

$$\Leftrightarrow (y+z)(y+z+2x)\{(z+x+2y)^2-4zx\}\{(x+y+2z)^2-4xy\}+$$

$$+(z+x)(z+x+2y)\{(x+y+2z)^2-4xy\}\{(y+z+2x)^2-4yz\}+$$

$$+(x+y)(x+y+2z)\{(y+z+2x)^2-4yz\}\{(z+x+2y)^2-4zx\}\ge$$

$$\ge 2\{(x+y+2z)^2-4xy\}\{(z+x+2y)^2-4zx\}\{(y+z+2x)^2-4yz\}$$

$$\Leftrightarrow 10\sum x^5y+10\sum xy^5+77\sum x^4y^2+77\sum x^2y^4+$$

$$+150\sum x^3y^3\overset{(2)}{\ge}118xyz\left(\sum x^3\right)+90xyz\left(\sum x^2y+\sum xy^2\right)++78x^2y^2z^2$$

$$\text{Now, } 59\sum x^4y^2+59\sum x^2y^4=$$

$$=59\{x^4(y^2+z^2)+y^4(z^2+x^2)+z^4(x^2+y^2)\}\underset{(iv)}{\overset{A-G}{\ge}}118xyz\left(\sum x^3\right)$$

$$\text{Now, } \forall\, u,v,w\in\mathbb{R}^+, \sum u^3+3uvw\overset{Shur}{\ge}\sum u^2v+\sum uv^2 \text{ and } \sum u^3\overset{A-G}{\ge}3uvw$$

$$\text{Adding the last 2, } 2\sum u^3\ge\sum u^2v+\sum uv^2 \quad (b)$$

$$(b) \Rightarrow 150\sum x^3y^3 \geq 75xyz\left(\sum x^2y + \sum xy^2\right) \quad (v)$$

$$\text{Again, } 15\sum x^4y^2 + 15\sum x^2y^4 \overset{A-G}{\geq} 30\sum x^3y^3$$

$$(vi) \geq 15xyz\left(\sum x^2y + \sum xy^2\right) \ (\text{by } (b))$$

$$\text{Also, } 3\sum x^4y^2 + 3\sum x^2y^4 \overset{A-G}{\geq} 18x^2y^2z^2 \quad (vii)$$

$$10\sum x^5y + 10\sum xy^5 \overset{A-G}{\geq} 60x^2y^2z^2 \quad (viii)$$

(iv)+(v)+(vi)+(vii)+(viii)⇒ (2) is true (proved)

PROBLEM 2.094-*Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam*

$$\sum \frac{\cos A\cos B}{\sin C} \leq \frac{\sqrt{3}}{2}$$

Since Δ ABC is acute then $\sin A, \sin B, \sin C > 0$. *So, the inequality is equivalent to:*

$$\sum \cos A\cos B\sin A\sin B \leq \frac{\sqrt{3}}{2}\sin A\sin B\sin C \Leftrightarrow$$

$$\Leftrightarrow \sum \sin 2A\sin 2B \leq 2\sqrt{3}\sin A\sin B\sin C$$

We have: $\sum \sin 2A\sin 2B \leq \frac{(\sum\sin 2A)^2}{3} = \frac{[4\sin A\sin B\sin C]^2}{3} \leq 2\sqrt{3}\sin A\sin B\sin C$

$\Leftrightarrow \sin A\sin B\sin C \leq \frac{3\sqrt{3}}{8}$, *this is true by AM-GM since:*

$$\sin A\sin B\sin C \leq \frac{(\sin A+\sin B+\sin C)^3}{27} \leq \frac{\left(\frac{3\sqrt{3}}{2}\right)^3}{27} = \frac{3\sqrt{3}}{8} \Rightarrow Q.E.D.$$

PROBLEM 2.095-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS = \frac{1}{4R^2}\{(b^2+c^2)^2 - 2b^2c^2\}a^2 +$$

$$+\frac{1}{4R^2}\{(c^2+a^2)^2 - 2c^2a^2\}b^2 + \frac{1}{4R^2}\{(a^2+b^2)^2 - 2a^2b^2\}c^2 \leq \frac{81}{4}(3R^4 - 16r^4)$$

$$\Leftrightarrow (b^2+c^2)^2a^2 + (c^2+a^2)^2b^2 + (a^2+b^2)^2c^2 \leq$$

$$\leq 81R^2(3R^4 - 16r^4) + 6a^2b^2c^2 \quad (1)$$

WLOG, we may assume $a \geq b \geq c$. *Then,* $a^2(b^2+c^2) \geq b^2(c^2+a^2) \geq c^2(a^2+b^2)$

$$b^2+c^2 \leq c^2+a^2 \leq a^2+b^2$$

$$\therefore LHS \text{ of } (1) \overset{Chebyshev}{\leq} \frac{1}{3}\left\{\sum a^2(b^2+c^2)\right\}\left\{\sum(b^2+c^2)\right\}$$

$$= \frac{4}{3}\left(\sum a^2b^2\right)\left(\sum a^2\right) \overset{Goldstone}{\leq} \frac{4}{3}(4R^2s^2)\left(\sum a^2\right)$$

$$\overset{Leibnitz}{\leq} \frac{4}{3}(4R^2s^2)(9R^2) = 48R^4s^2 \overset{?}{\leq} 81R^2(3R^4 - 16r^4) + 96R^2r^2s^2$$

$$\Leftrightarrow 16R^2s^2 \overset{?}{\leq} 27(3R^4 - 16r^4) + 32r^2s^2$$

$$\Leftrightarrow s^2(16R^2 - 32r^2) \overset{?}{\leq} 81R^4 - 432r^4 \quad (2)$$

$$\text{Now, LHS of } (2) \overset{Gerretsen}{\leq} (4R^2 + 4Rr + 3r^2)(16R^2 - 32r^2) \overset{?}{\leq} 81R^4 - 432r^4$$

$$\Leftrightarrow 17t^4 - 64t^3 + 80t^2 + 128t - 336 \geq 0 \ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\{(t-2)(17t^2+4t+28)+224\} \geq 0 \to true \because t = \frac{R}{r} \geq 2 \ (Euler)$$

PROBLEM 2.096-*Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia*

$$\sum \frac{1}{a \cdot w_a^2} \geq \frac{1}{R \cdot \Delta}; \begin{matrix} x = p-a \\ y = p-b \\ z = p-c \end{matrix} \Rightarrow x+y+z = p$$

$$1) \sum \frac{1}{a \cdot w_a^2} = \frac{1}{(y+z)\cdot\left(\frac{2}{2x+y+z}\cdot\sqrt{x(x+z)(y+x)\cdot\sum x}\right)^2} =$$

$$= \sum \frac{(2x+y+z)^2}{4x\prod(x+y)\cdot\sum x} \overset{Bergstrom}{\geq} \frac{(\sum(2xy+y+z))^2}{4\sum x\prod(x+y)} = \frac{16(x+y+z)^2}{4(x+y+z)^2\cdot\prod(x+y)} = \frac{4}{\prod(x+y)} = LHS$$

$$2) \frac{1}{R\cdot\Delta} = \frac{1}{\frac{abc}{4\Delta}\cdot\Delta} = \frac{4}{abc} = \frac{4}{\prod(x+y)} = RHS$$

$$1), 2) \sum \frac{1}{aw_a^2} \geq \frac{4}{\prod(x+y)} = \frac{1}{R\cdot\Delta}$$

PROBLEM 2.097-*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum \frac{\sqrt{AI}}{a} \overset{C-B-S}{\leq} \sqrt{\sum AI}\sqrt{\sum \frac{1}{a^2}}$$

$$= \sqrt{\sum AI}\sqrt{\frac{\sum a^2b^2}{a^2b^2c^2}} \overset{Goldstone}{\leq} \frac{2Rs}{4Rrs}\sqrt{\sum AI} = \frac{1}{2r}\sqrt{\sum AI} \overset{?}{\leq} \frac{\sqrt{2(R+r)}}{2r}$$

$$\Leftrightarrow \sum AI \overset{?}{\leq} 2(R+r) \ \ (1)$$

$$Now, \sum AI = r\sum\sqrt{\frac{bc}{(s-b)(s-c)}}$$

$$= \frac{r\sqrt{s}}{\sqrt{s(s-a)(s-b)(s-c)}}\sum\sqrt{bc}\sqrt{s-a} \overset{C-B-S}{\leq} \frac{r\sqrt{s}}{rs}\sqrt{\sum ab}\sqrt{3s-2s} = \sqrt{\sum ab}$$

$$= \sqrt{s^2+4Rr+r^2} \overset{Gerretsen}{\leq} \sqrt{4R^2+8Rr+4r^2} = \sqrt{4(R+r)^2} = 2(R+r)$$

$$\Rightarrow (1) \text{ is true (Proved)}$$

PROBLEM 2.098-*Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam*

$$AH\cdot BH + BH\cdot CH + CH\cdot AH = \sum 4R^2\cdot\cos A\cdot\cos B =$$

$$= 4R^2\left(\frac{p^2+r^2}{4R^2}-1\right) = p^2+r^2-4R^2 \leq 4R^2+4Rr+3r^2+r^2-4R^2$$

$$= 4R+4r^2 \leq 4Rr+2Rr = 6Rr \Rightarrow Q.E.D.$$

PROBLEM 2.099-*Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam*

We will prove that: $(a-b)^2(b-c)^2(c-a)^2 \leq \frac{27}{4}$. *WLOG, assume that* $c = \max\{a;b;c\}$

$$c \ge b \ge a \ge 0: (a-b)^2 \le b^2; (c-a)^2 \le c^2 \Rightarrow$$
$$\Rightarrow (a-b)^2(b-c)^2(c-a)^2 \le b^2c^2 \cdot (b-c)^2 = \frac{1}{4}(2bc)^2 \cdot (b^2-2bc+c^2)$$
$$\le \frac{(2bc+2bc+b^2-2bc+c^2)^3}{4\cdot 27} = \frac{(b+c)^6}{108} \le \frac{(a+b+c)^6}{108} = \frac{27}{4}$$
$$c^2 \ge a \ge b \ge 0: (a-b)^2 \le a^2; (b-c)^2 \le c^2 \Rightarrow (a-b)^2(b-c)^2(c-a)^2 \le$$
$$\le a^2c^2(c-a)^2 = \frac{1}{4}(2ac)^2 \cdot (a^2-2ac+c^2) \le \frac{\left(2ac+2ac+a^2-2ac+c^2\right)^3}{4\cdot 27}$$
$$= \frac{(a+c)^6}{108} \le \frac{(a+b+c)^6}{108} = \frac{27}{4}$$
$$\text{Hence: } (a-b)^2(b-c)^2(c-a)^2 \le \frac{27}{4} \Rightarrow |(a-b)(b-c)(c-a)| \le \frac{3\sqrt{3}}{2}$$
$$\text{The equality happens iff } (a;b;c) \sim \left(0; \frac{3-\sqrt{3}}{2}; \frac{3+\sqrt{3}}{2}\right)$$

PROBLEM 2.100-*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{In any } \Delta\ ABC \text{ with perimeter} = 3, 288r^2 \le \sum \frac{(a+b)^4}{a^2+b^2} \le \frac{2}{r^2}$$
$$a^2+b^2 \ge \frac{(a+b)^2}{2} \text{ etc}, \therefore \sum \frac{(a+b)^4}{a^2+b^2} \le 2\sum (a+b)^2 \le \frac{2}{r^2}$$
$$\Leftrightarrow \sum (a+b)^2 \le \frac{16s^4}{81r^2} \left(\because s^4 = \frac{81}{16} \text{ as } 2s = 3\right) \Leftrightarrow \sum a^2 + \sum ab \le \frac{8s^4}{81r^2}$$
$$\Leftrightarrow 8s^4 \ge 81r^2(3s^2 - 4Rr - r^2)$$
$$\Leftrightarrow 8s^4 + 324Rr^3 + 81r^4 \ge 243s^2r^2 \to (1)$$
$$\text{LHS of (1)} \overset{\text{Gerretsen}}{\ge} 8s^2(16Rr-5r^2) + 324Rr^3 + 81r^4 \overset{?}{\ge} 243s^2r^2$$
$$\Leftrightarrow s^2(128R - 256r) + 324Rr^2 + 81r^3 \overset{?}{\ge} 27s^2r \to (2)$$
$$\text{LHS of (2)} \overset{\text{Gerretsen}}{\ge} (16Rr-5r^2)(128R-256r) + 324Rr^2 + 81r^3$$
$$\text{and, RHS of (2)} \overset{\text{Gerretsen}}{\le} 27r(4R^2+4Rr+3r^2)$$
$$\therefore \text{in order to prove (2), it suffices to prove:}$$
$$(16Rr-5r^2)(128R-256r) + 324Rr^2 + 81r^3 \overset{?}{\ge} 27r(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow 97R^2 - 226Rr + 64r^2 \overset{?}{\ge} 0 \Leftrightarrow (R-2r)(97R-32r) \overset{?}{\ge} 0 \to \text{true}$$
$$\therefore R \ge 2r \text{ (Euler)} \Rightarrow \text{(2) is true} \therefore \frac{(a+b)^4}{a^2+b^2} \le \frac{2}{r^2}$$
$$\text{Again, } \frac{(a+b)^4}{a^2+b^2} \overset{\text{Bergstrom}}{\ge} \frac{\left(\sum (a+b)^2\right)^2}{2\sum a^2} \overset{\text{Leibniz}}{\ge} \frac{4\left(\sum a^2 + \sum ab\right)^2}{18R^2} \overset{?}{\ge} 288r^2$$
$$\Leftrightarrow \sum a^2 + \sum ab \overset{?}{\ge} 36Rr \Leftrightarrow 3s^2 \overset{?}{\ge} 40Rr + r^2 \to (3)$$
$$\text{LHS of (3)} \overset{\text{Gerretsen}}{\ge} 48Rr - 15r^2 \overset{?}{\ge} 40Rr + r^2 \Leftrightarrow 8Rr \overset{?}{\ge} 16r^2 \Leftrightarrow R \overset{?}{\ge} 2r$$
$$\to \text{true (Euler)} \Rightarrow \text{(3) is true} \therefore 288r^2 \le \sum \frac{(a+b)^4}{a^2+b^2}$$

PROBLEM 2.101-*Solution by Soumava Chakraborty-Kolkata-India*

$$a^4 + 2b^2c^2 = a^4 + b^2c^2 + b^2c^2 \overset{A-G}{\ge} 3\sqrt[3]{a^4b^4c^4} \Rightarrow \frac{1}{a^4+2b^2c^2} \le \frac{1}{3\sqrt[3]{a^4b^4c^4}} \quad (1)$$

$$\textit{Similarly, } \frac{1}{b^4+2c^2a^2} \overset{(2)}{\leq} \frac{1}{3\sqrt[3]{a^4b^4c^4}} \ \& \ \frac{1}{c^4+2a^2b^2} \overset{(3)}{\leq} \frac{1}{3\sqrt[3]{a^4b^4c^4}}$$

$$\text{(1)+(2)+(3)} \Rightarrow LHS \leq \sqrt[4]{\frac{1}{\sqrt[3]{a^4b^4c^4}}} = \frac{1}{\sqrt[3]{abc}} \overset{?}{\leq} \frac{\sqrt{3}}{6r} \Leftrightarrow \sqrt[3]{abc} \overset{(a)}{\geq} \frac{\sqrt{3}\sqrt{3}\cdot 2r}{\sqrt{3}} = 2\sqrt{3}r$$

$$\textit{Now, } \sqrt[3]{abc} = \sqrt[3]{4Rrs} \overset{Euler}{\geq} \sqrt[3]{4(2)rs}$$

$$\overset{s\geq 3\sqrt{3}r}{\geq} \sqrt[3]{4(2r)r(3\sqrt{3}r)} = \sqrt[3]{8\cdot 3\sqrt{3}r^3} = 2\sqrt{3}r \Rightarrow \textit{(a) is true (proved)}$$

PROBLEM 2.102-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$ab+bc+ca = p^2+r^2+4Rr, abc = 4Rrp \text{ and } \prod_{cyc}(p-a) = pr^2$$

$$\textit{again, } 9r(r+4R) \leq 3p^2 \leq (r+4R)^2$$

$$\sum_{cyc} bc(p-b)(p-c) = p^2\left(\sum_{cyc} ab\right) - p\sum_{cyc} ab(a+b) + \sum_{cyc} a^2b^2$$

$$= p^2\sum_{cyc} ab - p\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) + 3abcp + \left(\sum_{cyc} ab\right)^2 - 2abc\sum_{cyc} a$$

$$= r^2(r+4R)^2 + p^2r^2 \textit{ then}$$

$$\sum_{cyc} \sec^2\frac{A}{2} = \sum_{cyc} \frac{bc}{p(p-a)} = \frac{r^2(r+4R)^2+p^2r^2}{p(p-a)(p-b)(p-c)} = \left(\frac{r+4R}{p}\right)^2 + 1$$

$$\geq 3+1 = 4 \textit{ again, } \left(\frac{r+4R}{p}\right)^2 + 1 \leq \frac{2R}{r} \Leftrightarrow \frac{r(r+4R)^2}{2R-r} \leq p^2 \textit{ we will prove,}$$

$$3r(r+4R) \geq \frac{r(r+4R)^2}{2R-r} \Leftrightarrow 3(2R-r) \geq r+4R \Leftrightarrow 2(R-2r) \geq 0$$

which is true. Hence proved.

PROBLEM 2.103-*Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam*

$$\textit{The inequality is equivalent to: } 4 - \frac{2017m}{m+2017} - \frac{2017n}{n+2017} + \frac{m+n+2009}{2} \geq \frac{1}{\frac{1}{m}+\frac{1}{n}}$$

$$\textit{Applying AM-GM inequality: } 4 - \frac{2017m}{m+2017} - \frac{2017n}{n+2017} + \frac{m+n+2009}{2} \geq$$

$$\geq 4 - \frac{m+2017}{4} - \frac{n+2017}{4} + \frac{m+n+2009}{2} = \frac{m+n}{4}$$

$$\textit{So we need to prove that: } \frac{m+n}{4} \geq \frac{1}{\frac{1}{m}+\frac{1}{n}} \Leftrightarrow (m+n)^2 \geq 4mn \Leftrightarrow (m-n)^2 \geq 0 \textit{ (true)}$$

PROBLEM 2.104-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\sum_{cyc}(p-a)(p-b) = r(r+4R), abc = 4Rrp, \sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$$

$$\sin\frac{B}{2}=\sqrt{\frac{(p-a)(p-c)}{ca}}\ \textit{and}\ \sin\frac{C}{2}=\sqrt{\frac{(p-a)(p-b)}{ab}}$$

$$r\sum_{cyc}\frac{1}{\sin\frac{A}{2}}+\frac{abc}{2}\sum_{cyc}\frac{1}{\sqrt{abp(p-c)}}$$

$$\overset{Cauchy-Schwarz}{\leq} r\sqrt{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{1}{(p-a)(p-b)}\right)}+\frac{abc}{2}\sqrt{\left(\sum_{cyc}\frac{1}{ab}\right)\left(\sum_{cyc}\frac{1}{p(p-a)}\right)}$$

$$\leq r\sqrt{9R^2\cdot\frac{\sum_{cyc}(p-a)}{\prod_{cyc}(p-a)}}+\frac{abc}{2}\sqrt{\frac{2p}{4Rrp}\cdot\frac{\sum_{cyc}(p-a)(p-b)}{p\prod_{cyc}(p-a)}}$$

$$=r\cdot\sqrt{9R^2\frac{p}{pr^2}}+2Rrp\sqrt{\frac{1}{2Rr}\cdot\frac{r(r+4R)}{p^2r^2}}\leq 3R+3R=6R$$

PROBLEM 2.105-*Solution by Hoang Le Nhat Tung-Hanoi-Vietnam*

$$BC=a;CA=b;AB=c;S_{ABG}=S_{ACG}=S_{BCG}=\frac{S_{ABC}}{3}$$

$$\cot\widehat{GBA}+\cot\widehat{GCB}+\cot\widehat{GAC}=$$

$$=\frac{AB^2+BG^2-AG^2}{4S_{ABG}}+\frac{CG^2+BC^2-BG^2}{4S_{BGC}}+\frac{AG^2+AC^2-GA^2}{4S_{ACG}}$$

$$=\frac{3}{4}\left(\frac{a^2+b^2+c^2}{S_{ABC}}\right)=\frac{a^2+b^2+c^2}{4S_{ABC}}+\frac{a^2+b^2+c^2}{2S_{ABC}}\quad (1)$$

- $$\textit{Other: }S=\sqrt{p(p-a)(p-b)(p-c)}\leq\frac{ab+bc+ca}{4\sqrt{3}}\leq\frac{a^2+b^2+c^2}{4\sqrt{3}}$$

$$\Rightarrow\frac{a^2+b^2+c^2}{2S_{ABC}}\geq 2\sqrt{3}>3\quad (2)$$

$$(1),(2)\Rightarrow\cot\widehat{GBA}+\cot\widehat{GCB}+\cot\widehat{GAC}>\cot A+\cot B+\cot C+3$$

$$\left(\textit{Because }\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4S_{ABC}}\right)$$

PROBLEM 2.106-*Solution by proposer*

$$\cot 20^\circ\, 40^\circ\cot 80^\circ=\frac{\cos 20^\circ\cos 40^\circ\cos 80^\circ}{\sin 20^\circ\sin 40^\circ\sin 80^\circ}=\frac{\cos 80^\circ}{4\sin 20^\circ\cdot\frac{1}{2}(\cos 20^\circ-\cos 60^\circ)}=$$

$$=\frac{\cos 80^\circ}{2\sin 20^\circ\cos 20^\circ-\sin 20^\circ}=\frac{\cos 80^\circ}{2\sin 10^\circ\cos 30^\circ}=\frac{1}{2\cdot\frac{\sqrt{3}}{2}}=\frac{\sqrt{3}}{3}$$

$$\cot 20^\circ\cot 40^\circ\cot 80^\circ=\frac{\sqrt{3}}{3}\quad (1)$$

$$r\sum\frac{a^3}{r_a}=r\sum\frac{a^3(s-a)}{rs}=\frac{1}{s}\sum a^3(s-a)\quad (2)$$

$$\textit{We prove that: }\sum a^3(s-a)\leq abcs\Leftrightarrow\sum a^3(b+c-a)\leq abc(a+b+c)$$

$$\Leftrightarrow\sum a^2(a-b)(a-c)\geq\textit{(by Schur's inequality)}$$

$$\textit{By (2): }r\sum\frac{a^3}{r_a}\leq abc\quad (3)$$

$$(a\cot 20^\circ + b\cot 40^\circ + c\cot 80^\circ)^3 \overset{AM-GM}{>} 27abc\cot 20^\circ\cot 40^\circ\cot 80^\circ \overset{(1)}{=}$$
$$= 27\cdot\frac{\sqrt{3}}{3}abc \overset{(3)}{\geq} 9\sqrt{3}r\sum\frac{a^3}{r_a}$$

PROBLEM 2.107-*Solution by Soumitra Mandal-Chandar Nagore-India*

We know $x \geq \tan^{-1}x$ *and* $\tan^{-1}x \geq x + \frac{x^3}{3}$ *for all* $x \geq 0$

$$\left(\int_0^1 (\tan^{-1}x)^2 dx\right)\left(\int_0^1 \frac{dx}{\left(\tan^{-1}\frac{1}{x^2-x+1}\right)^2}\right)$$
$$\geq \left(\int_0^1 \left(x+\frac{x^3}{3}\right)^2 dx\right)\left(\int_0^1 (x^2-x+1)^2 dx\right)$$
$$= \left(\frac{1}{3}+\frac{1}{63}+\frac{2}{15}\right)\left(\frac{1}{5}+\frac{1}{3}+1-\frac{1}{2}-1+\frac{2}{3}\right) = \frac{152}{315}\cdot\frac{7}{10} > \frac{1}{4}\ \textit{(Proved)}$$

PROBLEM 2.108-*Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam*

By Cauchy's inequality we get: $\sum\frac{4(a+b)(a+c)}{(b+c)^2} \leq \sum\frac{4(a+b)(a+c)}{4bc} = \sum\frac{a(a+b+c)+bc}{bc}$
$$= \sum\frac{a^2bc+bc}{bc} = \sum(a^2+1) = 3+a^2+b^2+c^2 \Rightarrow Q.E.D.$$

PROBLEM 2.109-*Solution by proposer*

From Jordan's inequality: $\sin x \geq \frac{2x}{\pi}; x \geq 0 \Rightarrow \sin\left(\frac{x}{x^2+1}\right) \geq \frac{2x}{x^2+1}\cdot\frac{1}{\pi}$
$$\int_0^a \sin\left(\frac{x}{x^2+1}\right)dx \geq \frac{1}{\pi}\int_0^a \frac{2x}{x^2+1}dx = \frac{1}{\pi}\ln(a^2+1)$$
$$\pi\Omega(a) \geq \ln(a^2+1) \Rightarrow \pi b\Omega(a) \geq b\ln(a^2+1)$$
$$\sum \pi b\Omega(a) \geq \sum b\ln(a^2+1) = \sum(a^2+1)^b$$
$$\pi\sum b\Omega(a) \geq \ln\prod(a^2+1)^b;\ e^{\pi\sum b\Omega(a)} \geq \prod(a^2+1)^b$$
$$e^{\pi(b\Omega(a)+c\Omega(b)+a\Omega(c))} \geq (a^2+1)^b(b^2+1)^c(c^2+1)^a$$
Equality holds for $a=b=c=0$.

