PROBLEM 3.007-*Solution by George – Florin Șerban – Romania*

$$1=\lim_{n\to\infty}\frac{n}{\sqrt[n^2]{c_n}}$$

$$\ln 1=\lim_{n\to\infty}\sqrt[n^2]{\frac{n^{n^2}}{c_n}}=\lim_{n\to\infty}\frac{\ln\frac{n^{n^2}}{c_n}}{n^2}=\lim_{n\to\infty}\frac{\ln\frac{(n+1)^{(n+1)^2}}{c_{n+1}}-\ln\frac{n^{n^2}}{c_n}}{(n+1)^2-n^2}=$$

$$=\lim_{n\to\infty}\frac{\ln\frac{(n+1)^{(n+1)^2}c_n}{c_{n+1}n^{n^2}}}{2n+1},\qquad (Cesaro\ Stolz)$$

$$\ln l=\lim_{n\to\infty}\frac{\ln\frac{(n+1)^{(n+1)^2}c_n}{c_{n+1}n^{n^2}}}{2n+1}=\lim_{n\to\infty}\frac{\ln\frac{(n+2)^{(n+2)^2}c_{n+1}}{c_{n+2}(n+1)^{(n+1)^2}}-\ln\frac{(n+1)^{(n+1)^2}c_n}{c_{n+1}n^{n^2}}}{(2n+3)-(2n+1)}=$$

$$=\lim_{n\to\infty}\frac{\ln\frac{n^{n^2}(n+2)^{(n+2)^2}c_{n+1}^2}{(n+1)^{2(n+1)^2}c_nc_{n+2}}}{2},$$

$$\frac{c_{n+1}^2}{c_nc_{n+2}}=\frac{b_{n+1}^4}{b_{n+1}^2b_{n+2}^2}=\frac{b_{n+1}^2}{b_{n+2}^2}=\frac{1}{a_{n+2}^2}=\frac{1}{[a+(n+1)r]^2},$$

$$\frac{n^{n^2}(n+2)^{(n+2)^2}c_{n+1}^2}{(n+1)^{2(n+1)^2}c_nc_{n+2}}=\frac{n^{n^2}(n+2)^{n^2}(n+2)^{4n}(n+2)^4}{(n+1)^{2n^2}(n+1)^{4n}(n+1)^2[a+(n+1)r]^2}=$$

$$=\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n^2}\left(\frac{n+2}{n+1}\right)^{4n+2}\left(\frac{n+2}{a+(a+1)r}\right)^2,$$

$$\lim_{n\to\infty}\left(\frac{n^2+2n}{n^2+2n+1}\right)^2=\lim_{n\to\infty}\left\{\left(1+\frac{-1}{n^2+2n+1}\right)^{\frac{n^2+2n+1}{-1}}\right\}^{\frac{-n^2}{n^2+2n+1}}=e^{-1},$$

$$\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{4n+2}=\lim_{n\to\infty}\left\{\left(1+\frac{1}{n+1}\right)^{\frac{n+1}{1}}\right\}^{\frac{4n+2}{n+1}}=e^4,$$

$$\lim_{n\to\infty}\left(\frac{n+2}{a+(n+1)r}\right)^2=\left(\frac{1}{r}\right)^2=r^{-2},$$

$$\ln l=\lim_{n\to\infty}\frac{\ln\frac{n^{n^2}(n+2)^{(n+2)^2}c_{n+1}^2}{(n+1)^{2(n+1)^2}c_nc_{n+2}}}{2}=\frac{\ln e^{-1}\cdot e^4\cdot r^{-2}}{2}=\ln\sqrt{e^3\cdot r^{-2}},$$

$$l=\sqrt{e^3\cdot r^{-2}}=\frac{e\sqrt{e}}{r},\ \lim_{n\to\infty}\frac{n}{\sqrt[n^2]{c_n}}=\frac{e\sqrt{e}}{r}$$

PROBLEM 3.008-*Solution by Yen Tung Chung – Tainan – Taiwan*

$$\int\frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1}dx$$

$$=\int \frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x+1)(1-\sin x)e^x+(\cos x+1)(1-\sin x)}dx$$

$$=\int \frac{(2e^x+1)\sin x-\cos x-1}{(e^x+\cos x+1)(e^x-\sin x+1)}dx=\left(\int \frac{e^x-\sin x}{e^x+\cos x+1}-\frac{e^x-\cos x}{e^x-\sin x+1}\right)dx$$

$$=\ln|e^x+\cos x+1|-\ln|e^x-\sin x+1|+C=\ln\left|\frac{e^x+\cos x+1}{e^x-\sin x+1}\right|+C$$

PROBLEM 3.009-*Solution by Soumitra Mukherjee-Chandar Nagore-India*

For $n=3; \left(\frac{3!}{2}\right)^{2e}<e^6 \Leftrightarrow 3^{2e}<e^6 \Leftrightarrow 3^e<e^3$*, which is true,*

For $n=4; \left(\frac{4!}{2}\right)^{2e}<e^{14} \Leftrightarrow 12^{2e}<e^{14} \Leftrightarrow 12^e<e^7$*, which is also true,*

Let us assume that the statement is true for $n=k; \left(\frac{k!}{2}\right)^{2e}\le e^{k^2+k-6}$ *holds true.*

Now, $\left\{\frac{(k+1)!}{2}\right\}^{2e}=\left(\frac{k\times k!}{2}\right)^{2e}=\left(\frac{k}{2}\right)^{2e}\left(\frac{k!}{2}\right)^{2e}\le\left(\frac{k}{2}\right)^{2e}\cdot e^{k^2+k-6}$

we need to prove, $\left(\frac{k}{2}\right)^{2e}\cdot e^{k^2+k-6}\le e^{(k+1)^2+(k+1)-6}$

$$\Leftrightarrow \left(\frac{k}{2}\right)^{2e}\le e^{(k+1)^2-k+1}=e^2(k+1)\Leftrightarrow\left(\frac{k}{2}\right)^{e}\le e^{k+1} \qquad (1)$$

We need to prove statement (1);

Let $f(x)=e^{x+1}-\left(\frac{x}{2}\right)^e \ \forall\, x\ge 3$

$$f'(x)=e^{x+1}-\frac{e}{2}\left(\frac{x}{2}\right)^{e-1}\ge 0\ \forall\, x\ge 3.$$

f *increasing- on* $[3,\infty)$ *and* $f'(x)\ge 0\ \forall x\ge 3, f(x)\ge f(3)>0$

$$e^{x+1}>\frac{e}{2}\left(\frac{x}{2}\right)^{e-1}\quad \forall x\ge 3$$

hence, statement (1) is prove $\left\{\frac{(k+1)!}{2}\right\}^{2e}\le e^{(k+1)^2+(k+1)-6}$ *(proved).*

When $n=k$ *is true then* $n=k+1$ *is also true. So, by theory of Induction, we have,*

$\left(\frac{n!}{2}\right)^{2e}\le e^{n^2+n-6}$ *(proved).*

PROBLEM 3.010-*Solution by Henry Ricardo-New York -USA*

Let $u=\frac{e^x}{(1+e^x)\ln(1+e^x)}$*. Then*

$$du=\frac{e^x\ln(1+e^x)-e^{2x}}{(1+e^x)^2\ln^2(1+e^x)}dx,$$

so that we have

$$\int\frac{e^x\ln(1+e^x)-e^{2x}}{(1+e^x)^2\ln^2(1+e^x)}dx=\int 1\,du=u+C=\frac{e^x}{(1+e^x)\ln(1+e^x)}+C$$

[Since the denominator of the original integrand is $[(1+e^x)\ln(1+e^x)]^2$*, this suggested a possible antiderivative of the form* $f(x)/[(1+e^x)\ln(1+e^x)]$*.*
A little calculation indicated that $f(x)=e^x$*.]*

PROBLEM 3.011- ***Solution by proposer***

$$\textit{We have: } \frac{a}{3a^2+2b^2+c^2} \le \frac{1}{18}\left(\frac{2}{b}+\frac{1}{c}\right) \Leftrightarrow$$
$$3a^2b+6a^2c+2b^3+2c^3+4b^2c+bc^2 \ge$$
$$\ge 18\sqrt[18]{(a^2b)^3(a^2c)^6(b^3)^2(c^3)^2(b^2c)^4(bc^2)} = 18abc \textit{ therefore}$$
$$\begin{cases} \dfrac{a\ln x}{3a^2+2b^2+c^2} \le \dfrac{1}{18}\left(\dfrac{2}{b}+\dfrac{1}{c}\right)\ln x \\ \dfrac{b\ln y}{3b^2+2c^2+a^2} \le \dfrac{1}{18}\left(\dfrac{2}{c}+\dfrac{1}{a}\right)\ln y \\ \dfrac{c\,\mathrm{n}\,z}{3c^2+2a^2+b^2} \le \dfrac{1}{18}\left(\dfrac{2}{a}+\dfrac{1}{b}\right)\ln z \end{cases}$$
After addition we have:
$$\sum \ln x^{\frac{a}{3a^2+2b^2-c^2}} = \sum \frac{a\ln x}{3a^2+2b^2+c^2} \le \sum \frac{1}{18}\left(\frac{2}{b}+\frac{1}{c}\right)\ln x =$$
$$= \frac{1}{18}\sum \frac{2\ln z+\ln y}{a} = \sum \ln\left(\frac{c^2}{y}\right)^{\frac{1}{18a}}$$

PROBLEM 3.012- ***Solution by proposer***

$$f(x) = e^x + 2\sinh x + \sin x - \cos x + 2006$$
$$f'(x)+f(x) = 2e^x + 2(\sinh x + \cosh x) + 2\sin x + 2006 = 4e^x + 2\sin x + 2006 =$$
$$= 2(2e^x + \sin x + 1003) \textit{ so, } \int \frac{2e^x+\sin x+1003}{e^x+2\sinh x+\sin x} = \frac{1}{2}\int \frac{f'(x)+f(x)}{f(x)}dx =$$
$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{f'(x)}{f(x)}dx = \frac{x}{2} + \ln(e^x + 2\sinh x + \sin x - \cos x + 2006) + C$$

PROBLEM 3.013- ***Solution by Nicolae Papacu – Slobozia – Romania***

$$\textit{We have } t = 1 - xy = 1 - x(1-x) = 1 - x + x^2 \textit{ and}$$
$$1 - yx = 1 - (1-x)x = 1 - x + x^2 = t.$$
$$\textit{Because } x^{2016} = x, \textit{ we have } x^{2017} = x^2 \textit{ and then}$$
$$t = 1 - x + x^2 = 1 - x + x^{2017} = 1 - x(1 - x^{2016}) = 1 - (1 - x^{2016})x.$$
Because
$$1 - x^{2016} = (1 - x^6)\sum_{k=0}^{372} x^{6k}$$
$$\textit{and } 1 - x^6 = (1+x^3)(1-x^3) = (1 - x + x^2)(1+x)(1-x^3), \textit{ we have}$$
$$1 - x^{2016} = (1-x^6)\sum_{k=0}^{372} x^{6k} = (1-x+x^2)(1+x)(1-x^3)\sum_{k=0}^{372} x^{6k} = (1-x+x^2)z$$
and then
$$t = 1 - x + x^2 = 1 - x + x^{2017} = 1 - (1 - x^{2016})x = 1 - (1 - x + x^2)zx\,,$$
$$\textit{so } t = 1 - tzx, \textit{ wherefrom}$$
$$t(1+zx) = 1. \textit{ Analog } (1+zx)t = 1, \textit{ so } t = 1 - xy = 1 - yx \textit{ is invertible.}$$

PROBLEM 3.014-*Solution by Ravi Prakash-New Delhi-India*

$$S=\lim_{p\to\infty}\sum_{n=1}^{p}\sum_{m=1}^{p}\frac{1}{mn(m+n)}$$
$$=\frac{1}{1\cdot1\cdot2}+\frac{1}{1\cdot2\cdot3}+\frac{1}{1\cdot2\cdot4}+\dots+$$
$$+\frac{1}{2\cdot1\cdot3}+\frac{1}{2\cdot2\cdot4}+\frac{1}{2\cdot3\cdot5}+\dots+$$
$$+\frac{1}{3\cdot1\cdot4}+\frac{1}{3\cdot2\cdot5}+\frac{1}{3\cdot3\cdot6}+\dots$$

Let's sum up this double series diagonally:

$$S=\sum_{k=2}^{\infty}\cdot\sum_{m=1}^{k-1}\frac{1}{m(k-m)k}=\sum_{k=2}^{\infty}\frac{1}{k^2}\sum_{m=1}^{k-1}\left(\frac{1}{m}+\frac{1}{k-m}\right)=$$
$$=2\sum_{k=2}^{\infty}\frac{1}{k^2}\left(1+\frac{1}{2}+\dots+\frac{1}{k-1}\right)<2\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\left(1+\frac{1}{2}+\dots+\frac{1}{k}\right)$$
$$=2\left[\left(1-\frac{1}{2}\right)(1)+\left(\frac{1}{2}-\frac{1}{3}\right)\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)\left(1+\frac{1}{2}+\frac{1}{3}\right)+\dots\right]$$
$$=2\left[\frac{1}{1^2}+\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{2}\right)+\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}-1-\frac{1}{2}\right)+\dots\right]$$
$$=2\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right]<2\left(\frac{\pi^2}{6}\right)<\frac{\pi^3}{6}$$

PROBLEM 3.015-*Solution by Dana Heuberger – Romania*

We denote with (1) and (2) the equalities from the hypothesis.
For $a=b=1$, we obtain that $2=0$, so the ring that the characteristic 2.
So $\forall\,\alpha\in A,\alpha=-\alpha.$ *(3)*
For $a=1,b=x\in A$, from (2) we obtain:

$$(1+x)^{2k+1}\cdot(1+x)^2=1+x^{2k+2}\overset{(1),(3)}{\Leftrightarrow}(1+x^{2k})\cdot(1+x^2)=1+x^{2k+2}\Leftrightarrow$$
$$\Leftrightarrow x^{2k}+x^2=0\overset{(3)}{\Leftrightarrow}x^{2k}=x^2,\text{ so }\forall x\in A,x^{2k+1}=x^3.$$

Replacing x with $x+1$ in the preceding equality and using (1), we deduce:

$$\forall x\in A,(1+x)^3=(1+x)^{2k+1}=1+x^{2k}=1\overset{(3)}{\Leftrightarrow}1+x+x^2+x^3=$$
$$=1+x^2\overset{(3)}{\Leftrightarrow}x^3=x.$$

Replacing x with $x+1$ in the preceding equality and using (3), it follows:

$$\forall x\in A,(1+x)^3=1+x\Leftrightarrow\forall x\in A,1+x+x^2+x^3=1+x\Leftrightarrow$$
$$\Leftrightarrow\forall x\in A,x^2=x.$$

So, the ring is boolean, hence is commutative.

PROBLEM 3.016-*Solution by Quang Minh Tran - Ho Chi Minh City – VietNam*

$$\text{For all } \theta\in\left[\frac{\pi}{4},\frac{\pi}{3}\right], n\in\mathbb{N} \text{ we have:}$$

$$\left|\left\{\ln\left[\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\right]\right\}\right|\le|\cos\theta|\cdot\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)\le$$

$$\le\max_{x\in\left[\frac{\pi}{3},\frac{\pi}{4}\right]}\ln(1+\sin\theta\sec^2\theta)=M_1$$

$$\left|\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\right|\le\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\le(1+\cos\theta)^{\cot\theta}\le$$

$$\le\max_{x\in\left[\frac{\pi}{3},\frac{\pi}{4}\right]}(1+\cos\theta)^{\cot\theta}=M_2$$

$$\left|\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\cdot\sec^2\theta}\right|\le\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\cdot\sec^2\theta}\le(1+\cot\theta)^{\sin\theta\cdot\sec^2\theta}\le$$

$$\le\max_{x\in\left[\frac{\pi}{3},\frac{\pi}{4}\right]}(1+\cot\theta)^{\sin\theta\cdot\sec^2\theta}=M_3$$

$$\text{So for all } \theta\in\left[\frac{\pi}{4},\frac{\pi}{3}\right], n\in\mathbb{N} \text{ exists } M>0 \text{ such that:}$$

$$\left|\left\{\ln\left[\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\right]\right\}\cdot\left[\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\right]\cdot\left[\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\cdot\sec^2\theta}\right]\right|\le M$$

$$\text{For all } \theta\in\left[\frac{\pi}{4},\frac{\pi}{3}\right] \text{ we have:}$$

$$\lim_{n\to\infty}\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta}=0$$

Use Lebesgue dominated convergence theorem we have:

$$\lim_{n\to\infty}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta}d\theta=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}0\cdot d\theta=0$$

PROBLEM 3.017-*Solution by Ravi Prakash-New Delhi-India*

$$a_{n+1}=\frac{1}{2}a_n+\frac{n^2-2n-1}{n^2(n+1)^2}, \forall n\ge 1$$

$$=\frac{1}{2}a_n+\frac{2}{(n+1)^2}-\frac{1}{n^2}\Rightarrow a_{+1}-\frac{2}{(n+1)^2}=\frac{1}{2}\left(a_n-\frac{2}{n^2}\right)\ \forall n\ge 1$$

$$\text{Let } b_n=a_n-\frac{2}{n^2}$$

$$b_1=1-2=-1 \text{ and } b_{n+1}=\frac{1}{2}b_n\ \forall n\ge 1\Rightarrow b_n=\left(\frac{1}{2}\right)^{n-1}b_1=-\left(\frac{1}{2}\right)^{n-1}$$

$$\therefore a_n=\frac{1}{2n^2}-\left(\frac{1}{2}\right)^{n-1}\quad\forall n\ge 1$$

$$a_1=1, a_2=0, a_3=\frac{2}{a}-\frac{1}{4}=-\frac{1}{36}; a_4=\frac{2}{16}-\frac{1}{8}=0; a_5=\frac{2}{25}-\frac{1}{32}>0$$

As $2^n>n^2\ \forall n\ge 5$, *we get* $a_n>0\ \forall n\ge 5$. *Let* $s_n=a_1+a_2+\cdots+a_n$, *then*

$s_1 = a_1 = 1, s_2 = 1, s_3 = \frac{35}{36}, s_4 = \frac{35}{36}, s_5 = s_4 + a_5 > s_4$. *In fact* $s_{n+1} > s_n > s_5 > s_4$ $\forall n \geq 5$. *Thus,* s_n *is minimum when* $n = 3$ *or* 4.

PROBLEM 3.018-*Solution by proposer*

$$\text{Let } A = \begin{pmatrix} a & bc & 1 \\ 1 & cb & a \end{pmatrix}, A^T = \begin{pmatrix} a & 1 \\ b & c \\ c & b \\ 1 & a \end{pmatrix}, \text{ then } A \cdot A^T = \begin{pmatrix} 1+\sum_{cyc} a^2 & 2(a+bc) \\ 2(a+bc) & 1+\sum_{cyc} a^2 \end{pmatrix}$$

Applying Cauchy – Binet $\det(A \cdot A^T) \geq 0 \Rightarrow 1 + \sum_{cyc} a^2 \geq 2(a+bc)$

putting $b = \sqrt{\cos x}, c = \frac{1}{x}$ *and* $a = e^{\frac{\pi}{12}}$, *then* $\left(e^{\frac{\pi}{12}} - 1\right)^2 + \cos x + \frac{1}{x^2} \geq 2\frac{\sqrt{\cos x}}{x}$, *then integrating both sides*

$$\frac{\pi}{12}\left(e^{\frac{\pi}{12}} - 1\right)^2 + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{x^2} \geq 2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos x}}{x} dx$$

$$\Rightarrow \frac{\pi}{24}\left(e^{\frac{\pi}{12}} - 1\right)^2 + \frac{\sqrt{2}-1}{4} + \frac{152}{\pi^3} \geq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos x}}{x} dx$$

PROBLEM 3.019-*Solution by proposer*

$$p = 2, E(n,k,2) = 2\sum_{i=1}^{[\sqrt[k]{n}]} i^k - \sum_{i=1}^{[\sqrt[n]{2n}]} i^k - [\sqrt[k]{n}] + (2n+1)\left([\sqrt[k]{2n} - [\sqrt[k]{n}]]\right)$$

$$\text{and } \lim_{n\to\infty} \frac{[\sqrt[k]{n}]}{\sqrt[k]{n}} \Rightarrow \lim_{n\to\infty} \frac{[\sqrt[k]{2n}]}{\sqrt[k]{2n}} = \sqrt[k]{2}$$

$$\lim_{n\to\infty} \frac{1}{n\sqrt[k]{n}} \sum_{i=1}^{[\sqrt[n]{2n}]} i^k = \frac{2\sqrt[k]{2}}{k+1} \text{ and finally } \lim_{n\to\infty} \frac{E(n,k,2)}{n\sqrt[k]{n}} = \frac{2k}{k+1}\left(\sqrt[k]{2} - 1\right)$$

PROBLEM 3.020-*Solution by proposer*

$$\left(\sum_{k=1}^{n} \frac{1}{k(k+1)} x_k\right)^2 = \left(\sum_{k=1}^{n} \sqrt{\frac{1}{k(k+1)}} \sqrt{\frac{1}{k(k+1)}} x_k\right)^2 \leq$$

$$\leq \sum_{k=1}^{n} \left(\sqrt{\frac{1}{k(k+1)}}\right)^2 \sum_{k=1}^{n} \left(\sqrt{\frac{1}{k(k+1)}} x_k\right)^2 =$$

$$= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \sum_{k=1}^{n} \frac{x_k^2}{k(k+1)} = \frac{n}{n+1} \sum_{k=1}^{n} \frac{x_k^2}{k(k+1)}$$

PROBLEM 3.021-*Solution by Ravi Prakash - New Delhi – India*

$$For\ 0 < x < 1, 1 - x^2 + x^{2015} - x^{2016} = (1 - x)[1 + x + x^{2015}] > 0$$
$$Also,\ x^2 - x^{2015} + x^{2016} = x^2 + x^{2016} - x^{2015} \geq 2(x^{1009}) - x^{2015}$$
$$= x^{1009} + x^{1009} - x^{2015} > 0 \Rightarrow 1 - x^2 + x^{2015} - x^{2016} < 1$$
$$0 < 1 - x^2 + x^{2015} - x^{2016} < 1 \quad (1)$$
$$Also,\ x^{2015} - x^{2016} = x^{2015}(1 - x) > 0\ for\ 0 < x < 1$$
$$\Rightarrow 1 - x^2 + x^{2015} - x^{2016} > 1 - x^2, \quad (2) \quad 0 < x < 1. From\ (1)\ and\ (2)\ for\ 0 < x < 1$$
$$1 - x^2 < 1 - x^2 + x^{2015} - x^{2016} < 1 \Rightarrow 1 < \frac{1}{\sqrt{1 - x^2 + x^{2015} - x^{2016}}} < \frac{1}{\sqrt{1 - x^2}}$$
$$1 < \int_0^1 \frac{dx}{\sqrt{1 - x^2 + x^{2015} - x^{2016}}} < \frac{\pi}{2}$$

PROBLEM 3.022-*Solution by Soumava Chakraborty – Kolkata – India*

$$(r^2 + r_a r_b)(r^2 + r_b r_c)(r^2 + r_c r_a) = r^6 + r^4\left(\sum r_a r_b\right) + r^2 r_a r_b r_c \left(\sum r_a\right) + (r_a r_b r_c)^2$$
$$= r^6 + r^4 s^2 + r^2\left(\frac{S^2}{r}\right)(4R + r) + \left(\frac{S^2}{r}\right)^2$$
$$\left(\sum r_a r_b = s^2; r_a r_b r_c = \frac{S^2}{r}; \sum r_a = 4R + r\right) = r^6 + r^4 s^2 + r^3 s^2(4R + r) + r^2 s^4$$
$$(r^2 + r_a r_b)(r^2 + r_b r_c)(r^2 + r_c r_a) \geq \frac{1000}{27} r^2 S^2$$
$$\Leftrightarrow r^6 + r^4 s^2 + r^4 s^2 + 4Rr^3 s^2 + r^2 s^4 \geq \frac{1000}{27} r^4 s^2$$
$$\Leftrightarrow 27r^6 - 946r^4 s^2 + 27r^2 s^4 + 108Rr^3 s^2 \geq 0$$
$$\Leftrightarrow 27r^4 - 946r^2 s^2 + 27s^4 + 108Rrs^2 \geq 0 \quad (1)$$
$$Now,\ 27r^4 - 946r^2 s^2 + 27s^4 + 108rs^2 \geq 27r^4 - 946r^2 s^2 + 27s^4 + 216r^2 s^2$$
$$(R \geq 2r)$$
$$= 27r^4 - 730r^2 s^2 + 27s^4 = (27r^2 - s^2)(r^2 - 27s^2) = (s^2 - 27r^2)(27s^2 - r^2) \geq 0$$
$$\left(s^2 \overset{Gerretsen}{\geq} 16Rr - 5r^2 \overset{Euler}{\geq} 32r^2 - 5r^2 = 27r^2\right). (1)\ is\ proved\ (Done).$$

PROBLEM 3.023-*Solution by proposer*

$$We\ have: \frac{(y+z)(z+x)}{4yz} \geq \frac{x+z}{y+z} \Leftrightarrow (y - z)^2 \geq 0 \Rightarrow$$
$$\begin{cases} \left(\frac{(y+z)(z+x)}{4yz}\right)^a \geq \left(\frac{x+z}{y+z}\right)^a \\ \left(\frac{(z+x)(x+y)}{4zx}\right)^b \geq \left(\frac{y+x}{z+x}\right)^b \\ \left(\frac{(x+y)(y+z)}{4xy}\right)^c \geq \left(\frac{z+y}{x+y}\right)^c \end{cases}$$

After multiplication we obtain: $\prod\left(\frac{x+y}{2x}\right)^{b+c} \geq \prod(x+y)^{b-c}$

PROBLEM 3.024-*Solution by Hamza Mahmood-Lahore-Pakistan*

Compute $\sum_{n=2}^{\infty} \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right)$, *where* F_n *is the* nth *Fibonacci number with* $F_1 = 1$

The n^{th} *Fibonacci number can be expressed as:*

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \Rightarrow F_{2n} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right\}$$

$$\Rightarrow F_{2n} = \frac{1}{\sqrt{5}}\left\{\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right\}$$

$$\text{Let } a = \frac{3+\sqrt{5}}{2} \Rightarrow \frac{1}{a} = \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}$$

$$\Rightarrow F_{2n} = \frac{1}{\sqrt{5}}\left\{a^n - \left(\frac{1}{a}\right)^n\right\} = \frac{a^{2n}-1}{\sqrt{5}a^n} \Rightarrow \frac{F_{2n}+1}{F_{2n}-1} = \frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1}$$

$$\text{Now } \operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), x < 1$$

$$\text{Since } F_{2n} > 1 \text{ for } n \geq 2 \Rightarrow \frac{1}{F_{2n}} < 1 \Rightarrow \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right) = \frac{1}{2}\ln\left(\frac{F_{2n}+1}{F_{2n}-1}\right) = \frac{1}{2}\ln\left(\frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1}\right)$$

$$\text{Now } a + \left(-\frac{1}{a}\right) = a - \frac{1}{a} = \frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2} = \sqrt{5} \;\&\; a\left(-\frac{1}{a}\right) = -1$$

$$\text{So, } \frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1} = \frac{a^{2n}+\left(a+\left(-\frac{1}{a}\right)\right)a^n+(a)\left(-\frac{1}{a}\right)}{a^{2n}-\left(a+\left(-\frac{1}{a}\right)\right)a^n+(a)\left(-\frac{1}{a}\right)} = \frac{a^{2n}-\left(\frac{1}{a}+(-a)\right)a^n+\left(\frac{1}{a}\right)(-a)}{a^{2n}-\left(a+\left(-\frac{1}{a}\right)\right)a^n+(a)\left(-\frac{1}{a}\right)}$$

$$\Rightarrow \frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1} = \frac{\left(a^n-\frac{1}{a}\right)(a^n+a)}{(a^n-a)\left(a^n+\frac{1}{a}\right)}$$

$$x^2-(p+q)x+pq = (x-p)(x-q)$$

$$\Rightarrow \frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1} = \frac{\left(a^n-\frac{1}{a}\right)a(a^{n-1}+1)}{a(a^{n-1}-1)\left(a^n+\frac{1}{a}\right)} = \frac{(a^{n+1}-1)(a^{n-1}+1)}{(a^{n-1}-1)(a^{n+1}+1)}$$

$$\ln\left\{\frac{(a^{n+1}-1)(a^{n-1}+1)}{(a^{n-1}-1)(a^{n+1}+1)}\right\} = \ln(a^{n+1}-1) - \ln(a^{n-1}-1) + \ln(a^{n-1}+1) - \ln(a^{n+1}+1)$$

Now

$$\sum_{n=2}^{\infty} \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right) = \frac{1}{2}\sum_{n=2}^{\infty} \ln\left\{\frac{(a^{n+1}-1)(a^{n-1}+1)}{(a^{n-1}-1)(a^{n+1}+1)}\right\}$$

$$\sum_{n=2}^{\infty} \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right) = \frac{1}{2}[\{\ln(a^3-1)+\ln(a^4-1)+\ln(a^5-1)+\ln(a^6-1)+\cdots\}$$

$$-\{\ln(a-1)+\ln(a^2-1)+\ln(a^3-1)+\ln(a^4-1)+\cdots\}$$

$$+\{\ln(a+1)+\ln(a^2+1)+\ln(a^3+1)+\ln(a^4+1)+\ln(a^5+1)+\cdots\}$$

$$-\{\ln(a^3+1)+\ln(a^4+1)+\ln(a^5+1)+\cdots\}]$$

$$\sum_{n=2}^{\infty} \text{arctanh}\left(\frac{1}{F_{2n}}\right) = \frac{1}{2}\{-\ln(a-1) - \ln(a^2-1) + \ln(a+1) + \ln(a^2+1)\} =$$

$$= \frac{1}{2}\ln\left\{\frac{(a+1)(a^2+1)}{(a-1)(a^2-1)}\right\}$$

Since $a = \frac{3+\sqrt{5}}{2} \Rightarrow a+1 = \frac{5+\sqrt{5}}{2}, a-1 = \frac{1+\sqrt{5}}{2}, a^2 = \frac{7+3\sqrt{5}}{2}, a^2+1 = \frac{9+3\sqrt{5}}{2},$

$$a^2 - 1 = \frac{5+3\sqrt{5}}{2}$$

$$\frac{(a+1)(a^2+1)}{(a-1)(a^2-1)} = \frac{(5+\sqrt{5})(9+3\sqrt{5})}{(1+\sqrt{5})(5+3\sqrt{5})} = \frac{45+15\sqrt{5}+9\sqrt{5}+15}{5+3\sqrt{5}+5\sqrt{5}+15} = \frac{60+24\sqrt{5}}{20+8\sqrt{5}} = 3$$

$$\sum_{n=2}^{\infty} \text{arctanh}\left(\frac{1}{F_{2n}}\right) = \frac{1}{2}\ln 3 = \ln\sqrt{3}$$

PROBLEM 3.025-*Solution by Soumitra Moukherjee - Chandar Nagore – India*

$x_n = \sqrt[3n]{n!} \Rightarrow \ln x_n = \frac{1}{3n}\ln n!$. *Using Stirling's formula,*

$$\ln n! = \ln\sqrt{2\pi} + \left(n + \frac{1}{2}\right)\ln n - n + O\left(\frac{1}{n}\right)$$

$$\ln x_n = \frac{1}{3n}\ln n! = \frac{1}{3n}\ln\sqrt{2\pi} + \frac{1}{3}\left(1 + \frac{1}{2n}\right)\ln n - \frac{1}{3} + \frac{1}{3}O\left(\frac{1}{n^2}\right)$$

$$\ln x_{n+1} = \frac{1}{3(n+1)}\ln\sqrt{2\pi} + \frac{1}{3}\left(1 + \frac{1}{2(n+1)}\right)\ln(n+1) - \frac{1}{3} + \frac{1}{3}O\left(\frac{1}{(n+1)^2}\right)$$

$$\ln x_{n+1} - \ln x_n = -\frac{\ln 2\pi}{6n(n+1)} + \frac{1}{3}\ln\left(1 + \frac{1}{n}\right) + \frac{1}{6(n+1)}\ln(n+1) - \frac{1}{6n}\ln n + O\left(\frac{1}{n^2}\right)$$

Using Lagrange's Mean Value Theorem:

$\ln x_{n+1} - \ln x_n = (x_{n+1} - x_n)\frac{1}{c_n}$ *where* $c_n \in (x_n, x_{n+1})$

Also, $\lim_{n\to\infty} c_n = \frac{1}{e}$; $\sqrt[3]{n^2}(\ln x_{n+1} - \ln x_n)$

$= -\frac{\sqrt[3]{n^2}\ln 2\pi}{6n(n+1)} + \frac{\sqrt[3]{n^2}}{3}\ln\left(1+\frac{1}{n}\right) + \frac{\sqrt[3]{n^2}}{6(n+1)}\ln(n+1) - \frac{\sqrt[3]{n^2}}{6n}\ln + \sqrt[3]{n^2}\,O\left(\frac{1}{n^2}\right)$. *Now,*

$$\lim_{n\to\infty} \sqrt[3]{n^2}\,(\ln x_{n+1} - \ln x_n) = 0$$

$$\lim_{n\to\infty} \sqrt[3]{n^2}\,(x_{n+1} - x_n) = \frac{1}{e}\lim_{n\to\infty} \sqrt[3]{n^2}\,(\ln a_{n+1} - \ln x_n) = 0$$

$$\lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3n+3]{(n+1)!} - \sqrt[3n]{n!}\right) = 0$$

PROBLEM 3.026-*Solution by George – Florin Șerban – Romania*

Proposition: Let be the sequence:

$$(a_n)_{n\geq 1}, a_n > 0, \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1, \lim_{n\to\infty}\frac{a_n}{n} = a \in (0,\infty), \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n = b,$$

then $\lim_{n\to\infty}(a_{n+1} - a_n) = a\ln b$

Solution:

$$\left(\frac{a_{n+1}}{a_n}\right)^n = \left[\left(1 + \left(\frac{a_{n+1} - a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right]^{\frac{n}{a_n}(a_{n+1}-a_n)}$$

$$\ln\left(\frac{a_{n+1}}{n}\right)^n = \ln\left[1 + \left(\frac{a_{n+1} - a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right]^{\frac{n}{a_n}(a_{n+1}-a_n)},$$

$$\frac{a_n}{n}\ln\left(\frac{a_{n+1}}{a_n}\right)^n = (a_{n+1} - a_n)\ln\left[1 + \left(\frac{a_{n+1} - a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right],$$

$$\lim_{n\to\infty}\frac{a_{n+1} - a_n}{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} - \lim_{n\to\infty}\frac{a_n}{a_n} = 1 - 1 = 0,$$

$$Then\ \lim_{n\to\infty}\ln\left[\left(1 + \left(\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right] = \ln e = 1,\ then$$

$$\lim_{n\to\infty}\frac{a_n}{n}\ln\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty}(a_{n+1} - a_n)\ln\left[\left(1 + \left(\frac{a_{n+1} - a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right],$$

$$\lim_{n\to\infty}\frac{a_n}{n}\ln\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty}(a_{n+1} - a_n) = a(\ln b).$$

$$l = \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[2n+2]{(2n+1)!!}} - \frac{n}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}} =$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[2n+2]{(2n+1)!!}}\cdot\frac{\sqrt{n}}{\sqrt{n+1}} - \frac{n\sqrt{n}}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}}\cdot\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}},$$

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = 1,$$

$$l = \lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[2n+2]{(2n+1)!!}} - \frac{n\sqrt{n}}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}}\cdot\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = \lim_{n\to\infty}(a_{n+1} - a_n)\cdot\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}},$$

$$a_n = \frac{n\sqrt{n}}{\sqrt[2n]{(2n-1)!!}},\ \lim_{n\to\infty}\frac{a_n}{n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[2n]{(2n-1)!!}} = \sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}} =$$

$$= \sqrt{\lim_{n\to\infty}\frac{(n+1)^n(n+1)}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}},$$

$$\lim_{n\to\infty}\frac{a_n}{n} = \sqrt{\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n\cdot\frac{n+1}{2n+1}} = \sqrt{\frac{e}{2}},$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{n+1}\cdot\frac{n}{a_n}\cdot\frac{n+1}{n} = \sqrt{\frac{e}{2}}\cdot\sqrt{\frac{2}{e}}\cdot 1 = 1,$$

$$b = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[2n+2]{(2n+1)!!}}\cdot\frac{\sqrt[2n]{(2n-1)!!}}{n\sqrt{n}}\right)^n =$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[2n+2]{(2n+1)!!}}\right)^n, \lim_{n\to\infty}\frac{(n+1)\sqrt{n+1}}{n\sqrt{n}} = 1,$$

$$b = \lim_{n\to\infty}\left(\frac{\sqrt[2n]{(2n-1)!!}}{\sqrt{2n+1}}\right)^{\frac{n}{n+1}} = \sqrt{\lim_{n\to\infty}\left(\sqrt[n]{\frac{(2n-1)!!}{(2n+1)^n}}\right)^{\frac{n}{n+1}}} =$$

$$= \sqrt{\lim_{n\to\infty}\frac{(2n+1)!!}{(2n+3)^n(2n+3)}\cdot\frac{(2n+1)^n}{(2n-1)!!}},$$

$$b = \sqrt{\lim_{n\to\infty}\left(\frac{2n+1}{2n+3}\right)^n\cdot\frac{2n+1}{2n+3}} = \sqrt{\lim_{n\to\infty}\left(1+\frac{-2}{2n+3}\right)^{\frac{2n+3}{-2}\cdot\frac{-2n}{2n+3}}} = \sqrt{e^{-1}} = e^{-\frac{1}{2}},$$

$$\lim_{n\to\infty}\frac{2n+1}{2n+3} = 1,$$

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a\cdot\ln b = \sqrt{\frac{e}{2}}\cdot\ln e^{-\frac{1}{2}} = \frac{-1}{2}\sqrt{\frac{e}{2}},$$

$$l = \lim_{n\to\infty}(a_{n+1}-a_n)\cdot\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = \frac{-1}{2}\sqrt{\frac{e}{2}}\cdot 0 = 0.$$

PROBLEM 3.027-*Solution by Feti Sinani-Kosovo*

$$\Gamma(x)\sim\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)} \Rightarrow \Gamma(x) = \left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)} +$$

$$+o\left(\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}\right), x\to\infty$$

$$(x+1)\Gamma(x+2)^{-\frac{1}{2x+2}} = (x+1)e^{-\frac{\ln\Gamma(x+2)}{2x+2}} = (x+1)e^{-\frac{\ln\left(\left(\frac{x+1}{e}\right)^{x+1}\sqrt{2\pi(x+1)}(1+o(1))\right)}{2x+2}} =$$

$$= (x+1)\frac{\sqrt{e}}{\sqrt{x+1}}\left(1-\frac{\ln\sqrt{2\pi}}{x+1}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right)\left(1-\frac{\ln(x+1)}{4(x+1)}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right) =$$

$$= \sqrt{e}\sqrt{x}\left(1+\frac{1}{2x}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right)\left(1+o\left(\frac{1}{x^{\frac{1}{2}}}\right)\right) = \sqrt{e}\sqrt{x}+o(1)\quad (x\to\infty)$$

$$x\Gamma(x+1)^{-\frac{1}{2x}} = x\frac{\sqrt{e}}{\sqrt{x}}\left(1-\frac{\ln\sqrt{2\pi}}{x}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right)\left(1-\frac{\ln x}{4x}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right) = \sqrt{e}\sqrt{x}+o(1)$$

$$\therefore \lim_{x\to\infty}\left(\frac{x+1}{\Gamma(x+2)^{\frac{1}{2x+2}}}-\frac{x}{\Gamma(x+1)^{\frac{1}{2x}}}\right)^{\sqrt{x}} = \lim_{x\to\infty}e^{\sqrt{x}\ln(o(1)_+)} = e^{-\infty} = 0$$

PROBLEM 3.028-*Solution by Marian Ursărescu – Romania*

Let $g:[x,x+1]\to\mathbb{R}\ g(t)=t^{\frac{(m+1)(t+1)}{t}}$. *From Lagrange theorem we have:* $\exists x\in(x,x+1)$.

Such that $\frac{g(x+1)-g(x)}{x+1-x}=g'(c)\Rightarrow(x+1)^{\frac{(m+1)(x+2)}{x+1}}-x^{\frac{(m+1)(x+1)}{x}}=$

$$=c^{\frac{(m+1)(c+1)}{c}}\left[(m+1)\left(-\frac{1}{t^2}\right)\ln c+(m+1)\frac{c+1}{c}\cdot\frac{1}{c}\right]$$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x)}{x^m}=\lim_{x\to\infty}\frac{(m+1)c^{\frac{(m+1)(c+1)}{c}}\left[-\frac{\ln c}{c^2}+\frac{c+1}{c^2}\right]}{x^m}\quad (1)$$

Because $c\in(x,x+1)$ *and* $x\to\infty$ *we calculate this.*

$$\lim_{x\to\infty}\frac{(m+1)x^{(m+1)\left(1+\frac{1}{x}\right)}\left[-\frac{\ln x}{x^2}+\frac{x+1}{x^2}\right]}{x^m}=(m+1)\lim_{x\to\infty}x^{\frac{m+1}{x}}\cdot x\left[-\frac{\ln x}{x^2}+\frac{x+1}{x^2}\right]=$$

$$=(m+1)\lim_{x\to\infty}x^{\frac{m+1}{x}}\left[-\frac{\ln x}{x}+\frac{x+1}{x}\right]\quad (2)$$

$$\lim_{x\to\infty}x^{\frac{m+1}{x}}=e^{\lim_{x\to\infty}\frac{m+1}{x}\cdot\ln x}\overset{L'H}{=}e^{(m+1)\lim_{x\to\infty}\frac{1}{x}}=e^0=1\quad (3)$$

$$\lim_{x\to\infty}-\frac{\ln x}{x}\overset{L'H}{=}\lim_{x\to\infty}-\frac{1}{x}=0\quad (4)$$

$\lim_{x\to\infty}\frac{x+1}{x}=1$ *(5). From (1)+(2)+(3)+(4)+(5)* $\Rightarrow\lim_{x\to\infty}\frac{f(x)}{x^m}=m+1$.

PROBLEM 3.029-*Solution by Anas Adlany - Khemis Des Zemamra – Morroco*

Put $x+y=s$ *and* $p=xy$ *then* $s^2\ge 4p\Leftrightarrow\frac{s}{1+s^2}<\frac{s}{1+4p}$;

So it suffices to show that $\frac{2}{\pi}\arctan(x+y)\arctan\left(\frac{1}{x+y}\right)<\frac{s}{1+s^2}$

or $\arctan(s)-\frac{2}{\pi}(\arctan(s))^2<\frac{s}{1+s^2}$

But we know that $\begin{cases}\arctan(s)<s\\ \arctan(s)>\frac{s}{1+s^2}\end{cases}$

$\Rightarrow\arctan(s)-\frac{2}{\pi}(\arctan(s))^2<s-\frac{2}{\pi}\left(\frac{s}{1+s^2}\right)^2<\frac{s}{1+s^2}<\frac{s}{1+4p}$ *as desired.*

PROBLEM 3.030-*Solution by proposer*

First we show that for all $x,y,z>0$ *and* $m\in\mathbb{N}^*$ *we have*

$$x^{m+1}(y+z)+y^{m+1}(z+x)+z^{m+1}(x+y)\ge 2xyz(x^{m-1}+y^{m-1}+z^{m-1})$$

Proof.

$\frac{x^m}{y}+\frac{y^m}{x}\ge x^{m-1}+y^{n-1}(x,y>0)$. *If* $t=\frac{y}{x}>0\Rightarrow t^{m+1}+1\ge t^m+t\Leftrightarrow$

$$\Leftrightarrow(t-1)^2(t^{m-1}+t^{m-2}+\cdots+t+1)\ge 0$$

$$\sum\left(\frac{x^m}{y}+\frac{y^m}{x}\right)\ge\sum(x^{m-1}+y^{m-1})\Rightarrow\sum x^{m+1}(y+z)\ge 2xyz\sum x^{m-1}$$

If $k=k;y=k+1,z=k+2$ *then*

$$\sum_{k=1}^{n}\frac{k^{m-1}+(k+1)^{m-1}+(k+2)^{m-1}}{(2k+3)k^{m+1}+2(k+1)^{m+2}+(2k+1)(k+2)^{n+1}}=$$

$$=\sum_{k=1}^{n}\frac{k^{m-1}+(k+1)^{m-1}+(k+2)^{m-1}}{(k+1+k+2)k^{n+1}+(k+k+2)(k+1)^{n+1}+(k+k+1)(k+2)^{n+1}}\le$$

$$\le\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k}-\frac{1}{k+1}+\frac{1}{2(k+2)}\right)=\frac{1}{8}-\frac{1}{4(n+1)(n+2)}$$

PROBLEM 3.031-*Solution by Soumitra Mandal – Kolkata – India*

$AB=BA; AC=CA$ *and* $A^2B+C=ABC$

Now $A^2B+C=ABC\Rightarrow(A^2B+C)\cdot B=ABCB\Rightarrow A^2B^2+CB=ABCB$

$\Rightarrow A(AB)B+CB=ABCB\Rightarrow(AB)^2+CB=ABCB\ [\because AB=BA]$

$\Rightarrow CB=AB(CB-AB)$ *… (1)*

Again, $A^2B+C=ABC\Rightarrow B\cdot(A^2B+C)=BABC\Rightarrow BA^2B+BC=BABC$

$\Rightarrow(BA)AB+BC=BABC\Rightarrow(BA)^2+BC=BABC\ [\because AB=BA]$

$\Rightarrow BC=BA(BC-BA)\Rightarrow BC=AB(BC-AB)[\because AB=BA]$ *… (2)*

So, from (2) – (1): $BC-CB=AB(BC-CB)$

$\det(BC-CB)=\det(AB)\det(BC-CB)\Rightarrow\det(BC-CB)(1-\det(AB))=0$

Now, now $\det(AB)\neq 1$ *since if* $\det(AB)=1$ *the* $AB=I_n\Rightarrow B=A^{-1}$

so from relation $A^2B+C=ABC$ *we would have got* $A=O_n$ *but* $\det(A)\neq 0$*, hence a contradiction. So,* $\det(AB)=1$ *is neglected.*

$\therefore\det(BC-CB)=0\Rightarrow BC=CB$ *(proved)*

PROBLEM 3.032-*Solution by Marian Ursărescu – Romania*

Theorem: If $M,N\in M_n(\mathbb{C})$ *such that* $MN=I_n\Rightarrow NM=I_n$*, then* $MN=NM$

$x^2A+B=xAB\Rightarrow xAB-x^2-B=O_n\Rightarrow xA(B-xI_n)-B+xI_n=xI_n$

$\Rightarrow(xA-I_n)(B-xI_n)=xI_4\Rightarrow(B-xI_n)(xA-I_n)=xI_n\Rightarrow$

$\Rightarrow xBA-B-x^2A+xI_n=I_n\Rightarrow xBA=x^2A+B\Rightarrow AB=BA$ *and similarly* $BC=CB$ *and* $AC=CA$*. We must show this:*

$(x^2y^2z+2x^2y+x)A+(xy^2z^2+2y^2z+y)B+(x^2yz^2+2z^2x+z)C=3xyzABC$ *(1)*

$xAB=x^2A+B\Rightarrow xyzABC=x^2yzAC+yzBC=x^2y(z^2C+A)+z(y^2B+C)=$

$=x^2yz^2C+x^2yA+y^2zB+zC$ *(2)*

$yBC=y^2B+C\Rightarrow xyzABC=xy^2zAB+xzAC=$

$=y^2z(x^2A+B)+x(z^2C+A)=x^2y^2zA+y^2zB+xz^2C+xA$ *(3)*

$zCA=z^2C+A\Rightarrow xyz=xyz^2BC+xyAB=xz^2(y^2B+C)+y(x^2A+B)=$

$=xy^2z^2B+xz^2C+x^2yA+yB$ *(4)*

From (2)+(3)+(4)⇒ $3xyzABC=(x^2y^2z+2x^2y+x)A+(xy^2z^2+2y^2z+y)B+$
$+(x^2yz^2+2z^2x+z)C\Rightarrow$ *(1) its true.*

PROBLEM 3.033-*Solution by proposer*

$$\text{By the AM-GM inequality} \Rightarrow \begin{cases} \sqrt[3]{1\cdot\frac{2a}{a+b}\cdot\frac{3a}{a+b+c}} \le \frac{1+\frac{2a}{a+b}+\frac{3a}{a+b+c}}{3} \\ \sqrt[3]{1\cdot 1\cdot\frac{3b}{a+b+c}} \le \frac{1+1+\frac{3b}{a+b+c}}{3} \\ \sqrt[3]{1\cdot\frac{2b}{a+b}\cdot\frac{3c}{a+b+c}} \le \frac{1+\frac{2b}{a+b}+\frac{3c}{a+b+c}}{3} \end{cases}$$

After addition holds $a+\sqrt[3]{ab\frac{a+b}{2}}+\sqrt[3]{abc} \le \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}}$ *but*

$a+\sqrt{ab}+\sqrt[3]{abc} \le a+\sqrt[3]{ab\frac{a+b}{2}}+\sqrt[3]{abc}$ *therefore for* $a,b,c>0$ *holds:*

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \le \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}} \quad (1)$$

In (1) we take $c=a$ *and* $c=b$ *then:* $\frac{a+\sqrt{ab}+\sqrt[3]{a^2b}}{3} \le \sqrt[3]{a\frac{a+b}{2}\cdot\frac{2a+b}{3}}$ *and*

$\frac{a+\sqrt{ab}+\sqrt[3]{ab^2}}{3} \le \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+2b}{3}}$ *and after addition holds the desired inequality.*

PROBLEM 3.034-*Solution by SK Rejuan West Bengal – India*

Case I: If three at a,b,c *are equal ie* $a=b=c\in\mathbb{N}^*$

$\frac{a+1}{b}=\frac{b+1}{c}=\frac{c+1}{a}=\frac{a+1}{a}$*, which belongs to* $\mathbb{N}$ *[given]*

Now, $\frac{a+1}{a}\in\mathbb{N}$ *if* $a=1\Rightarrow a=1=b=c$

$\therefore (1,1,1)=(a,b,c)$ *is a solution*

Case II: If two of them are equal. Let $a=b(\ne c)$

$\frac{a+1}{b},\frac{b+1}{c},\frac{c+1}{a}\in\mathbb{N}$ *[given]. Now,* $\frac{a+1}{b}=\frac{a+1}{a}$*, it belongs to* $\mathbb{N}$ *if* $a=1\Rightarrow a=1=b$

From $\frac{b+1}{c}\in\mathbb{N}$ *we get,* $\frac{1+1}{c}\in\mathbb{N}$ $[\therefore b=1]\Rightarrow\frac{2}{c}\in\mathbb{N}\Rightarrow c=1$ *or* 2

but $c\ne(a=b)\Rightarrow c\ne 1\Rightarrow c=2$ *ie* $(a,b,c)=(1,1,2)$ *is a solution,*

similarly, by taking $a=c(\ne b)$ *we get* $(a,b,c)=(1,2,1)$ *is a solution*

and by taking $a\ne(b=c)$ *we get* $(a,b,c)=(2,1,1)$ *is a solution*

Case III: If three of them inequal, so in this case we get six possibilities ie

$a<b<c$ *or* $a<c<b$ *or* $b<c<a$ *or* $b<a<c$ *or* $c<a<b$ *or* $c<b<a$

Subcase I: When $a<b<c\Rightarrow(a+1)<b+1<(c+1)$ *(1)*

From (1) we get, $\frac{a+1}{b}<\frac{b+1}{b}=1+\frac{1}{b}$ $[\because b\in\mathbb{N}^*]$

$\because\frac{a+1}{b}\in\mathbb{N}$ *and* $\frac{a+1}{b}<1+\frac{1}{b}\Rightarrow\frac{a+1}{b}=1\Rightarrow a+1=b$

$$\left[\because b\in N^*\therefore b\ge 1\Rightarrow\frac{1}{b}\le 1\Rightarrow 1+\frac{1}{b}\le 2\Rightarrow\frac{a+1}{b}<2 \text{ and } \frac{a+1}{b}\in\mathbb{N}\right]$$

Subcase II: If $a<c<b\Rightarrow a+1<c+1<b+1$*. It is given* $\frac{a+1}{b}\in\mathbb{N}\Rightarrow b|(a+1)$

Also by own assumption $a<b$ *and by given condition* $b|(a+1)$

$\Rightarrow a$ *and* b *must be consecutive number in* $\mathbb{N}^*$ *and* $a<b$

$\because$ *a,b are consecutive numbers in $\mathbb{N}^*$ and $a<b$ so there exists no number c betwen a and b which also belongs to $\mathbb{N}^*$ ie for $a<c<b$ and*

$\frac{a+1}{b},\frac{b+1}{c},\frac{c+1}{b}\in\mathbb{N}$ *we get no solutions*

$\therefore$ *No solutions for the case $a<c<b$ and $\frac{a+1}{b},\frac{b+1}{c},\frac{c+1}{a}\in\mathbb{N}$.*

Subcase III: If $b<c<a$. In this case, by similar calculation at subcase I we get

$(a,b,c)=(3,1,2);\ (a,b,c)=(5,3,4)$

Subcase IV: If $b<a<c$

In this case, by similar calculation at subcase II we get $\exists$ no solution.

Subcase V: If $c<a<b$, in this case by similar calculation at subcase I we get,

$(a,b,c)=(2,3,1);\ (a,b,c)=(4,5,3)$

Subcase VI:

If $c<b<a$, in this case by similar calculation at subcase I we get, $\exists$ no solutions.

Similarly from (1) we get, $\frac{b+1}{c}<\frac{c+1}{c}=1+\frac{1}{c}$

$$\because \frac{b+1}{c}\in\mathbb{N} \text{ and } \frac{b+1}{c}<1+\frac{1}{c}\Rightarrow\frac{b+1}{c}=1\Rightarrow b+1=c\Rightarrow a+1+1=c\Rightarrow c=a+2$$

[by similar asignment]

Now, $\frac{c+1}{a}\in\mathbb{N}$ *[given* $\Rightarrow\frac{a+2+1}{a}\in\mathbb{N}\Rightarrow\frac{a+3}{a}\in\mathbb{N}\Rightarrow 1+\frac{3}{a}\in\mathbb{N}$

which is possible if $a=1$ or 3

$$\left.\begin{matrix} if\ a=1\Rightarrow b=2\Rightarrow c=3\\ if\ a=3\Rightarrow b=4\Rightarrow c=5\end{matrix}\right\}\begin{matrix}(a,b,c)=(1,2,3)\\(a,b,c)=(3,4,5)\end{matrix} \text{ is also solution}$$

Therefore the solutions are:

$$(a,b,c)\in\{(1,1,1),(1,1,2),(1,2,1),(2,1,1),(1,2,3),(3,4,5),(3,1,2),(5,3,4),(2,3,1),(4,5,3)\}$$

PROBLEM 3.035-*Solution by Soumitra Mandal – Kolkata – India*

Theorem: Let $(t,a)\in\mathbb{R}_+^*\times\mathbb{R}_+^*$. If $(a_n)_{n\ge1}\in S(\mathbb{R}_+^*)$ is a $B-(t+1,a)$ sequence, then $\left(\sqrt[n]{a_n}\right)_{n\ge1}$ is a $L-\left(t,a(t+1),e^{-(t+1)}\right)$ sequence.

$$\Omega=\lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!}-\sqrt[3n]{(2n-1)!!}\right)\sqrt[3]{n^2}=$$

$$=\left\{\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)\right\}\left(\lim_{n\to\infty}\sqrt[3]{n^2}\right) \text{ where } a_n=\sqrt[3]{(2n-1)!!}$$

for all $n\ge1$

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\left(\lim_{n\to\infty}\sqrt[3]{\frac{(2n+1)!!}{n\cdot(2n-1)!!}}\right)=$$

$$=\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\left(\lim_{n\to\infty}\sqrt[3]{\frac{1}{n}\cdot\frac{(2n+1)!}{2^n\cdot n!}\cdot\frac{2^{n-1}\cdot(n-1)!}{(2n-1)!}}\right)=$$

$$=\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\left(\lim_{n\to\infty}\sqrt[3]{2-\frac{1}{n}}\right)=\sqrt[3]{2}\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)$$

Hence, $(a_n)_{n\ge1}$ is a $B-\left(1,\sqrt[3]{2}\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\right)$ sequence so by the above theorem

$$\left(\sqrt[n]{a_n}\right)_{n\geq 1} \text{ is a } L-\left(0, \sqrt[3]{2}\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right)\cdot 1\cdot e^{-1}\right) \text{ sequence}$$

$$\Omega = \frac{\sqrt[3]{2}}{3}\left(\lim_{n\to\infty}\sqrt[3]{n^2}\right)\left(\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\right) = \frac{\sqrt[3]{2}}{e} \text{ (Ans:)}$$

PROBLEM 3.036-*Solution by Soumitra Mandal – Kolkata – India*

Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. If $< a_n >_{n\geq 1}\in S(\mathbb{R}_+^*)$ is a $B-(t+1,a)$ sequence then $<\sqrt[n]{a_n}>_{n\geq 1}$ is a $L-\left(t, a(t+1), e^{-(t+1)}\right)$ sequence.

$$\lim_{n\to\infty}(a_{n+1}-a_n) = r\in\mathbb{R}_+^*, u,v\in\mathbb{R} \text{ and } u+v=1$$

$$a_n! = a_1a_2\dots a_n \text{ and } G_n = \sqrt[n]{a_n!} \text{ where } n\in\mathbb{N}^*$$

$$\Omega = \lim_{n\to\infty}\left((n+1)^u\sqrt[n+1]{(G_{n+1})^v} - n^u\sqrt[n]{(G_n!)^v}\right)$$

$$= \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)^{u(n+1)}(G_{n+1})^v} - \sqrt[n]{n^{nu}(G_n!)^v}\right)$$

$$\text{Let } H_n = n^{nu}(G_n!)^v \text{ where } n\geq 1 \text{ and } u+v=1$$

$$\lim_{n\to\infty}\frac{H_{n+1}}{n\cdot H_n} = \left(\lim_{n\to\infty}\frac{1}{n^{1-u}}\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nu}\right)\left(\lim_{n\to\infty}\frac{G_{n+1}!}{G_n!}\right)^v$$

$$= e^u\left(\lim_{n\to\infty}\frac{G_{n+1}}{n}\right)^v \text{ since, } G_{n+1}! = G_{n+1}G_n!$$

$$= e^u\left\{\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\sqrt[n+1]{a_{n+1}!}\right)\right\}^v = e^u\left\{\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{a_{n+2}!}{a_{n+1}!}\right)\right\}^v$$

[Cauchy D – Alembert's Theorem]

$$= e^u\left(\lim_{n\to\infty}\frac{a_{n+2}}{n}\right)^v = e^u\left(\lim_{n\to\infty}\frac{a_{n+3}-a_{n+2}}{n+1-n}\right)^v = e^ur^v$$

Hence, $< H_n >_{n\geq 1}$ is a $B-(1,e^ur^v)$ sequence. By the above theorem it yields that $<\sqrt[n]{H_n}>_{n\geq 1}$ is a $L-(0,e^ur^v\cdot 1\cdot e^{-1})$ sequence i.e. $L-(0,e^{u-1}r^v)$ sequence.

$$\Omega = e^{u-1}r^v \text{ (Ans :)}$$

PROBLEM 3.037-*Solution by Soumitra Mandal – Kolkata – India*

Theorem: Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. If $< a_n >_{n\geq 1}\in S(\mathbb{R}_+^*)$ is a $B-(t+1,a)$ sequence then $<\sqrt[n]{a_n}>_{n\geq 1}$ is a $L-\left(t,a(t+1),e^{-(t+1)}\right)$ sequence.