PROBLEM 2.110-*Solution by Soumitra Mandal-Chandar Nagore-India*

We know $r(r+4R) \geq \sqrt{3}F$ *then* $\sum_{cyc}\frac{a^{2m+2}x^{m+1}}{(y+z)^{m+1}}$
$$\geq \frac{1}{3^m}\left\{\sum_{cyc}\frac{a^2x}{y+z}\right\}^{m+1} = \frac{1}{3^m}\left\{(x+y+z)\sum_{cyc}\frac{a^2}{y+z} - \sum_{cyc}a^2\right\}^{m+1}$$

$$\overset{Bergstrom}{\geq}\frac{1}{3^m}\left\{\frac{(a+b+c)^2}{2}-\sum_{cyc}a^2\right\}^{m+1}=\frac{1}{3^m}\left\{\frac{2(ab+bc+ca)-a^2-b^2-c^2}{2}\right\}^{m+1}$$

$$=\frac{1}{3^m}\left\{\frac{2(p^2+r^2+4Rr)-2(p^2-r^2-4Rr)}{2}\right\}^{m+1}=\frac{2^{m+1}}{3^m}(r^2+4Rr)^{m+1}$$

$$=\frac{2^{m+1}}{3^m}\left(\sqrt{3}F\right)^{m+1}=\frac{2^{m+1}}{\left(\sqrt{3}\right)^{m-1}}F^{m+1}\ \textit{(Proved)}$$

PROBLEM 2.111-*Solution by Soumitra Mandal-Chandar Nagore-India*

We know $r(r+4R)\geq\sqrt{3}F$ *and* $p^2\geq 3\sqrt{3}F$ *then*

$$\sum_{cyc}\frac{(y+z)^2a^4}{x^2}\geq\frac{1}{3}\left(\sum_{cyc}\frac{(y+z)a^2}{x}\right)^2=\frac{1}{3}\left\{(x+y+z)\sum_{cyc}\frac{a^2}{x}-\sum_{cyc}a^2\right\}^2$$

$$\overset{Bergstrom}{\geq}\frac{1}{3}\left\{(a+b+c)^2-\sum_{cyc}a^2\right\}^2=\frac{4}{3}\left(p^2+r(r+4R)\right)^2\geq\frac{4}{3}\left(3\sqrt{3}F+\sqrt{3}F\right)^2=64F^2$$

PROBLEM 2.112-*Solution by Marian Ursărescu-Romania*

First step: $\sum\frac{x}{y+z}\cdot\sin^2\frac{A}{2}=\sum\left(\frac{x+y+z-y-z}{y+z}\right)\sin^2\frac{A}{2}=$

$=(x+y+z)\sum\frac{1}{y+z}\cdot\sin^2\frac{A}{2}-\sum\sin^2\frac{A}{2}$ *(1)*

But from Cauchy inequality: $\sum\frac{1}{y+z}\cdot\sin^2\frac{A}{2}\geq\frac{\left(\sum\sin\frac{A}{2}\right)^2}{2(x+y+z)}$ *(2)*

From (1)+(2)$\Rightarrow\sum\frac{x}{y+z}\sin^2\frac{A}{2}\geq\frac{1}{2}\left(\sum\sin\frac{A}{2}\right)^2-\sum\sin^2\frac{A}{2}$ *(3)*

From (3) inequality becomes: $\frac{1}{2}\left(\sum\sin\frac{A}{2}\right)^2-\sum\sin^2\frac{A}{2}\geq\frac{F}{2\sqrt{3}R^2}$

But $F=pr$ *(4)*

$\sum\sin^2\frac{A}{2}=1-\frac{r}{2R}$ *(5)*

And $\sin^n\left(\frac{A}{2}\right)+\sin^n\left(\frac{B}{2}\right)+\sin^n\left(\frac{C}{2}\right)\geq\frac{3}{2^n},n\in\mathbb{N}^*$

In our case: $\sum\sin\frac{A}{2}\geq\frac{3}{2}$ *(6)*

From (4)+(5)+(6) inequality becomes: $\frac{9}{8}-1+\frac{r}{2R}\geq\frac{pr}{2\sqrt{3}R^2}$ *(7)*

But $p\leq\frac{3\sqrt{3}}{2}R\Rightarrow\frac{pr}{2\sqrt{3}R^2}\leq\frac{3r}{4R}$ *(8)*

From (7)+(8) we must show: $\frac{3r}{4R}\leq\frac{1}{8}+\frac{r}{2R}\Leftrightarrow\frac{r}{4R}\leq\frac{1}{8}\Leftrightarrow 2r\leq R$ *(true)*

PROBLEM 2.113-*Solution by proposer*

We have: $xy(x^2+y^2)\leq x^4+y^4\Leftrightarrow(x-y)^2(x^2+xy+y^2)\geq 0$ *or* $x^3y+xy^3\leq x^4+y^4$

and $2xz+2yz-2z^2\leq x^2+y^2\Leftrightarrow(x-z)^2+(y-z)^2\geq 0\Rightarrow$

$xy(x^2+y^2)(2xz+2yz-2z^2)\leq(x^2+y^2)(x^4+y^4)$ *or*

$2x^4yz+2x^2y^3z+2x^3y^2z+2xy^4z-2x^3yz^2-2xy^3z^2\leq(x^2+y^2)(x^4+y^4)\Rightarrow$

$$4xyz(x^3+y^3+z^3)=\sum_{cyclic}(2x^4yz+2x^2y^3z+2x^3y^2z+2xy^4z-2x^3yz^2-2xy^3z^2)\le$$
$$\le\sum_{cyclic}(x^2+y^2)(x^4+y^4)$$

PROBLEM 2.114-*Solution by proposer*

We have: $2x+2y-2z\le\frac{x^2+y^2}{z}\Leftrightarrow(x-z)^2+(y-z)^2\ge0$ *and* $x^2+y^2\le\frac{x^3}{y}+\frac{y^3}{x}\Leftrightarrow$
$\Leftrightarrow(x-y)^2(x^2+xy+y^2)\ge0$ *therefore* $\sum_{cyclic}(2x+2y-2z)+\sum_{cyclic}(x^2+y^2)\le$
$\le\sum_{cyclic}\frac{x^2+y^2}{z}+\sum_{cyclic}\left(\frac{x^3}{y}+\frac{y^3}{x}\right)$ *or* $2(\sum x^2+\sum x)\le\frac{1}{xyz}\sum(x^4+y^4)z+$
$$+\sum\frac{x^2+y^2}{z}\Rightarrow2\sum\left(x+\frac{1}{2}\right)^2\le\frac{3}{2}+\frac{1}{xyz}\sum(x^4+y^4)z+\sum\frac{x^2+y^2}{z}$$

PROBLEM 2.115-*Solution by Marian Ursărescu – Romania*

$\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_c}\ge3\sqrt[3]{\frac{(abc)^6}{r_ar_br_c}}\Rightarrow$ *we must show:* $3\sqrt[3]{\frac{(abc)^6}{r_ar_br_c}}\ge2^6\cdot3^3r^5\Leftrightarrow\frac{(abc)^6}{r_ar_br_c}\ge2^{18}3^6r^{15}$ *(1)*

But $abc=4sRr$ *and* $r_ar_br_c=s^2r$ *(2). From (1)+(2) we must show:*
$\frac{2^{12}s^6R^6r^6}{s^2r}\ge2^{18}\cdot3^6\cdot r^{15}\Leftrightarrow s^4R^6\ge6^6r^{10}\Leftrightarrow s^2R^3\ge6^3r^5$ *(3)*
But $R\ge2r\Rightarrow R^3\ge8r^3$ *and* $s^2\ge27r^2\Rightarrow s^2R^3\ge2^3\cdot3^3\cdot r^5\Rightarrow$ *(3) its true. Now we use Schur inequality:* $a^5(a-b)(a-c)+b^5(b-c)(b-a)+c^5(c-a)(c-b)\ge0\Leftrightarrow$
$\Leftrightarrow a^5(^2-ab-ac+bc)+\ b^5(b^2-ab-bc+ac)+c^5(c^2-4c-ac+a^b)\ge0\Leftrightarrow$
$\Leftrightarrow a^6(a-b-c)+b^6(b-c-a)+c^6(c-a-b)+a^5bc+ab^5c+abc^5\ge0\Leftrightarrow$
$\Leftrightarrow a^6(b+c-a)+b^6(a+c-b)+c^6(a+b-c)\le abc(a^4+b^4+c^4)\Leftrightarrow$
$\Leftrightarrow2a^6(s-a)+2b^6(s-b)+2c^6(s-c)\le abc(a^4+b^4+c^4)$. *But* $r_a=\frac{S}{s-a}\Rightarrow$
$\Rightarrow s-a=\frac{S}{r_a}\Rightarrow2S\left(\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_b}\right)\ge abc(a^4+b^4+c^4)$ *(4). But* $S=\frac{abc}{4R}$ *(5)* $\Rightarrow$ *From (4)+(5)*
$\Rightarrow\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_b}\le2R(a^4+b^4+c^4)$ *(6). From (6) we must show:*
$2R(a^4+b^4+c^4)\le108R^4(R-r)\Leftrightarrow a^4+b^4+c^4\le54R^3(R-r)$ *(7)*
But $a^4+b^4+c^4=2(s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2)$ *(8)*
From (7)+(8) we must show:
$s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2-27R^3(R-r)\le0$ *(9)*
Now, let $f(s^2)$ *a polygon of second degree* $\Rightarrow f(p^2)=(p^2-x_1)(p^2-x_2)$*, (9) its equivalent with* $\left[p^2-r(4R+3r)-\sqrt{8r^3(2R+r)+27R^3(R-r)}\right]\cdot$
$\cdot\left[s^2-r(4R+3r)+\sqrt{8r^3(2R+r)+27R^3(R-r)}\right]\le0$ *(10)*
(10) its true if $x_1\le p^2\le x_2,x_1,x_2$ *its square, then we must show:*
$r(4R+3r)-\sqrt{8r^3(2R+r)+27R^3(R-r)}\le s^2$ *(1)*
and $s^2\le r(4R+3r)-\sqrt{8r^3(2R+r)+27R^3(R-r)}$ *(2)*
For (1) using Gerretsen's inequality: $s^2\ge16Rr-5r^2$ *(11)* $\Leftrightarrow$

$$\Leftrightarrow 4Rr + 3r^2 - \sqrt{8r^3(2R+r) + 27R^3(R-r)} \le 16Rr - 5r^2 \Leftrightarrow$$
$$\Leftrightarrow 8r^2 = 12Rr \le \sqrt{8r^3(2R+r) + 27R^3(R-r)} \Leftrightarrow$$
$$\Leftrightarrow (8r^2 - 12Rr)^2 \le 8r^3(2R+r) + 27R^3(R-r) \Leftrightarrow R \ge 2r \quad \textit{(Euler). For (12) using again}$$
Gerretsen's inequality $s^2 \le 4R^2 + 3r^2 + 4Rr$
$$\textit{(12)} \Leftrightarrow 4R^2 + 3r^2 + 4Rr \le r(4R+3r) - \sqrt{8r^3(2R+r) + 27R^3(R-r)} \Leftrightarrow$$
$$\Leftrightarrow 8r^4 + 16Rr^3 - 27R^3r + 11R^4 \ge 0. \textit{ Let } x = \frac{R}{2r} \ge 1 \textit{ (Euler)} \Rightarrow \textit{we must show:}$$
$$22x^4 - 27x^3 + 4x + 1 \ge 0 \textit{ and with Horner and Rolle sequence} \Rightarrow$$
$$(x-1)\big(11x^3 + (x-1)(11x^2 + 6x + 1)\big) \ge 0 \textit{ true.}$$

PROBLEM 2.116-*Solution by Soumava Chakraborty-Kolkata-India*

$$1 = \frac{\sum a}{3} \therefore \textit{given inequality} \Leftrightarrow \left(\sum a\right)\left(\sum a^3\right) + 3\sum a^4 \ge 6\sum a^2b^2$$
$$\Leftrightarrow \sum a^4 + \sum a^3b + \sum ab^3 + 3\sum a^4 \ge 6\sum a^2\, b^2$$
$$\Leftrightarrow 4\sum a^4 + \sum a^3b + \sum ab^3 \ge 6\sum a^2b^2 \quad (1)$$
$$\textit{Now, } \sum a^3b + \sum ab^3 \underset{(a)}{\overset{A-G}{\ge}} 2\sum a^2b^2. \textit{ Also, } 4\sum a^4 \overset{(b)}{\ge} 4\sum a^2b^2$$
(a)+(b)⇒(1) is true (Proved)

PROBLEM 2.117-*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum\left(\cos^3\frac{A}{2}\right)^{-1} = \sum\frac{8\cos^3\frac{B-C}{2}}{\left(2\sin\frac{B+2}{2}\cos\frac{B-C}{2}\right)^3} \le \sum\frac{8}{(\sin B + \sin C)^3}$$
$$\left(\because 0 < \cos\frac{B-C}{2} \le 1 \textit{ as } -\frac{\pi}{2} < \frac{B-C}{2} < \frac{\pi}{2}\right)$$
$$= \sum\frac{64R^3}{(b+c)^3} = \sum\frac{64R^3}{b^3 + c^3 + 3bc(b+c)} \overset{(1)}{\le} \sum\frac{64R^3}{bc(b+c) + 3bc(b+c)}$$
$$= \sum\frac{16R^3}{bc(b+c)} = \frac{16R^3\sum a(c+a)(a+b)}{abc(a+b)(b+c)(c+a)} = \frac{16R^3\sum a\left(\sum ab + a^2\right)}{4Rrs(a+b)(b+c)(c+a)} =$$
$$= \frac{4R^2}{rs}\left(\frac{2s\sum ab + \sum a^3}{\pi(a+b)}\right). \textit{ Now, } \sum a^3 = 3abc + 2s\left(\sum a^2 - \sum ab\right) =$$
$$= 12Rr + 2s(s^2 - 12Rr - 3r^2) \overset{(a_1)}{=} 2s(s^2 - 6Rr - 3r^2)\ \&$$
$$\prod(a+b) = 2abc + \sum ab\,(2s-c) = 2s(s^2 + 4Rr + r^2) - 4Rrs \overset{(a_2)}{=}$$
$$= 2s(s^2 + 2Rr + r^2)$$
$$\textit{(1), (a}_2\textit{), (a}_3\textit{)} \Rightarrow \sum\left(\cos^3\frac{A}{2}\right)^{-1} \le \frac{4R^2}{rs}\cdot\frac{2s(2s^2 - 4Rr - 2r^2)}{2s(s^2 + 2Rr + r^2)} \overset{(?)}{\le} \frac{2R^2}{\sqrt{3}r^2} \Leftrightarrow$$
$$\Leftrightarrow 4\sqrt{3}r(s^2 - Rr - r^2) \overset{(?)}{\le} s(s^2 + 2Rr + r^2) \Leftrightarrow s^2(s^2 + 2Rr + r^2)^2 \overset{(?)}{\ge}$$
$$\ge 48r^2(s^2 - Rr - r^2)^2 \Leftrightarrow s^2\big(s^4 + r^2(2R+r)^2 + 2s^2(2Rr + r^2)\big) \overset{(?)}{\ge}$$
$$\ge 48r^2\big(s^4 + r^2(R+r)^2 - 2s^2(Rr + r^2)\big) \Leftrightarrow s^6 + 2s^4(2Rr + r^2) + s^2r^2(2R+r)^2 +$$

$$+96s^2r^2(Rr+r^2) \underset{(a_3)}{\overset{?}{\geq}} 48r^2s^4+48r^4(R+r)^2$$

Now, LHS of $(a_3) \overset{Gerretsen}{\geq} s^4(20Rr-3r^2)+s^2r^2\{(2R+r)^2+96(Rr+r^2)\} \overset{(?)}{\geq} 48r^2s^4+$
$+48r^4(R+r)^2 \Leftrightarrow s^4(20Rr-40r^2)+s^2r^2\{(2R+r)^2+96(Rr+r^2)\} \underset{(a_4)}{\overset{?}{\geq}}$

$$\geq \frac{1}{s^4r^2}+48r^4(R+r)^2$$

Now, LHS of $(a_4) \underset{(i)}{\overset{Gerretsen}{\geq}} s^2r^2(16R-5r)(20R-40r)+s^2r^2\{(2R+r)^2+96(Rr+r^2)\}$

and also, RHS of $(a_4) \underset{(ii)}{\overset{Gerretsen}{\geq}} 11s^2r^2(4R^2+4Rr+3r^2)+48r^4(R+r)^2$

(i) & (ii) $\Rightarrow$ *in order to prove* (a_4)*, it suffices to prove:*

$s^2\{(16R-5r)(20R-40r)+(2R+r)^2+96(Rr+r^2)-11(4R^2+4Rr+3r^2)\} \geq$
$\geq 48r^2(R+r)^2 \Leftrightarrow s^2(70R^2-171Rr+66r^2) \overset{(a_5)}{\geq}$
$\geq 12r^2(R+r)^2 \because 70R^2-171Rr+66r^2=(R-2r)(70R-31r)+4r^2>0$

$\therefore$ *LHS of* $(a_5) \overset{Gerretsen}{\geq} (16Rr-5r^2)(70R^2-171Rr+66r^2) \overset{?}{\geq} 12r^2(R+r)^2 \Leftrightarrow$
$\Leftrightarrow 1120t^3-3098t^2+1887t-342 \overset{?}{\geq} 0 \quad \left(t=\frac{R}{r}\right)$
$\Leftrightarrow (t-2)(1120t^2-858t+171) \overset{?}{\geq} 0 \to$ *true* $\because t \overset{Euler}{\geq} 2 \Rightarrow \sum\left(\cos^3\frac{A}{2}\right)^{-1} \leq \frac{2\sqrt{3}}{3}\left(\frac{R}{r}\right)^2$

Also, $\sum\frac{1}{\cos^3\frac{A}{2}} \overset{Radon}{\geq} \frac{3^4}{\left(\sum\cos\frac{A}{2}\right)^3} \overset{Jensen}{\geq} \frac{3^4}{\left(\frac{3\sqrt{3}}{2}\right)^3}$

$\left(\because f(x)=\cos\frac{x}{2}\right.$ *is concave on* $\left.(0,\pi)\right)=\frac{3^4\cdot 8}{3^3\cdot 3\sqrt{3}}=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3} \therefore \sum\frac{1}{\cos^3\frac{A}{2}} \geq \frac{8\sqrt{3}}{3}$

$\therefore$ *both bounds of (a) are proved. Now,* $\sum\cos^2\frac{A}{2}=\frac{1}{2}\sum(1+\cos A)=\frac{1}{2}\left(3+1+\frac{r}{R}\right) \overset{(b_1)}{=}$

$$=\frac{1}{2}\left(\frac{4R+r}{R}\right)$$

$(b_1) \Rightarrow \sum\cos^2\frac{A}{2} \geq \frac{9r^2}{R^2} \Leftrightarrow \frac{4R+r}{2R} \geq \frac{9r^2}{R^2} \Leftrightarrow R(4R+r) \geq 18r^2 \Leftrightarrow 4R^2+Rr-18r^2 \geq 0$
$\Leftrightarrow (R-2r)(4R+9r) \overset{?}{\geq} 0 \to$ *true* $\because R \overset{Euler}{\geq} 2r \therefore \sum\cos^2\frac{A}{2} \geq 9\left(\frac{r}{R}\right)^2$*. Also,* $(b_1) \Rightarrow$
$\Rightarrow \sum\cos^2\frac{A}{2} \leq \frac{9}{4} \Leftrightarrow \frac{4R+r}{2R} \leq \frac{9}{4} \Leftrightarrow 9R \geq 8R+2r \Leftrightarrow R \geq 2r \to$ *true (Euler)*
$\therefore \sum\cos^2\frac{A}{2} \leq \frac{9}{4} \therefore$ *both bounds of (b) are proved (Done)*