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a\in\mathbb{R}_+^* \text{ and } \lim_{n\to\infty}(b_{n+1}-b_n) = b\in\mathbb{R}_+^* \text{ and } u+v=1$$

$$\Omega = \lim_{n\to\infty}\left(a_{n+1}^u\sqrt[n+1]{\left(\prod_{k=1}^{n+1}b_k\right)^v} - a_n^u\sqrt[n]{\left(\prod_{k=1}^{n}b_k\right)^v}\right)$$

$$\text{Let } H_n = a_n^{nu}\left(\prod_{k=1}^n b_k\right)^v \text{ for all } n\geq 1 \text{ and } u+v=1$$

$$\lim_{n\to\infty}\frac{H_{n+1}}{n^{u+1}\cdot H_n} = \left(\lim_{n\to\infty}\frac{1}{n^{2u+v}}\right)\left(\lim_{n\to\infty}a_{n+1}^u\right)\left(\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^{nu}\right)\left(\lim_{n\to\infty}b_{n+1}^v\right)$$

$$= \left(\lim_{n\to\infty}\frac{1}{n^{2u+v}}\right)\left(\lim_{n\to\infty}(a_{n+1}a_n)^u\right)\left(\lim_{n\to\infty}b_{n+1}^v\right)$$

Applying Cauchy – D'Alembert's theorem

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \left(\lim_{n\to\infty}\frac{a_{n+1}}{n}\right)^u\left(\lim_{n\to\infty}\frac{a_n}{n}\right)^u\left(\lim_{n\to\infty}\frac{b_{n+1}}{n}\right)^v$$

$$= \left(\lim_{n\to\infty} \frac{a_{n+2}-a_{n+1}}{n+1-n}\right)^u \left(\lim_{n\to\infty} \frac{a_{n+1}-a_n}{n+1-n}\right)^u \left(\lim_{n\to\infty} \frac{b_{n+2}-b_{n+1}}{n+1-n}\right) = a^{2u}b^v$$

Hence $< H_n >_{n\ge1}$ *is* $B-(u+2, a^{2u}b^v)$ *sequence. Hence by the above theorem it yields* $< \sqrt[n]{H_n} >_{n\ge1}$ *as a* $L-\left(u+1, a^{2u}b^v(u+2)e^{-(u+2)}\right)$ *sequence or* $L-\left(u+1, a^{2u}b^v(3u+2v)\cdot e^{-(3u+2v)}\right)$ *sequence* $\because u+v=1$

$$\therefore \Omega = \frac{a^{2u}b^v(3u+2v)}{e^{3u+2v}} \quad \text{(Ans :)}$$

PROBLEM 3.038-*Solution by Soumitra Mandal – Kolkata – India*

Theorem: Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. *If* $< a_n >_{n\ge1}\in S(\mathbb{R}_+^*)$ *is a* $B-(t+1,a)$ *sequence then* $< \sqrt[n]{a_n} >_{n\ge1}$ *is a* $L-\left(t, a(t+1), e^{-(t+1)}\right)$ *sequence.*

$$\lim_{n\to\infty}(a_{n+1}-a_n) = r \in \mathbb{R}_+^*$$

$a_n! = a_1a_2\ldots a_n$ *and* $G_n = \sqrt[n]{a_n!}$ *for all* $n\in\mathbb{N}^*$

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{G_{n+1}!}} - \frac{n^2}{\sqrt[n]{G_n}}\right) = \lim_{n\to\infty}\left(\sqrt[n+1]{\frac{(n+1)^{2(n+1)}}{G_{n+1}!}} - \sqrt[n]{\frac{n^{2n}}{G_n!}}\right)$$

Let $H_n = \frac{n^{2n}}{G_n!}$ *for all* $n \ge 1$

$$\therefore \lim_{n\to\infty}\frac{H_{n+1}}{n\cdot H_n} = \left(\lim_{n\to\infty} n\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2(n+1)}\right)\left(\lim_{n\to\infty}\frac{G_n!}{G_{n+1}}\right)$$

$$= \left(\lim_{n\to\infty} n\right)e^2\left(\lim_{n\to\infty}\frac{1}{G_{n+1}}\right) = e^2\left(\lim_{n\to\infty} n\right)\left(\frac{1}{\lim_{n\to\infty}\sqrt[n+1]{a_{n+1}}}\right) = e^2\left(\lim_{n\to\infty} n\right)\left(\frac{1}{\lim_{n\to\infty}\frac{a_{n+2}!}{a_{n+1}!}}\right)$$

Applying Cauchy D-Alembert's Theorem

$$= e^2\left(\frac{1}{\lim_{n\to\infty}\frac{a_{n+2}}{n}}\right) = e^2\left(\frac{1}{\lim_{n\to\infty}\frac{a_{n+3}-a_{n+2}}{n+1-n}}\right) = \frac{e^2}{r}$$

hence $< H_n >_{n\ge1}$ *is a* $B-\left(1,\frac{e^2}{r}\right)$ *sequence. According to the above theorem it yields* $< \sqrt[n]{H_n} >_{n\ge1}$ *is a* $L-\left(0,\frac{e^2}{r}\cdot1\cdot e^{-1}\right)$ *sequence i.e.* $L-\left(0,\frac{e}{r}\right)$ *sequence.*

$$\Omega = \frac{e}{r} \quad \text{(Ans :)}$$

PROBLEM 3.039-*Solution by Soumitra Mandal – Kolkata – India*

Theorem: Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. *If* $< a_n >_{n\ge1}\in S(\mathbb{R}_+^*)$ *is a* $B-(t+1,a)$ *sequence then* $< \sqrt[n]{a_n} >_{a\ge1}$ *is a* $L-\left(t, a(t+1), e^{-(t+1)}\right)$ *sequence.*

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a\in\mathbb{R}_+^*;\ \lim_{n\to\infty}(b_{n+1}-b_n) = b\in\mathbb{R}_+^*$$

$$P_n = \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}, P_n! = \prod_{k=1}^n P_k$$

$$\Omega=\lim_{n\to\infty}\left(b_{n+1}^{u}\sqrt[n+1]{(P_{n+1}!)^{v}}-b_n^{u}\sqrt[n]{(P_n!)^{v}}\right)\ \textit{where } u+v=1$$

$$=\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}^{u(n+1)}(P_{n+1}!)^{v}}-\sqrt[n]{b_n^{un}(P_n!)^{v}}\right)=\lim_{n\to\infty}\left(\sqrt[n+1]{c_{n+1}}-\sqrt[n]{c_n}\right)$$

where $c_n=b_n^{un}(P_n!)^{v}$ *where* $n\in\mathbb{N}^*$

$$\therefore \lim_{n\to\infty}\frac{c_{n+1}}{n^{u+1}\cdot c_n}=\lim_{n\to\infty}\frac{1}{n^{u+1}}\frac{b_{n+1}^{u(n+1)}(P_{n+1}!)^{v}}{b_n^{un}(P_n!)^{v}}=$$

$$=\left(\lim_{n\to\infty}\frac{b_{n+1}^{u}}{n^{2u+v}}\right)\left(\lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\right)^{nu}\right)\left(\lim_{n\to\infty}(P_{n+1})^{v}\right)==\left(\lim_{n\to\infty}\frac{b_{n+1}^{u}}{n^{2u+v}}\right)\left(\lim_{n\to\infty}b_n^{u}\right)\left(\lim_{n\to\infty}(P_{n+1})^{v}\right)=$$

Applying Cauchy – D Alembert's Theorem

$$=\left(\lim_{n\to\infty}\left(\frac{b_{n+1}}{n}\right)^{u}\right)\left(\lim_{n\to\infty}\left(\frac{b_n}{n}\right)^{u}\right)\left(\lim_{n\to\infty}\left(\frac{P_{n+1}}{n}\right)^{v}\right)=$$

$$=\left(\lim_{n\to\infty}\left(\frac{b_{n+2}-b_{n+1}}{n+1-n}\right)^{u}\right)\left(\lim_{n\to\infty}\left(\frac{b_{n+1}-b_n}{n+1-n}\right)^{u}\right)\left(\lim_{n\to\infty}\frac{1}{n^{v}}\right)\left(\lim_{n\to\infty}\left(\frac{\sum_{k=1}^{n}a_k^2}{n}\right)^{\frac{v}{2}}\right)=$$

$$=b^{2u}\left(\lim_{n\to\infty}\frac{1}{n^{v}}\right)\left(\lim_{n\to\infty}\left(\prod_{k=1}^{n}a_k\right)^{\frac{v}{n}}\right)$$

since

$$\lim_{n\to\infty}\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}=\lim_{n\to\infty}\sqrt[n]{a_1a_2\dots a_n}=b^{2u}\left(\lim_{n\to\infty}\frac{1}{n^{v}}\right)\left(\lim_{n\to\infty}a_{n+1}^{v}\right)$$

Applying Cauchy – D' Alembert's Theorem

$$=b^{2u}\left(\lim_{n\to\infty}\left(\frac{a_{n+1}}{n}\right)^{v}\right)=b^{2u}\left(\lim_{n\to\infty}\left(\frac{a_{n+2}-a_{n+1}}{n+1-n}\right)^{v}\right)=b^{2u}a^{v}$$

Hence, $<c_n>_{n\geq1}$ *constitues a* $B-(u+2,b^{2u}a^{v})$ *sequence by the above theorem* $<\sqrt[n]{c_n}>_{n\geq1}$ *constitues* $L-\left(u+1,b^{2u}a^{v}(u+2)e^{-(u+2)}\right)$ *sequence or* $L-\left(u+1,b^{2u}a^{v}(3u+2v)e^{-(3u+2v)}\right)$ *sequence.*

$$\Omega=\frac{b^{2u}a^{v}(3u+2v)}{e^{3u+2v}}\quad \textit{(Ans:)}$$

PROBLEM 3.040-*Solution by Soumitra Mandal – Kolkata – India*

Theorem: Let $(t,a)\in\mathbb{R}_+\times\mathbb{R}_+^*$. *If* $<a_n>_{n\geq1}\in S(\mathbb{R}_+^*)$ *is a* $B-(t+1,a)$ *sequence then* $<\sqrt[n]{a_n}>_{n\geq1}$ *is a* $L-\left(t,a(t+1),e^{-(t+1)}\right)$ *sequence.*

$$\lim_{n\to\infty}(a_{n+1}-a_n)=r\in\mathbb{R}_+^*$$

for any $x\in\mathbb{R}_+^*$ *we denote* $M_n^{[x]}=\sqrt[x]{\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}}$ *and*

$M_n^{[x]}!=M_1^{[x]}M_2^{[x]}\dots M_n^{[x]}$ *for all* $n\in\mathbb{N}^*$

$$\Omega=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{M_{n+1}^{[x]}}}-\frac{n^2}{\sqrt[n]{M_n^{[x]}}}\right)=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{(n+1)^{2(n+1)}}{M_{n+1}^{[x]}!}}-\sqrt[n]{\frac{n^{2n}}{M_n^{[x]}!}}\right)=$$

$$= \lim_{n\to\infty}\left(\sqrt[n+1]{c_{n+1}} - \sqrt[n]{c_n}\right) \text{ where } c_n = \frac{n^{2n}}{M_n^{[x]}!} \text{ for all } n \geq 1$$

$$\lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n} = \left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{(n+1)^{2(n+1)}}{M_{n+1}^{[x]}!}\cdot\frac{M_n^{[x]}!}{n^{2n}}\right) =$$

$$= \left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}\right)\left(\lim_{n\to\infty}(n+1)^2\right)\left(\lim_{n\to\infty}\frac{1}{M_{n+1}^{[x]}}\right) =$$

$$= e^2\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\right)\left(\lim_{n\to\infty} n\right)\left(\lim_{n\to\infty}\frac{1}{\sqrt[x]{\frac{\sum_{k=1}^{n+1} a_k^x}{n+1}}}\right) = e^2\left(\lim_{n\to\infty} n\right)\left(\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{\prod_{k=1}^{n+1} a_k}}\right)$$

since

$$\lim_{n\to\infty}\sqrt[x]{\frac{\sum_{k=1}^{n} a_k^x}{n}} = \lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n} a_k} = e^2\left(\lim_{n\to\infty} n\right)\left(\lim_{n\to\infty}\frac{1}{\lim_{n\to\infty} a_{n+1}}\right)$$

Applying Cauchy – D Alembet's Theorem

$$= e^2\left(\frac{1}{\lim_{n\to\infty}\frac{a_{n+1}}{n}}\right) = \frac{e^2}{\lim_{n\to\infty}\frac{a_{n+2}-a_n}{n+1-n}} = \frac{e^2}{r}$$

hence, $< c_n >_{n\geq 1}$ *is a* $B - \left(0, \frac{e^2}{r}\right)$ *sequence and by the above theorem*

$< \sqrt[n]{c_n} >_{n\geq 1}$ *constitutes a* $L - \left(1, \frac{e^2}{r}\cdot e^{-1}\right)$ *sequence or,* $L - \left(1, \frac{e}{r}\right)$ *sequence.* $\Omega = \frac{e}{r}$ *(Ans :)*

PROBLEM 3.041-*Solution by proposer*

Let $f(2n+1) = \int_0^\infty u^{2n+1}e^{-u^2}\,du$

We now consider another similar integral as a function of v *such that:*

$$[f(2n+1)]^2 = \int_0^\infty u^{2n+1}e^{-u^2}\,du \cdot \int_0^\infty v^{2n+1}e^{-v^2}\,dv = \int_0^\infty\int_0^\infty (uv)^{2n+1}e^{-(u^2+v^2)}\,dudv$$

We now apply the change of variables: $u = r\cdot\cos\theta\,;\, v = r\cdot\sin\theta$

And our domain of integration is: $u \geq 0; v \geq 0 \Rightarrow 0 \leq \theta \leq \frac{\pi}{2}; r \geq 0$

$$\Rightarrow [f(2n+1)]^2 = \int_0^\infty\int_0^{\frac{\pi}{2}} [\sin(\theta)\cdot\cos(\theta)]^{2n+1}\, r^{4n+3}e^{-r^2}\,d\theta dr$$

$$= \int_0^{\frac{\pi}{2}} [\sin(\theta)\cdot\cos(\theta)]^{2n+1}\,d\theta\cdot f(2[2n+1]+1)$$

We now turn our attention back to $f(2n+1)$ *and apply the substitution:*

$$u^2 = x \Rightarrow du = \frac{1}{2}\cdot x^{-\frac{1}{2}}dx$$

$$\therefore \int_0^\infty u^{2n+1}e^{-u^2}du = \frac{1}{2}\int_0^\infty x^n e^{-x}\,dx = \frac{\Gamma(n+1)}{2} = \frac{n!}{2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} [\sin(\theta)\cdot\cos(\theta)]^{2n+1}\,d\theta = \frac{[f(2n+1)]^2}{f(2[2n+1]+1)} = \frac{1}{2}\cdot\frac{(n!)^2}{(2n+1)!}$$

Next:

$$\sum_0^\infty [\sin(\theta)\cdot\cos(\theta)]^{2n+1} = \frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2}$$

We consider: $\int \frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2}d\theta = -2\int\frac{\sin(2\theta)}{\sin^2(2\theta)-4}d\theta$

$$= -\int\frac{\sin(x)}{\sin^2(x)-4}dx = \int\frac{\sin(x)}{\cos^2(x)+3}dx$$

$$\cos(x) = y \Rightarrow dx = -\frac{dy}{\sin(x)}$$

$$\Rightarrow \int\frac{\sin(x)}{\cos^2(x)+3}dx = \int\frac{1}{3+y^2}dy = \frac{1}{\sqrt{3}}\arctan\left(\frac{y}{\sqrt{3}}\right)$$

Un-doing the substitutions yields:

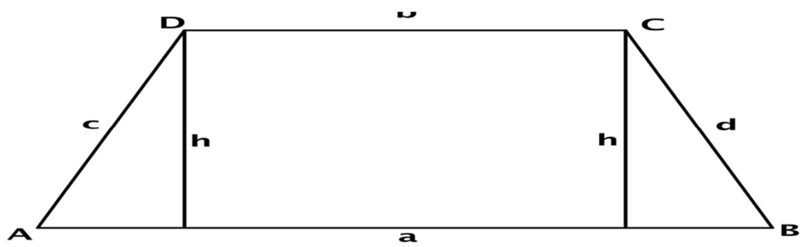
$$\int\frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2}d\theta = \frac{\arctan\left(\frac{\cos(2\theta)}{\sqrt{3}}\right)}{\sqrt{3}}$$

$$\therefore \int_0^{\frac{\pi}{2}}\frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2}d\theta = \frac{\pi}{3^{\frac{3}{2}}}$$

$$\sum_0^\infty \frac{1}{2}\cdot\frac{(n!)^2}{(2n+1)!} = \int_0^{\frac{\pi}{2}}\sum_0^\infty[\sin(\theta)\cdot\cos(\theta)]^{2n+1} = \int_0^{\frac{\pi}{2}}\frac{\sin(\theta)\cdot\cos(\theta)}{1-[\sin(\theta)\cdot\cos(\theta)]^2}d\theta = \frac{\pi}{3^{\frac{3}{2}}}$$

$$\therefore \frac{3^{\frac{3}{2}}}{2}\cdot\sum_0^\infty\frac{(n!)^2}{(2n+1)!} = \pi$$

PROBLEM 3.042-***Solution by SK Rejuan - West Bengal – India***



Let ABCD be the trapeze and h be the hight of it. Area $[ABCD] = \frac{1}{2}h(a+b)$

Now, from picture, $h < c$ *and* $h < b$

$\Rightarrow 2h < c + d = \frac{(a-b)(c+d)}{(a-b)}$ *[as* $a-b>0$*]* $\Rightarrow 2h < \frac{(a-b)(c+d)}{(a-b)} = \frac{4(a-b)(c+d)}{4(a-b)}$

$$\Rightarrow 2h < \frac{4(a-b)(c+d)}{(a-b)} < \frac{\{(a-b)+(c+d)\}^2}{4(a-b)} \text{ [by GM<AM]} \Rightarrow 2h < \frac{(a-b+c+d)^2}{4(a-b)}$$

$$\Rightarrow \frac{1}{4}(a+b)\cdot 2h < \frac{1}{4}(a+b)\frac{(a-b+c+d)^2}{4(a-b)}$$

$$[\because a+b>0]$$

$$\Rightarrow \frac{1}{2}h(a+b) < \frac{(a+b)(a-b+c+d)^2}{16(a-b)} \Rightarrow Area\,[ABCD] < \frac{(a+b)(a-b+c+d)^2}{16(a-b)}$$

PROBLEM 3.043-*Solution by Nguyen Phuc Tang – Hanoi – Vietnam*

We have: $LHS - RHS = \sqrt{\sum(a^2+2ab\cos(A-B))} - (a+b+c) + 2(a+b+c) -$

$$-\sum\sqrt{a^2+2ab\cos(A-B)+b^2} =$$

$$= \sum \frac{2ab[1-\cos(A-B)]}{\sqrt{a^2+2ab\cos(A-B)+b^2}+a+b} - \frac{\sum 2a\,[-\cos(A-B)+1]}{\sqrt{\sum(a^2+2ab\cos(A-B))}+(a+b+c)}$$

We prove that:

$$\sqrt{\sum(a^2+2ab\cos(A-B))}+a+b+c \ge \sqrt{(a^2+2ab\cos(A-B)+b^2}+a+b$$

$$\Leftrightarrow \sqrt{\sum(a^2+2ab\cos(A-B))} \ge \sqrt{(a^2+2ab\cos(A-B)+b^2}-c \quad (*)$$

$\oplus$ *case* $c \ge \sqrt{(a^2+2ab\cos(A-B))+b^2}$ *then (*) is true*

$\oplus$ *case* $c < \sqrt{(a^2+2ab\cos(A-B)+b^2}$

(*) $\Leftrightarrow 2bc\cos(B-C)+2ac\cos(A-C) \ge -2c\sqrt{(a^2+2ab\cos(A-B)+b^2)}$

$-[b\cos(B-C)+a\cos(A-C)] \le \sqrt{(a^2+2ab\cos(A-B))+b^2}$ *(**)*

if $b\cos(B-C)+a\cos(A-C) \ge 0 \Rightarrow$ *(**) is true*

if $b\cos(B-C)+a\cos(A-C) < 0 \Rightarrow \sin B\cos(B-C)+\sin A\cos(A-C) < 0$

$\Rightarrow \cos C\sin(A+B)+2\sin A\sin B\sin C < 0 \Rightarrow \cos C < 0 \Rightarrow C > A\ \&\ C > B$

(**) $\Leftrightarrow b^2(1-\cos^2(B-C))+c^2(1-\cos^2(A-C))+2ab[2\cos(A-B)-2\cos(A-C)\cos(B-C)] \ge 0$ *(***)*

(**) is true, because

$$2\cos(A-B)-2\cos(A-C)\cos(B-C) = \cos(A-B)-\cos(A+B-2C) =$$

$= 2\sin(C-A)\sin(C-B) > 0$. *Equality holds if* $a=b=c$.

PROBLEM 3.044-*Solution by proposer*

We prove that: $\sqrt{4n+1} \le \sqrt{n}+\sqrt{n+1} < \sqrt{4n+2}$

$$\sqrt{4n+3} \le \sqrt{n}+\sqrt{n+2} < \sqrt{4n+4}$$

$$\sqrt{4n+5} \le \sqrt{n}+\sqrt{n+3} < \sqrt{4n+6}$$

1) If $\sqrt{4n+2}, \sqrt{4n+4}, \sqrt{4n+6} \notin \mathbb{N}$ *then:* $[\sqrt{4n+1}] = [\sqrt{4n+2}]$;

$$[\sqrt{4n+3}] = [\sqrt{4n+4}]; [\sqrt{4n+5}] = [\sqrt{4n+6}]$$

2) If $\sqrt{4n+2}, \sqrt{4n+4}, \sqrt{4n+6} \in \mathbb{N} \Rightarrow [\sqrt{4n+2}] = [\sqrt{4n+1}]+1$;

$[\sqrt{4n+4}] = [\sqrt{4n+3}]+1; [\sqrt{4n+6}] = [\sqrt{4n+5}]+1$ *therefore*

$$[\sqrt{n}+\sqrt{n+1}] = [\sqrt{4n+1}];\ [\sqrt{n}+\sqrt{n+2}] = [\sqrt{4n+3}];\ [\sqrt{n}+\sqrt{n+3}] = [\sqrt{4n+5}]$$

After addition holds.

PROBLEM 3.045-*Solution by Hamza Mahmood – Lahore – Pakistan*

Let

$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{1}{1+\sqrt[2]{\tan(x_1)}\cdot\sqrt[3]{\tan(x_2)}\cdot\ldots\cdot\sqrt[n]{\tan(x_{n-1})}\cdot\sqrt[n+1]{\tan(x_n)}}dx_1dx_2\dots dx_{n-1}dx_n$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\prod_{k=2}^{n+1}\sqrt[k]{\cos(x_{k-1})}}{\prod_{k=1}^{n+1}\sqrt[k]{\cos(x_{k-1})}+\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}dx_1dx_2\dots dx_{n-1}dx_n\ \dots(A)$$

Now by substitution $x_i\to\frac{\pi}{2}-x_i$, we have:

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}{\prod_{k=1}^{n+1}\sqrt[k]{\sin(x_{k-1})}+\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}dx_1dx_2\dots dx_{n-1}dx_n\ \dots(B)$$

Adding (A) and (B) gives:

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\prod_{k=2}^{n+1}\sqrt[k]{\cos(x_{k-1})}+\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}{\prod_{k=1}^{n+1}\sqrt[k]{\cos(x_{k-1})}+\prod_{k=2}^{n+1}\sqrt[k]{\sin(x_{k-1})}}dx_1dx_2\dots dx_{n-1}dx_n=$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\dots\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(1)\,dx_1dx_2\dots dx_n\Rightarrow 2I=\left(\frac{\pi}{2}\right)^n\Rightarrow I=\frac{\pi^n}{2^{n+1}}$$

PROBLEM 3.046-*Solution by SK Rejuan-West Bengal-India*

Given $a,b,c\in\mathbb{R}^+$ such that $\sum a=1$.

Let us take a,b,c with the associated weight a^2,b^2,c^2 respectively, hence by applying $GM\geq HM$ we get,

$$\left(a^{a^2}\cdot b^{b^2}\cdot c^{c^2}\right)^{\frac{1}{a^2+b^2+c^2}}\geq\frac{a^2+b^2+c^2}{\frac{a^2}{a}+\frac{b^2}{b}+\frac{c^2}{c}}\quad[\because a,b,c\neq 0]$$

$$\Rightarrow\left(a^{a^2}\cdot b^{b^2}\cdot c^{c^2}\right)\geq\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a^2+b^2+c^2}$$

$$\Rightarrow a^{a^2}\cdot b^{b^2}\cdot c^{c^2}\geq(a^2+b^2+c^2)^{a^2+b^2+c^2}\quad [as \textstyle\sum a=1]\ [Proved]$$

PROBLEM 3.047-*Solution by Soumava Chakraborty-Kolkata-India*

a,b,c are distinct rational numbers such that $\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=0$

$$p.t\ \underbrace{\sqrt{\frac{(b-c)^4}{a^2}+\frac{(c-a)^4}{b^2}+\frac{(a-b)^4}{c^2}}}_{e}\ \text{is a rational no.}\because\frac{a}{b-c}+\frac{b}{c-a}=\frac{-c}{a-b}$$

$$\therefore \frac{-c}{a-b} = \frac{ac - a^2 + b^2 - bc}{(b-c)(c-a)} = \frac{c(a-b) - (a+b)(a-b)}{(b-c)(c-a)}$$
$$\Rightarrow \frac{c}{a-b} = \frac{(a-b)(a+b-c)}{(b-c)(c-a)} \Rightarrow \frac{(a-b)^2}{c} = \frac{(b-c)(c-a)}{a+b-c}$$
(If $a+b-c=0$, then $c=0$. But for e to be defined $c \neq 0 \therefore a+b-c \neq 0$)
$$\therefore \frac{(a-b)^4}{c^2} = \frac{(b-c)^2(c-a)^2}{(a+b-c)^2} \quad (1)$$
Similarly, $\frac{a}{b-c} = \frac{(b-c)(b+c-a)}{(c-a)(a-b)}$

If $b+c-a=0$, then $a=0$. But for e to be defined, $a \neq 0 \therefore b+c-a \neq 0$
$$\therefore \frac{(b-c)^4}{a^2} = \frac{(c-a)^2(a-b)^2}{(b+c-a)^2} \quad (2)$$
Also, similarly, $\frac{b}{c-a} = \frac{(c-a)(c+a-b)}{(a-b)(b-c)}$

If $c+a-b=0$, then $b=0$. But for e to be defined, $b \neq 0$.
$$\therefore c+a-b \neq 0 \therefore \frac{(c-a)^4}{b^2} = \frac{(a-b)^2(b-c)^2}{(c+a-b)^2} \quad (3)$$
$$(1)+(2)+(3) \Rightarrow e^2 = \frac{(c-a)^2(a-b)^2}{(b+c-a)^2} + \frac{(a-b)^2(b-c)^2}{(c+a-b)^2} + \frac{(b-c)^2(c-a)^2}{(a+b-c)^2} = x^2+y^2+z^2$$
where $x = \frac{(c-a)(a-b)}{b+c-a}, y = \frac{(a-b)(b-c)}{(c+a-b)}, z = \frac{(b-c)(c-a)}{a+b-c}$

Now, $\sum xy = \{\prod(a-b)\}\left[\frac{a-b}{(b+c-a)(c+a-b)} + \frac{b-c}{(c+a-b)(a+b-c)} + \frac{c-a}{(a+b-c)(b+c-a)}\right]$
$$\Rightarrow \sum xy = \left\{\prod(a-b)\left[\frac{(a-b)(a+b-c)+(b-c)(b+c-a)+(c-a)(c+a-b)}{(a+b-c)(b+c-a)(c+a-b)}\right]\right\}$$
$$= \left\{\prod(a-b)\right\}\left[\frac{a^2-b^2-ca+bc+b^2-c^2-ab+ca+c^2-a^2-bc+ab}{(a+b-c)(b+c-a)(c+a-b)}\right] = 0$$
$$\therefore e^2 = \sum x^2 = \sum x^2 + 0 = \sum x^2 + 2\sum xy \left(\because \sum xy = 0\right)$$
$$= (x+y+z)^2 \Rightarrow e = |x+y+z|$$
$$= \left|\frac{(c-a)(a-b)}{(b+c-a)} + \frac{(a-b)(b-c)}{(c+a-b)} + \frac{(b-c)(c-a)}{a+b-c}\right|,$$
which is obviously a rational number as a, b, c are rational numbers (Proved).

PROBLEM 3.048-*Solution by Soumava Chakraborty-Kolkata-India*

$$a^4+b^4+c^4+26abc \le 1 \Leftrightarrow a^4+b^4+c^4+26abc \le (a+b+c)^4$$
$$\Leftrightarrow 2a^3b+2ab^3+2b^3c+2bc^3+2c^3a+2ca^3+3a^2b^2+3b^2c^2+3c^2a^2+$$
$$+6a^2bc+6b^2ca+6c^2ab \ge 13abc$$
Now, $3\sum a^2b^2 \underset{(a)}{\ge} 3abc(a+b+c) (\because \sum x^2 \ge \sum xy)$

(true for $\forall\, a,b,c$)$= 3abc \quad (\because \sum a = 1)$

Also $6a^2bc+6b^2ca+6c^2ab = 6abc(\sum a) \underset{(b)}{=} 6abc$

(a), (b) $\Rightarrow$ it remains to prove:
$$a^3b+ab^3+b^3c+bc^3+c^3a+ca^3 \ge 2abc \quad (1)$$
Now, $a^3b+ab^3 = ab(a^2+b^2) \ge ab\cdot 2ab$ $(\because a^2+b^2 \ge 2ab$ *and* $a,b \ge 0)$

$\Rightarrow a^3b+ab^3 \underset{(c)}{\ge} 2a^2b^2$. *Similarly,* $b^3c+bc^3 \underset{(d)}{\ge} 2b^2c^2$ *and* $c^3a+ca^3 \underset{(e)}{\ge} 2c^2a^2$

$$(c)+(d)+(e)\Rightarrow \sum(a^3b+ab^3)\ge 2\sum a^2b^2\ge 2abc(a+b+c)=2abc$$
$\Rightarrow$ (1) is true (Proved)

PROBLEM 3.049-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Siendo ABC un triángulo $\wedge$ si $x,y,z\in R$, se cumple la siguiente desigualdad:
$x^2+y^2+z^2\ge 2xy\cos A+2yz\cos B+2zx\cos C$. *En otras palabras:*
$$x^2+y^2+z^2\ge xy\frac{b^2+c^2-a^2}{bc}+yz\frac{a^2+c^2-b^2}{ca}+zx\frac{a^2+b^2-c^2}{ab}\quad (A)$$
Aplicando para un triángulo de longitudes m_a, m_b, m_c, tenemos:
$$x^2+y^2+z^2\ge xy\frac{(m_b)^2+(m_c)^2-(m_a)^2}{m_bm_c}+\frac{(m_c)^2+(m_a)^2-(m_b)^2}{m_cm_a}yz+$$
$$+\frac{(m_a)^2+(m_b)^2-(m_c)^2}{m_am_b}zx$$
$$4(x^2+y^2+z^2)m_am_bm_c\ge$$
$$\ge xy(5a^2-b^2-c^2)m_a+yz(5b^2-c^2-a^2)m_b+zx(5c^2-a^2-b^2)m_c$$
Un caso particular, cuando $x=y=z=1$
$$12m_am_bm_c\ge(5a^2-b^2-c^2)m_a+(5b^2-c^2-a^2)m_b+(5c^2-a^2-b^2)m_c$$
$$12m_am_bm_c\ge 5a^2m_a-b^2m_a-c^2m_a+5b^2m_b-c^2m_b-a^2m_b-5c^2m_c-a^2m_c-b^2m_c$$
$$a^2(5m_a-m_b-m_c)+b^2(5m_b-m_c-m_a)+c^2(5m_c-m_a-m_b)\le 12m_am_bm_c$$

PROBLEM 3.050-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Si: $a,b,c>0$, se cumple la siguiente desigualdad: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}$ … (Desigualdad de Nesbitt). La desigualdad propuesta es equivalente: $\frac{\left(\frac{c}{a+b}\right)^2}{c^2a}+\frac{\left(\frac{a}{b+c}\right)^2}{a^2b}+\frac{\left(\frac{b}{c+a}\right)^2}{b^2c}\ge\frac{3}{4}$

Aplicando la desigualdad de Cauchy:
$$\frac{\left(\frac{c}{a+b}\right)^2}{c^2a}+\frac{\left(\frac{a}{b+c}\right)^2}{a^2b}+\frac{\left(\frac{b}{c+a}\right)^2}{b^2c}\ge\frac{\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2}{a^2b+b^2c+c^2a}\ge\frac{\frac{9}{4}}{3}=\frac{3}{4}\text{ … (LQQD)}$$

PROBLEM 3.051-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\Gamma(n+1)=n!,\ \lim_{n\to\infty}\frac{\sqrt[n]{\Gamma(n+1)}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}=$$
$$\overset{\substack{CAUCHY\\ D'ALEMBERT}}{\cong}\lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\right)=\frac{1}{e}$$
$$\text{Let } u_n=\left(\frac{\sqrt[n+1]{\Gamma(n+2)}}{\sqrt[n]{\Gamma(n+1)}}\right)^a \text{ where } n\in\mathbb{N}, \lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\frac{\sqrt[n+1]{(n+1)!}}{n+1}}{\frac{\sqrt[n]{n!}}{n}}\cdot\left(1+\frac{1}{n}\right)\right)^a=1$$
$$u_n\to 1 \text{ then } \frac{u_n-1}{\ln u_n}\to 1 \text{ as } n\to\infty$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\cdot\frac{1}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^a = e^a$$

$$\therefore \lim_{x\to\infty}\left(\left(f(x+1)\right)^a - \left(f(x)\right)^a\right)\cdot x^{1-a} = \lim_{\substack{n\to\infty\\ n\to\mathbb{N}^*}}\left(\left(f(n+1)\right)^a - \left(f(n)\right)^a\right)n^{1-a}$$

$$= \lim_{n\to\infty}\left(\left(\frac{\sqrt[n]{n!}}{n}\right)^a\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) = \frac{a}{e^a}$$

PROBLEM 3.052-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a} \overset{Bergstrom}{\geq} \frac{(a^3+b^3+c^3)^2}{\sum a^2+\sum a}$$

$$\overset{Chebyshev}{\geq} \frac{\left(\frac{1}{3}\sum a\sum a^2\right)^2}{\sum a^2+3} = \frac{t^2}{t+3}, \text{ where } t=\sum a^2 \therefore \text{it suffices to prove: } \frac{t^2}{t+3}\geq\frac{3}{2}$$

$$\Leftrightarrow 2t^2-3t-9\geq 0 \Leftrightarrow (2t+3)(t-3)\geq 0 \Leftrightarrow t\geq 3 \quad (1)$$

$$(\because 2t+3>3>0)$$

$$\text{Now, } t=\sum a^2 \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a\right)^2 = \frac{9}{3}=3 \Rightarrow t\geq 3 \Rightarrow (1) \text{ is true (Proved)}$$

PROBLEM 3.053-*Solution by Marian Ursărescu – Romania*

We use the following theorem: If $M,N\in M_n(\mathbb{C})$ such that $MN=I_n\Rightarrow NM=I_n$, then $MN=NM$. $x^2A+B=xAB\Rightarrow xAB-x^2A-B=O_n\Rightarrow xA(B-xI_n)-B+xI_n=xI_n\Rightarrow$ $\Rightarrow(xA-I_n)(B-xI_n)=xI_n\Rightarrow(B-xI_n)(xA-I_n)=I_n\Rightarrow xBA-B-x^2A+xI_n=I_n\Rightarrow$ $\Rightarrow xBA=x^2A+B\Rightarrow AB=BA$ *and similarly,* $BC=CB$ *and* $AC=CA$.

$$\left.\begin{aligned} x^2A+B=xAB\Rightarrow x^2AC+BC=xABC\\ y^2B+C=yBC\Rightarrow y^2AB+AC=yABC\\ z^2C+A=zCA\Rightarrow z^2BC+AB=zABC\end{aligned}\right\}\Rightarrow$$

$$(x^2+1)AC+(y^2+1)AB+(z^2+1)BC=(x+y+z)ABC \quad (1)$$

$$x^2A+B=xAB\Rightarrow AB=xA+\frac{1}{x}B$$

$$y^2B+C=yBC\Rightarrow BC=yB+\frac{1}{y}C \quad (2)$$

$$z^2C+A=zCA\Rightarrow CA=zC+\frac{1}{z}A$$

From (1)+(2) we have:

$$(y^2+1)\left(xA+\frac{1}{x}B\right)+(z^2+1)\left(yB+\frac{1}{y}C\right)+(x^2+1)\left(zC+\frac{1}{z}A\right)=(x+y+z)ABC\Leftrightarrow$$

$$\Leftrightarrow\left(x(y^2+1)+\frac{x^2+1}{z}\right)A+\left((z^2+1)y+\frac{y^2+1}{x}\right)B+\left((x^2+1)z+\frac{z^2+1}{y}\right)C=(x+y+z)ABC$$

PROBLEM 3.054-*Solution by proposer*

Applying Hölder's inequality:

$$\int_0^\infty f(t)\,g(t)dt\leq\left(\int_0^\infty\left(f(t)\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^\infty\left(g(t)\right)^q dt\right)^{\frac{1}{q}}$$

$$when\ \frac{1}{p}+\frac{1}{q}=1\ and\ p>1.\ Put\ p=\frac{1}{a}, q=\frac{1}{1-a}, f(t)=e^{-t}t^{x+a-1},$$

$$g(t)=e^{-t}t^{x-1}, \Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,dt\ we\ obtain\ \frac{\Gamma(x+a)}{x^a\Gamma(x)}\le 1$$

$$\big(\Gamma(x+1)=x\Gamma(x)\big).\ If\ in\ this\ case\ we\ take\ a\to 1-a, x\to x+a\ we\ obtain$$

$$\left(\frac{x}{x+a}\right)^{1-a}\le\frac{\Gamma(x+a)}{x^a\Gamma(x)}\ finally\ \left(\frac{x}{x+a}\right)^{1-a}\le\frac{\Gamma(x+a)}{x^a\Gamma(x)}\le 1.\ If\ a=\frac{1}{2}$$

$$then\ we\ have\ \sqrt{x}\le\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\le\sqrt{x+\frac{1}{2}}\ if\ x\to\sqrt{x}\Rightarrow$$

$$\sqrt[4]{x}\le\frac{\Gamma(\sqrt{x}+1)}{\Gamma\left(\sqrt{x}+\frac{1}{2}\right)}\le\sqrt{\sqrt{x}+\frac{1}{2}}\Rightarrow\sqrt{x}\le\sqrt{x}\le\left(\frac{\Gamma(\sqrt{x}+1)}{\Gamma\left(\sqrt{x}+\frac{1}{2}\right)}\right)^2\le\sqrt{x}+\frac{1}{2}\quad (1)$$

$$x\to\sqrt[4]{x};\ \sqrt[8]{x}\le\frac{\Gamma(\sqrt[4]{x}+1)}{\Gamma\left(\sqrt[4]{x}+\frac{1}{2}\right)}\le\sqrt{\sqrt[4]{x}+\frac{1}{2}}\Rightarrow\sqrt{x}\le\left(\frac{\Gamma(\sqrt[4]{x}+1)}{\Gamma\left(\sqrt[4]{x}+\frac{1}{2}\right)}\right)^4\le\sqrt{x}+\sqrt[4]{x}+\frac{1}{2}\quad (2)$$

After addition (1)+(2) holds.

PROBLEM 3.055-*Solution by Shivam Sharma-New Delhi-India*

$$\Rightarrow\frac{1}{2}\int_0^1\ln^3(x)\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n dx\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 x^n\ln^3(x)\,dx$$

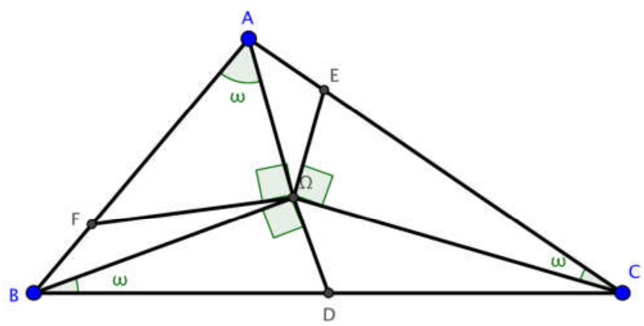
$$\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{\partial^3}{\partial n^3}\left[\int_0^1 x^n\,dx\right]\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{\partial^3}{\partial n^3}\left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\left[\frac{x^{n+1}\ln^3(x)}{n+1}-\frac{3x^{n+1}\ln^2(x)}{(n+1)^2}+\frac{6x^{n+1}\ln(x)}{(n+1)^3}-\frac{6x^{n+1}}{(n+1)^4}\right]_0^1$$

$$\Rightarrow -6\sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}(n+1)^4}\right)\Rightarrow -6\sum_{n=0}^{\infty}\left(\frac{1}{2^n n^4}\right),\ (OR)$$

$$I=-6Li_4\left(\frac{1}{2}\right)\ (Q.E.D.)$$

PROBLEM 3.056-*Solution by Marian Ursărescu – Romania*



$$\Delta B\Omega D\Rightarrow\cos\omega=\frac{B\Omega}{BD}\Rightarrow BD=\frac{B\Omega}{\cos\omega}\quad (1)$$

But from Sines Law we have:

$$A\Omega=\frac{b}{a}2R\sin\omega, B\Omega=\frac{c}{b}2R\sin\omega\wedge C\Omega=\frac{a}{c}2R\sin\Omega\quad (2)$$

$$\text{From (1)+(2)} \Rightarrow BD = \frac{c}{b} 2R\tan\omega \Rightarrow \frac{BD}{BC} = \frac{c}{AB} 2R\tan\omega \Rightarrow$$

$$\Rightarrow \frac{BD}{BC} + \frac{CE}{CA} + \frac{AF}{AB} = 2R\tan\omega\left(\frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}\right) = 2R\tan\omega\frac{(a^2+b^2+c^2)}{abc} \quad (3)$$

$$\text{But in any } \Delta ABC \text{ we have: } \cot\omega = \frac{a^2+b^2+c^2}{4S}, S = [ABC] \Rightarrow \tan\omega = \frac{4S}{a^2+b^2+c^2} \quad (4)$$

$$\text{From (3)+(4)} \Rightarrow \frac{BD}{BC} + \frac{CE}{CA} + \frac{AF}{AB} = \frac{2R\cdot 4S}{a^2+b^2+c^2}\cdot\frac{(a^2+b^2+c^2)}{abc} = 2\cdot\frac{4RS}{abc} = 2, \text{ because } abc = 4RS$$

PROBLEM 3.057-*Solution by Soumitra Mandal-Chandar Nagore-India*

WLOG let us assume $b > a$

Let us assume that the statement

$$\left(\frac{a+b}{2}\right)^n \le \frac{1}{n+1}\sum_{k=0}^{n} a^k b^{n-k} \le \frac{a^n+b^n}{2}$$

be $P(n)$. *Now,* $P(1): \frac{a+b}{2} \le \frac{a+b}{2} \le \frac{a+b}{2}$, *which is true*

so, $P(1)$ *is true.* $P(2): \left(\frac{a+b}{2}\right)^2 \le \frac{1}{3}(a^2+ab+b^2) \le \frac{a^2+b^2}{2}$

now, $\frac{a^2+ab+b^2}{3} \ge \left(\frac{a+b}{2}\right)^2 \Leftrightarrow \frac{(a-b)^2}{12} \ge 0$, *which is true*

again, $\frac{a^2+b^2}{2} \ge \frac{a^2+ab+b^2}{3} \Leftrightarrow \frac{(a-b)^2}{6} \ge 0$, *which is true.*

$\therefore P(2)$ *is established. Let us assume the statement is true for* $n = m$.

$\therefore P(m): \left(\frac{a+b}{2}\right)^m \le \frac{b^{m+1}-a^{m+1}}{(m+1)(b-a)} \le \frac{a^m+b^m}{2}$. *Similarly,* $P(m-1)$ *is also true.*

$$\therefore \left(\frac{a+b}{2}\right)^{m-1} \le \frac{b^m-a^m}{m(b-a)} \le \frac{a^{m-1}+b^{m-1}}{2}.$$

We need to prove, $n = m+1$. $\therefore \frac{a^{m+1}+b^{m+1}}{2} - \frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)}$

$$= \frac{m(b-a)(a^{m+1}+b^{m+1})}{2(m+2)(b-a)} - \frac{ab}{m+2}\cdot\frac{b^m-a^m}{b-a} \ge$$

$$\ge \frac{m}{2(m+2)}(a^{m+1}+b^{m+1}) - \frac{abm}{2(m+2)}(a^{m-1}+b^{m-1})$$

$$= \frac{m}{2(m+2)}(b-a)(b^m-a^m) \ge \frac{m^2}{2(m+2)}(b-a)^2\left(\frac{a+b}{2}\right)^{m-1} \ge 0$$

$$\therefore \frac{a^{m+1}+b^{m+1}}{2} \ge \frac{1}{m+2}\cdot\frac{b^{m+2}-a^{m+2}}{b-a}$$

now, $\frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)} - \left(\frac{a+b}{2}\right)^{m+1} = \frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)} - \left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}\right)^m$

$$\ge \frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)} - \frac{(a+b)(b^{m+1}-a^{m+1})}{2(m+1)(b-a)}$$

$$= \frac{2(m+1)(b^{m+2}-a^{m+2}) - (m+2)(a+b)(b^{m+1}-a^{m+1})}{2(b-a)(m+1)(m+2)}$$

$$= \frac{m(b^{m+1}+a^{m+1})}{2(m+1)(m+2)} - \frac{abm(b^{m-1}+a^{m-1})}{2(m+1)(m+2)} = \frac{m}{2(m+1)(m+2)}(b-a)(b^m-a^m)$$

$$\geq \frac{m^2}{2(m+1)(m+2)}(b-a)^2\left(\frac{a+b}{2}\right)^{m-1} \geq 0$$

$\therefore \frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)} \geq \left(\frac{a+b}{2}\right)^{m+1}$. *Hence* $\left(\frac{a+b}{2}\right)^{m+1} \leq \frac{1}{m+2}\sum_{k=0}^{m+1} a^k b^{m-k} \leq \frac{a^{m+1}+b^{m+1}}{2}$

$\therefore P(m+1)$ *is true. So, by theory of mathematical induction* $P(n)$ *is true.*

$$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{n+1}\sum_{k=0}^{n} a^k b^{n-k} \leq \frac{a^n+b^n}{2}$$

PROBLEM 3.058-*Solution by Abdilkadir Altintas – Afyon – Turkey*

Let $K(\theta)$ *be Kiepert perspector.* BXC, AYC *and* AZB *triangles drawn outwardly to the sides of* ABC. AX, BY *and* CZ *are concurrent at point*

$$K(\theta) = \left(\frac{1}{S_A+S_\theta}, \frac{1}{S_B+S_\theta}, \frac{1}{S_c+S_\theta}\right)$$

where S_A, S_B *and* S_C *are conway notations of* ABC. *If* $\theta = w$ *where* w *is Brocard angle than*

$S_w = \frac{a^2+b^2+c^2}{2}$; *X has barycentric coordintes*

$$X = (-a^2 : S_C + S_w : S_B + S_w).$$
$$Y = (S_C + S_w : -b^2 : S_A + S_w),$$
$$Z = (S_B + S_w : S_A + S_w : -c^2).$$

Simplifying we get: $X = (-a^2 : a^2+c^2 : a^2+b^2)$; $Y = (a^2+b^2 : -b^2 : b^2+c^2)$

$Z = (a^2+c^2 : b^2+c^2 : -c^2)$. *Using determinant to evaluate the area*

$$XYZ = \frac{3(a^2b^2+a^2c^2+b^2c^2)}{(a^2+b^2+c^2)^2}$$

PROBLEM 3.059-*Solution by Ravi Prakash-New Delhi-India*

For $a, b > 0$

$$\frac{a^3+b^3}{2} \geq \left(\frac{a+b}{2}\right)^3 \Rightarrow \frac{a^3+b^3}{(a+b)^3} \geq \frac{1}{4} \Rightarrow \frac{a^2-ab+b^2}{(a+b)^2} \geq \frac{1}{4} \Rightarrow \frac{(a^2-ab+b^2)^2}{(a+b)^4} \geq \frac{1}{16}$$

Similarly for other two expressios. Thus,

$$\frac{(a^2-ab+b^2)^2}{(a+b)^4} + \frac{(b^2-bc+c^2)^2}{(b+c)^4} + \frac{(c^2-ca+a^2)^2}{(c+a)^4} \geq \frac{3}{16}$$

PROBLEM 3.060-*Solution by Imad Zak-Saida-Lebanon*

$a, b, c > 0$ $/a+b+c=1$. *Prove that:* $\left(1+\frac{1}{2a+b}\right)^c \cdot \left(1+\frac{1}{2b+c}\right)^a \cdot \left(1+\frac{1}{2c+a}\right)^b \geq 2$

Let $f(x) = \ln\left(1+\frac{1}{x}\right)$ *for* $x > 0 \Rightarrow$

$f'(x) = -\frac{1}{x+x^2} < 0 \Rightarrow f$ *is decreasing … (1)*

$f''(x) = \frac{2x+1}{x^2(x+1)^2} > 0 \Rightarrow f$ *is convex … (2)*

Weighted Jensen's on $f(x) \Rightarrow \sum c \ln\left(1+\frac{1}{2a+b}\right) =$

$$\sum c\cdot f(2a+b)\ge (c+a+b)f\left(\frac{c(2a+b)+a(2b+c)+b(2c+a)}{a+b+c}\right)$$

$$=1\cdot f\left(\frac{3\sum ab}{1}\right)=f\left(3\sum ab\right)\ge f(1)=\ln 2$$

$$\textit{because } \sum ab\le \frac{p^2}{3}=\frac{1}{3}\Rightarrow 3\sum ab\le 1 \; f \textit{ is } \searrow$$

$$\therefore \sum c\cdot \ln\left(1+\frac{1}{2a+b}\right)\ge \ln 2\Leftrightarrow \sum \ln\left(1+\frac{1}{2a+b}\right)^c\ge \ln 2\Leftrightarrow$$

$$\ln\left(\prod\left(1+\frac{1}{2a+b}\right)^c\right)\ge \ln 2 \quad \textit{ln is} \nearrow\Rightarrow$$

$$\prod\left(1+\frac{1}{2a+b}\right)^c\ge 2 \textit{ Q.E.D equality holds for } a=b=c=\frac{1}{3}$$

PROBLEM 3.061-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\tan\frac{A}{2}=\frac{\Delta}{p(p-a)},\tan\frac{B}{2}=\frac{\Delta}{p(p-b)} \textit{ and } \tan\frac{C}{2}=\frac{\Delta}{p(p-c)}$$

a)
$$\sum_{cyc}\frac{\tan^3\frac{A}{2}}{m\tan\frac{B}{2}+n\tan\frac{C}{2}}\overset{HOLDER}{\ge}\frac{\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}{3(m+n)}$$

$$=\frac{\Delta^2}{\left(p(p-a)(p-b)(p-c)\right)^2}\cdot\frac{\left(\sum(p-a)(p-b)\right)^2}{3(m+n)}=\frac{r^2(r+4R)^2}{\Delta^2\cdot 3(m+n)}=\frac{(r+4R)^2}{3p^2(m+n)}$$

b)
$$\sum_{cyc}\frac{\tan\frac{A}{2}}{m+n\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}=\sum_{cyc}\frac{\tan^2\frac{A}{2}}{m\tan\frac{A}{2}+n\cdot\prod\tan\frac{A}{2}}$$

$$\overset{BERGSTROM}{\ge}\frac{\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}{m\sum\tan\frac{A}{2}+3n\prod\tan\frac{A}{2}}=\frac{\Delta^2}{\left(p(p-a)(p-b)(p-c)\right)^2}\cdot\frac{\left(\sum(p-a)(p-b)\right)^2}{\frac{\Delta}{p\prod(p-a)}m\left(\sum(p-a)(p-b)\right)+3n\frac{\Delta^3}{p^3\prod(p-a)}}$$

$$=\frac{1}{\Delta^2}\cdot\frac{r^2(r+4R)^2}{\frac{m}{\Delta}r(r+4R)+3n\frac{\Delta}{p^2}}=\frac{\Delta}{p^2}\cdot\frac{(r+4R)^2}{mr(r+4R)+3n\frac{\Delta^2}{p^2}}=\frac{(r+4R)^2}{p(m(r+4R)+3nr)} \quad \textit{(proved)}$$

c)
$$\sum_{cyc}\frac{\tan^3\frac{A}{2}}{m\cot\frac{B}{2}+n\cot\frac{C}{2}}=$$

$$\left(\prod_{cyc}\tan\frac{A}{2}\right)\sum_{cyc}\frac{\tan^2\frac{A}{2}}{m\tan\frac{C}{2}+n\tan\frac{B}{2}}\overset{BERGSTROM}{\ge}\left(\sum_{cyc}\tan\frac{A}{2}\right)\frac{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}{m+n}$$

$$=\frac{\Delta^3}{p^3\prod(p-a)}\cdot\frac{\Delta}{p\prod(p-a)}\cdot\frac{1}{m+n}\left(\sum_{cyc}(p-a)(p-b)\right)$$

$$=\frac{\Delta}{p^2}\cdot\frac{1}{\Delta}\cdot\frac{r(r+4R)}{m+n}=\frac{r(r+4R)}{(m+n)p^2} \quad \textit{(proved)}$$

d)
$$\sum_{cyc}\frac{\tan\frac{A}{2}}{\left(x+y\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}\right)^m}=\sum_{cyc}\frac{\tan^{m+1}\frac{A}{2}}{\left(x\tan\frac{A}{2}+y\prod\tan\frac{A}{2}\right)^m}$$

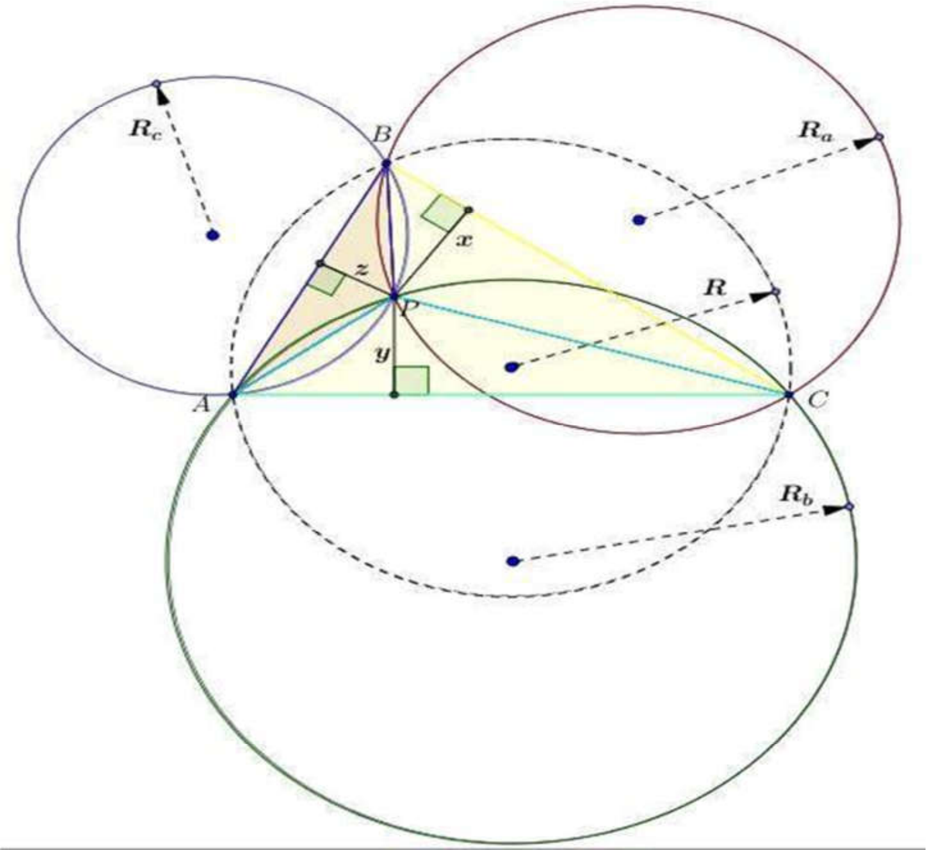
$$\overset{RADON}{\ge}\frac{\left(\sum\tan\frac{A}{2}\right)^{m+1}}{\left(x\sum_{cyc}\tan\frac{A}{2}+2y\prod\tan\frac{A}{2}\right)^m}=\frac{\left(\sum_{cyc}\frac{\Delta}{p(p-a)}\right)^{m+1}}{\left(x\sum_{cyc}\frac{\Delta}{p(p-a)}+3y\frac{\Delta^3}{p^3(p-a)}\right)^m}$$

$$=\frac{\Delta^{m+1}}{(p\prod(p-a))^{m+1}}\cdot\frac{\left(\sum_{cyc}(p-a)(p-b)\right)^{m+1}}{\left(\frac{\Delta}{p\prod(p-a)}x\sum_{cyc}(p-a)(p-b)+3y\frac{\Delta}{p^2}\right)^m}$$

$$=\frac{1}{\Delta^{m+1}}\cdot\frac{r^{m+1}(r+4R)^{m+1}}{\left(\frac{x}{\Delta}r(r+4R)+\frac{3y\Delta}{p^2}\right)^m}=\frac{1}{\Delta}\cdot\frac{r^{m+1}(r+4R)^{m+1}}{(xr(r+4R)+3yr^2)^m}=$$

$$=\frac{(r+4R)^{m+1}}{p(x(r+4R)+3yr)^m}$$

PROBLEM 3.062-*Solution by Kevin Soto Palacios – Huarmey – Peru*



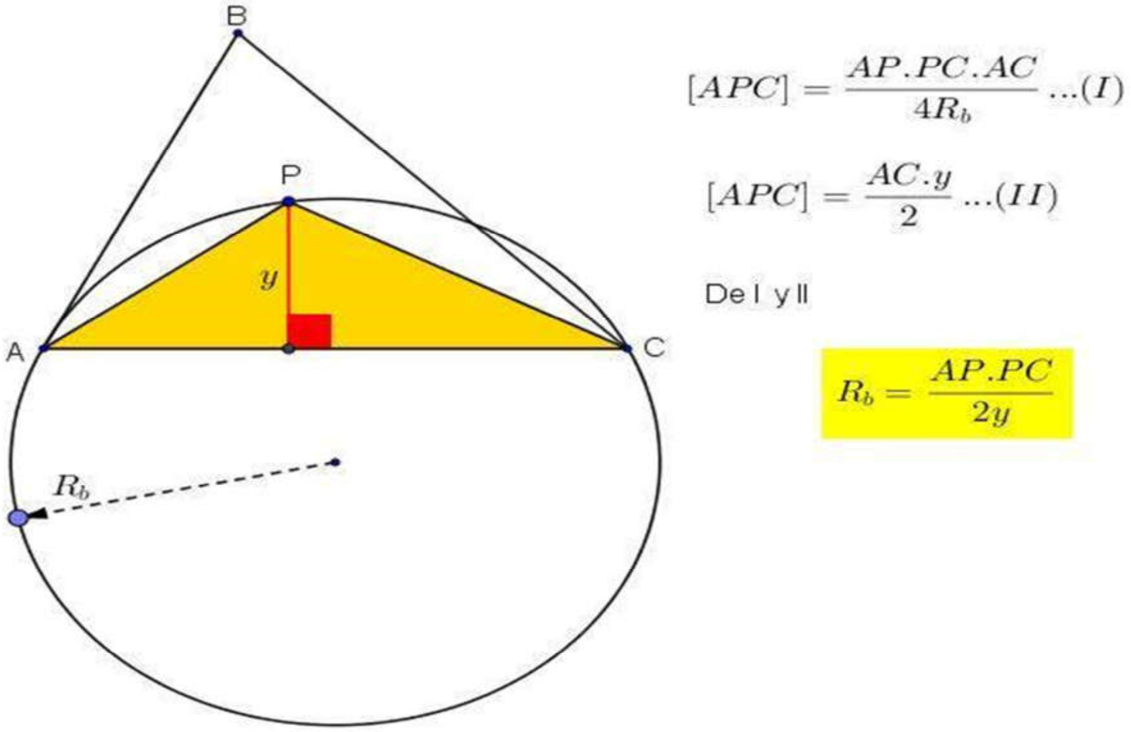
Dado un triángulo equilátero ABC y sea P un punto en este plano. Siendo R, R_a, R_b, R_c, respectivamete los radios de las circunferencias circunscritas ABC, BPC, CPA, APB, además x, y, z son las distancias de P a los lados BC, CA, AB.