PROBLEM 2.118-*Solution by proposer*

** By Cauchy-Schwarz's inequality we have:*

$$\left(\sqrt{2(b^8+c^8)}+2b^2c^2\right)^2 \leq 2(2(b^8+c^8)+4b^4c^4)=4(b^8+2b^4c^4+c^8)=4(b^4+c^4)^2$$
$$\Rightarrow \sqrt{2(b^8+c^8)}+2b^2c^2 \leq 2(b^4+c^4) \Leftrightarrow \sqrt{2(b^8+c^8)} \leq 2(b^4-b^2c^2+c^4) \Leftrightarrow$$
$$\Leftrightarrow \sqrt[4]{\frac{b^8+c^8}{2}} \leq \sqrt{b^4-b^2c^2+c^4} \Rightarrow$$

$$\Rightarrow \sqrt[4]{\frac{b^8+c^8}{2}} \le \sqrt{(2+\sqrt{3})(b^2-bc\sqrt{3}+c^2)\cdot(2-\sqrt{3})(b^2+bc\sqrt{3}+c^2)} \le$$

$$\le \frac{(2+\sqrt{3})(b^2-bc\sqrt{3}+c^2)+(2-\sqrt{3})(b^2+bc\sqrt{3}+c^2)}{2} = 2b^2-3bc+2c^2$$

$$\Leftrightarrow \sqrt[4]{\frac{b^8+c^8}{2}}+5bc \le 2(b^2+bc+c^2) \Leftrightarrow \frac{a^3}{\sqrt[4]{\frac{b^8+c^8}{2}}+5bc} \ge \frac{a^3}{2(b^2+bc+c^2)}$$

$$+ \textit{Similar: } \frac{b^3}{\sqrt[4]{\frac{c^8+a^8}{2}}+5ca} \ge \frac{b^3}{2(c^2+ca+a^2)};\ \frac{c^3}{\sqrt[4]{\frac{a^8+b^8}{2}}+5ab} \ge \frac{c^3}{2(a^2+b^2+ab)}$$

$$\Rightarrow P = \frac{a^3}{\sqrt[3]{\frac{b^8+c^8}{2}}+5bc}+\frac{b^3}{\sqrt[4]{\frac{c^8+a^8}{2}}+5ca}+\frac{c^3}{\sqrt[4]{\frac{a^8+b^8}{2}}+5ab}+\frac{(a+b)(b+c)(c+a)}{16}$$

$$\ge \frac{a^3}{2(b^2+bc+c^2)}+\frac{b^3}{2(c^2+ca+a^2)}+\frac{c^3}{2(a^2+b^2+ab)}+\frac{(a+b)(b+c)(c+a)}{16}$$

$$+ \textit{Using inequality: } 9(x+y)(y+z)(z+x) \ge 8(x+y+z)(xy+yz+zx)$$

$$\Rightarrow P \ge \frac{1}{2}\sum \frac{a^3}{b^2+bc+c^2}+\frac{\prod(b+c)}{16} \ge \frac{1}{2}\sum \frac{a^4}{ab^2+abc+ac^2}+\frac{\frac{8}{9}(\sum a)(\sum ab)}{16}$$

$$\Rightarrow P \ge \frac{1}{2}\cdot\frac{(\sum a^2)^2}{\sum(ab^2+abc+ac^2)}+\frac{(\sum a)(\sum ab)}{18} = \frac{9}{2(\sum a)(\sum ab)}+\frac{(\sum a)(\sum ab)}{18}\ \textit{(Cauchy-Schwarz)}$$

$$\Rightarrow P \ge 2\cdot\sqrt{\frac{9}{2(\sum a)(\sum ab)}\cdot\frac{(\sum a)(\sum ab)}{18}} = 2\cdot\sqrt{\frac{1}{4}} = 1 \Rightarrow P_{\min} = 1$$

(Because by AM-GM inequality and $a^2+b^2+c^2=3$)

$$\Rightarrow P_{\min} = 1 \Leftrightarrow \begin{cases} a=b=c \\ a^2+b^2+c^2=3 \\ (\sum a)(\sum ab)=9 \end{cases} \Leftrightarrow a=b=c=1.$$

PROBLEM 2.119-*Solution by proposer*

* *We have:* $b^6+c^6=(b^2+c^2)(b^4-b^2c^2+c^4)=(b^2+c^2)(b^2-bc\sqrt{3}+c^2)(b^2+bc\sqrt{3}+c^2)$

\- *Therefore, by AM-GM inequality:*

$$\sqrt[3]{4(b^6+c^6)} = 2\cdot\sqrt[3]{\frac{b^2+c^2}{2}(2-\sqrt{3})(b^2+bc\sqrt{3}+c^2)(2+\sqrt{3})(b^2-bc\sqrt{3}+c^2)} \le$$

$$\le 2\cdot\frac{\frac{b^2+c^2}{2}+(2-\sqrt{3})(b^2+bc\sqrt{3}+c^2)+(2+\sqrt{3})(b^2-bc\sqrt{3}+c^2)}{3} = 3b^2-4bc+3c^2$$

$$\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc} \ge \frac{a}{3b^2-4bc+3c^2+7bc} = \frac{a}{3(b^2+bc+c^2)}$$

$$+ \textit{Similar: } \frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca} \ge \frac{b}{3(c^2+ca+a^2)};\ \frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab} \ge \frac{c}{3(a^2+b^2+ab)}$$

$$\Rightarrow P=\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{(a+b)(b+c)(c+a)}{24}\geq$$

$$\geq\frac{a}{3(b^2+bc+c^2)}+\frac{b}{3(c^2+ca+a^2)}+\frac{c}{3(a^2+b^2+ab)}+\frac{(a+b)(b+c)(c+a)}{24}$$

+ *Using inequality:* $9(x+y)(y+z)(z+x)\geq 8(x+y+z)(xy+yz+zx)$

$$\Rightarrow P\geq\frac{1}{3}\sum\frac{a}{b^2+bc+c^2}+\frac{\prod(b+c)}{24}\geq\frac{1}{3}\sum\frac{a^2}{ab^2+abc+ac^2}+\frac{\frac{8}{9}(\sum a)(\sum ab)}{24}$$

$$\Rightarrow P\geq\frac{1}{3}\cdot\frac{(\sum a)^2}{\sum(ab^2+abc+ac^2)}+\frac{(\sum ab)}{9}=\frac{3}{(\sum a)(\sum ab)}+\frac{(\sum ab)}{9}\quad\textit{(Cauchy-Schwarz)}$$

$$\Rightarrow P\geq\frac{1}{\sum ab}+\frac{\sum ab}{9}\geq 2\cdot\sqrt{\frac{1}{\sum ab}\cdot\frac{\sum ab}{9}}=\frac{2}{3}\quad\textit{(Because by AM-GM and } a+b+c=3\textit{)}$$

$$\Rightarrow P_{\min}=\frac{2}{3}\Leftrightarrow\begin{cases}a=b=c\\a+b+c=3\end{cases}\Leftrightarrow a=b=c=1.$$

PROBLEM 2.120-*Solution by Marian Ursărescu-Romania*

$$B\sqrt[3]{\frac{a^2}{B^2}}+C\sqrt[3]{\frac{b^2}{C^2}}+A\sqrt[3]{\frac{c^2}{A^2}}\leq\sqrt[3]{\pi(a+b+c)^2}\Leftrightarrow$$

$$\frac{B}{\pi}\sqrt[3]{\frac{a^2}{B^2}}+\frac{C}{\pi}\sqrt[3]{\frac{b^2}{C^2}}+\frac{A}{\pi}\sqrt[3]{\frac{c^2}{A^2}}\leq\sqrt[3]{\frac{(a+b+c)^2}{\pi^2}}\quad(1)$$

Let $f:(0,\infty)\to\mathbb{R};f(x)=\sqrt[3]{x^2};\ f'(x)=\left(x^{\frac{2}{3}}\right)'=\frac{2}{3}x^{-\frac{1}{3}};f''(x)=-\frac{2}{9}x^{-\frac{4}{3}}<0\Rightarrow$

from Jensen's inequality $\Rightarrow$

$p_1f(x_1)+p_2f(x_2)+p_3f(x_3)\leq f(p_1x_1+p_2x_2+p_3x_3)$ *with* $p_1+p_2+p_3=1$

$$p_1=\frac{B}{\pi},p_2=\frac{C}{\pi},p_3=\frac{A}{\pi}$$

$$x_1=\frac{a}{B},x_2=\frac{b}{C},x_3=\frac{c}{A}\Rightarrow\frac{B}{\pi}\sqrt[3]{\left(\frac{a}{B}\right)^2}+\frac{C}{\pi}\sqrt[3]{\left(\frac{b}{C}\right)^2}+\frac{A}{\pi}\sqrt[3]{\left(\frac{c}{A}\right)^2}\leq\sqrt[3]{\left(\frac{a+b+c}{A+B+C}\right)^2}\Rightarrow$$

$$\frac{B}{\pi}\sqrt[3]{\left(\frac{a}{B}\right)^2}+\frac{C}{\pi}\sqrt[3]{\left(\frac{b}{C}\right)^2}+\frac{A}{\pi}\sqrt[3]{\left(\frac{c}{A}\right)^2}\leq\sqrt[3]{\frac{(a+b+c)^2}{\pi^2}}\textit{ then (1) is true}$$

PROBLEM 2.121-*Solution by proposer*

- *Because* $\begin{cases}x,y,z>0\\x+y+z=3\end{cases}\Rightarrow 0<x,y,z<3\Rightarrow 5-3\sqrt[3]{x}>0;5-2\sqrt[3]{y}>0;5-3\sqrt[3]{z}>0$

- *Be Cauchy – Schwarz inequality we have:*

$$\frac{x^4}{5-3\sqrt[3]{y}}+\frac{y^4}{5-3\sqrt[3]{z}}+\frac{z^4}{5-3\sqrt[3]{x}}+\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2}\geq\frac{(x^2+y^2+z^2)^2}{15-3(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z})}+\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2}\quad(1)$$

$$+ \textit{ Other, by AM-GM inequality: } \begin{cases} \sqrt{x}+\sqrt{x}+x^2 \geq 3\sqrt[3]{\sqrt{x}\cdot\sqrt{x}\cdot x^2} = 3x \\ \sqrt{y}+\sqrt{y}+y^2 \geq 3\sqrt[3]{\sqrt{y}\cdot\sqrt{y}\cdot y^2} = 3y \\ \sqrt{z}+\sqrt{z}+z^2 \geq 3\sqrt[3]{\sqrt{z}\cdot\sqrt{z}\cdot z^2} = 3z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2\sqrt{x} \geq 3x - x^2 \\ 2\sqrt{y} \geq 3y - y^2 \\ 2\sqrt{z} \geq 3z - z^2 \end{cases} \Rightarrow$$

$$\Rightarrow 2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) \geq 3(x+y+z) - (x^2+y^2+z^2) = (x+y+z)^2 - (x^2+y^2+z^2)$$

$\Leftrightarrow 2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) \geq 2(xy+yz+zx) \Leftrightarrow \sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx$ *(because* $x+y+z+3$*)* *(2)*

$$\begin{cases} \sqrt[3]{x}+\sqrt[3]{x}+\sqrt[3]{x}+x^2+x^2 \geq 5\sqrt[5]{x^5} = 5x \\ \sqrt[3]{y}+\sqrt[3]{y}+\sqrt[3]{y}+y^2+y^2 \geq 5\sqrt[5]{y^5} = 5y \\ \sqrt[3]{z}+\sqrt[3]{z}+\sqrt[3]{z}+z^2+z^2 \geq 5\sqrt[5]{z^5} = 5z \end{cases} \Leftrightarrow \begin{cases} 3\sqrt[3]{x} \geq 5x - 2x^2 \\ 3\sqrt[3]{y} \geq 5y - 2y^2 \\ 3\sqrt[3]{z} \geq 5z - 2z^2 \end{cases}$$

$$\Rightarrow 3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right) \geq 5(x+y+z) - 2(x^2+y^2+z^2) = 15 - 2(x^2+y^2+z^2)$$

$$\Leftrightarrow 15 - 3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right) \leq 2(x^2+y^2+z^2) \Leftrightarrow \frac{(x^2+y^2+z^2)^2}{15-3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)} \geq \frac{x^2+y^2+z^2}{2} \quad (3)$$

- Let (1), (2), (3):

$$\Rightarrow \frac{x^4}{5-3\sqrt[3]{y}} + \frac{y^4}{5-3\sqrt[3]{z}} + \frac{z^4}{5-3\sqrt[3]{x}} + \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2} \geq \frac{x^2+y^2+z^2}{2} + \frac{xy+yz+zx}{2} \quad (4)$$

$$+ \textit{ We have: } \frac{x^2+y^2+z^2}{2} + \frac{xy+yz+zx}{2} = \frac{(x+y+z)^2-(xy+yz+zx)}{2} \geq \frac{(x+y+z)^2-\frac{(x+y+z)^2}{3}}{2} = \frac{(x+y+z)^2}{3} = \frac{3^2}{3} = 3$$

(5)

- Let (4), (5): $\Rightarrow \frac{x^4}{5-3\sqrt[3]{y}} + \frac{y^4}{5-3\sqrt[3]{z}} + \frac{z^4}{5-3\sqrt[3]{x}} + \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2} \geq 3$ *and we get the result.*

+ Equality occurs if $x = y = z = 1$.

PROBLEM 2.122-*Solution by Omran Kouba-Damascus-Syria*

We will prove the stronger inequality: $|z_1 - z_3| + |z_2 - z_3| \leq 3 + \frac{1}{2}|z_1 + z_2|$ *(1)*

Consider x and y from $\left[0, \frac{\pi}{2}\right]$ *such that* $\frac{z_1}{z_3} = e^{4ix}$ *and* $\frac{z_2}{z_3} = e^{4iy}$. *With this notation (1) is equivalent to* $2\sin(2x) + 2\sin(2y) \leq 3 + |\cos(2x+2y)|$ *(2)*

Now, with $z = x + y \in [0, \pi]$, *clearly we have:*

$2\sin(2x) + 2\sin(2y) = 4\sin(z)\cos(x-y) \leq 4\sin z$ *(3)*

and $(3 + |\cos 2z|)^2 - (4\sin z)^2 = 9 + \cos^2 2z + 6|\cos 2z| - 8(1 - \cos 2z) =$

$= (1 + \cos 2z)^2 + 6(|\cos 2z| + \cos 2z) \geq 0$. *But,* $\sin z \geq 0$, *so the previous inequality implies:* $4\sin z \leq 3 + |\cos 2z|$, *thus (3) implies (2), and this is equivalent to (1). The proof of the stronger inequality (1) is completed.*

PROBLEM 2.123-*Solution by proposer*

A symmetric and invertible $\Rightarrow$ *eigen values* $\lambda_1, \lambda_2, \dots \lambda_n \in \mathbb{R}^*$. *Let be the polynomial*

$p(x) = x^2 + 1 \Rightarrow \det p(A) = p(\lambda_1)p(\lambda_2) - p(\lambda_n) = (\lambda_1^2 + 1)(\ {}_2^2 + 1) \dots .(\lambda_n^2 + 1)$ *(1)*

$$A^{-1} \text{ has the eigen values}$$
$$\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1} \in \mathbb{R}^* \Rightarrow \det p(A^{-1}) = (\lambda_1^{-2}+1)(\lambda_2^{-2}+1)\dots(\lambda_n^{-1}+1)$$
$$\Rightarrow \det p(A^{-1}) = (\lambda_1^{-2}+1)(\lambda_2^{-2}+1)\dots(\lambda_n^{-1}+1) = \frac{(\lambda_1^2+1)(\lambda_2^2+1)\dots(\lambda_n^2+1)}{\lambda_1^2\lambda_2^2\dots\lambda_n^2} \quad (2)$$
$$\text{But } A^2 + A^{-2} + 2I_n = (A^2+I_n)(A^{-2}+I_n) \Rightarrow \det(A^2+A^{-2}+2I_n) \Rightarrow$$
$$\det(A^2+A^{-2}+2I_n) = \det(A^2+I_n)\cdot\det(A^{-2}+I_n) \quad (3)$$
$$\text{From (1)+(2)+(3)} \Rightarrow \det(A^2+A^{-2}+2I_n) = \left[\frac{(\lambda_1^2+1)(\lambda_2^2+1)\dots(\lambda_n^2+1)}{\lambda_1\lambda_2\dots\lambda_n}\right]^2 \quad (4)$$
$$\text{But } \lambda_k^2 + 1 \geq 2\lambda_k \quad (5)$$
$$\text{But (4) + (5)} \Rightarrow \det(A^2+A^{-2}+2I_n) \geq \left(\frac{2^n\lambda_1\lambda_2\dots\lambda_n}{\lambda_1\lambda_2\dots\lambda_n}\right)^2 = 4^n$$

PROBLEM 2.124- ***Solution by Soumava Chakraborty-Kolkata-India***

$$\sum \frac{a^2}{b^2+c^2} \overset{A-G}{\leq} \sum \frac{a^2}{2bc} = \frac{\sum a^3}{2abc} = \frac{3abc + 2s(\sum a^2 - \sum ab)}{2\cdot 4Rrs} =$$
$$= \frac{2s(s^2-12Rr-3r^2)+12Rrs}{8Rrs} = \frac{2s(s^2-6Rr-3r^2)}{8Rrs} = \frac{s^2-6Rr-3r^2}{4Rr} \leq \frac{2R-r}{2r} \Leftrightarrow$$
$$\Leftrightarrow s^2 - 6Rr - 3r^2 \leq 2R(2R-r) = 4R^2 - 2Rr \Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \to \text{true}$$
$$\text{(Gerretsen)} \Rightarrow \sum \frac{a^2}{b^2+c^2} \leq \frac{2R-r}{2r}. \text{ Also, } \sum \frac{a^2}{b^2+c^2} \overset{Nesbitt}{\geq} \frac{3}{2} \text{ (Done).}$$

PROBLEM 2.125-*Solution by Soumava Chakraborty-Kolkata-India*

$$\left(\frac{h_a}{r_a}\right)^2 = \left(\frac{2\Delta}{a} \times \frac{s-a}{\Delta}\right)^2 = 4\frac{(s-a)^2}{a^2} \text{ etc } \therefore \text{ given inequality becomes:}$$
$$\sum a^4b^4(a^4+b^4) \overset{(1)}{\geq} 8a^2b^2c^2\{b^2c^2(s-a)^2 + c^2a^2(s-b)^2 + a^2b^2(s-c)^2\}.$$
$$\text{Let } s-a = x, s-b = y,$$
$$s-c = z \Rightarrow s = \sum x \therefore a = y+z, b = z+x, c = x+y \ \ (x,y,z > 0).$$
$$\text{Then (1) becomes:}$$
$$\sum_{cyc}[\{(y+z)(z+x)\}^4\{(y+z)^4 + (z+x)^4\}] \geq 8(x+y)^2(y+z)^2(z+x)^2 \cdot$$
$$\cdot\left[\sum_{cyc}\{x^2(z+x)^2(x+y)^2\}\right] \Leftrightarrow 2\sum x^{12} + 12\left(\sum x''y + \sum xy''\right) +$$
$$+26\left(\sum x^{10}y^2 + \sum x^2y^{10}\right) + 48xyz\left(\sum x^9\right) + 28\left(\sum x^9y^3 + \sum x^3y^9\right) +$$
$$+64xyz\left(\sum x^8y + \sum xy^8\right) + 25\left(\sum x^8y^4 + \sum x^4y^8\right) + 40xyz\left(\sum x^7y^2 + \sum x^2y^7\right)$$
$$+72xyz\left(\sum x^6y^3 + \sum x^3y^6\right) + 32\left(\sum x^7y^5 + \sum x^5y^7\right) + 40\sum x^6y^6 +$$
$$+144x^2y^2z^2\left(\sum x^3y^3\right) + 160xyz\left(\sum x^5y^4 + \sum x^4y^5\right) +$$
$$+36x^2y^2z^2\left(\sum x^4y^2 + \sum x^2y^4\right) \overset{(2)}{\geq} 4x^2y^2z^2\left(\sum x^6\right) + 80x^2y^2z^2\left(\sum x^5y + \sum xy^5\right)$$

$$+224x^3y^3z^3\left(\sum x^3\right)+312x^3y^3z^3\left(\sum x^2y+\sum xy^2\right)+636x^4y^4z^4$$

$$2\sum x^9=\sum(x^9+y^9)\overset{Chebyshev}{\geq}\sum\frac{1}{2}(x^2+y^2)(x^8+y^7)\overset{A-G}{\geq}\sum xy(x^7+y^7)=$$

$$=\sum y(x^8+z^8)\overset{Chebyshev}{\geq}\sum\frac{1}{2}y(x^2+z^2)(x^6+z^6)\overset{A-G}{\geq}\sum xyz(x\quad+z^6)=$$

$$=2xyz(\textstyle\sum x^6)\Rightarrow 4xyz(\sum x^9)\geq 4x^2y^2z^2(\sum x^6)\quad (a)$$

Again, $2\sum x^6y^6=\sum(x^6y^6+y^6z^6)\underset{(i)}{\overset{A-G}{\geq}}2\sum x^3z^3y^6=2\sum x^3y^3z^3(\sum x^3)\Rightarrow$

$$\Rightarrow 40\textstyle\sum x^6y^6\geq 40x^3y^3z^3(\sum x^3)\quad (b)$$

Also, $32(\sum x^7y^5+\sum x^5y^7)\overset{A-G}{\geq}64\sum x^6y^6\underset{(c)}{\overset{by\ (i)}{\geq}}64x^3y^3z^3(\sum x^3)$

Also, $25(\sum x^8y^4+\sum x^4y^8)\overset{A-G}{\geq}50\sum x^6y^6\underset{(d)}{\overset{by\ (i)}{\geq}}50x^3y^3z^3(\sum x^3)$

Also, $28(\sum x^9y^3+\sum x^3y^9)\overset{A-G}{\geq}56\sum x^6y^6\underset{(c)}{\overset{by\ (i)}{\geq}}56x^3y^3z^3(\sum x^3)$

Lastly, $7(\sum x^{10}y^2+\sum x^2y^{10})\overset{A-G}{\geq}14\sum x^6y^6\underset{(f)}{\overset{by\ (i)}{\geq}}14x^3y^3z^3(\sum x^3)$

Now, $144x^2y^2z^2(\sum x^3y^3)=72x^2y^2z^2(2\sum x^3y^3)\overset{(g)}{\geq}72x^2y^2z^2\cdot xyz(\sum x^2y+\sum xy^2)$

$$\left(\because 2\sum u^3\geq\sum u^2v+\sum uv^2\right)=72x^3y^3z^3\left(\sum x^2y+\sum xy^2\right)$$

Also, $160xyz(\sum x^5y^4+\sum x^4y^5)=160xyz\sum\{x^5(y^4+z^4)\}\overset{A-G}{\geq}320xyz(\sum x^5y^2z^2)=$

$$=320x^3y^3z^2\left(\sum x^3\right)=160x^3y^3z^3\left(2\sum x^3\right)\overset{(h)}{\geq}160x^3y^3z^3\left(\sum x^2y+\sum xy^2\right)$$