Probar que $xR_a + yR_b + zR_c \geq \frac{3}{2}R^2$

En un $\Delta\ ABC$ general, se cumple lo siguiente:

$$S_{\Delta BPC}=\frac{BC\cdot x}{2}=\frac{BP\cdot PC\cdot BC}{4R_a}\Leftrightarrow R_a=\frac{PB\cdot PC}{2x}$$

$$S_{\Delta CPA}=\frac{CA\cdot y}{2}=\frac{CP\cdot PA\cdot CA}{4R_b}\Leftrightarrow R_b=\frac{PC\cdot PA}{2y}$$
$$S_{\Delta CPA}=\frac{AB\cdot z}{2}=\frac{AP\cdot PB\cdot AB}{4R_b}\Leftrightarrow R_c=\frac{PA\cdot PB}{2z}$$



La desigualdad propuesta es equivalente: $PB\cdot PC+PC\cdot PA+PA\cdot PB\geq 3R^2$

Dado que es triángulo equilátero: $BC=CA=AB=l=2R\sin 60°=\sqrt{3}R$

Es suficiente probar: $PB\cdot PC+BPC\cdot PA+PA\cdot PB\geq l^2$

Para todo, α,β,γ *que satisface* $\alpha+\beta+\gamma=360°$ *se cumple la siguiente desigualdad* $\Rightarrow \cos\alpha+\cos\beta+\cos\gamma\geq -\frac{3}{2}$

En el triángulo APB*, por ley de cosenos tenemos*

$$l^2=PA^2+PB^2-2PA\cdot PB\cdot\cos\alpha\Leftrightarrow\cos\alpha=\frac{PA^2+PB^2-l^2}{2PA\cdot PB}$$

En el triángulo BPC*, por ley de cosenos tenemos*

$$l^2=PB^2+PC^2-2PB-2PB\cdot PC\cdot\cos\beta\Leftrightarrow\cos\beta=\frac{PB^2+PC^2-l^2}{2PB\cdot PC}$$

En el triángulo CPA*, por ley de cosenos tenemos*

$$l^2=PC^2+PA^2-2PC\cdot PA\cdot\cos\gamma\Leftrightarrow\cos\gamma=\frac{PC^2+PA^2-l^2}{2PC\cdot PA}$$

Por lo tanto

$$\frac{PA^2+PB^2-l^2}{2PA\cdot PB}+\frac{PB^2+PC^2-l^2}{2PB\cdot PC}+\frac{PC^2+PA^2-l^2}{2PC\cdot PA}+\frac{3}{2}\geq 0$$
$$\Leftrightarrow (PA^2+PB^2-l^2)PC+(PB^2+PC^2-l^2)PA+(PC^2+PA^2-l^2)PB+$$
$$+3(PA\cdot PB\cdot PC)\geq 0$$
$$\Leftrightarrow PA\cdot PB(PA+PB)+PB\cdot PC(PB+PC)+PC\cdot PA(PC+PA)+$$

$$+3PA\cdot PB\cdot PC-l^2(PA+PB+PC)\geq 0$$
$$\Leftrightarrow (PA+PB+PC)(PA\cdot PB+PB\cdot PC+PC\cdot PA)-l^2(PA+PB+PC)\geq 0$$
$$\Leftrightarrow (PA+PB+PC)(PA\cdot PB+PB\cdot PC+PC\cdot PA-l^2)\geq 0$$
Donde se deduce $\rightarrow PA\cdot PB+PB\cdot PC+PC\cdot PA\geq l^2$

PROBLEM 3.063-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Elevando al cuadrado la expresión, se tiene lo siguiente:
$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\left(\frac{bcd}{a}+\frac{cda}{b}\right)^2+\left(\frac{dab}{c}+\frac{abc}{d}\right)^2+2\left(\frac{bcd}{a}+\frac{cda}{b}\right)\left(\frac{dab}{c}+\frac{abc}{d}\right)$$
$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\left(\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+2c^2d^2\right)+\left(\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}+2a^2b^2\right)+$$
$$+2(b^2d^2+d^2a^2+b^2c^2+a^2c^2)$$
Ordenando la expresión convenienentemente:
$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}+$$
$$+2(a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2)$$
$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}+12\quad (A)$$
Desde que: $a,b,c,d>0$. *Por: MA* $\geq$ *MG*
$$\frac{1}{a^4}+\frac{1}{b^4}\geq\frac{2}{a^2b^2}\quad (I),$$
$$\frac{1}{b^4}+\frac{1}{c^4}\geq\frac{2}{b^2c^2}\quad (II),$$
$$\frac{1}{c^4}+\frac{1}{d^4}\geq\frac{2}{c^2d^2}\quad (III)$$
$$\frac{1}{d^4}+\frac{1}{a^4}\geq\frac{2}{d^2a^2}\quad (IV)$$
$$\frac{1}{a^4}+\frac{1}{c^4}\geq\frac{2}{a^2c^2}\quad (V)$$
$$\frac{1}{b^4}+\frac{1}{d^4}\geq\frac{2}{b^2d^2}\quad (VI)$$
Sumando: (I) + (II) + (III) + (IV) + (V) + (VI):
$$\Rightarrow\frac{3}{a^4}+\frac{3}{b^4}+\frac{3}{c^4}+\frac{3}{d^4}\geq\frac{2}{a^2b^2}+\frac{2}{b^2c^2}+\frac{2}{c^2d^2}+\frac{2}{d^2a^2}+\frac{2}{a^2c^2}+\frac{2}{b^2d^2}$$
Multiplicando $\times\ (abcd)^2$...
$$\Rightarrow 3\left(\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}\right)\geq 2(c^2d^2+d^2a^2+a^2b^2+b^2c^2+b^2d^2+a^2c^2)$$
$$\Rightarrow\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}\geq\frac{2}{3}(a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2)=$$
$$=\frac{2}{3}(6)=4.$$ *Finalmente tenemos en … (A):*
$$\left(\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\right)^2=\frac{b^2c^2d^2}{a^2}+\frac{c^2d^2a^2}{b^2}+\frac{d^2a^2b^2}{c^2}+\frac{a^2b^2c^2}{d^2}+12\geq 4+12=16$$
$$\Rightarrow\frac{bcd}{a}+\frac{cda}{b}+\frac{dab}{c}+\frac{abc}{d}\geq 4\ (LQQD)$$

PROBLEM 3.064-*Solution by Kevin Soto Palacios – Huarmey – Peru*

Tener en cuenta la siguiente identidad
$(x+y)^3=x^3+y^3+3xy(x+y)$, *donde* $x=a+b, y=c+d$

$$(a+b+c+d)^3=(a+b)^3+(c+d)^3+3(a+b)(c+d)(a+b+c+d)$$
$$(a+b+c+d)^3=a^3+b^3+3ab(a+b)+c^3+d^3+3cd(c+d)+12(a+b)(c+d)$$

La desigualdad propuesta es equivalente

$$\Leftrightarrow 3ab(a+b)+3ab(a+b)+12(a+b)(c+d)\le 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}\right)+48$$
$$\Leftrightarrow (a+b+c+d)^3\le a^3+b^3+c^3+d^3+3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}\right)+48$$
$$\Leftrightarrow 64\le a^3+b^3+c^3+d^3+3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}\right)+48$$
$$\Leftrightarrow 16\le a^3+b^3+c^3+d^3+3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}\right)$$

Como $a,b,c,d\ge 0$. Aplicando $MA\ge MG$

$$a^3+\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}\ge 4a,$$
$$b^3+\sqrt[3]{b}+\sqrt[3]{b}+\sqrt[3]{b}\ge 4b,$$
$$c^3+\sqrt[3]{c}+\sqrt[3]{c}+\sqrt[3]{c}\ge 4c,$$
$$d^3+\sqrt[3]{d}+\sqrt[3]{d}+\sqrt[3]{d}\ge 4d$$
$$\Rightarrow a^3+b^3+c^3+d^3+3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d}\right)\ge 4(a+b+c+d)=16$$

PROBLEM 3.065-*Solution by proposer*

Let X',Y',Z' be respectively images of X,Y,Z with the central symmetry M. Then

$$Vol(MXYZ)=Vol(MX'Y'Z').$$

Let A',B',C' be respectively intersections of the line AM with the sides BC,CA,AB. On the rays SA',SB',SC' we take points A_1,B_1,C_1, respectively such that

$SA_1=XM, SB_1=YM, SC_1=ZM$. Then the translation by vector $\overrightarrow{MS}$ transforms the tetahedron $MX'Y'Z'$ into the tetahedron $SA_1B_1C_1$. Therefore

$$Vol(MX'Y'Z')=Vol\ (SA_1B_1C_1)$$

Thus we have $Vol(MXYZ)=Vol(SA_1B_1C_1)$. Furthermore

$$\frac{Vol(SA_1B_1C_1)}{Vol(SA'B'C')}=\frac{SA_1}{SA'}\cdot\frac{SB_1}{SB'}\cdot\frac{SC_1}{SC'}=\frac{XM}{SA'}\cdot\frac{YM}{SB'}\cdot\frac{ZM}{SC'}=\frac{AM}{AA'}\cdot\frac{BM}{BB'}\cdot\frac{CM}{CC'}$$

From these above we deduce that: $\frac{Vol(MXYZ)}{Vol(SA'B'C')}=\left(1-\frac{MA'}{AA'}\right)\left(1-\frac{MB'}{BB'}\right)\left(1-\frac{MC'}{CC'}\right)$

Using the AM-GM inequality and note that: $\frac{MA'}{AA'}+\frac{MB'}{BB'}+\frac{MC'}{CC'}=1$*, we obtain*

$$\left(1-\frac{MA'}{AA'}\right)\left(1-\frac{MB'}{BB'}\right)\left(1-\frac{MC'}{CC'}\right)\le\left(\frac{3-\frac{MA'}{AA'}-\frac{MB'}{BB'}-\frac{MC'}{CC'}}{3}\right)^3=\frac{8}{27}$$

Thus $\frac{Vol(MXYZ)}{Vol\ (SA'B'C')}\le\frac{8}{27}$*. On the other hand:* $\frac{Vol(SA'B'C')}{Vol\ (SABC)}=\frac{Area(A'B'C')}{Area(ABC)}\le\frac{1}{4}$

Multiplying up these two inequalities we get: $\frac{Vol(MXYZ)}{Vol(SABC)}\le\frac{2}{27}$*, which is the desired result. The equality holds when the point M is the centroid of the triangle ABC. Now we will prove a result that has just been used above as* $\frac{Area(A'B'C')}{Area(ABC)}\le\frac{1}{4}$*. Indeed, this is equivalent to*

$$\frac{Area(AB'C')}{Area(ABC)}+\frac{Area(BC'A')}{Area(ABC)}+\frac{Area(CA'B')}{Area(ABC)}\ge\frac{3}{4},\ or\ \frac{AB'}{AC}\cdot\frac{AC'}{AB}+\frac{BC'}{BA}\cdot\frac{BA'}{BC}+\frac{CA'}{CB}\cdot\frac{CB'}{CA}\ge\frac{3}{4}.$$

Setting $\frac{A'B}{A'C}=x,\frac{B'C}{B'A}=y,\frac{C'A}{C'B}=z$*. By the Ceva's theorem, we have $xyz=1$. Then our inequality becomes:* $\frac{1}{1+y}\cdot\frac{z}{1+z}+\frac{1}{1+z}\cdot\frac{x}{1+x}+\frac{1}{1+x}\cdot\frac{y}{1+y}\ge\frac{3}{4}$*, or* $\frac{x(1+y)+y(1+z)+z(1+x)}{(1+x)(1+y)(1+z)}\ge\frac{3}{4}$*, or*

$$4(x+y+z+xy+yz+zx)\geq 3(1+x)(1+y)(1+z),\ or\ x+y+z+xy+yz+zx\geq 6$$

The last inequality is true by $x+y+z\geq 3\sqrt[3]{xyz}=3$, *and*

$xy+yz+zx\geq 3\sqrt[3]{x^2y^2z^2}=3$. *The proof is complete and we are done.*

PROBLEM 3.066-*Solution by Khalef Ahmad El Ruhemi-Jarash-Jordania*

$$I:=\sum_{n=1}^{\infty}\sum_{k=1}^{k=n}\frac{1}{n^3(2k-1)}$$

Define $I(x):=\sum_{n=1}^{\infty}\sum_{k=1}^{k=n}\frac{x^{2k-1}}{n^3(2k-1)}\Rightarrow I'(x)=\sum_{n=1}^{\infty}\sum_{k=1}^{k=n}\frac{x^{2k-2}}{n^3}=\sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\sum_{k=1}^{k=n}x^{2k-2}$

$$=\sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\left(\frac{1-x^{2n}}{1-x^2}\right)=\left(\frac{1}{1-x^2}\right)\cdot\sum_{n=1}^{\infty}\left(\frac{1-x^{2n}}{n^3}\right)$$

$$=\left(\frac{1}{1-x^2}\right)\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n^3}-\sum_{n=1}^{\infty}\frac{x^{2n}}{n^3}\right]=\frac{1}{(1-x^2)}[Li_3(1)-Li_3(x^2)]$$

Since $I(0)=0$, *then* $I=I(1)=\int_0^1 I'(x)\,dx$

$\therefore I=\int_0^1\frac{\left(Li_3(1)-Li_3(x^2)\right)}{(1-x^2)}dx$ *integrate by parts*

$$=-\frac{1}{2}\left(Li_3(1)-Li_3(x^2)\right)\ln\left(\frac{1-x}{1+x}\right)\Big|_0^1+\frac{1}{2}\cdot\int_0^1\left(\ln\left(\frac{1-x}{1+x}\right)\right)x-\frac{Li_2(x^2)}{x^2}x\ 2xdx$$

$$=\int_0^1\frac{\ln(1+x)}{x}\cdot Li_2(x^2)dx-\int_0^1\frac{\ln(1-x)}{x}\cdot Li_2(x^2)dx$$

$$=\int_0^1\frac{2\ln(1+x)}{x}\cdot\left(Li_2(x)+Li_2(-x)\right)dx-\int_0^1\frac{2\ln(1-x)}{x}\left(Li_2(x)+Li_2(-x)\right)dx$$

$$=2\cdot\int_0^1 Li_2(-x)\frac{\ln(1+x)}{x}dx-2\cdot\int_0^1\frac{\ln(1-x)\,x}{x}\cdot Li_2(x)dx+$$

$$+2\cdot\int_0^1\frac{\ln(1+x)}{x}Li_2(x)dx-2\cdot\int_0^1\frac{\ln(1-x)}{x}\cdot Li_2(-x)dx=I$$

$$\therefore I=-\left(Li_2(-x)\right)^2\Big|_0^1+\left(Li_2(x)\right)^2\Big|_0^1$$

$+2\cdot\int_0^1\frac{\ln(1+x)}{x}\cdot Li_2(x)dx-2\int_0^1\frac{\ln(1-x)}{x}Li_2(x)dx$ *integrate by parts*

$$=\left(Li_2(1)\right)^2-\left(Li_2(-1)\right)^2+2\cdot\int_0^1\frac{\ln(1+x)}{x}\cdot Li_2(x)dx$$

$$+2\left(Li_2(x)Li_2(-x)|_0^1-\int_0^1 Li_2(x)\times\frac{\ln(1+x)}{-x}dx\right)$$

$$= (Li_2(1))^2 - \left(Li_2(-1)\right)^2 + 2Li_2(1)Li_2(-1) + 4\int_0^1 \frac{\ln(1+x)Li_2(x)}{x}\cdot dx = I \quad (*)$$

But $\frac{\ln(1+x)}{x} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\cdot x^{k-1}}{k}$

$$\Rightarrow \int_0^1 \frac{\ln(1+x)Li_2(x)}{x}\cdot dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cdot\left(\int_0^1 x^{k-1}\cdot Li_2(x)dx\right) \text{ integrate by parts}$$

$$\int_0^1 x^{k-1}\cdot Li_2(x)\,dx = \frac{x^k}{k}Li_2(x)\Big|_0^1 - \int_0^1 \frac{x^k}{k}\times -\frac{\ln(1-x)}{x}dx$$

$$= \frac{Li_2(1)}{k} + \frac{1}{k}\cdot\int_0^1 x^{k-1}\ln(1-x)\,dx = \frac{Li_2(1)}{k} - \frac{H_k}{k^2}$$

$$\therefore \int_0^1 \frac{\ln(1+x)\,Li_2(x)}{x}\cdot dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cdot\left(\frac{Li_2(1)}{k} - \frac{H_k}{k^2}\right)$$

$$\therefore \int_0^1 \frac{\ln(1+x)\,Li_2(x)}{x}dx = Li_2(1)\cdot\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cdot H_k}{k^3}$$

$$= \varphi(2)\eta(2) - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}H_k}{k^3}$$

$$= \left(\frac{\pi^2}{6}\right)\left(1-\frac{1}{2}\right)\left(\frac{\pi^2}{6}\right) + \left(-\frac{11\pi^4}{360} + \frac{1}{12}\ln^4(2) - \frac{\pi^2\ln^2(2)}{12} + 2Li_4\left(\frac{1}{2}\right) + \frac{7}{4}\ln(2)\,\varphi(3)\right)$$

$$= \frac{\pi^4}{72} - \frac{11\pi^4}{360} + \frac{1}{12}\ln^4(2) - \frac{\pi^2\cdot\ln^2(2)}{12} + 2Li_4\left(\frac{1}{2}\right) + \frac{7}{4}\ln(2)\,\varphi(3)$$

going to (*), and using $Li_2(-1) = -\frac{1}{2}\varphi(2) = -\frac{\pi^2}{12}$

$$I = \frac{\pi^4}{36} - \frac{\pi^4}{144} - \frac{\pi^4}{36} + \frac{\pi^4}{18} - \frac{11\pi^4}{90} + \frac{1}{3}\ln^4(2) - \frac{\pi^2\cdot\ln^2(2)}{3}$$

$$+8Li_4\left(\frac{1}{2}\right) + 7\ln(2)\,\varphi(3)$$

$$= -\frac{53\pi^4}{720} - \frac{1}{3}\pi^2\cdot\ln^2(2) + \frac{1}{3}\ln^4(2) + 7\ln(2)\,\varphi(3) + 8Li_4\left(\frac{1}{2}\right)$$

$$\therefore \sum_{n=1}^{\infty}\sum_{k=1}^{k=n}\frac{1}{n^3(2k-1)} = -\frac{53}{720}\pi^4 - \frac{1}{3}\pi^2\cdot\ln^2(2) + \frac{1}{3}\ln^4(2) + 7\ln(2)\,\varphi(3) + 9Li_4\left(\frac{1}{2}\right)$$

PROBLEM 3.067-*Solution by Shivam Sharma – New Delhi – India*

Applying Abel's summation, with $a_n = H_n, b_n = \left(\zeta(8) - \frac{1}{1^8} - \frac{1}{2^8} - \frac{1}{3^8} - \cdots - \frac{1}{n^8}\right)$

$$S = \lim_{n\to\infty} H_n\left[\zeta(8) - \frac{1}{1^8} - \frac{1}{2^8} - \frac{1}{3^8} - \cdots - \frac{1}{(n+1)^8}\right] + \sum_{n=1}^{\infty}\left[\frac{H_{n+1}}{(n+1)^7} - \frac{1}{(n+1)^7}\right]$$

$$\Rightarrow 0 + \sum_{n=1}^{\infty}\left[\frac{H_{n+1}}{(n+1)^7} - \frac{1}{(n+1)^7}\right] \Rightarrow \sum_{n=1}^{\infty}\left[\frac{H_{n+1}}{(n+1)^7} - \frac{1}{(n+1)^7}\right] \Rightarrow \sum_{k=2}^{\infty}\left[\frac{H_k}{k^7} - \frac{1}{k^7}\right]$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{H_k}{k^7}-\sum_{k=1}^{\infty}\frac{1}{k^7}\dots.. \quad (1)$$

As we know,

$$\sum_{k=1}^{\infty}\frac{H_k}{k^m}=\frac{1}{2}\left[(m+2)\zeta(m+1)-\sum_{n=1}^{m-2}\{(\zeta(m-n))(\zeta(n+1))\}\right]$$

if $m=7$, *we get*

$$\sum_{n=1}^{\infty}\frac{H_k}{k^7}=\frac{\pi^8}{4200}-\zeta(3)\zeta(5)$$

Now put this result in equation (1), we get, $S=\frac{\pi^8}{4200}-\zeta(3)\zeta(5)-\zeta(7)$

PROBLEM 3.068-*Solution by Soumava Chakraborty-Kolkata-India*

Given inequality $\Leftrightarrow 4\sum a^2+4+3\sum a^2b^2+3a^2b^2c^2+2abc(\sum ab)\overset{(1)}{\geq} 2abc(\sum a)$

Now, $\forall m,n,p\in\mathbb{R}^*, \sum m^2-\sum mn=\frac{1}{2}[(m-n)^2+(n-p)^2+(p-m)^2]\geq 0$

$$\therefore 2\sum a^2b^2\overset{(2)}{\geq}2abc\left(\sum a\right)$$

(1), (2) $\Rightarrow$ *it suffices to prove:* $4\sum a^2+4+\sum a^2b^2+3a^2b^2c^2+2abc(\sum ab)\geq 0\Leftrightarrow$

$$\Leftrightarrow 3x^2+\left(2\sum ab\right)x+\left(\sum a^2b^2+4\sum a^2+4\right)\geq 0\ \ (x=abc)$$

Let $f(x)=3x^2+(2\sum ab)x+(\sum a^2b^2+4\sum a^2+4)$, *which is a quadratic in* x *as* $x=abc\neq 0\ \ (\because a,b,c\in\mathbb{R}^*)$

Discriminant Δ *of* $f(x)=4(\sum ab)^2-4\cdot 3(\sum a^2b^2+4\sum a^2+4)=$

$$=4\left\{\sum a^2b^2+2abc\left(\sum a\right)-3\sum a^2b^2-12\left(\sum a^2+1\right)\right\}=$$

$=4[2\{abc(\sum a)-\sum a^2b^2\}-12(\sum a^2+1)]<0$ *(using (2) &* $\because 12(\sum a^2+1)>0$*)*

$\therefore f(x)>0$ *(*$\because f(x)$ *never touches* x *– axis as roots of* $f(x)=0$ *are imaginary)* $\therefore f(x)\geq 0$ *(Done)*

PROBLEM 3.069-*Solution by Ravi Prakash-New Delhi-India*

As $a>1, 0<\frac{1}{a}<1\Rightarrow 0<1-\frac{1}{a}<1$. *Now,* $\frac{1-\left(1-\frac{1}{a}\right)^n}{\frac{1}{a}}=\frac{1-\left(1-\frac{1}{a}\right)^n}{1-\left(1-\frac{1}{a}\right)}$

$$=1+\left(1-\frac{1}{a}\right)+\left(1-\frac{1}{a}\right)^2+\dots+\left(1-\frac{1}{a}\right)^{n-1}$$

$$\geq\underbrace{\left(1-\frac{1}{a}\right)^{n-1}+\left(1-\frac{1}{a}\right)^{n-1}+\dots+\left(1-\frac{1}{a}\right)^{n-1}}_{\text{"}n\text{" times}}$$

$$\Rightarrow 1-\left(1-\frac{1}{a}\right)^n\geq\frac{n}{a}\left(1-\frac{1}{a}\right)^{n-1}\Rightarrow\left(1-\frac{1}{a}\right)^n+\frac{n}{a}\left(1-\frac{1}{a}\right)^{n-1}\leq 1$$

$$\Rightarrow\left(1-\frac{1}{a}+\frac{n}{a}\right)\left(1-\frac{1}{a}\right)^{n-1}\leq 1\Rightarrow(a-1+n)(a-1)^{n-1}\leq a^n$$

PROBLEM 3.070-*Solution by Ravi Prakash-New Delhi-India*

Let

$$I=\int_a^b \frac{f(x)g(x)}{1+g(x)}dx \qquad (1)$$

$$=\int_a^b \frac{f(a+b-x)g(a+b-x)}{1+g(a+b-x)}dx=\int_a^b \frac{f(x)\left(\frac{1}{g(x)}\right)}{1+\frac{1}{g(x)}}dx$$

$$I=\int_a^b \frac{f(x)}{g(x)+1}dx \qquad (2)$$

Adding (1) and (2) we get

$$2I=\int_a^b \frac{f(x)(g(x)+1)}{g(x)+1}dx=\int_a^b f(x)\,dx \Rightarrow I=\frac{1}{2}\int_a^b f(x)dx$$

PROBLEM 3.071-*Solution by Khalef Ruhemi-Jarash-Iordania*

$$I:=\int_0^1 \ln\left[\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\right]\cdot\frac{x\,dx}{1-x^2}$$

Let $x=\sin\theta,\ dx=\cos\theta\,d\theta$

$$\therefore I=\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \ln\left(\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}\right)^2\cdot\frac{\sin\theta\cos\theta}{\cos^2\theta}d\theta=\int_0^{\frac{\pi}{2}} \ln\left(\frac{\tan\theta+1}{\tan\theta-1}\right)^2\cdot\tan(\theta)\,d\theta$$

let $\tan\theta=x;\ \theta=\tan^{-1}(x);\ d\theta=\frac{dx}{1+x^2}$

$$\therefore I=\int_0^\infty \frac{x}{1+x^2}\cdot\ln\left(\frac{1+x}{1-x}\right)^2\cdot dx=\int_0^\infty \frac{2x}{1+x^2}\cdot\ln\left|\frac{1+x}{1-x}\right|\cdot dx,\ \textit{let}\ x=\frac{1}{y}, dx=-\frac{dy}{y^2}$$

$$\therefore I=\int_0^\infty \frac{\frac{2}{x}}{\left(1+\frac{1}{x^2}\right)x^2}\ln\left|\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right|\cdot dx=\int_0^\infty \frac{2}{x(1+x^2)},\ln\left|\frac{1+x}{1-x}\right|dx$$

$$=\int_0^\infty 2\ln\left|\frac{1+x}{1-x}\right|\cdot\left(\frac{1}{x}-\frac{x}{1+x^2}\right)\cdot dx$$

$$=\int_0^\infty \frac{2\ln\left|\frac{1+x}{x}\right|}{x}\cdot dx-\int_0^\infty \frac{2x}{1+x^2}\ln\left|\frac{1+x}{1-x}\right|dx$$

$$\therefore I=2\int_0^\infty \frac{\ln\left|\frac{1+x}{1-x}\right|dx}{x}-I\ \Rightarrow I=\int_0^\infty \underbrace{\frac{\ln\left|\frac{1+x}{1-x}\right|dx}{x}}_{\text{integrating by parts}} \qquad (*)$$

$$I=\lim_{a\to 0^+}\ln\left|\frac{1+x}{1-x}\right|\cdot\ln x\Big|_a^{\frac{1}{a}}-\int_0^\infty(\ln x)\cdot\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

$$= \lim_{a\to 0^+}\left(\ln\left(\frac{\left(1+\frac{1}{a}\right)}{\left(1-\frac{1}{a}\right)}\right)\right)\ln\left(\frac{1}{a}\right) - \ln\left(\left|\frac{1+a}{1-a}\right|\ln(a)\right) - \int_0^\infty \ln x\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx.$$

$$= \lim_{a\to 0^+} -2\ln\left|\frac{1+a}{1-a}\right|\cdot\ln(\theta) - \int_0^\infty \ln x\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

But $\lim_{a\to 0^+} -2\ln(a)\ln\left|\frac{1+a}{1-a}\right| = -2\lim_{a\to 0^+}(\ln(a)\ln(1+a) - \ln(a)\ln(1-a))$

$$= -2(0-0) = 0$$

$$\therefore I = -\int_0^\infty \ln(x)\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

$$\therefore -I = \int_0^1 (\ln x)\cdot\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx + \int_1^\infty \ln(x)\underbrace{\left(\frac{1}{1+x}+\frac{1}{1-x}\right)}_{let\ x=\frac{1}{y}, dx=-\frac{dy}{yz}}dx$$

$$\therefore -I = \int_0^1 \frac{\ln x}{1+x}dx + \int_0^1 \frac{\ln x}{1-x}dx + \int_0^1 -\ln(x)\cdot\left(\frac{1}{1+x}-\frac{1}{1-x}\right)\cdot\frac{dx}{x}$$

$$-I = \int_0^1 \frac{\ln x\,dx}{1+x} + \int_0^1 \frac{\ln x\,dx}{1-x} + \int_0^1 \ln x\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

$$-I = \int_0^1 \frac{\ln x\,dx}{1+x} + \int_0^1 \frac{\ln x\,dx}{1-x} + \int_0^1 \frac{\ln x}{1+x}dx + \int_0^1 \frac{\ln x\,dx}{1-x}$$