Lastly,

$$44xyz\left(\sum x^6y^3+\sum x^3y^6\right)=44xyz\left\{\sum y^3(x^6+z^6)\right\}\overset{(i)}{\geq}44xyz\sum\{y^3x^2z^2(x^2+z^2)\}$$

$$=44x^3y^3z^3\left(\sum x^2y+\sum xy^2\right)$$

PROBLEM 2.126-*Solution by Bogdan Fustei-Romania*

the medians m_a,m_b,m_c *of* $\Delta\,ABC$ *can be also the sides of a triangle of medians denoted* m_{a1},m_{b1},m_{c1}. *But* $\left.\begin{matrix}m_{a1}=\frac{3}{4}a\\ m_{b1}=\frac{3}{4}b\\ m_{c1}=\frac{3}{4}c\end{matrix}\right\}$ *we will write the inequality from enunciation for*

$$m_{a1},m_{b1},m_{c1}:\frac{1}{\frac{9}{16}ab}+\frac{1}{\frac{9}{16}bc}+\frac{1}{\frac{9}{16}ac}+\frac{3}{\frac{9}{16}(ab+bc+ac)}\geq\frac{4\cdot\frac{4}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{\frac{3}{4}(a+b+c)}$$

$$\frac{16}{9}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)+\frac{16}{3}\cdot\frac{1}{ab+bc+ac}\geq\frac{\frac{16}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{\frac{3}{4}(a+b+c)}$$

$$\frac{16}{9}\cdot\sum\frac{1}{ab}+\frac{16}{3}\cdot\frac{1}{ab+bc+ac}\geq\frac{16}{3}\cdot\frac{4}{3}\cdot\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{a+b+c}\Bigg|:\frac{16}{3}$$

$$\frac{1}{3}\cdot\sum\frac{1}{ab}+\frac{1}{ab+bc+ac}\geq\frac{4}{3}\cdot\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{a+b+c}$$

$$But\sum\frac{1}{ab}=\frac{1}{2Rr};\Rightarrow\frac{1}{6Rr}+\frac{1}{ab+bc+ac}\geq\frac{4}{3}\cdot\frac{\frac{ab+bc+ac}{abc}}{2s}$$

$$\frac{1}{6Rr}+\frac{1}{ab+bc+ac}\geq\frac{4}{3}\cdot\frac{ab+bc+ac}{4RS\cdot 2s}\,[abc=4RS]$$

$$\frac{1}{6Rr}+\frac{1}{ab+bc+ac}\geq\frac{ab+bc+ac}{6RSs}$$

$$\frac{6Rr+ab+bc+ac}{6Rr(ab+bc+ac)}\geq\frac{ab+bc+ac}{6Rrs^2}\Leftrightarrow\frac{6Rr+ab+bc+ac}{ab+bc+ac}\geq\frac{ab+bc+ac}{s^2}$$

$$s^2(6Rr+ab+bc+ac)\geq(ab+bc+ac)^2.\ But\ ab+bc+ac=2R(h_a+h_b+h_c)$$

$$2Rs^2(h_a+h_b+h_c+3r)\geq 4R^2(h_a+h_b+h_c)^2$$

$$\frac{s^2}{2R}\geq\frac{(h_a+h_b+h_c)^2}{h_a+h_b+h_c+3r};\ But\ h_a+h_b+h_c=r\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)$$

$$(h_a+h_b+h_c)^2=r^2\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2$$

$$3r+h_a+h_b+h_c=r\left(9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)$$

$$\frac{s^2}{2R}\geq\frac{r^2\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2}{r\left(9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)}\Rightarrow\frac{s^2}{2Rr}\geq\frac{\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2}{9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}}$$

$$\sum\frac{h_a}{r_a}=\sum\frac{bc(s-a)}{2Rrs}=\frac{s(s^2+4Rr+r^2)-12Rrs}{2Rrs}=\frac{s^2-8Rr+r^2}{2Rr}\overset{(1)}{=}$$

$$\frac{\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2}{9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}}=\frac{(s^2+4Rr+r^2)^2}{4R^2r^2}\cdot\frac{2Rr}{(s^2+10Rr+r^2)}=\frac{(s^2+4Rr+r^2)^2}{2Rr(s^2+16Rr+r^2)}\leq$$

$$\leq\frac{s^2}{2Rr}\Leftrightarrow s^4+s^2(10Rr+r^2)\geq s^4+r^2(4R+r)^2+s^2(8Rr+r^2)\Leftrightarrow$$

$$\Leftrightarrow s^2(2R-r)\geq r(4R+r)^2\overset{(2)}{=}s^2\geq 16Rr-5r^2\text{ – Gerretsen's inequality}\Rightarrow$$

$$\Rightarrow s^2(2R-r)\geq(16Rr-5r^2)(2R-r)\text{ – true. We will prove that:}$$

$$(16Rr-5r^2)(2R-r)\geq r(4R+r)^2$$

$$r(16R-5r)(2R-r)\geq r(4R+r)^2\Leftrightarrow 32R^2-10Rr-16Rr+5r^2\geq 16R^2+r^2+8Rr$$

$$16R^2+4r^2\geq 34Rr\Rightarrow 8R^2+2r^2\geq 17Rr\Rightarrow 8R^2-17Rr+2r^2\geq 0\Leftrightarrow$$

$$\Leftrightarrow(R-2r)(8R-r)\geq 0\ R-2r\geq 0\Rightarrow R\geq 2r\text{ – Euler's inequality}$$

$8R>r$ – *true. So, (2) true;* $\Rightarrow\frac{s^2}{2Rr}\geq\frac{\left(6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)^2}{9+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}}$ – *true* $\Rightarrow$ *inequality from enunciation is true,*

namely: $\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_am_c}+\frac{3}{m_am_b+m_bm_c+m_cm_a}\geq 4\cdot\frac{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}{m_a+m_b+m_c}$

PROBLEM 2.127-*Solution by proposer*

Let be the equation $x^2 - 2\sin\frac{\pi}{x}x + 1 = 0$ *which has the roots* $z_1 = \sin\frac{\pi}{x} - i\cos\frac{\pi}{x}$

$$z_2 = \sin\frac{\pi}{x} + i\cos\frac{\pi}{x}$$

$$(B - z_1A)(B - z_2A) = B^2 + z_1z_2A^2 - z_1AB - z_2BA =$$

$$= B^2 + A^2 - \left(2\sin\frac{\pi}{x} - z_2\right)AB - z_2BA =$$

$$B^2 + A^2 - 2\sin\frac{\pi}{x}AB + z_2AB - z_2BA = z_2(AB - BA)$$

$$\det\underbrace{(B - z_1A)(B - z_2A)}_{\geq 0} = \det\big(z_2(AB - BA)\big) \Rightarrow$$

$$z_2^n\det(AB - BA) \geq 0; \det(AB - BA) \neq 0 \Rightarrow$$

$$z_2^n\det(AB - BA) > 0 \Rightarrow z_2^n \in \mathbb{R} \Rightarrow$$

$$\left(\sin\frac{\pi}{x} + i\cos\frac{\pi}{x}\right)^n \in \mathbb{R} \Rightarrow \left(\cos\left(\frac{\pi}{2} - \frac{\pi}{x}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{x}\right)\right)^n \in \mathbb{R}$$

$$\Rightarrow \cos n\left(\frac{\pi}{2} - \frac{\pi}{x}\right) + i\sin n\left(\frac{\pi}{2} - \frac{\pi}{x}\right) \in \mathbb{R} \Rightarrow \sin n\left(\frac{\pi}{2} - \frac{\pi}{x}\right) = 0 \Rightarrow n\left(\frac{\pi}{2} - \frac{\pi}{x}\right) = k\pi \Rightarrow$$

$$\Rightarrow n\left(\frac{x-2}{2}\right) = k \Rightarrow nx - 2n = 2k \Rightarrow nx = 2(n + k) \Rightarrow nx \text{ is even.}$$

PROBLEM 2.128-*Solution by proposer*

- By Cauchy – Schwarz inequality we have:

$$\left(\sqrt{2(y^4 + z^4)} + 2yz\right)^2 \leq 2(2(y^4 + z^4) + 4y^2z^2) = 4(y^4 + 2y^2z^2 + z^4) = 4(y^2 + z^2)^2$$

$$\Rightarrow \sqrt{2(y^4 + z^4)} + yz \leq 2y^2 - yz + 2z^2 \Leftrightarrow \frac{x^3}{y\left(\sqrt{2(y^4 + z^4)} + yz\right)^2} \geq \frac{x^3}{y(2y^2 - yz + 2z^2)^2}$$

$$+ \text{ Similar: } \frac{y^3}{z\left(\sqrt{2(z^4+x^4)}+zx\right)^2} \geq \frac{y^3}{z(2z^2-zx+2x^2)^2}; \frac{z^3}{z\left(\sqrt{2(z^4+x^4)}+zx\right)^2} \geq \frac{z^3}{x(2x^2-xy+2y^2)^2}$$

$$- \text{ Therefore: } \frac{x^3}{y\left(\sqrt{2(y^4+z^4)}+yz\right)^2} + \frac{y^3}{z\left(\sqrt{2(z^4+x^4)}+zx\right)^2} + \frac{z^3}{x\left(\sqrt{2(x^4+y^4)}+xy\right)^2} \geq$$

$$\geq \frac{x^3}{y(2y^2-yz+2z^2)^2} + \frac{y^3}{z(2z^2-zx+2x^2)^2} + \frac{z^3}{x(2x^2-xy+2y^2)^2} \quad (1)$$

- By Cauchy – Schwarz inequality:

$$\sum\frac{x^3}{y(2y^2-yz+2z^2)^2} = \sum\frac{\left(\frac{x^2}{2y^2-yz+2z^2}\right)^2}{xy} \geq \frac{\left(\sum\frac{x^2}{2y^2-yz+2z^2}\right)^2}{\sum xy} \quad (2)$$

$$+ \text{ Other, } \sum\frac{x^2}{2y^2-yz+2z^2} = \sum\frac{x^4}{2x^2y^2-x^2yz+2x^2z^2} \geq \frac{\left(\sum x^2\right)^2}{\sum(2x^2y^2-x^2yz+2x^2z^2)} \geq 1$$

$$\Leftrightarrow \left(\sum x^2\right)^2 \geq 4\sum x^2y^2 - xyz\sum x \Leftrightarrow \sum x^4 + xyz\sum x \geq 2\sum x^2y^2 \quad (3)$$

$$+ \text{ By Schur and AM-GM inequality: } \sum x^2(x - y)(x - z) \geq 0 \Rightarrow \sum x^4 + xyz\sum x \geq \sum xy\,(x^2 + y^2)$$

$$\sum xy(x^2 + y^2) \geq \sum xy \cdot 2xy = 2\sum x^2y^2 \Rightarrow \sum x^4 + xyz\sum x \geq 2\sum x^2y^2 \Rightarrow (3) \text{ True.}$$

$$+ \text{ Let (2), (3): } \Rightarrow \sum\frac{x^3}{y(2y^2-yz+2z^2)^2} \geq \frac{1}{\sum xy}. \text{ Let (1): } \Rightarrow \sum\frac{x^3}{y\left(\sqrt{2(y^4+z^4)}+yz\right)^2} \geq \frac{1}{\sum xy} \quad (4)$$

- By AM-GM inequality:

$$\begin{cases}\sqrt[4]{x}+\sqrt[4]{x}+\sqrt[4]{x}+\sqrt[4]{x}+x^3+x^2\ge 6\sqrt[6]{x\cdot x^3\cdot x^2}=6x\\ \sqrt[4]{y}+\sqrt[4]{y}+\sqrt[4]{y}+\sqrt[4]{y}+y^3+y^2\ge 6\sqrt[6]{y\cdot y^3\cdot y^2}=6y\\ \sqrt[4]{z}+\sqrt[4]{z}+\sqrt[4]{z}+\sqrt[4]{z}+z^3+z^2\ge 6\sqrt[6]{z\cdot z^3\cdot z^2}=6z\end{cases}\Leftrightarrow\begin{cases}4\cdot\sqrt[4]{x}\ge 6x-x^2-x^3\\ 4\cdot\sqrt[4]{y}\ge 6y-y\quad -y^3\\ 4\cdot\sqrt[4]{z}\ge 6z-z^2-z^3\end{cases}$$

$$\Rightarrow 4\left(\sum\sqrt[4]{x}\right)\ge 6\sum x-\sum x^2-\sum x^3=6\cdot 3-\left(\sum x\right)^2+2\sum xy-\sum x^3=2\sum xy+9-\sum x^3 \quad (5)$$

+ *Other, because* $x+y+z=3; x,y,z>0\Rightarrow\sum(x-3)(x-1)^2\le 0\Leftrightarrow\sum(x-3)(x^2-2x+1)\le 0$

$$\Leftrightarrow\sum x^3-5\sum x^2+7\sum x-9\le 0\Leftrightarrow\sum x^3\le 5\sum x^2-7\sum x+9$$
$$=5\cdot 3^2-10\sum xy-7\cdot 3+9$$

$\Leftrightarrow\sum x^3\le 33-10\sum xy$. *Let (5):* $\Rightarrow 4\left(\sum\sqrt[4]{x}\right)\ge 2\sum xy+9-\left(33-10\sum xy\right)\Leftrightarrow$
$$\sum\sqrt[4]{x}\ge 3\sum xy-6\quad (6)$$

- *Let (4), (6):* $\Rightarrow P\ge\frac{1}{\sum xy}+\frac{3\sum xy-6}{27}=\frac{1}{\sum xy}+\sum xy-\frac{2}{9}\ge 2\sqrt{\sum xy\cdot\frac{\sum xy}{9}}-\frac{2}{9}=\frac{2}{3}-\frac{2}{9}=\frac{4}{9}$

$\Rightarrow P\ge\frac{4}{9}\Rightarrow P_{\min}=\frac{4}{9}$. *Equality occurs if:* $\begin{cases}x=y=z>0\\ x+y+z=3\end{cases}\Leftrightarrow x=y=z=1.$

PROBLEM 2.129-*Solution by proposer*

− *By Cauchy-Schwarz inequality we have:*

$$\left(\sqrt{2(b^8+c^8)}+2b^2c^2\right)^2\le 2\left(2(b^8+c^8)+4b^4c^4\right)=4(b^8+2b^4c^4+c^8)=4(b^4+c^4)^2$$

$$\Rightarrow\sqrt{2(b^8+c^8)}+2b^2c^2\le 2(b^4+c^4)\Leftrightarrow\sqrt[4]{\frac{b^8+c^8}{2}}\le\sqrt{b^4-b^2c^2+c^4}$$
$$=\sqrt{(b^2+c^2)^2-\left(bc\sqrt{3}\right)^2}$$

$$\Leftrightarrow\sqrt[4]{\frac{b^8+c^8}{2}}=\sqrt{\left(2+\sqrt{3}\right)\left(b^2-bc\sqrt{3}+c^2\right)\left(2-\sqrt{3}\right)\left(b^2+bc\sqrt{3}+c^2\right)}\le$$
$$\le\frac{\left(2+\sqrt{3}\right)\left(b^2-bc\sqrt{3}+c^2\right)+\left(2-\sqrt{3}\right)\left(b^2+bc\sqrt{3}+c^2\right)}{2}=\frac{4b^2-6bc+4c^2}{2}$$
$$=2b^2-3bc+2c^2$$

$$\Rightarrow\sqrt[4]{\frac{b^8+c^8}{2}}+5bc\le 2(b^2+bc+c^2)\Leftrightarrow\frac{a^3}{b^2\left(\sqrt[4]{\frac{b^8+c^8}{2}}+5bc\right)}\ge\frac{a^3}{2b^2(b^2+bc+c^2)}$$

PROBLEM 2.130-*Solution by proposer*

- *By Cauchy-Schwarz inequality and* $a+b+c=3$*, we have:*

$$\frac{a^3}{b^2(b^2+bc+c^2)}+\frac{b^3}{c^2(c^2+ca+a^2)}+\frac{c^3}{a^2(a^2+ab+b^2)}$$

$$=\frac{\left(\frac{a^2}{b}\right)^2}{a(b^2+bc+c^2)}+\frac{\left(\frac{b^2}{c}\right)^2}{b(c^2+ca+a^2)}+\frac{\left(\frac{c^2}{a}\right)^2}{c(a^2+ab+b^2)}$$
$$\geq\frac{\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2}{a(b^2+bc+c^2)+b(c^2+ca+a^2)+c(a^2+ab+b^2)}$$
$$\Rightarrow\frac{a^3}{b^2(b^2+bc+c^2)}+\frac{b^3}{c^2(c^2+ca+a^2)}+\frac{c^3}{a^2(a^2+ab+b^2)}\geq\frac{\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2}{(a+b+c)(ab+bc+ca)}=\frac{\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2}{3(ab+bc+ca)}\quad (2)$$

- Using Cauchy-Schwarz inequality: $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{a^4}{a^2b}+\frac{b^4}{b^2c}+\frac{c^4}{c^2a}\geq\frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a}$ *(3)*

- By Bunhiacopxki we have:

$$(a\cdot ab+b\cdot bc+c\cdot ca)^2\leq(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)\leq(a^2+b^2+c^2)\cdot\frac{(a^2+b^2+c^2)^2}{3}$$
$$\Rightarrow(a^2b+b^2c+c^2a)^2\leq\frac{(a^2+b^2+c^2)^3}{3}\Leftrightarrow a^2b+b^2c+c^2a\leq\sqrt{\frac{(a^2+b^2+c^2)^3}{3}}$$

+ Let (3):

$$\Rightarrow\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a^2+b^2+c^2)^2}{\sqrt{\frac{(a^2+b^2+c^2)^2}{3}}}=\sqrt{3(a^2+b^2+c^2)}\Leftrightarrow\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2\geq 3(a^2+b^2+c^2)\quad (4)$$

- Let (2), (4): $\Rightarrow\frac{a^3}{b^2(b^2+bc+c^2)}+\frac{b^3}{c^2(c^2+ca+a^2)}+\frac{c^3}{a^2(a^2+ab+b^2)}\geq\frac{3(a^2+b^2+c^2)}{3(ab+bc+ca)}$

$$\Rightarrow\frac{a^3}{b^2(b^2+bc+c^2)}+\frac{b^3}{c^2(c^2+ca+a^2)}+\frac{c^3}{a^2(a^2+ab+b^2)}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq$$
$$\geq\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 2\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}\cdot\frac{ab+bc+ca}{a^2+b^2+c^2}}=2$$

⇒ (1) True and we get the result.

+ Equality occurs if: $\begin{cases}a=b=c\\a+b+c=3\end{cases}\Leftrightarrow a=b=c=1.$

PROBLEM 2.131-*Solution by proposer*

In Milne's Inequality we take the pairs:

$(a,1-b),(b,1-c),(c,1-a)\rightarrow 3\sum\frac{a-ab}{1+a-b}\leq 2.$

But:

$$\sum\frac{a-ab}{1+a-b}=\frac{\sum(a-ab)(1+b-c)(1+c-a)}{\prod(2a+c)}=\frac{(\sum ab)^2-2\sum a^2b-\sum ab^2}{9abc+4\sum ab^2+2\sum a^2b}.$$

So, after calculations :

$$3\left(\sum ab\right)^2\leq 18abc+10\sum ab(a+b)+\sum ab^2$$

Using that $\sum ab(a+b)=\sum ab-3abc$*, it results q.**e.d.***

PROBLEM 2.132- *Solution by proposer*

$$First,\ observe\ that\ 1-\frac{(1+ab)(ab-a-b-1)}{(a^2+1)(b^2+1)}=\frac{a+1}{a^2+1}+\frac{b+1}{b^2+1}.$$

Now

$$,\frac{a+1}{a^2+1}+\frac{b+1}{b^2+1}=\frac{1}{a+1-\frac{2a}{a+1}}+\frac{1}{b+1-\frac{2b}{b+1}}\geq\frac{4}{a+b+2-2\left(\frac{a^2}{a^2+a}+\frac{b^2}{b^2+b}\right)}\geq\frac{4(a^2+b^2+a+b)}{(2+a+b)(a^2+b^2+a+b)-2(a+b)^2},$$

by C-B-S

PROBLEM 2.133-*Solution by Michael Sterghiou-Greece*

$$2\sqrt{abc}\sum_{cyc}\frac{a}{1+a^2}\leq\frac{1+\sum_{cyc}a^2}{3+\sum_{cyc}a^2}\quad (1)$$

$$Let\ \left(\sum_{cyc}a,\sum_{cyc}ab,abc\right)=(p,q,r): p=1, q\leq\frac{1}{3},\sum_{cyc}a^2=1-2q$$

$$f(a)=\frac{a}{1+a^2},\ has\ f''(a)=\frac{2a(a^2-3)}{(a^2+1)^3}<0\ for\ 0<a<1\ hence$$

$$\sum_{cyc}\frac{a}{1+a^2}\overset{Jensen}{\leq}3\cdot\frac{\frac{1}{3}}{1+\left(\frac{1}{3}\right)^2}=\frac{9}{10}.\ Also\ r\leq\left(\frac{a}{3}\right)^{\frac{3}{2}}\ so\ it\ is\ enough\ to\ show\ the\ stronger\ than\ (1)$$

$$inequality\ 2r^{\frac{1}{2}}\sum_{cyc}\frac{a}{a^2+1}\leq 2\left(\frac{a}{3}\right)^{\frac{3}{4}}\cdot\frac{9}{10}\leq\frac{1-q}{2-q}\ or$$

$$f(q)=-3^{\frac{5}{4}}q^{\frac{7}{4}}+2\cdot 3^{\frac{5}{4}}q^{\frac{3}{4}}+5q-5\leq 0$$

$$f'(q)=-\frac{21}{4}3^{\frac{1}{4}}q^{\frac{3}{4}}+\frac{9\cdot 3^{\frac{1}{4}}}{2q^{\frac{1}{4}}}+5>0\ because\ -\frac{21}{4}\cdot 3^{\frac{1}{4}}\cdot\left(\frac{1}{3}\right)^{\frac{3}{4}}<5\ hence\ f(q)\uparrow and$$

$$f(q)\leq f\left(\frac{1}{3}\right) for\leq\frac{1}{3}\ or\ f(q)<\frac{5}{3}\left(\sqrt{3}-2\right)<0$$

PROBLEM 2.134-*Solution by proposer*

$$The\ function\ f:(0,1)\to(0,\infty), f(x)=\sqrt{\frac{1-x}{(1-a)(1-b)(1-c)(1-d)}}\ is\ concave.$$

$$We\ know\ that\ \tan\frac{A}{2}=\sqrt{\frac{(1-a)(1-d)}{(1-c)(1-b)}}=f(1-[(1-a)(1-d)]^2).\ Applying\ Jensen\to$$

$$\sum\tan\frac{A}{2}\leq 4f\left(\frac{4-\sum[(1-a)(1-d)]^2}{4}\right)=2\sqrt{\frac{\sum[(1-a)(1-d)]^2}{(1-a)(1-b)(1-c)(1-d)}}$$

$$But\ \sum[(1-a)(1-d)]^2<\left[\sum(1-a)(1-d)\right]^2=[(a+c)(b+d)]^2.$$

$$So, \sum\tan\frac{A}{2}<\frac{2(a+c)(b+d)}{\sqrt{\prod(1-a)}}.$$

Now,

$$\sum\tan\frac{A}{2}=\sqrt{\frac{(1-a)(1-d)}{(1-c)(1-b)}}+\sqrt{\frac{(1-b)(1-c)}{(1-a)(1-d)}}+\sqrt{\frac{(1-c)(1-d)}{(1-a)(1-b)}}+\sqrt{\frac{(1-a)(1-b)}{(1-c)(1-d)}},$$

Which is bigger or equal than 4. Using this → q.e.d.