$\therefore -I = 2\cdot\int_0^1 \frac{\ln x dx}{1+x} + 2\int_0^1 \frac{\ln x dx}{1-x}$ *integrating by parts*

$$\Rightarrow -\frac{1}{2}I = \ln x\ln(1+x)|_0^1 - \int_0^1 \frac{\ln(1+x)}{x}dx - \ln x\ln(1-x)|_0^1 + \int_0^1 \frac{\ln(1-x)}{x}dx$$

$$\therefore -\frac{1}{2}I = -Li_2(x)|_0^1 + Li_2(-x)|_0^1$$

$$= -Li_2(1) + Li_2(-1) = \varphi(2) - \eta(2) = -\frac{\pi^2}{6} - \frac{1}{2}\left(\frac{\pi^2}{6}\right) = -\frac{\pi^2}{6} - \frac{\pi^2}{12} = -\frac{3\pi^2}{12} = -\frac{\pi^2}{4}$$

$\therefore -\frac{1}{2}I = -\frac{\pi^2}{4} \Rightarrow I = \frac{\pi^2}{2}$, *(Note:* $\eta(2) = \frac{1}{2}\varphi(2) = \frac{\pi^2}{12}$*)*

$$\therefore \int_0^1 \ln\left[\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\right]\cdot\frac{x\cdot dx}{1-x^2} = \frac{\pi^2}{2}$$

PROBLEM 3.072-*Solution by proposer*

For $x,z,t>0$ *we have:* $\frac{x}{x^2+tz} \le \frac{1}{4}\left(\frac{1}{t}+\frac{1}{z}\right) \Leftrightarrow t(x-z)^2 + z(x-t)^2 \ge 0$

therefore $\int_a^b \frac{xdt}{x^2+tz} \le \frac{1}{4}\int_a^b\left(\frac{1}{t}+\frac{1}{z}\right)dt \Rightarrow \frac{x}{z}\ln\frac{x^2+bz}{x^2+az} \le \frac{1}{4}\ln\frac{b}{a} + \frac{b-a}{4z} \Rightarrow$

$$\Rightarrow \sum_{cyclic} \frac{x}{z}\ln\frac{x^2+bz}{x^2+az} \le \sum_{cyclic}\left(\frac{1}{4}\ln\frac{b}{a}+\frac{b-a}{4z}\right) = \frac{3}{4}\ln\frac{b}{a}+\frac{b-a}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

PROBLEM 3.073-*Solution by SK Rejuan-West Bengal-India*

$a,b,c \in C$. *The given equation is,*

$$x^3-(a+b+c)x^2+(ab+bc+ca-1)x+b-abc=0$$
$$\Rightarrow x^3-bx^2-(a+c)^2x^2+b(a+c)x+(ac-1)x+b-abc=0$$
$$\Rightarrow x^2(x-b)-(a+c)x(x-b)+(ac-1)(x-b)=0$$
$$\Rightarrow (x-b)\{x^2-(a+c)x+(ac-1)\}=0$$

Either $x=b$

Or, $x=\frac{(a+c)\pm\sqrt{(a+c)^2-4(ac-1)}}{2}=\frac{(a+c)\pm\sqrt{(a+c)^2-4ac+4}}{2}=\frac{(a+c)^2\pm\sqrt{(a-c)^2+4}}{2}$

$\therefore$ *Solutions of the given equation* $x=b,\frac{(a+c)\pm\sqrt{(a-c)^2+4}}{2}$

PROBLEM 3.074-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a\in\mathbb{R}_+, u+v=1 \text{ then}$$

$$\lim_{n\to\infty}\left((n+1)^u\sqrt[n+1]{\left(a_{n+1}f(n+1)\right)^v}-n^u\sqrt[n]{\left(a_nf(n)\right)^v}\right)$$

$$=\lim_{n\to\infty}\left(\sqrt[n]{\left(\frac{a_nf(n)}{n^n}\right)^v}\cdot\frac{w_n-1}{\ln w_n}\cdot\ln w_n^n\right) \text{ where } w_n=\left(1+\frac{1}{n}\right)^u\frac{\sqrt[n+1]{\left(a_{n+1}f(n+1)\right)^v}}{\sqrt[n]{\left(a_nf(n)\right)^v}}$$

$$\text{now, } \lim_{n\to\infty}\sqrt[n]{\left(\frac{a_nf(n)}{n^n}\right)^v}\overset{D'ALEMBERT}{=}\lim_{n\to\infty}\left(\frac{a_{n+1}}{na_n}\cdot\frac{\frac{f(n+1)}{f(n)}}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{n}{n+1}\right)^v=\left(\frac{a}{e}\right)^v$$

$$\lim_{n\to\infty}w_n=\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{\left(\frac{a_{n+1}f(n+1)}{(n+1)^{n+1}}\right)^v}}{\sqrt[n]{\left(\frac{a_nf(n)}{n^n}\right)^v}}=1, \text{ so, } \lim_{n\to\infty}\frac{w_n-1}{\ln w_n}=1$$

$$\lim_{n\to\infty}w_n^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nu}\cdot\left(\frac{a_{n+1}}{na_n}\cdot\frac{f(n+1)}{f(n)}\cdot\frac{1}{\sqrt[n+1]{\left(\frac{a_{n+1}f(n+1)}{(n+1)^{n+1}}\right)^v}}\cdot\frac{n}{n+1}\right)^v=e$$

$$\therefore \lim_{n\to\infty}\left((n+1)^u\sqrt[n+1]{\left(a_{n+1}f(n+1)\right)^v}-n^u\sqrt[n]{\left(a_nf(n)\right)^v}\right)$$

$$=\left(\frac{a}{e}\right)^v\cdot 1\cdot\ln e=\left(\frac{a}{e}\right)^v \text{ (Proved)}$$

PROBLEM 3.075-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{D'ALEMBERT}{\triangleq}\lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\right)=\frac{1}{e}$$

$$\text{we know, } \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\frac{1}{e}$$

$$\lim_{n\to\infty}\left(a_{n+1}\sqrt[n+1]{(n+1)!}-b_n\sqrt[n]{n!}\right)$$

$$=\lim_{n\to\infty}\left((a_{n+1}-a)\sqrt[n+1]{(n+1)!}-(b_n-a)\sqrt[n]{n!}+a\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\right)$$

$$=\lim_{n\to\infty}\left(n(a_{n+1}-a_n)\frac{\sqrt[n+1]{(n+1)!}}{n+1}\left(1+\frac{1}{n}\right)-n(b_n-a)\frac{\sqrt[n]{n!}}{n}+\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\right)$$

$$=\frac{a+b-c}{e}$$

PROBLEM 3.076-*Solution by Ali Shather-Nasyria-Iraq*

$$S=\sum_{n=1}^{\infty}\frac{H_{2n+1}}{n^2}=\sum_{n=1}^{\infty}\frac{H_{2n}}{n^2}+\sum_{n=1}^{\infty}\frac{1}{n^2(2n+1)}=S_1+S_2$$

$$S_1=\sum_{n=1}^{\infty}\frac{H_{2n}}{n^2}=4\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^2}=4\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(\frac{1+(-1)^n}{2}\right)$$

$$=2\sum_{n=1}^{\infty}\frac{H_n}{n^2}+2\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^2}=4\xi(3)-\frac{5}{4}\xi(3)=\frac{11}{4}\xi(3)$$

$$S_2=\sum_{n=1}^{\infty}\frac{1}{n^2(2n+1)}=\sum_{n=1}^{\infty}\frac{1}{n^2}+4\sum_{n=1}^{\infty}\frac{1}{2n+1}-2\sum_{n=1}^{\infty}\frac{1}{n}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}-2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{1}{2}}\right)=\xi(2)-2H_{\frac{1}{2}}=\xi(2)-2(2-2\ln 2)=\xi(2)-4+4\ln 2$$

$$\therefore S=S_1+S_2=\frac{11}{4}\xi(3)+\xi(2)+4\ln 2-4$$

PROBLEM 3.077-*Solution by Abdelhak Maoukouf-Casablanca-Morocco*

$$L=\prod_{n=1}^{\infty}\left(e\left(\frac{n}{n+1}\right)^n\sqrt{\frac{n}{n+1}}\right)=\lim_{p\to\infty}\prod_{n=1}^{p}\left(e\left(\frac{n}{n+1}\right)^n\sqrt{\frac{n}{n+1}}\right)$$

$$=\lim_{p\to\infty}e^p\prod_{n=1}^{p}\left(\frac{n^n}{(n+1)^{n+1}}\sqrt{n(n+1)}\right)=\lim_{p\to\infty}e^p\prod_{n=1}^{p}\left(\frac{n^n}{(n+1)^{n+1}}\right)\prod_{n=1}^{p}\left(\sqrt{n(n+1)}\right)$$

$$=\lim_{p\to\infty}e^p\frac{1^1}{(p+1)^{p+1}}\sqrt{p!\,(p+1)!}=\lim_{n\to\infty}\frac{e^p p!}{(p+1)^{p+\frac{1}{2}}}=\lim_{p\to\infty}\frac{e^p\left(\frac{p}{e}\right)^p\sqrt{2\pi p}}{(p+1)^{p+\frac{1}{2}}}$$

$$=\lim_{p\to\infty}\sqrt{2\pi\frac{p}{p+1}}\left(\frac{p}{p+1}\right)^p=\lim_{p\to\infty}\sqrt{2\pi\frac{p}{p+1}}\left(\frac{1}{\left(1+\frac{1}{p}\right)^p}\right)=\frac{\sqrt{2\pi}}{e}\because\lim_{p\to\infty}\left(1+\frac{1}{p}\right)^p=e$$

PROBLEM 3.078-*Solution by Abdelhak Maoukouf-Casablanca-Morocco*

$$\Omega=\lim_{+\infty} n\left(\sqrt[2n+2]{(2n+1)!!}-\sqrt[n]{(2n-1)!!}\right)\left(\sqrt[2n+2]{(n+1)!}-\sqrt[2n]{n!}\right)$$

$$=\lim_{+\infty} n\left(\sqrt[2n]{\frac{(2n+1)!}{2^n\times n!}}-\sqrt[2n]{\frac{(2n-1)!}{2^{n-1}(n-1)!}}\right)\left(\sqrt[2n]{(n+1)!}-\sqrt[2n]{n!}\right)$$

$$=\lim_{+\infty} n\left(\frac{(2n-1)!}{2^{n-1}(n-1)!}\right)^{\frac{1}{2n}}(n!)^{\frac{1}{2n}}\left(\left(\frac{(2n+1)!}{2^n\, n!}\right)\times\left(\frac{2^{n-1}(n-1)!}{(2n-1)!}\right)^{\frac{1}{2n}}\right)^{\frac{1}{2n}}\cdot$$

$$\cdot\left((n+1)^{\frac{1}{2n}}\times\frac{1}{(n!)^{\frac{1}{2n}}}-1\right)$$

$$=\lim_{+\infty} n\left(\frac{(2n-1)!\times n}{2^{n-1}}\right)^{\frac{1}{2n}}\left(\left(\frac{2n\times(2n+1)}{2\times n}\right)^{\frac{1}{2n}}-1\right)\left((n+1)^{\frac{1}{2n}}-1\right)$$

$$=\lim_{+\infty} n\left(\frac{(2n)!}{2^n}\right)^{\frac{1}{2n}}\left(\sqrt[2n]{2n+1}-1\right)\left(\sqrt[2n]{n+1}-1\right)$$

$$=\lim_{+\infty}\frac{n}{\sqrt{2}}\cdot\left(\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}\right)^{\frac{1}{2n}}\left(\frac{e^{\frac{\ln(2n+1)}{2n}}-1}{\frac{\ln(2n+1)}{2n}}\right)\left(\frac{e^{\frac{\ln(n+1)}{2n}}-1}{\frac{\ln(n+1)}{2n}}\right)\times$$

$$\times\frac{\ln(2n+1)\ln(n+1)}{4n^2}$$

$$\sim\lim_{+\infty}\frac{n}{\sqrt{2}}\times\left(\frac{2n}{e}\right)\times\frac{\ln(2n+1)\ln(n+1)}{4n^2}=\lim_{+\infty}\frac{\sqrt{2}}{4e}\times\ln(2n+1)\ln(n+1)\to+\infty$$

PROBLEM 3.079-*Solution by proposer*

$$If\ x,t>0\ then\ \frac{t^3}{x+t}\ge\frac{5t^2-x^2}{8}\Leftrightarrow(x-t)^2(3t+x)\ge 0;\frac{1}{x+t}\ge\frac{5}{8t}-\frac{x^2}{8t^3}$$

$$\int_a^b\frac{dt}{x+t}\ge\int_a^b\left(\frac{5}{8t}-\frac{x^2}{8t^3}\right)dt\Leftrightarrow\ln\frac{b+x}{a+x}\ge\frac{5}{8}\ln\frac{b}{a}+\frac{x^2}{16}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\Rightarrow$$

$$\sum_{cyclic}\ln\frac{b+x}{a+x}\ge\sum_{cyclic}\left(\frac{5}{8}\ln\frac{b}{a}+\frac{x^2}{16}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\right)=\frac{15}{8}\ln\frac{b}{a}+\frac{1}{16}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)(x^2+y^2+z^2)$$

PROBLEM 3.080-*Solution by Remus Florin Stanca-Romania*

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)}=\lim_{x\to\infty}\frac{f(x+1)}{x+1}\cdot\frac{x}{f(x)}\cdot\frac{x+1}{x}=1$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{f(x+1)}{f(x)}\right)^x = \lim_{x\to\infty}\left(\frac{f(x+1)-f(x)}{f(x)}+1\right)^{\frac{f(x)}{f(x+1)-f(x)}\cdot x\cdot\frac{f(x+1)-f(x)}{f(x)}} =$$

$$= e^{\lim_{x\to\infty}\frac{x}{f(x)}\cdot(f(x+1)-f(x))} = b \Rightarrow e^{\frac{\Omega}{a}} = b \Rightarrow \frac{\Omega}{a} = \ln(b) \Rightarrow \Omega = a\ln(b).$$

PROBLEM 3.081-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$B_n(t) = n^{1-t}\left(\frac{(n+1)^{2t}}{\left(\sqrt[n+1]{(n+1)!}\right)^t} - \frac{n^{2t}}{\left(\sqrt[n]{n!}\right)^t}\right)$$

$$= \left(\left(\frac{n}{\sqrt[n]{n!}}\right)^t\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) \text{ where } u_n = \left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}\cdot\frac{\sqrt[n]{n!}}{n^2}\right)^t \ \forall n\in\mathbb{N}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} \overset{CAUCHY-D'ALEMBERT}{=} \lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty}u_n = \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\left(1+\frac{1}{n}\right)\right)^t = 1, \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1$$

$$\lim_{n\to\infty}u_n^n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\left(1+\frac{1}{n}\right)\cdot\left(1+\frac{1}{n}\right)^{2n}\right)^t = e^t$$

$$\therefore \lim_{n\to\infty}B_n(t) = \lim_{n\to\infty}\left(\left(\frac{n}{\sqrt[n]{n!}}\right)^t\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) = e^t\cdot\ln e^t = te^t \ \textit{(Ans :)}$$

PROBLEM 3.082-*Solution by Kays Tomy-Nador-Tunisia*

$$I_n = \int_0^{\frac{\pi}{2}}\sin^2 x\left(\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)+\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx$$

Let $J_n = \int_0^{\frac{\pi}{2}}\sin^2 x\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)dx.$

And $T_n = \int_0^{\frac{\pi}{2}}\sin^2 x\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)dx$

$\Rightarrow I_n = J_n + T_n$. *As* $\cos^2 x + \sin^2 x = 1$. *Then we have*

$$(*)\begin{cases} J_n = \int_0^{\frac{\pi}{2}}\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right) - dx - \int_0^{\frac{\pi}{2}}\cos^3 x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)dx \\ T_n = \int_0^{\frac{\pi}{2}}\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)dx - \int_0^{\frac{\pi}{2}}\cos^2 x\sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)dx \end{cases}$$

Let us denote $K_n = \int_0^{\frac{\pi}{2}}\cos^3 x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right)dx$

By substitution variable $x = \frac{\pi}{2} - U \Rightarrow \sin x = \cos U$ *and* $\cos x = \sin u$ *and* $dx = -du$

$$\Rightarrow K_n = -\int_{\frac{\pi}{2}}^{0} \sin^3 u \cos^{2n}\left(\frac{\pi}{2}\cos u\right) du$$

$$\Rightarrow K_n = \int_0^{\frac{\pi}{2}} (1-\cos^2 u)\sin(u)\cos^{2n+1}\left(\frac{\pi}{2}\cos(u)\right) du = T_n \ (**)$$

Then combining (*) and (**) we get $I_n = \int_0^{\frac{\pi}{2}} \cos x \cos^{2n+1}\left(\frac{\pi}{2}\sin x\right) dx$

substituting $U = \frac{\pi}{2}\sin x \Rightarrow du = \frac{\pi}{2}\cos x\, dx \Rightarrow I_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n+1}(u)\, du$

Then $I_n = \frac{2}{\pi} W_{2n+1}$*; with* w_{2n+1} *is the Wallis integral for der* $2n+1$

We know that $w_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}$*. Finally we get* $I_n = \frac{2^{2n+1}(n!)^2}{\pi\,(2n+1)!}$

PROBLEM 3.083-*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum (b+c-2a)^2 = b^2+c^2+4a^2+2bc-4ca-4ab+c^2+a^2+4b^2+$$
$$+2ca-4ab-4bc+a^2+b^2+4c^2+2ab-4bc-4ca$$
$$\overset{(1)}{=} 6\sum a^2 - 6\sum ab$$

(1)⇒ given inequality is $R(6\sum a^2 - 6\sum ab) \le 4(R-2r)\sum a^2$

$$\Leftrightarrow (R+4r)\sum a^2 \le 3R\left(\sum ab\right) \Leftrightarrow (2R+8r)(s^2-4Rr-r^2) \le 3R(s^2+4Rr+r^2)$$

$$\Leftrightarrow Rs^2 + (5R+8r)(4R+r)r \ge 8rs^2 \quad (2)$$

Now, LHS of (2) $\overset{Gerretsen}{\ge} Rr(16R-5r) + (5R+8r)(4R+r)r \overset{?}{\ge} 8rs^2$

$$\Leftrightarrow R(16R-5r) + (5R+8r)(4R+r) \overset{?}{\ge} 8s^2 \Leftrightarrow 8s^2 \overset{?}{\le} 36R^2 + 32Rr + 8r^2$$

$$\Leftrightarrow 2s^2 \overset{?}{\le} 9R^2 + 8Rr + 2r^2 \quad (3)$$

Now, LHS of (3) $\overset{Gerrestsen}{\le} 8R^2 + 8Rr + 6r^2 \overset{?}{\le} 9R^2 + 8Rr + 2r^2$

$\Leftrightarrow R^2 \overset{?}{\ge} 4r^2 \Leftrightarrow R \overset{?}{\ge} 2r \rightarrow$ *true by Euler (Proved)*

PROBLEM 3.084-*Solution by Khalef Ruhemi-Jarash-Jordan*

Evaluate $I := \int_0^1 \int_0^1 \frac{(\ln(x)\ln(y))^s}{1-xy} \cdot dx \cdot dy \quad (*)$

$$I = \int_0^1 \int_0^1 \frac{\left(\ln\left(\frac{1}{x}\right) \cdot \ln\left(\frac{1}{y}\right)\right)^s \cdot dx \cdot dy}{1-xy} = \int_0^1 \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^s \cdot \left(\ln\left(\frac{1}{y}\right)\right)^s \cdot \sum_{n=0}^{\infty} x^n \cdot y^n \cdot dx \cdot dy$$

$$= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^s \cdot x^n \cdot \left(\ln\left(\frac{1}{y}\right)\right)^s \cdot y^n \cdot dx \cdot dy$$

$$= \sum_{n=0}^{\infty} \left(\int_0^1 \left(\ln\left(\frac{1}{y}\right)\right)^s \cdot y^n \cdot dy \cdot \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^s x^n \cdot dx\right)$$

$$= \sum_{n=0}^{\infty}\left(\int_0^1 x^n \cdot \left(\ln\left(\frac{1}{x}\right)\right)^s dx\right)^2 = I \quad (1)$$

To find $\int_0^1 x^n \left(\ln\left(\frac{1}{x}\right)\right)^s dx \coloneqq A$, *let* $\ln\left(\frac{1}{x}\right) = v \Rightarrow x = e^{-v} \Rightarrow dx = -e^{-v}dv$

$$\therefore A = \int_0^{\infty} v^s \cdot e^{-(1+n)v} dv = \frac{\Gamma(1+s)}{(1+n)^{1+s}} \therefore I = \sum_{n=0}^{\infty} \frac{\Gamma(1+s)}{(1+n)^{2(1+s)}}$$

$$\therefore I = \sum_{n=0}^{\infty} \frac{\Gamma^2(1+s)}{n^{2(1+s)}} = \Gamma^2(1+s) \cdot \sum_{n=0}^{\infty} \frac{1}{n^{2+2s}} = \Gamma^2(1+s)\mathcal{G}(2+2s), s > -\frac{1}{2}$$

$$\therefore \int_0^1 \int_0^1 \frac{(\ln(x)\ln(y))^s}{1-xy} dx\, dy = \Gamma^2(1+s)\mathcal{G}(2+2s), s > -\frac{1}{2}$$

PROBLEM 3.085-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\lim_{n\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{k+1}{x+1}} - \left(\Gamma(x+1)\right)^{\frac{k+1}{x}}\right) \cdot \left(\Gamma(x+1)\right)^{-\frac{k}{x}}$$

$$= \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}}\left(\left(\Gamma(n+2)\right)^{\frac{k+1}{n+1}} - \left(\Gamma(n+1)\right)^{\frac{k+1}{n}}\right) \cdot \left(\Gamma(n+1)\right)^{-\frac{k}{n}}$$

$$= \lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{k+1} - \left(\sqrt[n]{n!}\right)^{k+1}\right) \cdot \left(\sqrt[n]{n!}\right)^{-k}$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n\right) \text{ where } u_n = \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{k+1} \quad \forall n \in \mathbb{N}$$

Now, $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}, \lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n}{\sqrt[n]{n!}} \cdot \frac{n+1}{n}\right)^{k+1} = 1$

$$\therefore \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1 \cdot \lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{n}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^{k+1} = e^{k+1}$$

$$\therefore \lim_{n\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{k+1}{x+1}} - \left(\Gamma(x+1)\right)^{\frac{k+1}{x}}\right) \cdot \left(\Gamma(x+1)\right)^{-\frac{k}{x}} = \frac{\ln e^{k+1}}{e} = \frac{k+1}{e}$$

PROBLEM 3.086-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$f(x) + f(-x) = 0, g(x) + g(-x) = 0$$

$$\Omega = \int_{-a}^{a} f(x) \ln\left(b^{g(x)} + c^{g(x)}\right) dx = \int_{-a}^{a} f(a-a-x) \ln\left(b^{g(a-a-x)} + c^{g(a-a-x)}\right) dx$$

$$= \int_{-a}^{a} f(-x) \ln\left(b^{g(-x)} + c^{g(-x)}\right) dx = -\int_{-a}^{a} f(x) \ln\left(b^{-g(x)} + c^{-g(x)}\right) dx$$

$$= -\int_{-a}^{a} f(x) \ln\left(b^{g(x)} + c^{g(x)}\right) + \ln(bc) \int_{-a}^{a} f(x)\, g(x) dx$$

$$\Rightarrow 2\Omega = \ln(bc) \int_{-a}^{a} f(x)\, g(x) dx = 2\ln(bc) \int_{0}^{a} f(x)\, g(x) dx$$

$$\Omega = \ln(bc)\int_0^a f(x)g(x)\,dx \quad \textit{(Proved)}$$

PROBLEM 3.087-***Solution by Anas Adlany-El Zemamra-Morocco***

$$B = \int_a^b f(x)\arctan\big(g(x)\big)\ln\big(1+e^{h(x)}\big)\,dx \quad (u = a+b-x \Rightarrow du = -dx)$$

$$= -\int_a^b f(a+b-u)\arctan\big(g(a+b-x)\big)\ln\big(1+e^{h(a+b-u)}\big)\,du$$

$$= \int_a^b -f(x)\arctan\big(g(x)\big)\ln\left(\frac{1+e^{h(x)}}{e^{h(x)}}\right) =$$

$$-\int_a^b f(x)\arctan\big(g(x)\big)\left[\ln\big(1+e^{h(x)}\big)-h(x)\right]dx = -B + \int_a^b f(x)\,h(x)\arctan\big(g(x)\big)\,dx \Rightarrow$$

$$\int_a^b f(x)\arctan\big(g(x)\big)\ln\big(1+e^{h(x)}\big)\,dx = \frac{1}{2}\int_a^b f(x)\,h(x)\arctan\big(g(x)\big)\,dx$$

PROBLEM 3.088-***Solution by Thanasis Xenos-Greece***

$$I = \int_0^1 \frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx \quad (1)$$

$$t = 1-x$$

$$I = \int_1^0 \frac{\sqrt{t}+\sqrt{1-t}}{1+\sqrt{2(1-t)}}f(1-t)(-dt)$$

$$I = \int_0^1 \frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2(1-x)}}f(x)dx \quad (2)$$

$$(1)+(2) \Rightarrow 2I = \int_0^1\left(\sqrt{x}+\sqrt{1-x}\right)\cdot\left(\frac{1}{1+\sqrt{2x}}+\frac{1}{1+\sqrt{2(1-x)}}\right)f(x)dx$$

$$2I = \int_0^1\left(\sqrt{x}+\sqrt{1-x}\right)\cdot\frac{2+\sqrt{2x}+\sqrt{2(1-x)}}{\left(1+\sqrt{2x}\right)\cdot\left(1+\sqrt{2(1-x)}\right)}f(x)dx$$

$$2I = \sqrt{2}\int_0^1 \frac{\left(\sqrt{x}+\sqrt{1-x}\right)\left(\sqrt{2}+\sqrt{x}+\sqrt{1-x}\right)}{\left(1+\sqrt{2x}\right)\cdot\left(1+\sqrt{2(1-x)}\right)}f(x)dx \ (3)$$

$$\left(\sqrt{x}+\sqrt{1-x}\right)\cdot\left(\sqrt{2}+\sqrt{x}+\sqrt{1-x}\right)$$

$$= \sqrt{2x}+x+\sqrt{x(1-x)}+\sqrt{2(1-x)}+\sqrt{x(1+x)}+1-x$$

$$= 1+\sqrt{2x}+\sqrt{2(1-x)}+2\sqrt{x(1-x)} = \left(1+\sqrt{2x}\right)\left(1+\sqrt{2(1-x)}\right)$$

$$(3) \Rightarrow 2I = \sqrt{2}\cdot\int_0^1 f(x)\,dx \Rightarrow I = \frac{\sqrt{2}}{2}\cdot\int_0^1 f(x)\,dx$$

PROBLEM 3.089-*Solution by Ali Shather-Nasyria-Iraq*

$$I=\int_0^1\left(\ln(x)\ln(1-x)+Li_2(x)\right)\left(\frac{Li_2(x)}{x(1-x)}-\frac{\zeta(2)}{1-x}\right)dx$$

$$I=\int_0^1\frac{\ln(x)\ln(1-x)Li_2(x)}{x}dx+\int_0^1\frac{\ln(x)\ln(1-x)Li_2(x)}{1-x}-$$

$$-\zeta(2)\int_0^1\frac{\ln(x)\ln(1-x)}{1-x}dx+\int_0^1\frac{Li_2^2(x)}{x}dx+\int_0^1\frac{Li_2^2(x)-\zeta(2)Li_2(x)}{1-x}dx$$

combining the second and the las term, we get

$$\int_0^1\frac{Li_2^2(x)-Li_2(x)[\zeta(2)-\ln(x)\ln(1-x)]}{1-x}dx=\int_0^1\frac{Li_2^2(x)-Li_2(x)[Li_2(x)+Li_2(1-x)]}{1-x}dx=$$

$$=-\int_0^1\frac{Li_2(x)Li_2(1-x)}{1-x}dx=-\int_0^1\frac{Li_2(1-x)Li_2(x)}{x}dx=$$

$$=-\int_0^1\frac{Li_2(x)}{x}[\zeta(2)-\ln(x)\ln(1-x)-Li_2(x)]dx=-\zeta(2)\int_0^1\frac{Li_2(x)}{x}dx+$$

$$+\int_0^1\frac{\ln(x)\ln(1-x)Li_2(x)}{x}dx+\int_0^1\frac{Li_2^2(x)}{x}dx$$

$$\therefore I=\int_0^1\frac{\ln(x)\ln(1-x)Li_2(x)}{x}dx-\zeta(2)\int_0^1\frac{\ln(x)\ln(1-x)}{1-x}dx+\int_0^1\frac{Li_2^2(x)}{x}dx-$$

$$-\zeta(2)\int_0^1\frac{Li_2(x)}{x}dx+\int_0^1\frac{\ln(x)\ln(1-x)Li_2(x)}{x}dx+\int_0^1\frac{Li_2^2(x)}{x}dx$$

$$I=2\int_0^1\frac{\ln(x)\ln(1-x)Li_2(x)}{x}dx-\zeta(2)\int_0^1\frac{\ln(x)\ln(1-x)}{1-x}dx+2\int_0^1\frac{Li_2^2(x)}{x}dx-\zeta(2)\int_0^1\frac{Li_2(x)}{x}dx$$

$$I=[-Li_2^2(x)\ln(x)]_0^1+\int_0^1\frac{Li_2^2(x)}{x}-\zeta(2)\int_0^1\frac{\ln(1-x)\ln(x)}{x}dx+2\int_0^1\frac{Li_2^2(x)}{x}dx-\zeta(2)\zeta(3)$$

$$I=3\int_0^1\frac{Li_2^2(x)}{x}dx+\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\ln(x)\,dx-\zeta(2)\zeta(3)=$$

$$=3\int_0^1\frac{Li_2^2(x)}{x}dx-\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^3}-\zeta(2)\zeta(3)=3\int_0^1\frac{Li_2^2(x)}{x}dx-2\zeta(2)\zeta(3)$$

$$\int_0^1\frac{Li_2^2(x)}{x}dx=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(nk)^2}\int_0^1 x^{n+k-1}dx=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(nk)^2(n+k)}=$$

$$=\sum_{n=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{\infty}\frac{k}{n^2(n+k)}=\sum_{n=1}^{\infty}\frac{1}{k^3}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{n(n+k)}\right]$$

$$=\sum_{k=1}^{\infty}\frac{1}{k^3}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{H_k}{k}\right]=\zeta(3)\zeta(2)-\sum_{n=1}^{\infty}\frac{H_k}{k^4}=\zeta(3)\zeta(2)-\big(3\zeta(5)-\zeta(3)\zeta(2)\big)=$$

$$=2\zeta(3)\zeta(2)-3\zeta(5)$$

$$\therefore I=3\big(2\zeta(2)\zeta(3)-3\zeta(5)\big)-2\zeta(2)\zeta(3)=4\zeta(2)\zeta(3)-9\zeta(5)$$