PROBLEM 2.135-*Solution by Amit Dutta-Jamshedpur-India*

Using Cauchy – Schwarz's Inequality:

$$(a_1^2+a_2^2+\cdots+a_n^2)^2(b_1^2+b_2^2+\cdots+b_n^2)\ge(a_1b_1+a_2b_2+\cdots+a_nb_n)^2$$

Putting $a_i=\frac{x_i}{\sqrt{yi}}$ *and* $b_i=\sqrt{y_i}$, *we have:*

$$\left(\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+\cdots+\frac{x_n^2}{y_n}\right)(y_1+y_2+\cdots+y_n)\ge(x_1+x_2+\cdots+x_n)^2$$

$$\Rightarrow\left(\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+\cdots+\frac{x_n^2}{y_n}\right)\ge\frac{(x_1+x_2+\cdots+x_n)^2}{(y_1+y_2+\cdots+y_n)}\to \text{Titu's Lemma}$$

Using this inequality, putting $x_1=\sqrt{\log_a b}\,,y_1=(a+b+c)$

$$x_2=\sqrt{\log_b c}\,,y_2=(b+c+d)$$

$$x_3=\sqrt{\log_c d}\,,y_3=c+d+a$$

$$x_4=\sqrt{\log_d a}\,,y_4=d+a+b$$

We have,

$$\left\{\frac{\log_a b}{a+b+c}+\frac{\log_b c}{b+c+d}+\frac{\log_c d}{c+d+a}+\frac{\log_d a}{d+a+b}\right\}\ge\frac{\left(\sqrt{\log_a b}+\sqrt{\log_b c}+\sqrt{\log_c d}+\sqrt{\log_d a}\right)^2}{3(a+b+c+d)}$$

AM-GM

$$\sqrt{\log_a b}+\sqrt{\log_b c}+\sqrt{\log_c d}+\sqrt{\log_d a}\ge 4$$

$$\therefore\frac{\log_a b}{a+b+c}+\frac{\log_b c}{b+c+d}+\frac{\log_c d}{c+d+a}+\frac{\log_d a}{d+a+b}\ge\frac{16}{3(a+b+c+d)}$$

PROBLEM 2.136-*Solution by proposer*

** Let* $x,y,z>0$, *we will prove that inequality:*

$$x^4+y^4+z^4+xyz(x+y+z)\ge xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\quad (1)$$

$$(1)\Leftrightarrow x^4+y^4+z^4+xyz(x+y+z)-xy(x^2+y^2)-yz(y^2+z^2)-zx(z^2+x^2)\ge 0$$

$$\Leftrightarrow x^2(x^2-xy-xz+yz)+y^2(y^2-yz-yx+zx)+z^2(z^2-zx-zy+xy)\ge 0$$

$$\Leftrightarrow x^2(x-y)(x-z)+y^2(y-z)(y-x)+z^2(z-x)(z-y)\ge 0\quad (2)$$

- Supposed $x\ge y\ge z>0$

+ We have: $\begin{cases}z\le x\\ z\le y\end{cases}\Leftrightarrow\begin{cases}z-x\le 0\\ z-y\le 0\end{cases}\Rightarrow(z-x)(z-y)\ge 0\Rightarrow z^2(z-x)(z-y)\ge 0\quad (3)$

+ Other: $x^2(x-y)(x-z)+y^2(y-z)(y-x)$

$$=(x-y)[x^2(x-z)-y^2(y-z)]=(x-y)[(x^3-y^3)-z(x^2-y^2)]$$

$$=(x-y)[(x-y)(x^2+xy+y^2)-z(x-y)(x+y)]=(x-y)^2(x^2+xy+y^2-zx-zy)\ge 0\quad (4)$$

(because $x\ge y\ge z>0, x^2+xy+y^2-zx-zy=x(x-z)+y(x-z)+y^2\ge y^2>0$ *and* $(x-y)^2\ge 0$*)*

- Let (3), (4): $\Rightarrow x^2(x-y)(x-z)+y^2(y-z)(y-x)+z^2(z-x)(z-y)\ge 0$

$\Rightarrow$ *Inequality (2) true* $\Rightarrow$ *(1) true.*

** We have:* $x^6+y^4=(x^2+y^2)(x^4-x^2y^2+y^4)=(x^2+y^2)\left(x^2-xy\sqrt{3}+y^2\right)\left(x^2+xy\sqrt{3}+y^2\right)$

- Therefore, by AM-GM inequality: $\sqrt[3]{\frac{x^6+y^6}{2}}=\sqrt[3]{\frac{(x^2+y^2)\left(x^2-xy\sqrt{3}+y^2\right)\left(x^2+xy\sqrt{3}+y^2\right)}{2}}$

$$= \sqrt[3]{\frac{(x^2+y^2)}{2}\cdot(2+\sqrt{3})(x^2-xy\sqrt{3}+y^2)(2-\sqrt{3})(x^2+xy\sqrt{3}+y^2)}$$

$$\le \frac{\frac{x^2+y^2}{2}+(2+\sqrt{3})(x^2-xy\sqrt{3}+y^2)+(2-\sqrt{3})(x^2+xy\sqrt{3}+y^2)}{3} = \frac{3x^2-3xy+3y^2}{2}$$

$\Rightarrow \sqrt[3]{\frac{x^6+y^6}{2}} \le \frac{3x^2-4xy+3y^2}{2}$. *Similar:* $\sqrt[3]{\frac{y^6+z^6}{2}} \le \frac{3y^2-4yz+3z^2}{2}$; $\sqrt[3]{\frac{z^6+x^6}{2}} \le \frac{3z^2-4zx+3x^2}{2}$

$$\Rightarrow P = \sqrt[3]{\frac{x^6+y^6}{2}}+\sqrt[3]{\frac{y^6+z^6}{2}}+\sqrt[3]{\frac{z^6+x^6}{2}}$$

$$\le \frac{3x^2-4xy+3y^2}{2}+\frac{3y^2-4yz+3z^2}{2}+\frac{3z^2-4zx+3x^2}{2}$$

$\Leftrightarrow P \le 3(x^2+y^2+z^2)-2(xy+yz+zx)$ *(5)*

**We will prove:* $3(x^2+y^2+z^2)-2(xy+yz+zx) \le 3$ *(6)*

$\Leftrightarrow 3(x^2+y^2+z^2)-2(xy+yz+zx) \le \frac{3(x^4+y^4+z^4)}{xy+yz+zx}$ $(x^4+y^4+z^4 = xy+yz+zx$ *then* $\frac{x^4+y^4+z^4}{xy+yz+zx} = 1)$

$$\Leftrightarrow \big(3(x^2+y^2+z^2)-2(xy+yz+zx)\big)(xy+yz+zx) \le 3(x^4+y^4+z^4)$$

$$\Leftrightarrow 3(x^2+y^2+z^2)(xy+yz+zx) \le 3(x^4+y^4+z^4)+2(xy+yz+zx)^2$$

$$\Leftrightarrow 3xy(x^2+y^2)+3yz(y^2+z^2)+3zx(z^2+x^2)+3xyz(x+y+z) \le$$

$$\le 3(x^4+y^4+z^4)+2(x^2y^2+y^2z^2+z^2z^2)+4xyz(x+y+z)$$

$$\Leftrightarrow 3(x^4+y^4+z^4)+xyz(x+y+z)+2(x^2y^2+y^2z^2+z^2z^2) \ge 3xy(x^2+y^2)+3yz(y^2+z^2)+3zx(z^2+x^2) \quad (7)$$

- By AM-GM inequality for 2 real numbers:

$$x^2y^2+y^2z^2+z^2z^2 = \frac{x^2(y^2+z^2)}{2}+\frac{y^2(z^2+x^2)}{2}+\frac{z^2(x^2+y^2)}{2}$$

$$\ge \frac{x^2\cdot 2yz}{2}+\frac{y^2\cdot 2zx}{2}+\frac{z^2\cdot 2xy}{2}$$

$$\Rightarrow x^2y^2+y^2z^2+z^2z^2 \ge xyz(x+y+z) \quad (8)$$

$$\Rightarrow 3(x^4+y^4+z^4)+xyz(x+y+z)+2(x^2y^2+y^2z^2+z^2z^2) \ge 3\big(x^4+y^4+z^4+xyz(x+y+z)\big) \quad (9)$$

- Let (1), (9):

$$\Rightarrow 3(x^4+y^4+z^4)+xyz(x+y+z)+2(x^2y^2+y^2z^2+z^2z^2)$$

$$\ge 3xy(x^2+y^2)+3yz(y^2+z^2)+3zx(z^2+x^2)$$

$\Rightarrow$ *(7) True* $\Rightarrow$ *Inequality (6) true- Let (5), (6):* $\Rightarrow P \le 3 \Rightarrow P_{max} = 3$

+ Equality occurs if: $\Leftrightarrow \begin{cases} x,y,z > 0 \\ x^4+y^4+z^4 = xy+yz+zx \\ x = y = z \\ x^2 = y^2+z^2 \end{cases} \Leftrightarrow x = y = z = 1$

PROBLEM 2.137-*Solution by proposer*

** We have:*

$$b^6+c^6 = (b^2+c^2)(b^4-b^2c^2+c^4) = (b^2+c^2)\left[(b^2+c^2)^2-\left(bc\sqrt{3}\right)^2\right]$$

$$=(b^2+c^2)\left(b^2-bc\sqrt{3}+c^2\right)\left(b^2+bc\sqrt{3}+c^2\right)$$

- By inequality AM-GM for three positive real numbers:

$$\sqrt[3]{4(b^6+c^6)}=\sqrt[3]{(b^2+c^2)\cdot 2\left(2+\sqrt{3}\right)\left(b^2-bc\sqrt{3}+c^2\right)\cdot 2\left(2-\sqrt{3}\right)\left(b^2+bc\sqrt{3}+c^2\right)}\le$$

$$\le\frac{(b^2+c^2)+2\left(2+\sqrt{3}\right)\left(b^2-bc\sqrt{3}+c^2\right)+2\left(2-\sqrt{3}\right)\left(b^2+bc\sqrt{3}+c^2\right)}{3}$$

$$=\frac{9b^2-12bc+9c^2}{3}$$

$$\Leftrightarrow\sqrt[3]{4(b^6+c^6)}\le 3b^2-4bc+3c^2\Leftrightarrow\sqrt[3]{4(b^6+c^6)}+7bc\le 3b^2+3bc+3c^2$$

$$\Leftrightarrow\frac{1}{\sqrt[3]{4(b^6+c^6)}+7bc}\ge\frac{1}{3(b^2+bc+c^2)}\Leftrightarrow\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}\ge\frac{a}{3(b^2+bc+c^2)}\quad (2)$$

$$+\text{ Similar: }\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}\ge\frac{b}{3(c^2+ca+a^2)}\quad (3)$$

$$\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\ge\frac{c}{3(a^2+ab+b^2)}\quad (4)$$

- Then (2), (3), (4): $\Rightarrow\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\ge$

$$\ge\frac{a}{3(b^2+bc+c^2)}+\frac{b}{3(c^2+ca+a^2)}+\frac{c}{3(a^2+ab+b^2)}\quad (5)$$

− Other, by Cauchy-Schwarz we have:

$$\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}$$

$$=\frac{a^2}{ab^2+abc+ac^2}+\frac{b^2}{bc^2+bca+ba^2}+\frac{c^2}{ca^2+cab+cb^2}\ge$$

$$\ge\frac{(a+b+c+)^2}{(ab^2+abc+ac^2)+(bc^2+bca+ba^2)+(ca^2+cab+cb^2)}\quad (6)$$

- That $\frac{(a+b+c)^2}{(ab^2+abc+ac^2)+(bc^2+bca+ba^2)+(ca^2+cab+cb^2)}$

$$=\frac{(a+b+c)^2}{ab(a+b)+bc(b+c)+ca(c+a)+3abc}=\frac{(a+b+c)^2}{(a+b+c)(ab+bc+ca)}=\frac{a+b+c}{ab+bc+ca}\quad (7)$$

- Then (6), (7): $\Rightarrow\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}\ge\frac{a+b+c}{ab+bc+ca}\quad (8)$

+ And $a+b+c=3$. *Then (8):*

$$\Rightarrow\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}\ge\frac{3}{ab+bc+ca}\quad (9)$$

- Then (5), (9): $\Rightarrow\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\ge\frac{1}{ab+bc+ca}\quad (10)$

- By AM-GM for five positive real numbers:

$$\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}+a^2+a^2\ge 5\sqrt[5]{\sqrt[3]{a}\cdot\sqrt[3]{a}\cdot\sqrt[3]{a}\cdot a^2\cdot a^2}=5\sqrt[5]{a^5}=5a$$

$$\Leftrightarrow 3\cdot\sqrt[3]{a}+2a^2\ge 5a\Leftrightarrow 3\sqrt[3]{a}\ge 5a-2a^2\quad (11)$$

$$+\text{ Similar: }\sqrt[3]{b}\ge 5b-2b^2;\,3\sqrt[3]{c}\ge 5c-2c^2\quad (12)$$

− Then (11), (12): $\Rightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 5(a+b+c)-2(a^2+b^2+c^2)$

$$\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 15-2(a^2+b^2+c^2)\ \ (a+b+c=3)$$

$$\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 18-2(a^2+b^2+c^2)=2(a+b+c)^2-2(a^2+b^2+c^2)$$

(Because $a+b+c=3\Rightarrow 2(a+b+c)^2=18$*)*

$$\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 2(a^2+b^2+c^2+2ab+2bc+2ca)-2(a^2+b^2+c^2)$$

$$\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\geq 4(ab+bc+ca)\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\geq 4(ab+bc+ca)-3$$

$$\Leftrightarrow \frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq\frac{4(ab+bc+ca)-3}{36}\Leftrightarrow\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq\frac{ab+bc+ca}{9}-\frac{1}{2}\quad (13)$$

- Then (10), (13):

$$\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq$$

$$\geq\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12}\quad (14)$$

- By AM-GM we have:

$$\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}\geq 2\cdot\sqrt{\frac{1}{ab+bc+ca}\cdot\frac{ab+bc+ca}{9}}=2\sqrt{\frac{1}{9}}=\frac{2}{3}$$

$$\Rightarrow\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12}\geq\frac{2}{3}-\frac{1}{12}=\frac{7}{12}\Leftrightarrow\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12}\geq\frac{7}{12}\quad (15)$$

- Then (14), (15):

$$\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq\frac{7}{12}$$

⇒ Inequality (1) True and we get the result

$$+\text{ Equality occurs if: }\begin{cases}a,b,c>0; a+b+c=3\\ a=b=c\\ \frac{1}{b^2+bc+c^2}=\frac{1}{c^2+ca+a^2}=\frac{1}{a^2+ab+b^2}\\ \sqrt[3]{a}=a^2;\sqrt[3]{b}=b^2;\sqrt[3]{c}=c^2\\ \frac{1}{ab+bc+ca}=\frac{ab+bc+ca}{9}\end{cases}\Leftrightarrow a=b=c=1.$$

PROBLEM 2.138-*Solution by Heikichi Ezakiya-Jakarta-Indonesia*

$$\text{Let}:\varphi=\frac{a^2}{\sqrt{5(b^4+4)}}+\frac{b^2}{\sqrt{5(a^4+4)}}+\frac{c^2}{\sqrt{5(a^4+4)}}=\frac{1}{\sqrt{5}}\left(\frac{a^2}{\sqrt{b^4+4}}+\frac{b^2}{\sqrt{c^4+4}}+\frac{c^2}{\sqrt{a^4+4}}\right)$$

$$\text{Using CBS: }\varphi\geq\frac{1}{\sqrt{5}}\cdot\frac{(a+b+c)^2}{\left(\sqrt{a^4+4}+\sqrt{b^4+4}+\sqrt{c^4+4}\right)}=\varphi^{(1)}$$

Using QM-AM for $\left(\sqrt{a^4+4}+\sqrt{b^4+4}+\sqrt{c^4+4}\right)$:

$$\sqrt{\frac{a^4+b^4+c^4+12}{3}}\geq\frac{\sqrt{a^4+4}+\sqrt{b^4+4}+\sqrt{c^4+4}}{3}\Leftrightarrow\frac{1}{\left(\sqrt{a^4+4}+\sqrt{b^4+4}+\sqrt{c^4+4}\right)}\geq\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{a^4+b^4+c^4+12}}\text{ so,}$$

$$\varphi^{(1)}\geq\frac{1}{\sqrt{15}}\cdot\frac{1}{\sqrt{a^4+b^4+c^4+12}}=\varphi^{(2)}\quad (\#)$$

Using QM-AM for $a^2+b^2+c^2$

$$\sqrt{\frac{a^2+b^2+c^2}{3}}\geq\frac{a+b+c}{3}\Leftrightarrow a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}$$

because $a+b+c$*, then:* $a^2+b^2+c^2\geq 3\quad (1)$

Using QM-AM for $a^4+b^4+c^4$

$$\sqrt{\frac{a^4+b^4+c^4}{3}}\geq\frac{a^2+b^2+c^2}{3}\Leftrightarrow a^4+b^4+c^4\geq\frac{(a^2+b^2+c^2)}{3}\quad (2)$$

From (1) & (2): $a^4+b^4+c^4\geq\frac{(3)^2}{3}=3$

From (#), if we choose $a^4+b^4+c^4=3$*, # becomes equal, then*

$$\varphi^{(1)} \geq \varphi^{(2)} = \frac{1}{\sqrt{15}} \cdot \frac{(a+b+c)^2}{\sqrt{3+12}} = \frac{(a+b+c)^2}{15}$$

Because $a+b+c=3$, *then:* $\varphi \geq \varphi^{(1)} \geq \varphi^{(2)} = \frac{(3)^2}{15} = \frac{3}{5}$

PROBLEM 2.139-*Solution by Soumava Chakraborty-Kolkata-India*

In any $\Delta ABC, \sum \frac{w_b w_c}{w_a} \geq \sum \frac{h_b h_c}{h_a}$. *Firstly,*

$$\prod(a+b) = 2abc + \sum ab\,(2s-c) = 2s(s^2+4Rr+r^2) - 4Rrs \overset{(1)}{=} 2s(s^2+2Rr+r^2)$$

$$\text{Also, } \sum(s-b)(s-c) = \sum(s^2 - s(b+c) + bc) = 3s^2 - 4s^2 + \sum ab \overset{(2)}{=} 4Rr + r^2$$

$$\text{Also, } \prod w_a = \prod\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right) = \frac{8(16R^2r^2s^2)}{\prod(b+c)}\left(\frac{s}{4R}\right) \overset{by\ (1)}{=} \frac{128R^2r^2s^2}{2s(s^2+2Rr+r^2)}\left(\frac{s}{4R}\right) \overset{(3)}{=} \frac{16Rr^2s^2}{s^2+2Rr+r^2}$$

$$\text{Now, } \sum\frac{h_b h_c}{h_a} = \sum\frac{(h_b h_c)^2}{h_a h_b h_c} = \left(\frac{8R^3}{a^2b^2c^2}\right)\sum\left(\frac{ca}{2R}\cdot\frac{b}{2R}\right)^2 = \left(\frac{8R^3}{a^2b^2c^2}\right)\left(\frac{a^2b^2c^2}{16R^4}\right)\sum a^2 \overset{(a)}{=} \frac{\sum a^2}{2R}$$

$$\text{Now, } \sum\frac{w_b w_c}{w_a} = \left(\frac{1}{\prod w_a}\right)\sum w_b^2 w_a^2 \overset{by\ (3)}{=} \left(\frac{s^2+2Rr+r^2}{16Rr^2s^2}\right)\sum\left[\frac{4c^2a^2}{(c+a)^2}\cdot\frac{s(s-b)}{ca}\cdot\frac{4a^2b^2}{(a+b)^2}\cdot\frac{s(s-c)}{ab}\right]$$

$$= \left(\frac{s^2+2Rr+r^2}{16Rr^2s^2}\right)\cdot\frac{16\cdot 4Rrs}{(\prod(a+b))^2}\left[\sum a\,(s-b)(s-c)(b+c)^2\right]$$

$$= \left(\frac{s^2+2Rr+r^2}{16Rr^2s^2}\right)\cdot\frac{64Rrs\cdot r^2s}{4s^2(s^2+2Rr+r^2)^2}\left[\sum\frac{a(b+c)^2}{s-a}\right] \overset{(4)}{=} \left(\frac{r}{s^2+2Rr+r^2}\right)\left[\sum\frac{a(b+c)^2}{s-a}\right]$$

$$\text{Now, } \sum\frac{a(b+c)^2}{s-a} = \sum\frac{a(s+s-a)^2}{s-a} = \sum\frac{as^2+a(s-a)^2+2as(s-a)}{s-a} = s^2\sum\frac{a-s+s}{s-a} + \sum a\,(s-a) + 2s(2s)$$

$$= s^2\left(-3+4+\frac{s}{r^2s}\sum(s-b)(s-c)\right) + s(2s) - 2(s^2-4Rr-r^2)$$

$$\overset{by\ (2)}{=} s^2\left(1+\frac{4R+r}{r}\right) + 2(4Rr+r^2) \overset{(5)}{=} \frac{s^2(4R+2r)+2r^2(4R+r)}{r}$$

$$(4),(5) \Rightarrow \sum\frac{w_b w_c}{w_a} \overset{(b)}{=} \frac{s^2(4R+2r)+2r^2(4R+r)}{s^2+2Rr+r^2}$$

$$(a),(b) \Rightarrow \text{given inequality} \Leftrightarrow \frac{s^2(4R+2r)+2r^2(4R+r)}{s^2+2Rr+r^2} \geq \frac{\sum a^2}{2R} = \frac{s^2-4Rr-r^2}{R}$$

$$\Leftrightarrow s^2(4R^2+2Rr) + 2Rr^2(4R+r) \geq (s^2-4Rr-r^2)(s^2+2Rr+r^2)$$

$$\Leftrightarrow s^2(4R^2+2Rr) + 2Rr^2(4R+r) \geq s^4 - 2Rrs^2 - r^2(4R+r)(2R+r)$$

$$\Leftrightarrow s^2(4R^2+4Rr) + r^2(4R+r)^2 \overset{(c)}{\geq} s^4$$

$$\text{Now, RHS of (4)} \overset{Gerretsen}{\leq} s^2(4R^2+4Rr+3r^2) \overset{?}{\leq} s^2(4R^2+4Rr) + r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow (4R+r)^2 \geq 3s^2 \to \text{true (Trucht) (Proved)}$$

PROBLEM 2.140-*Solution by Rade Krenkov-Sturmica-Macedonia*

From Cauchy – Schwarz inequality we have:

$$\left(\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}\right)(b^3a+c^3a+c^3b+a^3b+a^3c+b^3c) \geq 4(a^2+b^2+c^2)^2 \quad (1)$$

$$\left(\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}\right)(abc^2+acb^2+bca^2+bac^2+cab^2+cba^2) \geq 4(ab+bc+ca)^2$$

$$\left(\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}\right)(abc^2+bca^2+cab^2) \geq 2(ab+bc+ca)^2 \quad (2)$$

From (1) and (2) we get:

$$\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)(ab+bc+ca)(a^2+b^2+c^2)$$
$$\ge 4(a^2+b^2+c^2)^2+2(ab+bc+ca)^2$$

Now, we have that:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\ge\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

PROBLEM 2.141-*Solution by Michael Sterghiou-Greece*

$$\sum_{cyc}\frac{a^4}{b^4(2ab-\sqrt{c}+2)}\ge\frac{\sum_{cyc}a^2}{3}\quad (1)$$

Let $\left(\sum_{cyc}a,\sum_{cyc}ab,abc\right)=(p,q,r).\ p=3.\ \sum_{cyc}a^3=9-2q$

$$LHS\ of\ (1)\ge\frac{\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2}{\sum_{cyc}(2ab-\sqrt{c}+2)}\ [BCS]\ge\frac{\left(\sum_{cyc}\frac{a}{b}\right)^4}{9\left[2q+6-\sum_{cyc}\sqrt{a}\right]}\ [again]\ BCS\quad (2)$$

It suffices that (2) $\ge\frac{9-2q}{3}$. *But it holds that*

$\sum_{cyc}\frac{a}{b}\ge\frac{p}{r^{\frac{1}{3}}}$ *(AM-GM) and* $\sum_{cyc}\sqrt{a}\ge q$ *(as* $p=3$*)*

The last one: as $\sum_{cyc}a^2+2\sum_{cyc}ab=\left(\sum_{cyc}a\right)^2=9$ *it suffices that*

$\sum_{cyc}a^2+2\sum_{cyc}\sqrt{a}\ge 9$. *But*

$$\sum_{cyc}\left(a^2+\sqrt{a}+\sqrt{a}\right)\overset{AM-GM}{\ge}3\cdot\sum_{cyc}a=9$$

Therefore we have to show that $\frac{81}{9r^{\frac{4}{3}}(q+6)}\ge\frac{9-2q}{3}$ *or*

$f(q)=\left(\frac{q}{3}\right)^2(q+6)(9-2q)-27\le 0$ *because this stronger inequality arises from the fact that* $r\le\left(\frac{q}{3}\right)^{\frac{3}{2}}$. *But* $f(q)=\frac{1}{9}(3-q)(2q^3+9q^2-27q-81)$ *and* $q\le 3$,

$9(q)=q(2q^2+9q-27)\le 81$ *as* $q\le 3$ *and* $2q^2+9q-27\le 27$

PROBLEM 2.142-*Solution by Marian Ursărescu-Romania*

$a^4-2a+b^2+2=a^4-2a^2+1+2a^2+b^2+2^{-2a}=(a^2-1)^2+a^2-2a+1+a^2+$
$+b^2=(a^2-1)^2+(a-1)^2+a^2+b^2\ge a^2+b^2\ge 2ab$, *with equality for* $a=b=1$.

Inequality becomes: $\frac{a^2b^2}{2ab}+\frac{b^2c^2}{2bc}+\frac{a^2c^2}{2ac}\le\frac{a^2+b^2+c^2+3}{4}\Leftrightarrow$

$a^2+b^2+c^2+3\ge 2(ab+bc+ac),\forall a,b,c>0$ *with* $abc=1$ *(1)*

$a^2+b^2\ge 2ab$ *(2)*; $c^2+1\ge 2c$ *(3)*; $2+2c\ge 2ac+2bc$ *(4)* $\Leftrightarrow 1+\frac{1}{ab}\ge\frac{1}{b}+\frac{1}{a}\Leftrightarrow$

$\Leftrightarrow ab+1\ge a+b\Leftrightarrow (a-1)(b-1)\ge 0$,

true because we can choose two numbers so that $a,b\ge 1$ *or* $a,b\le 1$. *From (2)+(3)+(4)*$\Rightarrow$

$a^2+b^2+c^2+1+2+2c\ge$

$\ge 2ab+2c+2ac+2bc\Rightarrow a^2+b^2+c^2+3\ge 2(ab+bc+ac)\Rightarrow$ *then (1) its true.*

PROBLEM 2.143-*Solution by Rade Krenkov-Strumica-Macedonia*

Using Cauchy – Schwarz inequality we have: $(3x^2+yz)(3x^2+x^2)\geq\left(3x^2+x\sqrt{yz}\right)^2$. *Now,*

$$2x\sqrt{3x^2+yz}\geq 3x^2+\sqrt{yz}\quad(1)$$

we get: $$2y\sqrt{2y^2+zx}\geq 3y^2+\sqrt{zx}\quad(2)$$

$$2z\sqrt{3z^2+xy}\geq 3z^2+\sqrt{xy}\quad(3)$$

From (1), (2) and (3) we get: $2\left(x\sqrt{3x^2+yz}+y\sqrt{3y^2+zx}+z\sqrt{3z^2+xy}\right)\geq$

$$\geq 2(x^2+y^2+z^2)+\left(x^2+y^2+z^2+\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right).$$

It is enough to prove that:

$x^2+y^2+z^2+x\sqrt{yz}+y\sqrt{zx}+z\sqrt{xy}\geq 2(xy+yz+zx)$. *Introducing substitution*

$$x=a^2,y=b^2,z=c^2$$

we get: $a^4+b^4+c^4+abc(a+b+c)\geq 2(a^2b^2+b^2c^2+c^2a^2)$.

Using Schur's inequality we have:

$$\sum_{cyc}a^4+abc\sum_{cyc}a=\left(\sum_{cyc}a^3+3abc\right)\cdot\sum_{cyc}a-\left(\sum_{cyc}a^3b+\sum_{cyc}ab^3\right)$$

$$\sum_{cyc}a^4+abc\sum_{cyc}a\geq\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\sum_{cyc}a-2abc\sum_{cyc}a-\left(\sum_{cyc}a^3b+\sum_{cyc}ab^3\right)$$

$$\sum_{cyc}a^4+abc\sum_{cyc}a\geq 2\sum_{cyc}a^2b^2$$

PROBLEM 2.144-*Solution by Marian Ursărescu-Romania*

$$s=\frac{a+b+c}{2}$$

$$\left(\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}}\right)^2=\frac{\frac{(s-b)(s-c)}{bc}}{\frac{(s-a)(s-c)}{s(s-b)}}=\frac{s(s-b)^2}{bc(s-a)}=1 \text{ we must show: }\sum\frac{s(s-b)^2}{bc(s-a)}\geq\frac{9}{4}\Leftrightarrow\sum\frac{as(s-b)^2}{abc(s-a)}\geq\frac{9}{4}\Leftrightarrow$$

$$\Leftrightarrow\frac{a(s-b)^2}{s-a}\geq\frac{9}{4}\cdot\frac{abc}{s}\Leftrightarrow\sum\frac{a^2(s-b)^2}{a(s-a)}\geq\frac{9abc}{4s}\quad(1)$$

From Bergstrom inequality we have: $\sum\frac{a(s-b)^2}{a(s-a)}\geq\frac{(\sum a(s-b))^2}{\sum a(s-a)}\quad(2)$

From (1)+(2) we must show this: $\frac{\left(2s^2-(ab+bc+ac)\right)^2}{2s^2-(a^2+b^2+c^2)}\geq\frac{9abc}{4s}\quad(3)$

But we have $abc=4sRr\quad(4)$

$$ab+bc+ac=s^2+r^2+4Rr\quad(5)$$

$$a^2+b^2+c^2=2(s^2-r^2-4Rr)\quad(6)$$

From (3)+(4)+(5)+(6) we must show: $\frac{\left(2s^2-s^2-r^2-4Rr\right)^2}{2s^2-2s^2+2r^2+8Rr}\geq\frac{94sRr}{4s}\Leftrightarrow\frac{\left(s^2-r^2-4Rr\right)^2}{2r(4R+r)}\geq 9Rr\Leftrightarrow$

$$(s^2-r^2-4Rr)^2\geq 18Rr^2(4R+r)\quad(7)$$

From Gerretsen inequality we have: $s^2\geq 16Rr-5r^2\quad(8)$

From (7)+(8) we must show: $(12Rr-6r^2)^2\geq 18Rr^2(4R+r)\Leftrightarrow$

$$\Leftrightarrow 36r^2(2R-r)^2 \geq 18Rr^2(4Rr+r) \Leftrightarrow 2(2R-r)^2 \geq R(4R+r) \Leftrightarrow$$
$$\Leftrightarrow 8R^2-8Rr+2r^2 \geq 4R^2+Rr \Leftrightarrow 4R^2-9Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(4R-r) \geq 0$$

true, because from Euler $R \geq 2r$.

PROBLEM 2.145-*Solution by Soumitra Mandal-Chandar Nagore-India*

Let $x=e^m, y=e^n$ and $e^p=z$ where $m,n,p>0$

Let $f(m)=\frac{1}{1+e^m}$ *for all* $m>0, f'(m)=-\frac{e^m}{(1+e^m)^2}, f''(m)=\frac{e^m(e^m-1)}{(1+e^m)^3}>0$

hence f is convexe function, $\therefore \sum_{cyc}\frac{1}{1+e^m} \geq \frac{3}{1+e^{\frac{m+n+p}{3}}}$

$$\Rightarrow \sum_{cyc}\frac{1}{1+x} \geq \frac{3}{1+\sqrt[3]{xyz}} \Rightarrow \frac{1}{3}\sum_{cyc}\int_a^b\int_a^b\int_a^b \frac{1}{1+x}dx \geq \int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{1+\sqrt[3]{xyz}}$$

$$\Rightarrow \frac{(a-b)^2}{3}\sum_{cyc}[\log(x+1)]_{x=a}^{x=b} \geq \int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{1+\sqrt[3]{xyz}}$$

$$\Rightarrow \log\left(\frac{b+1}{a+1}\right)^{(a-b)^2} \geq \int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{1+\sqrt[3]{xyz}}$$

PROBLEM 2.146-*Solution by Ravi Prakash-New Delhi-India*

$$AB=\begin{pmatrix}2 & 1 & 1\\ 0 & -1 & 1\\ 0 & 0 & -1\end{pmatrix}$$

$$\det(AB)=2 \Rightarrow \det(A)\det(B)=2 \neq 0 \Rightarrow \det(A)\neq 0, \det(B)\neq 0$$

$\therefore A^{-1}, B^{-1}$ *both exist. Now,* $(BA)^2-3I_3=A^{-1}A((BA)^2-3I_3)BB^{-1}=$

$=A^{-1}[A(BA)^2B-3AB]B^{-1}=A^{-1}[(AB)^3-3(AB)]B^{-1}$ *(1)*

Characteristic equation of AB

$$\begin{vmatrix}2-t & 1 & 1\\ 0 & -1-t & 1\\ 0 & 0 & -1-t\end{vmatrix}=0 \Rightarrow (1+t)^2(2-t)=0 \Rightarrow (1+2t+t^2)(2-t)=0 \Rightarrow$$

$\Rightarrow 2+4t+2t^2-t-2t^2-t^3=0$ *or* $2+3t-t^3=0$. *As AB satisfies this equation*

$2I_3=(AB)^3-(3AB)$ *(2)*

From (1), (2): $(BA)^2-3I_3=A^{-1}(2I_3)B^{-1}=2A^{-1}B^{-1}$

$$\det((BA)^2-3I_3)=8\det(A^{-1})\det(B^{-1})=\frac{8}{\det(A)\det(B)}=\frac{8}{\det(BA)}=\frac{8}{2}=4$$

PROBLEM 2.147-*Solution by Tran Hong-Vietnam*

Let $g(x)=f(x)-[x^2+x+1]; \forall x \in \mathbb{R}$ *since f continuous* $\Rightarrow$ *g continuos on* $\mathbb{R} \Rightarrow$

$$\Rightarrow g(x)+2g(2x)+g(ax)=0, \forall x \in \mathbb{R} \Rightarrow g\left(\frac{x}{2^{n+2}}\right)+2g\left(\frac{x}{2^{n+1}}\right)+g\left(\frac{x}{2^n}\right)=0, \forall x \in \mathbb{R}$$

$$\Rightarrow \lim_{n\to\infty}\left[g\left(\frac{x}{2^{n+2}}\right)+2g\left(\frac{x}{2^{n+1}}\right)+g\left(\frac{x}{2^n}\right)\right]=0; (\forall)x \in \mathbb{R} \Rightarrow g(0)+2g(0)+g(0)=0 \Rightarrow$$

$$\Rightarrow 3g(0)=0 \Rightarrow g(0)=0; [g(x)+g(2x)]+[g(2x)+g(x)]=0;$$

$$Let\ h(x) = g(x) = g(x); h(0) = 0 \Rightarrow h(x) + h(2x) = 0 \Rightarrow$$
$$h(2x) = (-1)^n h\left(\frac{x}{2^n}\right); \forall n \in \mathbb{N}$$
$$\Rightarrow h(x) = \lim_{n\to\infty}(-1)6n\, h\left(\frac{x^2}{2^n}\right) = 0 \Rightarrow g(2x) = \lim_{n\to\infty}(-1)^n\, g\left(\frac{x}{2^n}\right) = 0$$
$$f(x) = x^2 + x + 1$$

PROBLEM 2.148-*Solution by Remus Florin Stanca-Romania*

We prove by using Mathematical induction that $x_n > 0, \forall n \in \mathbb{N}$.

1) We prove that $P(0): x_0 > 0$ *is true (true)*

2) We suppose that $P(n): x_n > 0$ *is true.*

3) We prove by using $P(n)$ *that* $P(n+1): x_{n+1} > 0$ *is true.*

$$x_n > 0 \Rightarrow x_{n+1} \in \left(0; \frac{\pi}{2}\right] \Rightarrow x_{n+1} > 0 \Rightarrow x_n > 0; \forall n \in \mathbb{N}$$

$$x_{n+1} = \arctan\frac{x_n}{x_n+1}\ and\ because\ x_n > 0; \forall n \in \mathbb{N} \Rightarrow x_n \in \left(0; \frac{\pi}{2}\right]$$

$$We\ study\ the\ sign\ of\ x_1 - x_0 = \arctan\frac{x_0}{x_0+1} - x_0$$

$$Let\ f: \left(0; \frac{\pi}{2}\right] \to \mathbb{R}\ such\ that\ f(x) = \arctan\frac{x}{x+1} - x$$

$$\Rightarrow f'(x) = \frac{1}{1+\frac{x^2}{(x+1)^2}} \cdot \frac{1}{(x+1)^2} - 1 = \frac{1}{2x^2+2x+1} - 1 \Rightarrow f'(x) < 0$$

$$\Rightarrow f(x)\ is\ a\ decreasing\ function\ f(0) = 0 \Rightarrow f(x) < 0\ for\ x \in \left(0, \frac{\pi}{2}\right] \Rightarrow x_1 < x_0$$

We prove by using the Mathematical induction that $x_n > x_{n+1}$

1) We proved that $P(0): x_0 > x_1$ *is true*

2) We suppose that $P(n): x_n > x_{n+1}$ *is true*

3) We prove that $P(n+1): x_{n+1} > x_{n+2}$ *is true by using* $P(n)$

$$x_{n+1} - x_{n+2} = \arctan\frac{x_n}{x_n+1} - \arctan\frac{x_{n+1}}{x_{n+1}+1}$$

$$We\ prove\ that\ the\ function\ f: \left(0; \frac{\pi}{2}\right] \to \mathbb{R}, f(x) = \arctan\frac{x}{x+1}\ is\ an\ increasing\ function.$$

$$f(x) = \frac{1}{1+\frac{x^2}{(x+1)^2}} \cdot \frac{1}{(x+1)^2} > 0 \Rightarrow true\ so\ x_{n+1} - x_{n+2} > 0$$

$$\succ x_{n+1} > x_{n+2} \succ x_n > x_{n+1}\ for\ n \in \mathbb{N}.$$

$$x_n \in \left(0; \frac{\pi}{2}\right]\ and\ x_n\ is\ a\ decreasing\ sequence \Rightarrow -l = \lim_{n\to\infty} x_n\ such\ that\ l \in \mathbb{R}$$

$$\Rightarrow l = \arctan\frac{l}{l+1}, l \in \left(0, \frac{\pi}{2}\right]\ the\ funciton\ f(l) = \text{arctn}\frac{l}{l+1} - l\ is\ a\ decreasing\ function\ so\ l = 0$$

is the unique solution.

$$\Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow \lim_{n\to\infty} nx_n = \lim_{n\to\infty}\frac{n}{\frac{1}{x_n}} \overset{Stolz-Cesaro}{=} \lim_{n\to\infty}\frac{1}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} =$$

$$= \lim_{n\to\infty}\frac{x_{n+1}x_n}{x_n - x_{n+1}} = \lim_{n\to\infty}\frac{\arctan\frac{x_n}{x_n+1}x_n}{x_n - \arctan\frac{x_n}{x_n+1}}$$

$$\lim_{n\to 0}\frac{\arctan\frac{x}{x+1}\cdot x}{x-\arctan\frac{x}{x+1}}\overset{L'H}{\underset{\frac{0}{0}}{=}}\lim_{x\to 0}\frac{\arctan\frac{x}{x+1}+x\cdot\frac{1}{1+\frac{x^2}{(x+1)^2}}\cdot\frac{1}{(x+1)^2}}{1-\frac{1}{1+\frac{x^2}{(x+1)^2}\cdot\frac{1}{(x+1)^2}}}=$$

$$=\lim_{x\to 0}\frac{\left(\arctan\frac{x}{x+1}\right)(2x^2+2x+1)-x}{2x^2+2x}=\lim_{x\to 0}\frac{\arctan\frac{x}{x+1}}{\frac{x}{x+1}\cdot(x+1)(2x+2)}(2x^2+2x+1)=$$

$$=\frac{1}{2x+2}=\frac{1}{2}-\frac{1}{2}=0$$

PROBLEM 2.149-*Solution by Remus Florin Stanca-Romania*

We prove that $x_n>1,\forall n\in\mathbb{N}$ *by using the Mathematical induction:*

1) we prove $P(0): x_0>1$ *(true)*

2) we suppose that $P(n): x_n>1$ *is true*

3) we prove $P(n+1): x_{n+1}>1$ *by using* $P(n)$

$$x_n>1\Rightarrow\frac{2x_n}{x_n+1}>1\Rightarrow\ln\left(\frac{2x_n}{x_n+1}\right)+1>1\Rightarrow x_{n+1}>1\Rightarrow x_n>1\ \forall n\in\mathbb{N}$$

We study the sign of $x_1-x_0=1+\ln\left(\frac{2x_0}{1+x_0}\right)-x_0$

Let $f:(1;\ +\infty)\to\mathbb{R}; f(x)=1+\ln\left(\frac{2x}{1+x}\right)-x$

$f'(x)=\frac{1+x}{2x}\cdot\frac{2}{(x+1)^2}-1=\frac{1}{x(x+1)}-1<0\Rightarrow f$ *is a decreasing function*

$f(1)=0\succ f(x)<0$ *for* $x>1\succ x_1<x_0$

$g(x)=\frac{2x}{2x+1}$ *is an increasing function so* $x_{n+1}<x_n$

$x_{n+1}<x_n$ *and* $x_n>1\succ l=\lim_{n\to\infty}x_n\in\mathbb{R}$

$l=1+\ln\left(\frac{2l}{l+1}\right)\Rightarrow f(l)=1+\ln\left(\frac{2l}{l+1}\right)-f$ *is a decreasing function* $\Rightarrow$

$l=1$ *is an unique solution* $\Rightarrow\lim_{n\to\infty}x_n=1$

$$\lim_{n\to\infty}n\ln x_n=\lim_{n\to\infty}\frac{n}{\frac{1}{\ln x_n}}=\lim_{n\to\infty}\frac{1}{\frac{1}{\ln x_{n+1}}-\frac{1}{\ln x_n}}=$$

$$=\lim_{n\to\infty}\frac{1}{\frac{1}{\ln\left(1+\ln\frac{2x_n}{x_n+1}\right)}-\frac{1}{\ln x_n}}=\lim_{n\to\infty}\frac{\ln x_n\ln\left(1+\ln\frac{2x_n}{x_n+1}\right)}{\ln x_n-\ln\left(1+\ln\frac{2x_n}{x_n+1}\right)}$$

$$\lim_{x\to 1}\frac{\ln x\ln\left(1+\ln\frac{2x}{x+1}\right)}{\ln x-\ln\left(1+\ln\frac{2x}{x+1}\right)}=\lim_{x\to 1}\frac{\frac{\ln x}{x-1}\cdot(x-1)\cdot\frac{\ln\left(1+\ln\frac{2x}{x+1}\right)}{\ln\frac{2x}{x+1}}\cdot\ln\frac{2x}{x+1}}{\ln\left(\frac{x}{1+\ln\frac{2x}{x+1}}-1+1\right)}\cdot$$

$$\cdot\frac{x-1-\ln\frac{2x}{x+1}}{1+\ln\frac{2x}{x+1}}\cdot\frac{1+\ln\frac{2x}{x+1}}{x-1-\ln\frac{2x}{x+1}}=\lim_{x\to 1}\frac{(x-1)\ln\frac{2x}{x+1}}{x-1-\ln\frac{2x}{x+1}}=\lim_{x\to 1}\frac{(x-1)\ln\left(\frac{x-1}{x+1}+1\right)}{x-1-\ln\frac{2x}{x+1}}$$

$$=\lim_{x\to 1}\frac{(x-1)\cdot\frac{\ln\left(\frac{x-1}{x+1}+1\right)}{\frac{x-1}{x+1}}}{x+1-\frac{\ln\left(\frac{x-1}{x+1}+1\right)}{\frac{x-1}{x+1}}}=\lim_{x\to 1}\frac{x-1}{x}=0\Rightarrow\lim_{n\to\infty}n\ln x_n=0$$