PROBLEM 3.090-*Solution by Khalef Ruhemi-Jarash-Jordan*

$$I=\int_0^1\ln\big(\Gamma(x)\big)\sin(2\pi kx)\,dx\ \ldots.\#$$

$$\textit{Notice that } \int_0^1\frac{1-t^{x-1}}{1-t}\cdot dt=\frac{\Gamma'(x)}{\Gamma(x)}-\Gamma'(1)$$

$$\therefore\int_0^1\left(\frac{1}{1-t}\right)\left(\int_1^x(1-t^{v-1})dv\right)\cdot dt=\int_1^x\left(\frac{\Gamma'(v)}{\Gamma(v)}-\Gamma'(1)\right)dv$$

$$=\ln\big(\Gamma(v)\big)-v\Gamma(1)|_1^x=\ln\big(\Gamma(x)\big)-x\Gamma'(1)+\Gamma'(1)$$

$$=\int_0^1\left(\frac{1}{1-t}\right)\cdot\left(v-\frac{t^{v-1}}{\ln(t)}\right)\cdot dt=\int_0^1\left(\frac{1}{1-t}\right)\cdot\left(x-\frac{t^{x-1}}{\ln(t)}-1+\frac{1}{\ln(t)}\right)dt$$

$$=\int_0^1\left(\frac{1-t^{x-1}-(1-x)\ln(t)}{(1-t)\ln(t)}\right)\cdot dt=\ln\big(\Gamma(x)\big)+\Gamma'(1)(1-x)$$

$$\therefore\ln\big(\Gamma(x)\big)=\Gamma'(1)(x-1)+\int_0^1\frac{1-t^{x-1}-(1-x)\ln(t)}{(1-t)\ln(t)}\cdot dt\quad (1)$$

$$\therefore\ln\big(\Gamma(x)\big)\sin(2\pi kx)=\Gamma'(1)(x-1)\sin(2\pi kx)$$
$$+\int_0^1\frac{\sin(2\pi kx)-\ln(t)\sin(2\pi kx)+\ln(t)x\sin(2\pi kx)-t^{x-1}\cdot\sin(2\pi kx)}{(1-t)\ln(t)}\cdot dt\quad (2)$$

$$\therefore\int_0^1\ln\big(\Gamma(x)\big)\sin(2\pi kx)\,dx=\Gamma'(1)\int_0^1(x-1)\sin(2\pi kx)\,dx$$

$$+\int_0^1\frac{dt}{(1-t)\ln(t)}\cdot\left[\begin{array}{c}\int_0^1\sin(2\pi kx)\,dx-\ln(t)\int_0^1\sin(2\pi kx)\,dx\\+\ln(t)\int_0^1x\sin(2\pi kx)\,dx-\frac{1}{t}\int_0^1t^x\sin(2\pi kx)\,dx\end{array}\right]\quad (3)$$

$$\int_0^1\sin(2\pi kx)\,dx=\cos\frac{(2\pi kx)}{2\pi k}\cdot|_1^0=\frac{1-1}{2\pi k}=0$$

$$\int_0^1x\sin(2\pi kx)\,dx=x\cos\frac{(2\pi kx)}{2\pi k}|_1^0+\frac{1}{2\pi k}\cdot\int_0^1\cos(2\pi kx)\,dx$$

$$=-\frac{\cos(2\pi k)}{2\pi k}+\left(\frac{1}{4\pi^2k^2}\right)\sin(2\pi kx)|_0^1=-\frac{1}{2\pi k}$$

$$\int_0^1 t^x \sin(2\pi kx)\,dx = \int_0^1 e^{(\ln(t))x}\cdot\sin(2\pi kx)\,dx$$

$$= \frac{e^{\ln(t)x}}{\ln(t)}\cdot\sin(2\pi kx)\,|_0^1 - \frac{2\pi k}{\ln(t)}\int_0^1 e^{\ln(t)x}\cdot\cos(2\pi kx)$$

$$= -\frac{2\pi k}{\ln(t)}\cdot\left(\frac{e^{\ln(t)x}}{\ln(t)}\cdot\cos(2\pi kx)\,|_0^1 + \frac{2\pi k}{\ln(t)}\int_0^1 e^{\ln(t)x}\cdot\sin(2\pi kx)\,dx\right)$$

$$= -\frac{2\pi k}{\ln(t)}\cdot\left(\frac{(t-1)}{\ln(t)} + \frac{2\pi k}{\ln(t)}\int_0^1 e^{\ln(t)x}\sin(2\pi kx)\,dx\right)$$

$$\therefore \int_0^1 t^x \sin(2\pi kx)\,dx = \frac{2\pi k(1-t)}{\ln^2(t)} - \frac{4\pi^2k^2}{\ln^2(t)}\cdot\int_0^1 t^x\sin(2\pi kx)\,dx$$

$$\therefore \int_0^1 t^x \sin(2\pi kx)\,dx = \frac{\frac{2\pi k(1-t)}{\ln^2(t)}}{1+\frac{4\pi^2k^2}{\ln^2(t)}} = \frac{2\pi k(1-t)}{4\pi^2k^2+\ln^2(t)}$$

$$\therefore \int_0^1 t^x \sin(2\pi kx)\,dx = \frac{2\pi k(1-t)}{4\pi^2k^2+\ln^2(t)}$$

$$\therefore I = \int_0^1\left(-\frac{\ln(t)}{2\pi k} - \frac{2\pi k(1-t)}{t(4\pi^2k^2+\ln^2(t))}\right)\cdot\frac{dt}{(1-t)\ln(t)} - \frac{\Gamma'(1)}{2\pi k} \quad (4)$$

$$\therefore I = -\frac{\Gamma'(1)}{B} - \int_0^1\left(\frac{\ln(x)}{B} + \frac{B(1-x)}{x(B^2+\ln^2 x)}\right)\cdot\frac{dx}{(1-x)\ln(x)},\ B := 2\pi k$$

$$\therefore I = -\frac{\Gamma'(1)}{B} - \int_0^1\left(\frac{1}{B(1-x)} + \frac{1}{B\ln(x)} + \frac{B}{x\ln x\,(B^2+\ln^2 x)} - \frac{1}{B\ln(x)}\right)dx$$

$$= -\frac{\Gamma'(1)}{B} - \frac{1}{B}\int_0^1\left(\frac{1}{1-x}+\frac{1}{\ln x}\right)dx + \int_0^1 \frac{1}{\ln(x)}\left(\frac{1}{B} - \frac{B}{x(B^2+\ln^2 x)}\right)dx$$

$$= -\frac{\Gamma'(1)}{B} + \frac{\Gamma'(1)}{B} + \int_0^1 \frac{1}{\ln(x)}\left(\frac{1}{B} - \frac{B}{x(B^2+\ln^2 x)}\right)dx$$

$$\therefore I = \int_0^1 \frac{1}{\ln(x)}\left(\frac{1}{B} - \frac{B}{x(B^2+\ln^2 x)}\right)dx \quad (6)$$

$$Let\ \ln\left(\frac{1}{x}\right) = y \Rightarrow x = e^{-y} \Rightarrow dx = -e^{-y}\cdot dy$$

$$\therefore I = \int_0^\infty -\frac{1}{y}\left(\frac{1}{B} - \frac{B}{e^{-y}(B^2+y^2)}\right)e^{-y}\cdot dy$$

$$= \int_0^\infty \left(\frac{-e^{-y}}{By} + \frac{B}{y(B^2+y^2)}\right) dy = \int_0^1 \left(\frac{1}{By} - \frac{e^{-y}}{By} - \frac{y}{B(B^2+y^2)}\right) dy$$

$$= \frac{1}{B}\int_0^\infty \left(\frac{1-e^{-x}}{x} - \frac{x}{B^2+x^2}\right) dx = I \quad (7)$$

$$Let\ x = By \Rightarrow dx = Bdy$$

$$\therefore I = \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1-e^{-Bx}}{x} - \frac{x}{1+x^2}\right) dx$$

$$\therefore I = \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1-e^{-BX}}{x} + \frac{1}{1+x} - \frac{x}{1+x^2} - \frac{1}{1+x}\right) dx$$

$$= \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1}{1+x} - \frac{x}{1+x^2}\right) dx + \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1-e^{-Bx}}{x} - \frac{1}{1+x}\right) dx$$

$$Since \int_0^\infty \left(\frac{1}{1+x} - \frac{x}{1+x^2}\right) dx = \ln\left(\frac{1+x}{\sqrt{1+x^2}}\right)\Big|_0^\infty = \lim_{n\to\infty} \ln\left(\frac{\frac{1}{x}+1}{\sqrt{\frac{1}{x^2}+1}}\right) = \ln(1) = 0$$

$$\therefore I = \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1-e^{-Bx}}{x} - \frac{1}{1+x}\right) dx$$

$$\therefore I = \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1}{x} - \frac{1}{1+x} - \frac{e^{-x}}{x} + \frac{e^{-x}}{x} - \frac{e^{-Bx}}{x}\right) dx$$

$$= \left(\frac{1}{B}\right)\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{1}{x} dx + \left(\frac{1}{B}\right)\int_0^\infty \frac{e^{-x}-e^{-Bx}}{x} dx = -\frac{\Gamma'(1)}{B} + \left(\frac{1}{B}\right)\int_0^\infty \frac{e^{-x}-e^{-Bx}}{x} dx = I \quad (8)$$

$$Let\ F(A) := \int_0^\infty \frac{e^{-x}-e^{-Ax}}{x} dx, A > 0 \Rightarrow F(1) = 0 \Rightarrow F'(A) = \int_0^\infty e^{-Ax} dx = \frac{1}{A}$$

$$\therefore F(A) = \int_0^A \frac{dx}{x} = \ln(A) \Rightarrow I = \frac{\gamma}{B} + \frac{\ln(B)}{B}$$

$$\therefore I = \int_0^1 \ln(\Gamma(x)) \sin(2\pi kx)\, dx = \frac{\gamma + \ln(2\pi k)}{2\pi k}$$

PROBLEM 3.091-***Solution by Abdallah El Farisi-Bechar-Algerie***

$$\int_{-a}^{a} f(x)\ln(1+e^{g(x)})\arctan(h(x))\,dx = -\int_{-a}^{a} f(x)\ln(1+e^{-g(x)})\arctan(h(x))\,dx$$

$$= -\int_{-a}^{a} f(x)\left(\ln(1+e^{g(x)}) - g(x)\right)\arctan(h(x))\,dx$$

$$= -\int_{-a}^{a} f(x)\left(\ln\left(1+e^{g(x)}\right)\right)\arctan\left(h(x)\right)dx + \int_{-a}^{a} f(x)\,g(x)\arctan\left(h(x)\right)dx$$

$$= -\int_{-a}^{a} f(x)\left(\ln\left(1+e^{g(x)}\right)\right)\arctan\left(h(x)\right)dx + 2\int_{0}^{a} f(x)g(x)\arctan\left(h(x)\right)dx$$

$$\int_{-a}^{a} f(x)\ln\left(1+e^{g(x)}\right)\arctan\left(h(x)\right)dx = \int_{0}^{a} f(x)g(x)\arctan\left(h(x)\right)$$

PROBLEM 3.092-***Solution by Soumitra Mandal-Chandar Nagore-India***

$$\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$$

$$\Omega_n = \lim_{n\to\infty}\sqrt[3]{n^2}\left(\sqrt[3(n+1)]{(n+1)!} - \sqrt[3n]{n!}\right)$$

$$= \lim_{n\to\infty}\left(\sqrt[3]{\frac{\sqrt[3]{n!}}{n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) \textit{ where } u_n = \frac{\sqrt[3(n+1)]{(n+1)!}}{\sqrt[3n]{n!}} \textit{ for all } n\in\mathbb{N}$$

$$\therefore u_n = \frac{\sqrt[3(n+1)]{(n+1)!}}{\sqrt[3n]{n!}} = \frac{\sqrt[3(n+1)]{(n+1)!}}{\sqrt[3]{n+1}}\cdot\frac{\sqrt[3]{n}}{\sqrt[3n]{n!}}\cdot\sqrt[3]{1+\frac{1}{n}} \textit{ then } \lim_{n\to\infty} u_n = 1$$

$$\textit{now, } u_n \to 1 \textit{ then } \frac{u_n-1}{\ln u_n} \to 1 \textit{ for all } n\to\infty$$

$$u_n^n = \left(\frac{\sqrt[3(n+1)]{(n+1)!}}{\sqrt[3n]{n!}}\right)^n = \sqrt[3]{\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}} = \sqrt[3]{\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}}$$

$$\therefore \lim_{n\to\infty} u_n^n = \sqrt[3]{e} \textit{ and } \Omega_n = \frac{1}{\sqrt[3]{e}}\cdot 1\cdot\ln\sqrt[3]{e} = \frac{1}{3\sqrt[3]{e}}$$

PROBLEM 3.093-***Solution by Soumitra Mandal-Chandar Nagore-India***

a. Let $\lim_{n\to\infty}(b_n - u\cdot a_n) = v$ *now let* $\lim_{n\to\infty} a_n = x > 0$ *because* $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a > 0$ *then*

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a \Rightarrow \frac{x}{x}\cdot\frac{1}{\infty} = a \Rightarrow a = 0\textit{, which is false. Then } \lim_{n\to\infty} a_n = \infty$$

$$\textit{now, } \lim_{n\to\infty}(b_n - u\cdot a_n) = v \Rightarrow \lim_{n\to\infty}\left(\frac{b_n}{a_n} - u\right) = v\lim_{n\to\infty}\frac{1}{a_n} = 0 \textit{ then}$$

$$\Rightarrow \lim_{n\to\infty}\frac{b_n}{a_n} = u\textit{. Now, } \lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}$$

$$\overset{Cauchy\ D'Alembert}{=} \lim_{n\to\infty}\left(\frac{b_{n+1}}{a_{n+1}}\cdot\frac{a_n}{b_n}\cdot\frac{a_{n+1}}{n\cdot a_n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{n}{n+1}\right) = \left(u\cdot\frac{1}{u}\cdot a\cdot\frac{1}{e}\right) = \frac{a}{e}$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{b_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) \textit{ where } u_n = \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}$$

$$u_n = \left(\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\frac{n+1}{n}\right) \Rightarrow \lim_{n\to\infty} u_n = 1 \textit{ then } \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1$$

$$\therefore u_n^n=\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^n=\left(\frac{b_{n+1}}{a_{n+1}}\cdot\frac{a_n}{b_n}\cdot\frac{a_{n+1}}{n\cdot a_n}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{n}{n+1}\right)$$

$$\therefore \lim_{n\to\infty}u_n^n=\left(u\cdot\frac{1}{u}\cdot a\cdot\frac{e}{a}\right)=e,\ then$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)=\left(\frac{a}{e}\cdot 1\cdot\ln e\right)=\frac{a}{e}$$

$$b.\ \lim_{n\to\infty}\frac{n}{\sqrt[n]{b_n}}=\frac{e}{a}\ then\ \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)\ where\ u_n=\left(1+\frac{1}{n}\right)^2\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}for\ all\ n\in\mathbb{N}$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^2\cdot\frac{\sqrt[n]{b_n}}{n}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{n}{n+1}\right)=1\ then\ \lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{2n}\cdot\frac{b_n}{a_n}\cdot\frac{a_{n+1}}{b_{n+1}}\cdot\frac{n\cdot a_n}{a_{n+1}}\left(1+\frac{1}{n}\right)\cdot\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\right)=\left(e^2\cdot u\cdot\frac{1}{u}\cdot\frac{1}{a}\cdot\frac{a}{e}\right)=e$$

then

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)=\left(\frac{e}{a}\cdot 1\cdot\ln e\right)=\frac{e}{a}$$

PROBLEM 3.094-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}\ then\ \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)\ where\ u_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}for\ all\ n\in\mathbb{N}$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{n}\right)=1\ then\ \frac{u_n-1}{\ln u_n}\to 1\ as\ n\to\infty$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)=e$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\left(\frac{1}{e}\cdot 1\cdot\ln e\right)=\frac{1}{e}$$

$$\lim_{n\to\infty}\left(s_n\sqrt[n+1]{(n+1)!}-\frac{\pi^2}{6}\sqrt[n]{n!}\right)$$

$$=\lim_{n\to\infty}\left(s_n-\frac{\pi^2}{6}\right)\sqrt[n+1]{(n+1)!}+\frac{\pi^2}{6}\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)$$

$$=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{n}\cdot n\left(s_n-\frac{\pi^2}{6}\right)+\frac{\pi^2}{6e}$$

$$=\frac{1}{e}\lim_{n\to\infty}\left(s_n-\frac{\pi^2}{6}\right)+\frac{\pi^2}{6e}=\frac{1}{e}\lim_{n\to\infty}\frac{s_{n+1}-s_n}{\frac{1}{(n+1)}-\frac{1}{n}}+\frac{\pi^2}{6e}=\frac{\pi^2}{6e}\ (Ans:)$$

PROBLEM 3.095-*Solution by proposer*

** Hence (1), by AM-GM inequality for three positive real numbers we have:*

$$3=\frac{1}{\sqrt{a^3}}+\frac{1}{\sqrt{b^3}}+\frac{1}{\sqrt{c^3}}\geq 3\cdot\sqrt[3]{\frac{1}{\sqrt{a^3}\cdot\sqrt{b^3}\cdot\sqrt{c^3}}}=\frac{3}{\sqrt[6]{(abc)^3}}=\frac{3}{\sqrt{abc}}\Leftrightarrow 3\geq\frac{3}{\sqrt{abc}}\Leftrightarrow$$

$$\Leftrightarrow\sqrt{abc}\geq 1\Leftrightarrow abc\geq 1$$

Hence (2):$\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{a^3c+b^3a+c^3b}{abc}\leq\frac{a^3c+b^3a+c^3b}{1}=a^3c+b^3a+c^3b$ *(3)*

- By AM-GM inequality, we have:

$$a^3c+b^3a+c^3b=a^3ac+b^2ba+c^2cb\leq\frac{a^3+(ac)^2}{2}+\frac{b^4+(ba)^2}{2}+\frac{c^4+(cb)^2}{2}$$

$$\Leftrightarrow a^3c+b^3a+c^3b\leq\frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2}\quad (4)$$

- Hence (3), (4): $\Rightarrow\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\leq\frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2}$ *(5)*

- Other, by AM-GM inequality:

$$\frac{a^6+a^6+1}{2}+\frac{b^6+b^6+1}{2}+\frac{c^6+c^6+1}{2}\geq\frac{3\sqrt[3]{a^6\cdot a^6\cdot 1}}{2}+\frac{3\sqrt[3]{b^6\cdot b^6\cdot 1}}{2}+\frac{3\sqrt[3]{c^6\cdot c^6\cdot 1}}{2}=\frac{3(a^4+b^4+c^4)}{2}$$

$$\Leftrightarrow a^6+b^6+c^6+\frac{3}{2}\geq\frac{3(a^4+b^4+c^4)}{2}\Leftrightarrow 2(a^6+b^6+c^6)+3\geq 3(a^4+b^4+c^4)\quad (6)$$

$$(a^3b^3+a^3b^3+1)+(b^3c^3+b^3c^3+1)+(c^3a^3+c^3a^3+1)\geq$$

$$\geq 3\sqrt[3]{(a^3b^3)(a^3b^3)\cdot 1}+3\sqrt[3]{(b^3c^3)(b^3c^3)\cdot 1}+3\sqrt[3]{(c^3a^3)(c^3a^3)\cdot 1}=3(a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow 2(a^3b^3+b^3c^3+c^3a^3)+3\geq 3(a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow 4(a^3b^3+b^3c^3+c^3a^3)+6\geq 6(a^2b^2+b^2c^2+c^2a^2)\quad (7)$$

- Let (6), (7): $\Rightarrow 2(a^6+b^6+c^6)+3+4(a^3b^3+b^3c^3+c^3a^3)+6\geq$

$$\geq 3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow 2(a^6+b^6+c^6+2a^3b^3+2b^3c^3+2c^3a^3)+9\geq$$

$$\geq 3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow 2(a^3+b^3+c^3)^2\geq 3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)-9\quad (8)$$

- By AM-GM inequality and (2). We have:

$$3(a^2b^2+b^2c^2+c^2a^2)\geq 3\cdot 3\cdot\sqrt[3]{(a^2b^2)(b^2c^2)(c^2a^2)}=9\sqrt[3]{(abc)^4}\geq 9\sqrt[3]{1^4}=9$$

$$\Leftrightarrow 6(a^2b^2+b^2c^2+c^2a^2)-9\geq 3(a^2b^2+b^2c^2+c^2a^2)\quad (9)$$

- Let (8),(9):$\Rightarrow 2(a^3+b^3+c^3)^2\geq 3(a^4+b^4+c^4)+3(a^2b^2+b^2c^2+c^2a^2)$

$$\Leftrightarrow 2(a^3+b^3+c^3)^2\geq 3(a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow \frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2} \le \frac{(a^3+b^3+c^3)^2}{3} \quad (10)$$

- (5), (10):$\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \le \frac{(a^3+b^3+c^3)^2}{3} \quad (11)$

- (1), (11):$\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} = \frac{(a^3+b^3+c^3)^2}{3}$ *occurs if:* $\begin{cases} a=b=c>0 \\ \frac{1}{\sqrt{a^3}}+\frac{1}{\sqrt{b^3}}+\frac{1}{\sqrt{c^3}}=3 \end{cases} \Leftrightarrow a=b=c=1.$

Solution of equation is: $(a,b,c)=(1,1,1)$.

PROBLEM 3.096-*Solution by Shivam Sharma-New Delhi-India*

Let,

$$L=\lim_{n\to\infty}\left(s_n\sqrt[n+1]{(2n+1)!!}-\frac{\pi^2}{6}\sqrt[n]{(2n-1)!!}\right)$$

As we know, $(2n+1)!!=\frac{(2n+1)!}{2^n n!}$, $(2n-1)!=\frac{(2n)!}{2^n n!}$. *Using this, we get,*

$$\Rightarrow \lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)\left(\frac{(2n+1)!}{2^n n!}\right)^{\frac{1}{n+1}}-\frac{\pi^2}{6}\left(\frac{(2n)!}{2^n n!}\right)^{\frac{1}{n}}\right] \Rightarrow \frac{\pi^2}{6}\left[\lim_{n\to\infty}\left\{\left(\frac{(2n+1)!}{2^n n!}\right)^{\frac{1}{n+1}}-\left(\frac{(2n)!}{2^n n!}\right)\right\}\right]$$

Now, applying Stirling's formula, we get,

$$\Rightarrow \frac{\pi^2}{6}\left[\lim_{n\to\infty}\left\{\left(\frac{\left(\frac{2n+1}{e}\right)^{2n+1}\sqrt{2\pi(2n+1)}}{2^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}\right)^{\frac{1}{n+1}}-\left(\frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}}{2^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}\right)^{\frac{1}{n}}\right\}\right]$$

Now, appling Cauchy D'Alembert, we get,

$$L=\frac{\pi^2}{3e}-\frac{2}{e},\ or\ L=\frac{\pi^2-6}{3e}$$

PROBLEM 3.097-*Solution by proposer*

We have: $\begin{cases} \frac{(y+z)(z+x)}{4yz} \ge \frac{x+z}{y+z} \Leftrightarrow (y-z)^2 \ge 0 \\ \frac{(y+z)(z+x)}{4xz} \ge \frac{y+z}{x+z} \Leftrightarrow (z-x)^2 \ge 0 \end{cases}$

After addition we obtain: $\frac{(x+y)(y+z)(z+x)}{4xyz} \ge \frac{x+z}{y+z}+\frac{y+z}{x+z} \ge 2$ *and*

$$\begin{cases} \left(\frac{(x+y)(y+z)(z+x)}{4xyz}\right)^a \ge \left(\frac{x+z}{y+z}+\frac{y+z}{x+z}\right)^a \ge 2^a \\ \left(\frac{(x+y)(y+z)(z+x)}{4xyz}\right)^b \ge \left(\frac{y+x}{z+x}+\frac{z+x}{y+x}\right)^b \ge 2^b \\ \left(\frac{(x+y)(y+z)(z+x)}{4xyz}\right)^c \ge \left(\frac{z+y}{x+y}+\frac{x+y}{z+y}\right)^c \ge 2^c \end{cases}$$

After multiplication we obtain the desired inequalities.

PROBLEM 3.098-***Solution by Soumitra Mandal-Chandar Nagore-India***

$$Let\ x = a + b - z \Rightarrow dx = -dz;\ when\ x = a, z = b; x = b, z = a$$

$$Let\ I = \int_a^b \frac{g(x)}{1+f(x)}dx = \int_a^b \frac{g(a+b-z)(-dz)}{1+f(a+b-z)} = \int_a^b \frac{g(z)dz}{1+\frac{1}{f(z)}} = \int_a^b \frac{f(z)g(z)}{1+f(z)}dz$$

$$= \int_a^b g(z)dz - \int_a^b \frac{g(z)}{1+f(z)}dz \Rightarrow 2I = \int_a^b g(z)dz \Rightarrow I = \frac{1}{2}\int_a^b g(x)\,dx$$

PROBLEM 3.099-***Solution by Soumava Chakraborty-Kolkata-India***

$$Proof\ of\ (a)\ l_a^2 = \frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc} = \frac{bc(b+c+a)(b+c-a)}{(b+c)^2} = \frac{bc\{(b+c)^2-a^2\}}{(b+c)^2} = bc - \frac{a^2bc}{(b+c)^2}$$

$$\therefore \frac{l_a^2}{h_a^2} = bc\cdot\frac{4R^2}{b^2c^2} - \frac{a^4bc}{4\Delta^2(b+c)^2} = 4R^2\cdot\frac{1}{bc} - \frac{4Rrs}{4r^2S^2}\cdot\frac{a^3}{(b+c)^2} =$$

$$\overset{(1)}{=} 4R^2\left(\frac{1}{bc}\right) - \frac{R}{rs}\cdot\frac{a^3}{(b+c)^2}$$

$$Similarly, \frac{l_b^2}{h_b^2} \overset{(2)}{=} 4R^2\left(\frac{1}{ca}\right) - \frac{R}{rs}\cdot\frac{b^3}{(c+a)^2}\ \&\ \frac{l_c^2}{h_c^2} \overset{(3)}{=} 4R^2\left(\frac{1}{ab}\right) - \frac{R}{rs}\cdot\frac{c^3}{(a+b)^2}$$

$$(1)+(2)+(3) \Rightarrow \sum\frac{l_a^2}{h_a^2} = \frac{4R^2}{4Rrs}(2S) + \frac{R}{rs}\sum\frac{(2s-a-2s)^3}{(2s-a)^2} =$$

$$= \frac{2R}{r} + \frac{R}{rs}\sum\frac{(2s-a)^3 - 8s^3 - 3(2s-a)^2\cdot 2S + 3(2s-a)4S^2}{(2s-a)^2} =$$

$$= \frac{2R}{r} + \frac{R}{rs}\sum(2s-a) - \frac{3R}{rs}(2S)(3) + \frac{12RS^2}{rs}\sum\frac{1}{b+c} - \frac{8Rs^3}{rs}\sum\frac{1}{(b+c)^2} =$$

$$\overset{(4)}{=} \frac{2R}{r} + \frac{4RS}{rs} - \frac{18R}{r} + \frac{12RS}{r}\sum\frac{1}{b+c} - \frac{8RS^2}{r}\sum\frac{1}{(b+c)^2}$$

$$Now,\ (a+b)(b+c)(c+a) = 2abc + \sum ab\,(2S-c) =$$

$$= 2s(s^2+4Rr+r^2) - 4Rrs \overset{(5)}{=} 2s(s^2+2Rr+r^2)$$

$$(5) \Rightarrow \frac{2RS}{r}\sum\frac{1}{b+c} = \frac{12RS}{r}\cdot\frac{\sum(c+a)(a+b)}{2s(s^2+2Rr+r^2)} = \frac{12RS[(\sum a^2+2\sum ab)+\sum ab]}{2s(s^2+2Rr+r^2)r} \overset{(i)}{=} \frac{16R(5s^2+4Rr+r^2)}{r(s^2+2Rr+r^2)}$$

$$Now, \sum(c+a)^2(a+b)^2 = \sum\left(a^2+\sum ab\right)^2 = \sum\left\{a^4 + \left(\sum ab\right)^2 + 2\left(\sum ab\right)a^2\right\} =$$

$$= \sum a^4 + 3\left(\sum ab\right)^2 + 2\left(\sum ab\right)\left(\sum a^2\right) = \left(\sum a^2\right)^2 - 2\left\{\left(\sum ab\right)^2 - 2abc(2s)\right\} +$$

$$+3\left(\sum ab\right)^2 + 2\left(\sum ab\right)\left(\sum a^2\right) = \left(\sum a^2\right)^2 + \left(\sum ab\right)^2 + 2\left(\sum ab\right)\left(\sum a^2\right) +$$

$$+32Rrs^2 = \left(\sum a^2 + \sum ab\right)^2 + 32Rrs^2 = (3s^2-4Rr-r^2)^2 + 32Rrs^2 =$$

$$= 9s^4 - 6s^2(4Rr+r^2) + 32Rrs^2 + r^2(4R+r)^2 \overset{(6)}{=} 9s^4 + r^2(4R+r)^2 + s^2(8Rr-6r^2)$$

$$(5), (6) \Rightarrow \frac{-8RS^2}{r}\sum\frac{1}{(b+c)^2} = \frac{[9s^4+r^2(4R+r)^2+s^2(8Rr-6r^2)]}{r\cdot 4s^2(s^2+2Rr+r^2)^2} \overset{(ii)}{=} \frac{-2R[9S^4+r^2(4R+r)^2+s^2(8Rr-6r^2)]}{r(s^2+2Rr+r^2)^2}$$

$$(i),(ii), (4) \Rightarrow \sum\frac{l_a^2}{h_a^2} = \frac{-12R}{r} + \frac{6R(5s^2+4Rr+r^2)}{r(s^2+2Rr+r^2)} - \frac{2R[9s^4+r^2(4R+r)^2+s^2(8Rr-6r^2)]}{r(s^2+2Rr+r^2)^2}$$

$$=\frac{-12R(s^2+2Rr+r^2)^2+6R(5s^2+4Rr+r^2)(s^2+2Rr+r^2)}{r(s^2+2Rr+r^2)^2}-$$

$$-\frac{2R[9s^4+r^2(4R+r)^2+s^2(8Rr-6r^2)]}{r(s^2+2Rr+r^2)^2}\overset{(7)}{=}\frac{RS^2(20Rr+24r^2)-Rr^2(32R^2+28Rr+8r^2)}{r(s^2+2Rr+r^2)^2}$$

$$Now, \frac{2l_al_bl_c}{h_ah_bh_c}+1\overset{by\,(5)}{=}\frac{2\cdot 8R^3}{16R^2r^2s^2}\cdot\frac{8\cdot 16R^2r^2s^2\left(\frac{s}{4R}\right)}{2s(s^2+2Rr+r^2)}+1=\frac{16R^2}{s^2+2Rr+r^2}+1\overset{(8)}{=}\frac{16R^2+s^2+2Rr+r^2}{s^2+2Rr+r^2}$$

$$\because RS^2(20Rr+24r^2)-Rr^2(32R^2+28Rr+8r^2)\overset{Gerretsen}{\geq}$$

$$\geq Rr^2[(20R+24r)(16R-5r)-(32R^2+28Rr+8r^2)]=$$

$$=Rr^2(288R^2+256Rr-128r^2)=Rr^2\{288R^2+192Rr+64r(R-2r)\}>0,$$

∴ (7), (8) ⇒ given inequality is equivalent to:

$$R(20R+24r)s^2-Rr(32R^2+28Rr+8r^2)$$

$$\geq(s^2+2Rr+r^2)(s^2+16R^2+2Rr+r^2)\Leftrightarrow s^2(4R^2+20Rr-2r^2)\overset{(9)}{\geq}$$

$$\geq s^4+64R^3r+48R^2r^2+12Rr^3+r^4$$

Now, the fundamental triangle inequality (Rouche) $\Rightarrow s^2\geq m-n\Rightarrow s^2-m+n\overset{(a)}{\geq}0$ &