PROBLEM 2.150-*Solution by proposer*

Suppose $\exists g,h\in\mathbb{Z}$ *such that* $f=g\cdot h$, *grade* $f,h\ge 1$. $f(0)=g(0)=h(0)\Rightarrow$
$\Rightarrow a_0=g(0)\cdot h(0)$. *But* a_0 *being prime* $\Rightarrow g(0)=1$ *or* $h(0)=1$.
Suppose $g(0)=1\Rightarrow g(x)=b_kx^k+\dots+b_1x+1$
Let be $x_1,x_2,\dots,x_k$ *the roots of* f. *From the last Viete relationship* $\Rightarrow$
$\Rightarrow|x_1x_2\dots x_n|=\left|\frac{(-1)^k}{b_k}\right|=\frac{1}{|b_k|}\le 1$, *because* $b_k\in\mathbb{Z}$
$|x_1x_2\dots x_k|\le 1\Rightarrow\exists p\in\{1,2,\dots,k\}$ *such that* $|x_p|\le 1$.
But x_p *is root and for* $f\Rightarrow a_nx_p^n+a_{n-1}x_p^{n-1}+\dots+a_1x_p+a_0=0\Rightarrow$
$\Rightarrow|a_0|=|a_nx_p^n+a_{n-1}x_p^{n-1}+\dots+a_1x_p|\le|a_n||x_p|^n+\dots+|a_1||x_p|\le$
$\le|a_1|+|a_2|+\dots+|a_n|\Rightarrow|a_0|\le|a_1|+|a_2|+\dots+|a_n|\le n^2\Rightarrow a_0\le n^2$ *false*$\Rightarrow$
$\Rightarrow f$ *is irreducible over* $\mathbb{Z}$

PROBLEM 2.151-*Solution by Soumava Chakraborty-Kolkata-India*

$$a^4+b^4\le 2(a^2-ab+b^2)^2\Leftrightarrow a^4+b^4+6a^2b^2-4ab(a^2+b^2)\ge 0$$
$$\Leftrightarrow(a^2+b^2)^2+4a^2b^2-4ab(a^2+b^2)\ge 0\Leftrightarrow(a^2+b^2-2ab)^2\ge 0\to true$$
$$\therefore\sqrt{a^4+b^4}\le\sqrt{2}|a^2-ab+b^2|=\sqrt{2}(a^2-ab+b^2)$$
$$\left(\because a^2-ab+b^2=\frac{1}{4}(a+b)^2+\frac{3}{4}(a-b)^2>0\right)$$
$$\Rightarrow\sqrt{2(a^4+b^4)}\le 2a^2-2ab+2b^2\Rightarrow\sqrt{2(a^4+b^4)}+7ab\le 2a^2+5ab+2b^2=$$
$$=(2a+b)(a+2b)\overset{G-A}{\le}\frac{(2a+b+a+2b)^2}{4}=\frac{9}{4}(a+b)^2\Rightarrow$$
$$\Rightarrow\frac{1}{\sqrt{2(a^4+b^4)}+7ab}\ge\frac{4}{9}\cdot\frac{1}{(a+b)^2}\Rightarrow\frac{c}{\sqrt{2(a^4+b^4)}+7ab}\overset{(1)}{\ge}\frac{4}{9}\cdot\frac{c}{(a+b)^2}$$

Similarly, $\frac{a}{\sqrt{2(b^4+c^4)}+7bc}\overset{(2)}{\ge}\frac{4}{9}\cdot\frac{a}{(b+c)^2}$ & $\frac{b}{\sqrt{2(c^4+a^4)}+7ca}\overset{(3)}{\ge}\frac{4}{9}\cdot\frac{b}{(c+a)^2}$

(1)+(2)+(3) $\Rightarrow$ *LHS* $\overset{(4)}{\ge}\frac{4}{9}\sum\frac{a}{(b+c)^2}$

WLOG, we may assume $a\ge b\ge c$

Now, $\frac{a}{b+c}\ge\frac{b}{c+a}\Leftrightarrow a^2+ac\ge b^2+bc\Leftrightarrow(a-b)(a+b+c)\ge 0\to$ *true* $\because a\ge b\ge c$

$\therefore \frac{a}{b+c} \geq \frac{b}{c+a}$. *Similarly,* $\frac{b}{c+a} \geq \frac{c}{a+b} \Rightarrow \frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b}$ *& also,* $\because a \geq b \geq c, \frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$

$\therefore$ *by Chebyshev & using (4),* $LHS \geq \frac{4}{9} \cdot \frac{1}{3}\left(\sum \frac{a}{b+c}\right)\left(\sum \frac{1}{b+c}\right) \overset{Nesbitt}{\geq} \frac{4}{9} \cdot \frac{1}{3} \cdot \frac{3}{2}\left(\sum \frac{1}{b+c}\right) \overset{Bergstrom}{\geq}$

$\geq \frac{2}{9} \cdot \frac{9}{2\sum a} = \frac{1}{\sum a} = \frac{1}{3}$ *(Proved)*

PROBLEM 2.152-*Solution by Soumava Chakraborty-Kolkata-India*

Firstly, $a^4 + b^4 \leq 2(a^2 - ab + b^2)^2 \Leftrightarrow a^4 + b^4 + 6a^2b^2 - 4ab(a^2 + b^2) \geq 0 \Leftrightarrow$

$\Leftrightarrow (a^2 + b^2)^2 + 4a^2b^2 - 4ab(a^2 + b^2) \geq 0 \Leftrightarrow (a^2 + b^2 - 2ab)^2 \geq 0 \rightarrow true$

$$\therefore \sqrt{a^4 + b^4} \leq \sqrt{2}|a^2 - ab + b^2| = \sqrt{2}(a^2 - ab + b^2)$$

$$\left(\because a^2 - ab + b^2 = \frac{1}{4}(a+b)^2 + \frac{3}{4}(a-b)^2 > 0\right)$$

$$\therefore \sqrt{2(a^4 + b^4)} \leq 2a^2 - 2ab + 2b^2 \Rightarrow \frac{1}{\sqrt{2(a^4 + b^4)}} \geq \frac{1}{2}\left(\frac{1}{a^2 + b^2 - ab}\right)$$

$$\therefore \frac{1}{\sqrt{2(a^4 + b^4)}} + \frac{a^2}{b} \geq \frac{1}{2}\left(\frac{1}{a^2 + b^2 - ab}\right) + \frac{a^2}{2b} + \frac{a^2}{2b}$$

$$\overset{A-G}{\geq} 3\sqrt[3]{\frac{a^4}{8b^2(a^2 + b^2 - ab)}} = \frac{3}{2}\sqrt[3]{\frac{a^6}{(a^2 + b^2 - ab)ab \cdot ab}}$$

$$= \frac{3a^2}{2} \cdot \frac{1}{\sqrt[3]{(a^2 + b^2 - ab)ab \cdot ab}} \overset{A-G}{\geq} \frac{3a^2}{2} \cdot \frac{3}{a^2 + b^2 - ab + 2ab}$$

$$= \frac{9a^2}{2(a^2 + b^2 + ab)} \therefore \frac{1}{\sqrt{2(a^4 + b^4)}} + \frac{a^2}{b} \overset{(1)}{\geq} \frac{9}{2}\left(\frac{a^2}{a^2 + b^2 + ab}\right)$$

Similarly, $\frac{1}{\sqrt{2(b^4+c^4)}} + \frac{b^2}{c} \overset{(2)}{\geq} \frac{9}{2}\left(\frac{b^2}{b^2+c^2+bc}\right)$ & $\frac{1}{\sqrt{2(c^4+a^4)}} + \frac{c^2}{a} \overset{(3)}{\geq} \frac{9}{2}\left(\frac{c^2}{c^2+a^2+ca}\right)$

(1)+(2)+(3) $\Rightarrow p \geq \frac{9}{2}\left(\frac{a^2}{a^2+b^2+ab} + \frac{b^2}{b^2+c^2+bc} + \frac{c^2}{c^2+a^2+ca}\right) \overset{?}{\geq} \frac{9}{2} \Leftrightarrow$

$\Leftrightarrow a^2(b^2 + c^2 + bc)(c^2 + a^2 + ca) + b^2(c^2 + a^2 + ca)(a^2 + b^2 + ab) +$

$+c^2(a^2 + b^2 + ab)(b^2 + c^2 + bc) - (a^2 + b^2 + ab)(b^2 + c^2 + bc)(c^2 + a^2 + ca) \overset{?}{\geq} 0$

$\Leftrightarrow a^4b^2 + b^4c^2 + c^4a^2 \overset{?}{\geq} a^2b^3c + b^2c^3a + c^2a^3b \rightarrow true$

$\because x^2 + y^2 + z^2 \geq xy + yz + zx$ *where* $x = a^2b, y = b^2c, z = c^2a \Rightarrow P \geq \frac{9}{2}$ *equality at*

$a = b = c = 1 \Rightarrow P_{min} = \frac{9}{2}$ *(where* $a = b = c = 1$*) (Answer)*

PROBLEM 2.153-*Solution by Soumava Chakraborty-Kolkata-India*

$$2\left(\frac{x^3}{y^2} + \frac{y^3}{x^2}\right) \overset{(1)}{=} \sqrt[4]{8(x^4 + y^4)} + 2\sqrt{xy} \ \& \ 16x^5 - 20x^3 + 5\sqrt{xy} \overset{(2)}{=} \sqrt{\frac{y+1}{2}}$$

Of course, $x, y \neq 0$ & $xy > 0 \Rightarrow x, y < 0$ *or* $x, y > 0$. *If* $x, y < 0$, *then LHS of (1)* < 0, *but RHS of (1)* $> 0 \Rightarrow x, y < 0$ *is impossible* $\therefore x, y > 0$. *Now,* $x^4 + y^4 \leq 2(x^2 - xy + y^2)^2$

$\Leftrightarrow x^4 + y^4 + 6x^2y^2 - 6xy(x^2 + y^2) \geq 0 \Leftrightarrow (x^2 + y^2)^2 + 4x^2y^2 - 4xy(x^2 + y^2) \geq 0 \Leftrightarrow$

$\Leftrightarrow (x^2 + y^2 - 2xy)^2 \geq 0 \rightarrow true$

$$\therefore \sqrt{x^4+y^4} \le \sqrt{2}|x^2-xy+y^2| = \sqrt{2}(x^2-xy+y^2)$$
$$\left(\because x^2-xy+y^2 = \frac{1}{4}(x+y)^2 + \frac{3}{4}(x-y)^2 > 0\right)$$
$$\Rightarrow \sqrt[4]{\frac{x^4+y^4}{2}} \le \sqrt{x^2-xy+y^2} \Rightarrow \sqrt[4]{8(x^4+y^4)} \le 2\sqrt{x^2-xy+y^2} \Rightarrow$$
$$\Rightarrow \sqrt[4]{8(x^4+y^4)} + 2\sqrt{xy} \le 2\left(\sqrt{x^2-xy+y^2}+\sqrt{xy}\right) \overset{CBS}{\le} 2\sqrt{2}\sqrt{x^2-xy+y^2+xy} =$$
$$= 2\sqrt{2}\sqrt{x^2+y^2} \underset{(a)}{\overset{?}{\le}} \frac{2(x^2+y^2)^2}{xy(x+y)} \Leftrightarrow \frac{(x^2+y^2)^4}{x^2y^2(x+y)^2} \overset{?}{\ge} 2(x^2+y^2) \Leftrightarrow$$
$$\Leftrightarrow (x^2+y^2)^3 \underset{(b)}{\overset{?}{\ge}} 2x^2y^2(x+y)^2$$

Now, $(x^2+y^2)^3 = (x^2+y^2)(x^2+y^2)^2 \overset{Chebyshev}{\ge} \frac{1}{2}(x+y)^2(x^2+y^2) \overset{A-G}{\ge}$

$\ge \frac{1}{2}(x+y)^2 \cdot 4x^2y^2 = 2x^2y^2(x+y)^2 \Rightarrow$ *(b) is true* $\Rightarrow$ *(a) is true* $\Rightarrow$

$\Rightarrow$ *RHS of (1)* $\overset{(i)}{\le} \frac{2(x^2+y^2)^2}{xy(x+y)}$, *equality at* $x=y$.

Again, LHS of (1)= $2\left(\frac{x^4}{xy^2}+\frac{y^4}{x^2y}\right) \underset{(ii)}{\overset{Bergstrom}{\ge}} 2\frac{(x^2+y^2)^2}{xy(x+y)}$, *equality at* $x=y$.

(i), (ii) $\Rightarrow$ *LHS of (1) = RHS of (1) =* $\frac{2(x^2+y^2)^2}{xy(x+y)}$ *&* $\because$ *respective equalities occur at* $x=y$

$\therefore x=y$

Putting $y=x$ *in (2), we get:* $16x^5-20x^3+5x = \sqrt{\frac{x+1}{2}} \Rightarrow$

$$\Rightarrow 16x^5-20x^3+5x-1 = \sqrt{\frac{x+1}{2}}-1 \Rightarrow (x-1)(4x^2+2x-1)^2 =$$

$= \frac{\frac{x+1}{2}-1}{\sqrt{\frac{x+1}{2}}+1} = \frac{x-1}{2\left(1+\sqrt{\frac{x+1}{2}}\right)}$. *One possibility is* $x=1 \Rightarrow x=y=1$ *is a solution when*

$x \ne 1, 2(4x^2+2x-1)^2\left(1+\sqrt{\frac{x+1}{2}}\right) = 1$. *Let* $\sqrt{\frac{x+1}{2}} = t$. *Then, we have:*

$$(2+2t)(4(2t^2-1)^2+2(2t^2-1)-1)^2-1=0 \Rightarrow$$
$$\Rightarrow (2t+1)(8t^3-6t+1)(32t^5+16t^4-32t^3-12t^2+6t+1)=0$$

The equations yield two acceptable solutions: $t = \cos^2\frac{2\pi}{9} \Rightarrow \sqrt{\frac{x+1}{2}} = \cos\frac{2\pi}{9} \Rightarrow x = \cos\frac{4\pi}{9}$

& $t \approx .84125 \Rightarrow x \approx .415415$ $\therefore$ *all possible solutions are:*

$$(x=y=1), \left(x=y=\cos\frac{4\pi}{9}\right), (x=y\approx .415415)$$

PROBLEM 2.154-*Solution by Tran Hong-Vietnam*

$$\sum\left(3\sqrt[3]{x}+x\right) = \sum\left(\sqrt[3]{x}+\sqrt[3]{x}+\sqrt[3]{x}+x\right) \overset{Cauchy}{\ge} \sum 4\sqrt[3]{x} \Leftrightarrow \sum\sqrt[3]{x} \ge \frac{4\sum\sqrt{x}-3}{3}$$

$$P \geq \frac{1}{3} \cdot \frac{\sum \sqrt{x}}{\sum \sqrt{x}} + \frac{8}{3} \sum x^2 \geq \frac{1}{3} + \frac{8}{3} \cdot \frac{(x+y+z)^2}{3} = \frac{1}{3} + 8 = \frac{25}{3}$$

$$\Rightarrow P_{\min} = \tfrac{25}{3} \Leftrightarrow a = b = c = 1.$$

PROBLEM 2.155-*Solution by proposer*

** Lemma: Let a, b, c be positive real numbers we have inequality:*

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2) \quad (2)$$

(2): $a^4 + b^4 + c^4 + abc(a+b+c) \geq ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2)$

$$\Leftrightarrow a^4 + b^4 + c^4 + abc(a+b+c) - ab(a^2+b^2) - bc(b^2+c^2) - ca(c^2+a^2) \geq 0$$

$$\Leftrightarrow a^2(a^2 - ab - ac + bc) + b^2(b^2 - bc - ba + ca) + c^2(c^2 - ca - cb + ab) \geq 0$$

$$\Leftrightarrow a^2(a-b)(a-c) + b^2(b-a)(b-c) + c^2(c-a)(c-b) \geq 0 \quad (3)$$

- Supposed $a \geq b \geq c > 0$.

+ We have: $\begin{cases} c \leq a \\ c \leq b \end{cases} \Leftrightarrow \begin{cases} c - a \leq 0 \\ c - b \leq 0 \end{cases} \Rightarrow (c-a)(c-b) \geq 0 \Leftrightarrow c^2(c-a)(c-b) \geq 0 \quad (4)$

+ Let: $a^2(a-b)(a-c) + b^2(b-a)(b-c) = (a-b)[a^2(a-c) - b^2(b-c)]$

$$\Leftrightarrow a^2(a-b)(a-c) + b^2(b-a)(b-c) = (a-b)[(a^3-b^3) - c(a^2-b^2)]$$

$$= (a-b)[(a-b)(a^2+ab+b^2) - c(a-b)(a+b)]$$

$$= (a-b)(a-b)(a^2+ab+b^2-ac-bc) = (a-b)^2(a^2+ab+b^2-ac-bc) \quad (5)$$

$-$ *Because* $a \geq b \geq c > 0$ *then* $a - c \geq 0; b - c \geq 0$

+ Hence: $a^2 + ab + b^2 - ac - bc = a(a-c) + b(b-c) + ab \geq ab > 0; (a-b)^2 \geq 0; \forall a, b \in \mathbb{R}$

$\Rightarrow (a-b)^2(a^2+ab+b^2-ac-bc) \geq 0$. *Let (5):* $\Rightarrow a^2(a-b)(a-c) + b^2(b-a)(b-c) \geq 0 \quad$ *(6)*

- Let (4), (6): $\Rightarrow a^2(a-b)(a-c) + b^2(b-a)(b-c) + c^2(c-a)(c-b) \geq 0$

$\Rightarrow$ *Inequality (3) true* $\Rightarrow$ *(2) true and lemma get the result.*

** Let* $(a, b, c) = (x, y, z)$*:*

$$\Rightarrow x^4 + y^4 + z^4 + xyz(x+y+z) \geq xy(x^2+y^2) + yz(y^2+z^2) + zx(z^2+x^2) \quad (7)$$

- By AM-GM inequality we have:

$$xy(x^2+y^2) + yz(y^2+z^2) + zx(z^2+x^2) \geq xy \cdot 2xy + yz \cdot 2yz + zx \cdot 2zx$$

$$= 2(x^2y^2 + y^2z^2 + z^2x^2)$$

$$\Leftrightarrow xy(x^2+y^2) + yz(y^2+z^2) + zx(z^2+x^2) \geq 2(x^2y^2 + y^2z^2 + z^2x^2) \quad (8)$$

- Let (7), (8): $\Rightarrow x^4 + y^4 + z^4 + xyz(x+y+z) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$

$$\Leftrightarrow x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2) \geq 4(x^2y^2 + y^2z^2 + z^2x^2) - xyz(x+y+z)$$

$$\Leftrightarrow (x^2+y^2+z^2)^2 \geq 4(x^2y^2 + y^2z^2 + z^2x^2) - xyz(x+y+z)$$

$$\Leftrightarrow \frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2) - xyz(x+y+z)} \geq 1 \quad (9)$$

** By Cauchy Schwarz inequality we have:*

$$\frac{x^3}{(2y^2 - yz + 2z^2)^2} + \frac{y^3}{(2z^2 - zx + 2x^2)^2} + \frac{z^3}{(2x^2 - xy + 2y^2)^2}$$

$$= \frac{\left(\frac{x^2}{2y^2 - yz + 2z^2}\right)^2}{x} + \frac{\left(\frac{y^2}{2z^2 - zx + 2x^2}\right)^2}{y} + \frac{\left(\frac{z^2}{2x^2 - xy + 2y^2}\right)^2}{z} \geq$$

$$\geq \frac{\left(\frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2}\right)^2}{x+y+z} \quad (10)$$

- Other: $\frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2}$

$$=\frac{x^4}{2x^2y^2-x^2yz+2x^2z^2}+\frac{y^4}{2y^2z^2-y^2zx+2y^2x^2}+\frac{z^4}{2z^2x^2-z^2xy+2z^2y^2}\geq$$

$$\geq \frac{(x^2+y^2+z^2)^2}{(2x^2y^2-x^2yz+2x^2z^2)+(2y^2z^2-y^2zx+2y^2x^2)+(2z^2+x^2-z^2xy+2z^2y^2)}$$

$$\Leftrightarrow \frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2}\geq \frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)} \quad (11)$$

- Let (9), (11):$\Rightarrow \frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2}\geq 1 \quad (12)$

- Let (10), (12):

$$\Rightarrow \frac{x^3}{(2y^2-yz+2z^2)^2}+\frac{y^3}{(2z^2-zx+2x^2)^2}+\frac{z^3}{(2x^2-xy+2y^2)^2}\geq \frac{1}{x+y+z}$$

$$\Rightarrow P=\frac{x^3}{(2y^2-yz+2z^2)^2}+\frac{y^3}{(2z^2-zx+2x^2)^2}+\frac{z^3}{(2x^2-xy+2y^2)^2}+\frac{xy+yz+zx}{3}\geq$$

$$\geq \frac{1}{x+y+z}+\frac{xy+yz+zx}{3} \quad (13)$$

- By inequality: $(mn+np+pm)^2\geq 3mnp(m+n+p)$ *and AM-GM inequality and:* $xyz=1$.

We have:

$$\frac{1}{x+y+z}+\frac{xy+yz+zx}{3}=\left(\frac{1}{x+y+z}+\frac{xy+yz+zx}{9}+\frac{xy+yz+zx}{9}\right)+\frac{xy+yz+zx}{9}$$

$$\geq 3\cdot\sqrt[3]{\frac{1}{x+y+z}\cdot\frac{xy+yz+zx}{9}\cdot\frac{xy+yz+zx}{9}}+\frac{3\cdot\sqrt[3]{xy\cdot yz\cdot zx}}{9}$$

$$\Rightarrow \frac{1}{x+y+z}+\frac{xy+yz+zx}{3}\geq 3\cdot\sqrt[3]{\frac{(xy+yz+zx)^2}{81(x+y+z)}}+\frac{3\sqrt[3]{(xyz)^2}}{9}\geq 3\sqrt[3]{\frac{3xyz(x+y+z)}{81(x+y+z)}}+\frac{3\cdot 1}{9}$$

$$\Rightarrow \frac{1}{x+y+z}+\frac{xy+yz+zx}{3}\geq 3\sqrt[3]{\frac{3\cdot 1}{81}}+\frac{3}{9}=1+\frac{1}{3}=\frac{4}{3}\Leftrightarrow \frac{1}{x+y+z}+\frac{xy+yz+zx}{3}\geq\frac{4}{3} \quad (14)$$

- Let (13), (14): $\Rightarrow P\geq\frac{4}{3}\Rightarrow P_{\min}=\frac{4}{3}$. *Equality occurs if:*

$$\Leftrightarrow \begin{cases} x=y=z>0 \\ xyz=1 \\ \frac{1}{2y^2-yz+2z^2}=\frac{1}{2z^2-zx+2x^2}=\frac{1}{2x^2-xy+2y^2} \\ \frac{1}{x+y+z}=\frac{xy+yz+zx}{9} \end{cases}\Leftrightarrow x=y=z=1.$$

PROBLEM 2.156-*Solution by Ravi Prakash-New Delhi-India*

Let $P(x)$ *be a polynomial of degree* m *where* $m\in\mathbb{N}$.

If $m=1$, *let* $P(x)=ax+b, a\neq 0$, *then* $ax+b=a\left(x+\sqrt{x^2+1}\right)+b\ \forall x\in\mathbb{R}$

$$\Rightarrow a\sqrt{x^2+1}=0,\forall x\in\mathbb{R}\Rightarrow a=0$$

A contradiction.

Assume $m\geq 2$.

Choose a sequence $m_1 > m_2 > \cdots > m_m$ *of positive integers such that*

$$m_{k+1} > m_k + \sqrt{m_k^2 + 1} \text{ for } 1 \le k \le m-1.$$

For $1 \le r \le m$

$$P(m_r) = P\left(m_r + \sqrt{m_r^2 + 1}\right) \text{ (given)}$$

By the Rolle's theorem $\exists \alpha_r \in \left(m_r, m_r + \sqrt{m_r^2 + 1}\right)$ *such that*

$P'(\alpha_r) = 0 \quad (1 \le r \le m) \Rightarrow P'(x)$ *has at least* m *zeros. But* $P'(x)$ *is a polynomial of degree* $(m-1)$. *A contradiction.*

$\therefore$ *there is no polynomial of degree* ≥ 1, *satisfying the given condition.*

Thus, $P(x)$ *satisfies the given condition if and only if* $P(x)$ *is a constant*

PROBLEM 2.157-*Solution by Tran Hong-Vietnam*

$$f'(x)(f(x) + x^2 + 2x + a) = 1$$

$$\Rightarrow f'(x) > 0 \ \forall x \ge 0 \ \text{(because } f(x) \ge 0 \ \forall x \ge 0, a > 1) \Rightarrow f(x) \nearrow \text{ on } [0, +\infty)$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = l \in [0, +\infty) \text{ or } \lim_{x\to+\infty} f(x) = +\infty.$$

$$\text{If } \lim_{x\to+\infty} f(x) = +\infty \Rightarrow \lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty} \frac{1}{f(x) + x^2 + 2x + a} = 0$$

$$\Rightarrow \exists \alpha > 0: 0 < f'(x) \le \frac{1}{x^2 + 1} \ (\forall x \ge \alpha) \Rightarrow 0 < \int_0^x f'(t)\,dt \le \int_0^x \frac{1}{t^2 + 1}\,dt$$

$$\Rightarrow 0 < f(x) - f(\alpha) \le \tan^{-1}(x) - \tan^{-1}(\alpha) \Rightarrow f(x) \le \tan^{-1}(x) + f(\alpha) - \tan^{-1}(\alpha)$$

$$\Rightarrow \lim_{x\to+\infty} f(x) \le \frac{\pi}{2} + f(\alpha) - \tan^{-1}(\alpha)$$

which is contrary with $\lim_{x\to+\infty} f(x) = +\infty$. *So, we have* $\lim_{x\to+\infty} f(x) = l \in [0, +\infty)$.

PROBLEM 2.158-*Solution by Tran Hong-Vietnam*

With $n = 1$: $a_1(a_1 + a_2)(a_1^2 + a_2^2) \ldots \left(a_1^{2^k} + a_2^{2^k}\right)$

$$\overset{(a_2 = a_1)}{=} 2a_1^2 \cdot 2a_1^2 \ldots .2a_1^{2^k} \ge 0 \text{ (true)}$$

Suppose it true with $1, 2, \ldots, n$; *we prove with* $n+1$;

Let: $a_{n+1} = \max\{a_i | i = 1, 2, \ldots, n+1\} \Rightarrow a_{n+1} \ge a_1$

$$U_{n+1} = \sum_{i=1}^{n+1} a_i\,(a_i + a_{i+1})(a_i^2 + a_{i+1}^2) \ldots \left(a_i^{2^k} + a_i^{2^k}\right)$$

$$= \sum_{i=1}^{n} a_i\,(a_i + a_{i+1})(a_i^2 + a_{i+1}^2) \ldots \left(a_i^{2^k} + a_{i+1}^{2k}\right) + a_{n+1}(a_{n+1} + a_{n+2})(a_{n+1}^2 + a_{n+2}^2) \ldots$$

$$\ldots \left(a_{n+2}^{2^k} + a_{n+2}^{2k}\right)$$

$$U_{n+1} = \sum_{i=1}^{n} a_i\,(a_i + a_{i+1})(a_i^2 + a_{i+1}^2) \ldots \left(a_i^{2k} + a_i^{2^k}\right) +$$

$$+ a_{n+1}(a_{n+1} + a_1)(a_{n+1}^2 + a_1^2) \ldots \left(a_{n+1}^{2^k} + a_1^{2k}\right);$$

If $a_{n+1} \ge a_1 \ge 0$ *then* $U_{n+1} \ge 0$

If $a_1 \le a_{n+1} \le 0$ *then* $U_{n+1} \ge 0$

If $a_{n+1} \ge 0 > a_1$ *then we have:*

$$U_{n+1} = \sum_{i=1}^{n} a_i\,(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)\dots\left(a_i^{2^k} + a_i^{2^k}\right) +$$

$$+\frac{a_{n+1}}{a_{n+1} - a_1}\cdot(a_{n+1}^2 - a_1^2)(a_{n+1}^2 + a_1^2)\dots\left(a_{n+1}^{2k} + a_1^{2^k}\right)$$

$$= \sum_{i=1}^{n} a_i\,(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)\dots\left(a_i^{2^k} + a_i^{2^k}\right) + a_{n+1}\cdot\frac{a_{n+1}^{2^{k+1}} - a_1^{2^{k+1}}}{a_{n+1} - a_1} \ge 0$$

$\Rightarrow$ *Proved. Equality* $\Leftrightarrow a_i = 0 \quad (i = 1,2,\dots,n)$

PROBLEM 2.159-*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum a \sum a^2 \overset{(1)}{\ge} 2\sum (b+c)h_a^2$$

$$(1) \Leftrightarrow 2s(s^2 - 4Rr - r^2) \ge \sum(2s - a)\frac{b^2c^2}{4R^2}$$

$$\Leftrightarrow 4R^2\cdot 2s(s^2 - 4Rr - r^2) \ge 2s\left\{\left(\sum ab\right)^2 - 16Rrs^2\right\} - 4Rrs\left(\sum ab\right)$$

$$\Leftrightarrow 4R^2(s^2 - 4Rr - r^2) \ge (s^2 + 4Rr + r^2)^2 - 16Rrs^2 - 2Rr(s^2 + 4Rr + r^2)$$

$$\Leftrightarrow s^4 - s^2(4R^2 + 10Rr - 2r^2) + 16R^3r + 12R^2r^2 + 6Rr^3 + r^4 \overset{(2)}{\le} 0$$

Now, $s^2 \ge m - n \Rightarrow s^2 - m + n \overset{(2)}{\ge} 0$ & $s^2 \le m + n \Rightarrow s^2 - m - n \overset{(b)}{\le} 0$, *where*

$$m = 2R^2 + 10Rr - r^2 \;\&\; n = 2(R - 2r)\sqrt{R^2 - 2Rr}$$

(a).(b) $\Rightarrow s^4 - 2ms^2 + m^2 - n^2 \le 0 \Rightarrow$

$$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + 64R^3r + 48R^2r^2 + 12Rr^3 + r^4 \overset{(3)}{\le} 0$$

(2), (3) $\Rightarrow$ *in order to prove (2), it suffices to show:*

$$-s^2(4R^2 + 10Rr - 2r^2) + 16R^3r + 12R^2r^2 + 6Rr^3 \le -s^2(4R^2 + 20Rr - 2r^2) +$$

$$+64R^3r + 48R^2r^2 + 12Rr^3 \Leftrightarrow 5Rs^2 \overset{(4)}{\le} 24R^3 + 18R^2r + 3Rr^2$$

Now, LHS of (4) $\overset{Gerretsen}{\le} 5R(4R^2 + 4Rr + 3r^2) \overset{?}{\le} 24R^3 + 18R^2r + 3Rr^2$

$\Leftrightarrow 2R^2 - Rr - 6r^2 \overset{?}{\le} 0 \Leftrightarrow (R - 2r)(2R + 3r) \overset{?}{\ge} 0 \to$ *true* $\because R \overset{Euler}{\ge} 2r$ *(proved)*

PROBLEM 2.160-*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum a \ge \sum \frac{1}{a} \Leftrightarrow \frac{abc(\sum a)}{\sum ab} \overset{(1)}{\ge} 1$$

Now, $3(a^3b + b^3c + c^3a) = 3abc\left(\frac{a^3}{ac} + \frac{b^3}{ab} + \frac{c^3}{bc}\right) \ge$

$$\overset{Holder}{\ge} 3abc\frac{(\sum a)^3}{3(\sum ab)} = \frac{abc(\sum a)}{\sum ab}\cdot\left(\sum a\right)^2 \overset{by\,(1)}{\ge} \left(\sum a\right)^2$$

PROBLEM 2.161-*Solution by Ravi Prakash-New Delhi-India*

$$(1-a+a^2)(1-b+b^2)-(1-ab+a^2b^2)=1-a+a^2-b+ab-a^2b+b^2-$$
$$-ab^2+a^2b^2-[1-ab+a^2b^2]$$
$$=(a+b)^2-(a+b)-ab(a+b)=(a+b)[a+b-ab-1]=$$
$$=-(a+b)(1-a)(1-b)\le 0$$

Equality when $a=b=0$ *or* $a=1$ *or* $b=1$

$$\therefore (1-a+a^2)(1-b+b^2)\le 1-ab+a^2b^2$$

Equality when $a=b=0$ *or* $a=1$ *or* $b=1$.

$$\Rightarrow (1-a+a^2)(1-b+b^2)(1-c+c^2)\le(1-ab+a^2b^2)(1-c+c^2)\le$$

$\le 1-abc+a^2b^2c^2$. *Equality when* $a=b=c=0$ *or when at least two of* a,b,c *are equal to* 1. *Next,* $(1-a+a^2)^2(1-b+b^2)^2(1-c+c^2)^2=$

$$=[(1-a+a^2)(1-b+b^2)][(1-b+b^2)(1-c+c^2)][(1-c+c^2)(1-a+a^2)]$$
$$=[(1-ab+a^2b^2)-(a+b)(1-a)(1-b)]$$
$$[(1-bc+b^2c^2)-(b+c)(1-b)(1-c)]$$
$$[(1-ca+c^2+a^2)-(c+a)(1-c)(1-a)]$$
$$\le(1-ab+a^2b^2)(1-bc+b^2c^2)(1-ca+c^2a^2)$$

with equality if $(a+b)(1-a)(1-b)=0$

$$(b+c)(1-b)(1-c)=0;\ (c+a)(1-a)(1-c)=0$$

$\Leftrightarrow a=b=0,c=0$ *or* $a=b=0,c=1$ *or* $a=1,b=c=0$ *or* $a=0,b=1,c=0$ *or* $a=1,b=1$ *or* $a=1,c=1$ *or* $b=1,c=1$

PROBLEM 2.162-*Solution by Soumava Chakraborty-Kolkata-India*

Firstly, $\sum w_a^2\overset{(1)}{\le}\sum s(s-a)=s^2$. *Now, LHS* $\overset{Radon}{\ge}\frac{\left(x\sum a^2+y\sum m_a^2\right)^{m+1}}{\left(z\sum w_a^2+t\sum h_a^2\right)^m}=\frac{\left(x\sum a^2+\frac{3}{4}y\sum a^2\right)^{m+1}}{\left(z\sum w_a^2+t\sum h_a^2\right)^m}\ge$

$\ge\frac{\left(\sum a^2\right)^{m+1}(4x+3y)^{m+1}}{4^{m+1}(z+t)^m\left(\sum w_a^2\right)^m}$ $(\because h_a\le w_a$ *etc*, $\Rightarrow h\sum h_a^2\le\sum w_a^2)\ge\frac{\left(\sum a^2\right)^{m+1}(4x+3y)^{m+1}}{(4s^2)^m\cdot 4(z+t)^m}$ *(using (1))*

$$\ge\frac{\left(\sum a^2\right)^{m+1}(4x+3y)^{m+1}}{\left(3\sum a^2\right)^m\cdot 4(z+t)^m}\quad\left(\because 4s^2=\left(\sum a\right)^2\le 3\sum a^2\right)$$

$$=\frac{\left(\sum a^2\right)(4x+3y)^{m+1}}{3^m\cdot 4(z+t)^m}\overset{\substack{Ionescu\\Weitzenbock}}{\ge}\frac{4\sqrt{3}S(4x+3y)^{m+1}}{3^m\cdot 4(z+t)^m}=\frac{(4x+3y)^{m+1}}{3^m(z+t)^m}\sqrt{3}S\quad \textit{(Proved)}$$

PROBLEM 2.163-*Solution by Soumava Chakraborty-Kolkata-India*

Firstly, $\sum w_a^2\overset{(1)}{\le}\sum s(s-a)=s^2$. *Now, LHS* $\overset{Radon}{\ge}\frac{\left(x\sum a^2+y\sum a^2\right)^{m+1}}{\left(z\sum w_a^2+t\sum w_a^2\right)^m}=\frac{(x+y)^{m+1}\left(\sum a^2\right)^{m+1}}{(z+t)^t\left(\sum w_a^2\right)^m}$

$$\overset{by\ (1)}{\ge}\frac{(x+y)^{m+1}\left(\sum a^2\right)^{m+1}}{(z+t)^m(s^2)^m}\ge\frac{(x+y)^{m+1}\left(\sum a^2\right)^{m+1}}{(z+t)^m\left(\frac{3}{4}\sum a^2\right)^m}\left(\because s^2\le\frac{3\sum a^2}{4}\right)$$

$$=\frac{4^m(x+y)^{m+1}\left(\sum a^2\right)^{m+1}}{(z+t)^m3^m\cdot\left(\sum a^2\right)^m}=\frac{4^m(x+y)^{m+1}\sum a^2}{(z+t)^m\cdot 3^m}$$

$$\overset{\substack{Ionesecu-\\Wetizenbock}}{\ge}\frac{4^m(x+y)^{m+1}\cdot 4\sqrt{3}}{(z+t)^m\cdot 3^m}=\frac{4^{m+1}(x+y)^{m+1}3^{\frac{1}{2}}S}{3^m(z+t)^m}=\frac{4^{m+1}(x+y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}S\ \textit{(Proved)}$$

PROBLEM 2.164-*Solution by Soumava Chakraborty-Kolkata-India*

$$a^2+b^2-ab=\frac{1}{4}(a+b)^2+\frac{3}{4}(a-b)^2\geq\frac{1}{4}(a+b)^2\Rightarrow\sqrt{a^2+b^2-ab}\overset{(1)}{\geq}\frac{a+b}{2}$$

Similarly, $\sqrt{b^2+c^2-bc}\overset{(2)}{\geq}\frac{b+c}{2}$ & $\sqrt{c^2+a^2-ca}\overset{(3)}{\geq}\frac{c+a}{2}$

(1), (2), (3) $\Rightarrow LHS\geq\frac{\sum(a+b)^2}{2}\overset{?}{\geq}2\sum ab\Leftrightarrow 2\sum a^2+2\sum ab\overset{?}{\geq}4\sum ab\Leftrightarrow\sum a^2\overset{?}{\geq}\sum ab$

$\rightarrow$ *true (Proved)*

PROBLEM 2.165-*Solution by Michael Sterghiou-Greece*

If $a,b,c\geq 0$ *then:*

$\sum_{cyc}(a+b)\sqrt{a^2+b^2}\geq\left(2\sqrt{3}-1\right)\sum_{cyc}ab$ *(1)*

$$\sum_{cyc}(a+b)\sqrt{a^2+b^2}\overset{AM-GM}{\geq}\sum_{cyc}2\sqrt{ab}\cdot\sqrt{2ab}=\sum_{cyc}2\sqrt{2}ab=2\sqrt{2}\cdot\sum_{cyc}ab\geq\left(2\sqrt{3}-1\right)\sum_{cyc}ab$$

$$\left.\begin{matrix}\sqrt{2}>1\\ \sqrt{3}>1\end{matrix}\right\}\rightarrow\sqrt{2}+\sqrt{3}>2\rightarrow 1-\frac{2}{\sqrt{2}+\sqrt{3}}>0\rightarrow 1-2\cdot\frac{\left(\sqrt{3}\right)^2\left(\sqrt{2}\right)^2}{\sqrt{3}+\sqrt{2}}>0\rightarrow 1\cdot\left(\sqrt{3}-\sqrt{2}\right)>0$$

$\rightarrow 2\sqrt{2}-\left(2\sqrt{3}-1\right)>0.$ *Done.*

UNDERGRADUATE PROBLEMS-SOLUTIONS

PROBLEM 3.001-*Solution by Tran Quang Minh – Nguyen Thi Linh – Ho Chi Minh – Vietnam*

We denote $f(\alpha)=\frac{\alpha-2}{5}\int_2^7 \arctan^5 x\cdot dx-\int_2^\alpha \arctan^5 x\cdot dx$ *with* $\alpha\in[2,7]$, *we have:*

$f''(\alpha)=-\frac{5\arctan^4\alpha}{\alpha^2+1}<0$ *for all* $\alpha\in[2,7]$, *so for all* $\alpha\in[2,7]$ *we have inequality:*

$$f(\alpha)\ge \min\{f(2),f(7)\}=0.\ Or$$

$$\frac{\alpha-2}{5}\int_2^7 \arctan^5 x\cdot dx\ge\int_2^\alpha \arctan^5 x\cdot dx\ \text{ for all } \alpha\in[2,7].$$

PROBLEM 3.002-*Solution by Tran Quang Minh – Nguyen Thi Linh – Ho Chi Minh – Vietnam*

We have one lemma.

Lemma 1. If $x,y,z\in(0,+\infty)$ *then:*

$$x^5+y^5+z^5+x^3yz+xy^3z+xyz^3\ge x^4(y+z)+y^4(z+x)+z^4(x+y)\quad (1)$$

Proof. We normalize $x+y+z=1$ *and denote* $xy+yz+zx=q, xyz=r$ *then:*

$$(1)\quad\Leftrightarrow(-12q+7)r+8q^2-6q+1\ge 0$$

Use $r\ge\max\left\{0,\frac{4q-1}{9}\right\}$ *we will have* $(-12q+7)r+8q^2-6q+1\ge 0$

Back to the problem:

From Lemma, denote $x=t^a, y=t^b, z=t^c$, *we have:*

$$\sum t^{5a}+\sum t^{3a+b+c}\ge\sum\left(t^{4a+b}+t^{4a+c}\right)$$

$$\text{or } \sum t^{5a-1}+\sum t^{3a+b+c-1}\ge\sum\left(t^{4a+b-1}+t^{4a+c-1}\right)$$

Take integral from 0 *to* 1 *we have:*

$$\int_0^1\sum t^{5a-1}\,dt+\int_0^1\sum t^{3a+b+c-1}\,dt\ge\int_0^1\sum\left(t^{4a+b-1}+t^{4a+c-1}\right)dt$$

$$Or\ \sum\frac{1}{5a}+\sum\frac{1}{3a+b+c}\ge\sum\left(\frac{1}{4a+b}+\frac{1}{4a+c}\right)$$

PROBLEM 3.003-*Solution by proposer*

$$If\ X,Y\in M_2(C)\ and\ f:C\to C, f(t)=\det(X+tY)=t^2\det Y+at+\det X;\ a\in C$$

$$f(1)+f(-1)=2(\det X+\det Y)\Rightarrow\det(X+Y)+\det(X-Y)=2(\det X+\det Y)$$

$$Let\ be\ X=A^2+B^2, Y=AB+BA\Rightarrow A^2+B^2+AB+BA=(A+B)^2$$

$$A^2+B^2+AB+BA=(A+B)^2$$

$$(\det(A+B))^2+(\det(A-B))^2=2\det(A^2+B^2)+2\det(AB+BA)\Rightarrow$$

$$\frac{1}{2}((\det(A+B))^2+(\det(A-B))^2-2\det(AB+BA))=\det(A^2+B^2)=$$

$$=\det(A+iB)\,(A-iB)=\det(A+iB)\det(A-iB)=\det(A+iB)\,\overline{\det(A+\imath B)}=$$

$$=(\alpha+i\beta)(\alpha-i\beta)=\alpha^2+\beta^2\ge 0\quad(\alpha,\beta\in R)$$

PROBLEM 3.004-*Solution by Tran Quang Minh – Nguyen Thi Linh – Ho Chi Minh – Vietnam*

If $b \in [a, c]$ *we will have:*

$$\begin{cases} f(x) \le g(x) = \dfrac{f(b) - f(a)}{b - a}(x - a) + f(a), x \in [a, b] \\ f(x) \le h(x) = \dfrac{f(c) - f(b)}{c - b}(x - b) + f(b), x \in [b, c] \end{cases}$$

then

$$2\int_a^c f(x)\,dx \le 2\left[\int_a^b g(x)\,dx + \int_b^c h(x)\,dx\right] = (b - a)[f(b) + f(a)] + (c - b)[f(c) + f(b)]$$

PROBLEM 3.005-*Solution by proposer*

Let be $g(x) = \frac{y}{\ln y}, y \in [2,3], g'(y) = \frac{\ln y - 1}{\ln^2 y} \Rightarrow$

y	2	e	3
$g'(x)$	$- - - - -$	0	$- - - - -$
$g(y)$	$\frac{2}{\ln 2}$ ↘	e	↗ $\frac{3}{\ln 3}$

But $9 > 8 \Rightarrow 3^2 > 2^3 \Rightarrow 2\ln 3 > 3\ln 2 \Rightarrow \frac{2}{\ln 2} > \frac{3}{\ln 3}$

$$\Rightarrow Im(g) = \left[e, \frac{2}{\ln 2}\right] \Rightarrow e \le g(y) \le \frac{2}{\ln 2} \Rightarrow e \le \frac{y}{\ln y} \le \frac{2}{\ln 2} \Rightarrow$$

$\Rightarrow e\ln y \le y \le \frac{2}{\ln 2}\ln y$. *In these we take* $y = \left(1 + \frac{1}{x}\right)^x, x \ge 1 \Rightarrow$

$$ex\ln\left(1 + \frac{1}{x}\right) \le \left(1 + \frac{1}{x}\right)^x \le \frac{2x}{\ln 2}\ln\left(1 + \frac{1}{x}\right)$$

PROBLEM 3.006-*Solution by proposer*

We have: $2a + 2b - 2c \le \frac{a^2+b^2}{c} \Leftrightarrow (a - c)^2 + (b - c)^2 \ge 0$ *therefore*

$$\begin{cases} (2a + 2b - 2c)\ln x \le \frac{a^2+b^2}{c}\ln x \\ (2b + 2c - 2a)\ln y \le \frac{b^2+c^2}{a}\ln y \\ (2c + 2a - 2b)\ln z \le \frac{c^2+a^2}{b}\ln z \end{cases}$$

. After addition we obtain:

$$\sum (2a + 2b - 2c)\ln x = \sum 2a(\ln x - \ln y + \ln z) = \sum \ln\left(\frac{xy}{y}\right)^{2a} \le$$

$$\le \sum \frac{a^2 + b^2}{c}\ln x = \sum \ln x^{\frac{a^2+b^2}{c}}$$