$s^2\leq m+n\Rightarrow s^2-m-n\overset{(b)}{\leq}0$, *where* $m=2R^2+10Rr-r^2$ &

$$n=2(R-2r)\sqrt{R^2-2Rr}$$

(a).(b)$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-2s^2(2R^2+10Rr-r^2)+$

$+(2R^2+10Rr-r^2)^2-4(R-2r)^2(R^2-2Rr)\leq 0\Rightarrow s^4+64R^3r+48R^2r^2+12Rr^3+$

$+r^4\overset{(c)}{\leq}s^2(4R^2+20Rr-2r^2)\Rightarrow$ *(9) is true (proved)*

∵ (c) is analogous with the fundamental triangle inequality & ∵ given inequality is equivalent to (c), hence, given inequality is equivalent to the fundamental triangle inequality

Proof of (b) $m_a^2m_b^2m_c^2=\frac{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)}{64}\overset{(1)}{=}$

$=\frac{1}{64}\{-4\sum a^6+6(\sum s^4b^2+\sum a^2b^2)+3a^2b^2c^2\}$. *Now,*

$$\sum a^6=\left(\sum a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)=$$

$$=\left(\sum a^2\right)^3-3\left(\sum a^2-c^2\right)\left(\sum a^2-a^2\right)\left(\sum a^2-b^2\right)=$$

$$=\left(\sum a^2\right)^3-3\left\{\left(\sum a^2\right)^3-\left(\sum a^2\right)^3+\left(\sum a^2\right)\left(\sum a^2b^2\right)-a^2b^2c^2\right\}$$

$\overset{(2)}{=}(\sum a^2)^3-3(\sum a^2)(\sum a^2b^2)+3a^2b^2c^2$. *Also,* $\sum a^4b^2+\sum a^2b^4=\sum a^2b^2(\sum a^2-c^2)=$

$$\overset{(3)}{=}\left(\sum a^2\right)\left(\sum a^2b^2\right)-3a^2b^2c^2$$

(1), (2), (3) $\Rightarrow m_a^2m_b^2m_c^2=\frac{1}{64}\begin{Bmatrix}-4(\sum a^2)^3+12(\sum a^2)(\sum a^2b^2)-12a^2b^2c^2+\\ +6(\sum a^2)(\sum a^2b^2)-18a^2b^2c^2+3a^2b^2c^2\end{Bmatrix}$

$$=\frac{1}{64}\left\{-4\left(\sum a^2\right)^3+18\left(\sum a^2\right)\left(\sum a^2b^2\right)-27a^2b^2c^2\right\}=$$

$$=\frac{1}{64}\begin{bmatrix}-32(s^2-4Rr-r^2)^3+18\cdot 2(s^2-4Rr-r^2)\cdot\\ \{(s^2+4Rr+r^2\}-2abc(2s)-432R^2r^2s^2\end{bmatrix}=$$

$$\overset{(4)}{=}\frac{1}{16}\begin{Bmatrix}s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-\\ -64R^3r^3-48R^2r^4-12Rr^5-r^6\end{Bmatrix}$$

$$Now,\ 4\sum a^2b^2-\sum a^4=6\sum a^2b^2-\left(\sum a^2\right)^2=6\{(\sum ab)^2-2abc(2s)\}-\left(\sum a^2\right)^2$$
$$=4\{(s^2+4Rr+r^2)^2-(s^2-4Rr-r^2)^2\}+2(s^2+4Rr+r^2)^2-96Rrs^2=$$
$$=4(2s^2)(8Rr+2r^2)+2\left(s^4+r^2(4R+r)^2+2s^2(4Rr+r^2)\right)-96Rrs^2$$
$$\overset{(5)}{=}2s^4-s^2(16Rr-20r^2)+2r^2(4R+r)^2$$
$$Now,\ \sum\frac{m_a^2}{h_a^2}-1=\sum\frac{2b^2+2c^2-a^2}{4}\cdot\frac{a^2}{4\Delta^2}-1=\frac{4\sum a^2b^2-\sum a^4}{16\Delta^2}-1=$$
$$=\frac{s^4-s^2(8Rr-10r^2)+r^2(4R+r)^2-8r^2s^2}{8\Delta^2}\ (by\ (5))=\frac{s^4+r^2(4R+r)^2-s^2(8Rr-2r^2)}{8\Delta^2}$$
$$\therefore\left(\sum\frac{m_a^2}{h_a^2}-1\right)^2$$
$$\overset{(6)}{=}\frac{1}{64\Delta^2}\begin{bmatrix}s^8-s^6(16Rr-4r^2)+s^4(96R^2r^2+16Rr^3+6r^4)-\\-s^2(256R^3r^3+64R^2r^4-16Rr^5-4r^6)+\\+256R^4r^4+256R^3r^5+96R^2r^6+16Rr^7+r^8\end{bmatrix}$$
$$Also,\ \left(\frac{2m_am_bm_c}{h_ah_bh_c}\right)^2=\left(\frac{28R^3}{16R^2r^2s^2}\right)^2\cdot m_a^2m_b^2m_c^2$$
$$\overset{(7)}{=}\frac{R^2}{16\Delta^4}\begin{Bmatrix}s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-\\-64R^3r^3-48R^2r^4-12Rr^5-r^6\end{Bmatrix}\ (by\ (4))$$

(6), (7) ⇒ given inequality is equivalent to:

$$s^8-s^6(16Rr-4r^2)+s^4(96R^2r^2+16Rr^3+6r^4)-$$
$$-s^2(256R^3r^3+64R^2r^4-16Rr^5-4r^6)+256R^4r^4+256R^3r^5+96R^2r^6+$$
$$+16Rr^7+r^8\le 4R^2\begin{Bmatrix}s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-\\-64R^3r^3-48R^2r^4-12Rr^5-r^6\end{Bmatrix}$$
$$\Leftrightarrow s^8-s^6(4R^2+16Rr-4r^2)+s^4(48R^3r-36R^2r^2-16Rr^3+6r^4)+$$
$$+s^2(240R^4r^2+224R^3r^3+68R^2r^4+16Rr^5+4r^6)+256R^5r^3+448R^4r^4+$$
$$+304R^3r^5+100R^2r^6+16Rr^7+r^8\le 0\Leftrightarrow$$
$$\Leftrightarrow\{s^4-(4R^2+20Rr-2r^2)s^2+64R^3r+48R^2r^2+12Rr^3+r^4\}$$
$$\{s^4+s^2(4Rr+2r^2)+4R^2r^2+4Rr^3+r^4\}\le 0\Leftrightarrow$$
$$\Leftrightarrow s^4+64R^3r+48R^2r^2+12Rr^3+r^4\le s^2(4R^2+20Rr-2r^2)$$

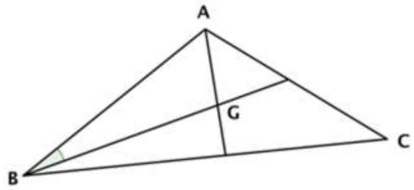
But, the above is inequality (c) proved in the proof of (a) earlier.

⇒ given inequality is true (Proved)

∵ given inequlality reduces to inequality (c) & (c) is analogous to the fundamental inequality of the triangle, hence, this given inequality is equivalent to the fundamental inequality of the triangle (Done).

PROBLEM 3.100-*Solution by proposer*

Let G be the centroid of ΔABC.



$$AG=\frac{2}{3}m_a;\ BG=\frac{2}{3}m_b$$

$$1 > \cos(\widehat{GBA}) = \frac{GB^2 + AB^2 - GA^2}{2GB \cdot AB} = \frac{\left(\frac{2}{3}m_b\right)^2 + c^2 - \left(\frac{2}{3}m_a\right)^2}{2 \cdot \frac{2}{3}m_b \cdot c} =$$

$$= \frac{9c^2 + 4m_b^2 - 4m_a^2}{12cm_b} = \frac{9c^2 + 2a^2 + 2c^2 - b^2 - 2b^2 - 2c^2 + a^2}{12cm_b} =$$

$$= \frac{9c^2 + 3a^2 - 3b^2}{12cm_b} = \frac{3c^2 + a^2 - b^2}{4cm_b}$$

$$3c^2 + a^2 - b^2 < 4cm_b \quad (1)$$

Analogous:

$$3a^2 + b^2 - c^2 < 4am_c$$

$$3a^2 + b^2 - c^2 < 4am_c \quad (2)$$

$$3b^2 + c^2 - a^2 < 4bm_a \quad (3)$$

By adding (1); (2); (3): $3(a^2 + b^2 + c^2) < 4(am_c + bm_a + cm_b)$

PROBLEM 3.101-*Solution by Rovsen Pirguliyev-Sumgait-Azerbaidian*

Lemma: if $x > q$*, then prove:* $\sin\frac{\pi}{x} > \frac{3}{\sqrt{x^2+9}}$

Proof: $x > 2 \Rightarrow \frac{\pi}{x} < \frac{\pi}{2} \Rightarrow \tan\frac{\pi}{x} > \frac{\pi}{x}$*, we have* $\frac{\pi}{x} > \frac{3}{x} \Rightarrow \tan\frac{\pi}{x} > \frac{3}{x}$ *(*)*

$$\cos x = \sqrt{\frac{1}{1 + \tan^2 x}} < \sqrt{\frac{1}{1 + \frac{\pi^2}{x^2}}} \overset{(x)}{<} \frac{x}{\sqrt{x^2 + 9}} \Rightarrow \sin\frac{\pi}{x} > \frac{3}{\sqrt{x^2 + 9}}$$

it is known that: if $x > q$*, then* $\sqrt{x^2 + 9}\sin\frac{\pi}{x} > 3 \Rightarrow x \to 3x$*, we have:* $\sin\frac{\pi}{3x} > \frac{1}{\sqrt{x^2+1}}$

$$x\sin\frac{\pi}{3x} > x \cdot \frac{1}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} \quad (*)$$

$$3\sqrt{2} + \int_1^a x\sin\frac{\pi}{3x}dx + \int_1^b x\sin\frac{\pi}{3x}dx + \int_1^c x\sin\frac{\pi}{3x} \overset{(*)}{>}$$

$$> 3\sqrt{2} + \int_1^a \frac{x}{\sqrt{x^2 + 1}}dx + \int_1^b \frac{x}{\sqrt{x^2 + 1}}dx + \int_1^c \frac{x}{\sqrt{x^2 + 1}}dx =$$

$$= 3\sqrt{2} + \sqrt{x^2 + 1}\,\Big|_1^a + \sqrt{x^2 + 1}\Big|_1^b + \sqrt{x^2 + 1}\Big|_1^c =$$

$$= 3\sqrt{2} + \sqrt{a^2 + 1} - \sqrt{2} + \sqrt{b^2 + 1} - \sqrt{2} + \sqrt{c^2 + 1} - \sqrt{2} =$$

$$= \sqrt{a^2 + 1} + \sqrt{b^2 + 1} + \sqrt{c^2 + 1} > \sqrt{3 + a^2 + b^2 + c^2}$$

PROBLEM 3.102-*Solution by Hoang Le Nhat Tung-Hanoi-Vietnam*

By AM-GM

$$n^{n(x_1^2 - x_2)} + n^{n(x_2^2 - x_3)} + \cdots + n^{n(x_n^2 - x_1)} \geq n\sqrt[n]{(n^n)^{x_1^2 - x_2 + x_2^2 - x_3 + \cdots + x_n^2 - x_1}}$$

$$= n\sqrt[n]{(n^n)^{(x_1^2-x_1)+(x_2^2-x_2)+\dots+(x_n^2-x_n)}} = n\sqrt[n]{(n^n)^{\left(x_1-\frac{1}{2}\right)^2+\left(x_1-\frac{1}{2}\right)^2+\dots+\left(x_n-\frac{1}{2}\right)^2-\left(\frac{1}{4}+\dots+\frac{1}{4}\right)}}$$

$$\geq n\sqrt[n]{(n^n)^{-\frac{1}{4}n}} = n\sqrt[n]{(n^n)^{-\frac{n}{4}}} = \frac{n}{\sqrt[4]{n^n}} \Rightarrow n^{n(x_1^2-x_2)} + \dots + n^{n(x_n^2-x_1)} \geq \frac{n}{\sqrt[4]{n^n}}$$

$$\Rightarrow x_1 = x_2 = \dots = x_n = \frac{1}{2}$$

PROBLEM 3.103-*Solution by proposer*

$$\left|\sqrt{\cos A}\right| = \left|\sqrt{|(\cos A - \cos B) + \cos B|}\right| \leq$$

$$\leq \sqrt{|\cos A - \cos B| + |\cos B|} \leq \sqrt{|\cos A - \cos B|} + \sqrt{|\cos B|}$$

because if $x, y \geq 0$ *then* $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$

$$\sqrt{|\cos A|} - \sqrt{|\cos B|} \leq \sqrt{|\cos A - \cos B|}$$

$$\left|\sqrt{\cos B}\right| = \left|\sqrt{|(\cos B - \cos A) + \cos A|}\right| \leq$$

$$\leq \sqrt{|\cos B - \cos A| + |\cos A|} \leq \sqrt{|\cos A - \cos B|} + \sqrt{|\cos A|}$$

$$-\left(\sqrt{|\cos A|} - \sqrt{|\cos B|}\right) \leq \sqrt{|\cos A - \cos B|} \quad (2)$$

By (1); (2): $\sqrt{|\cos A - \cos B|} \geq \left|\sqrt{|\cos A|} - \sqrt{|\cos B|}\right|$

By squaring: $|\cos A - \cos B| \geq |\cos A| + |\cos B| - 2\sqrt{|\cos A \cos B|}$

$$\left|2\sin\frac{B-A}{2}\cos\frac{C}{2}\right| \geq |\cos A| + |\cos B| - 2\sqrt{|\cos A \cos B|}$$

$$2\sqrt{|\cos A \cos B|} + 2\left|\cos\frac{A}{2}\sin\frac{B-A}{2}\right| \geq |\cos A| + |\cos B|$$

$$2\sum\left(\sqrt{|\cos A \cos B|} + \left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|\right) \geq \sum(|\cos A| + |\cos B|)$$

$$2\sum\left(\sqrt{|\cos A \cos B|} + \left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|\right) \geq 2\sum|\cos A|$$

$$|\cos A| + |\cos B| + |\cos C| \leq \sum\left(\sqrt{|\cos A \cos B|} + \left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|\right)$$

PROBLEM 3.104-*Solution by Soumitra Mandal-Chandar Nagore-India*

We have, $x_i^2 + x_i x_{i+1} + x_{i+1}^2 \geq \frac{3}{4}(x_i + x_{i+1})^2$

$$\sum_{i=1}^{n}\frac{\frac{x_i}{x_{i+1}} + \frac{x_{i+1}}{x_i} + 1}{\sqrt{x_i^2 + x_i x_{i+1} + x_{i+1}^2}} = \sum_{i=1}^{n}\frac{\sqrt{x_i^2 + x_i x_{i+1} + x_{i+1}^2}}{x_i x_{i+1}} \geq \frac{\sqrt{3}}{2}\sum_{i=1}^{n}\frac{x_i + x_{i+1}}{x_i x_{i+1}}$$

$$\overset{AM\geq GM}{\geq} \sqrt{3}\sum_{i=1}^{n}\frac{1}{\sqrt{x_i x_{i+1}}} \overset{AM\geq GM}{\geq} \frac{n\sqrt{3}}{\sqrt[n]{\prod_{i=1}^{n} x_i}} = n\sqrt{3}$$

PROBLEM 3.105-*Solution by Soumava Chakraborty-Kolkata-India*

$$LHS \overset{Holder}{\geq} \frac{(\sum A)^3}{3\sum a}\cdot\frac{(\sum A)^3}{3\sum a}\cdot\frac{(\sum A)^3}{3\sum a}=\frac{(\sum A)^9}{27(2s)^3}=\frac{\pi^9}{216s^3}$$

PROBLEM 3.106-*Solution by proposer*

We make some transformations, let $k_a=\tan\frac{A}{4}\Rightarrow\cos\frac{A}{4}=\frac{1}{\sqrt{k_a^2+1}}$ *and we know*

$$\tan\frac{A}{4}=\frac{AI-(s-a)}{r}=\frac{\sqrt{2Rr}\sqrt{\frac{2(s-a)}{a}}-(s-a)}{r}=|Ravi|=$$

$$=\frac{2\sqrt{\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}\cdot\sqrt{\frac{xyz}{x+y+z}}\cdot x}-x\sqrt{y+z}}{\sqrt{\frac{xyz}{x+y+z}}\cdot\sqrt{y+z}}=\frac{\sqrt{(x+y)(x+z)}-\sqrt{x(x+y+z)}}{\sqrt{yz}}$$

$$=\frac{\sqrt{yz}}{\sqrt{(x+y)(x+z)}+\sqrt{x(x+y+z)}}$$

$$\Rightarrow\sum_{cyc}\cos\frac{A}{4}=\sum_{cyc}\frac{1}{\sqrt{\left(\frac{\sqrt{yz}}{\sqrt{(x+y)(x+z)}+\sqrt{x(x+y+z)}}\right)^2+1}}=$$

$$=\sum_{cyc}\frac{1}{\sqrt{\frac{\left(\sqrt{(x+y)(x+z)}-\sqrt{x(x+y+z)}\right)^2}{yz}+1}}$$

$$=\sum_{cyc}\sqrt{\frac{yz}{(x+y+z)x+(x+y)(x+z)+yz-2\sqrt{x(x+y+z)(x+y)(x+z)}}}$$

$$=\frac{1}{\sqrt{2}}\sum_{cyc}\sqrt{\frac{yz}{(x+y)(x+z)-\sqrt{x(x+y+z)(x+y)(x+z)}}}$$

$$=\frac{1}{\sqrt{2}}\sum_{cyc}\sqrt{\frac{\sqrt{(x+y)(x+z)}+\sqrt{x(x+y+z)}}{\sqrt{(x+y)(x+z)}}}=\frac{1}{\sqrt{2}}\sum_{cyc}\sqrt{1+\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}}$$

$$\leq\sqrt{\frac{3}{2}\left(3+\sum_{cyc}\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}\right)}\leq\sqrt{\frac{3}{2}\left(3+\sqrt{3\sum_{cyc}\frac{x(x+y+z)}{(x+y)(x+z)}}\right)}$$

$$= \sqrt{\frac{3}{2}\left(3+\sqrt{6\left(1+\frac{xyz}{(x+y)(y+z)(x+z)}\right)}\right)} = \sqrt{\frac{9}{2}+3\sqrt{\frac{3}{2}+\frac{3}{8}\cdot\frac{r}{R}}}$$

Hence, we need to prove: $\sqrt{\frac{9}{2}+3\sqrt{\frac{3}{2}+\frac{3}{8}\cdot t}} \le \frac{\sqrt{3}-1}{4\sqrt{6}}t+\frac{11\sqrt{3}+37}{8\sqrt{6}}$

$$\Leftrightarrow \left(\left(\frac{\sqrt{3}-1}{4\sqrt{6}}t+\frac{11\sqrt{3}+37}{8\sqrt{6}}\right)^2-\frac{9}{2}\right)^2-\frac{27}{2}\left(1+\frac{1}{4}t\right)\ge 0$$

$$\Leftrightarrow \frac{(2t-1)^2\left(t^2\left(28-16\sqrt{3}\right)+t\left(208\sqrt{3}-316\right)+713-1628\sqrt{3}\right)}{36864}\ge 0 \Leftrightarrow$$

$$\left(t^2\left(28-16\sqrt{3}\right)+t\left(208\sqrt{3}-316\right)-713+1628\sqrt{3}\right)\ge 0 \quad (***)$$

$$D_t=\left(208\sqrt{3}-316\right)^2-4\left(-713+1628\sqrt{3}\right)\left(28-16\sqrt{3}\right)=13824\left(45-26\sqrt{3}\right)<0 \Rightarrow$$

$$(***)\ \forall\, t$$

PROBLEM 3.107-*Solution by Soumava Chakraborty-Kolkata-India*

$$2\sqrt{3}\left(\frac{r}{R}\right)^2 \overset{(1)}{\le} \frac{\sum\sin^4 A}{\sum\sin^3 A} \overset{(2)}{\le} \frac{\sqrt{3}}{4}\left(\frac{R}{r}\right)^2\left(1-\frac{r}{R}\right)$$

$$\textit{Firstly, } \sum a^4=\left(\sum a^2\right)^2-2\left\{\left(\sum ab\right)^2-2abc(2s)\right\}$$

$$=4(s^2-4Rr-r^2)^2-2(s^2+4Rr+r^2)^2+32Rrs^2$$

$$=2\left(s^4+r^2(4R+r)^2-2s^2(4Rr+r^2)\right)+2(2s^2)(-8Rr-2r^2)+32Rrs^2$$

$$=2s^4+2r^2(4R+r)^2-4s^2(4Rr+r^2)-8s^2r^2$$

$$\overset{(i)}{=} 2s^4+2r^2(4R+r)^2-4s^2(4Rr+3r^2)$$

$$\textit{Also, } \sum a^3=3abc+2s(s^2-12Rr-3r^2) \overset{(ii)}{=} 2s(s^2-6Rr-3r^2)$$

$$\textit{Using } \sin A=\frac{a}{2R}\ etc, \frac{\sum\sin^4 A}{\sum\sin^3 A}=\left(\frac{1}{2R}\right)\left(\frac{\sum a^4}{\sum a^3}\right) \overset{by\ (ii)}{=} \frac{\sum a^4}{4Rs(s^2-6Rr-3r^2)} \overset{?}{\le} \frac{\sqrt{3}}{4}\left(\frac{R}{r}\right)^2\left(1-\frac{r}{R}\right)$$

$$\Leftrightarrow r^2\sum a^4 \underset{(2a)}{\overset{?}{\le}} \sqrt{3}R\cdot R(R-r)s(s^2-6Rr-3r^2)$$

$$\textit{RHS of (2a)} \overset{Mitrinovic}{\ge} \frac{2s^2(s^2-6Rr-3r^2)R(R-r)}{3} \overset{?}{\ge} r^2\sum a^4$$

$$\Leftrightarrow 6r^2s^4+6r^4(4R+r)^2-12s^2r^2(4Rr+3r^2) \underset{by\ (i)}{\overset{?}{\le}} 2s^2(s^2-6Rr-3r^2)R(R-r)$$

$$\Leftrightarrow s^4(R^2-Rr-3r^2)+6s^2r^3(4R+3r) \overset{?}{\ge} 3s^2Rr(R-r)(2R+r)+3r^4(4R+r)^2$$

$$\Leftrightarrow s^4(R^2-Rr-2r^2)+6s^2r^3(4R+3r) \underset{(2b)}{\overset{?}{\ge}} r^2s^4+3s^2Rr(R-r)(2R+r)+3r^4(4R+r)^2$$

$$\textit{LHS of (2b)} \overset{Gerretsen}{\ge} s^2(16Rr-5r^2)(R^2-Rr-2r^2)+6s^2r^3(4R+3r)$$

$$\textit{Also, RHS} \overset{Gerretsen}{\le} r^2s^2(4R^2+4Rr+3r^2)+3s^2Rr(R-r)(2R+r)+3r^4(4R+r)^2$$

$\therefore$ *in order to prove (2b), it suffices to prove:*

$$s^2(16R-5r)(R^2-Rr-2r^2)+6s^2r^2(4R+3r)\ge$$

$$rs^2(4R^2+4Rr+3r^2)+3s^2R(R-r)(2R+r)+3r^3(4R+r)^2$$

$$\Leftrightarrow s^2\{(16R-5r)(R^2-Rr-2r^2)-3R(R-r)(2R+r)+6r^2(4R+3r)$$

$$-r(4R^2+4Rr+3r^2)\}$$

$$\Leftrightarrow s^2(10R^3-22R^2r-4Rr^2+25r^3) \overset{(2c)}{\geq} 3r^3(4R+r)^2$$

$\because 10R^3-22R^2r-4Rr^2+25r^3=(R-2r)\{(R-2r)(10R+18r)+28r^2\}+9r^3>0$ *as* $R\geq 2r$ *(Euler),*

$$\therefore LHS\ of\ (2c) \overset{Gerretsen}{\geq} (16Rr-5r^2)(10R^3-22R^2r-4Rr^2+25r^3) \overset{?}{\geq} 3r^3(4R+r)^2$$

$$\Leftrightarrow 80t^4-201t^3-t^2+198t-64 \overset{?}{\geq} 0\ (where\ t=\frac{R}{r})$$

$$\Leftrightarrow (t-2)\{(t-2)(80t^2+119t+155)+342\} \overset{?}{\geq} 0 \to true \because t \overset{Euler}{\geq} 2$$

$\Rightarrow$ *(2a) is true* $\Rightarrow$ *(2) is true*

$$Also, \frac{\sum\sin^4 A}{\sum\sin^3 A}=\left(\frac{1}{2R}\right)\left(\frac{s^4+r^2(4R+r)^2-2s^2(4Rr+3r^2)}{s(s^2-6Rr-3r^2)}\right)\geq 2\sqrt{3}\left(\frac{r}{R}\right)^2$$

$$\Leftrightarrow s^4+r^2(4R+r)^2-2s^2(4Rr+3r^2) \overset{(1a)}{\geq} \frac{4s(s^2-6Rr-3r^2)\sqrt{3}r^2}{R}$$

$$Now,\ RHS\ of\ (1a) \overset{Mitrinovic}{\leq} 18r^2(s^2-6Rr-3r^2) \overset{?}{\leq} s^4+r^2(4R+r)^2-2s^2(4Rr+3r^2)$$

$$\Leftrightarrow s^4+r^2(4R+r)^2+54r^3(2R+r) \underset{(1b)}{\overset{?}{\geq}} 18r^2s^2+s^2(8Rr+6r^2)$$

$$Now,\ LHS\ of\ (1b) \overset{Gerretsen}{\geq} s^2(16Rr-5r^2)+r^2(4R+r)^2+54r^3(2R+r) \overset{?}{\geq} s^2(8Rr+24r^2)$$

$$\Leftrightarrow s^2(8R-29r)+r(4R+r)^2+54r^2(2R+r) \overset{?}{\geq} 0$$

$$\Leftrightarrow s^2(8R-16r)+r(4R+r)^2+54r^2(2R+r) \underset{(1c)}{\overset{?}{\geq}} 13rs^2$$

$$Now,\ LHS\ of\ (1c) \overset{Gerretsen}{\geq} (16Rr-5r^2)(8R-16r)+r(4R+r)^2+54r^2(2R+r)$$

$$Also,\ RHS\ of\ (1c) \overset{Gerretsen}{\leq} 13r(4R^2+4Rr+3r^2)$$

$\therefore$ *in order to prove (1c), it suffices to prove:*

$$(16R-5r)(8R-16r)+(4R+r)^2+54r(2R+r)\geq 13(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 23R^2-58Rr+24\geq 0 \Leftrightarrow (R-2r)(23R-12r)\geq 0 \to true \because R\geq 2r$$

$\Rightarrow$ *(1a) is true* $\Rightarrow$ *(1) is true*

PROBLEM 3.108-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{3}{16} \overset{(1)}{\leq} \sum\cos^4 A \overset{(2)}{\leq} 6\left(\frac{r}{R}\right)^2-\left(\frac{123}{8}\right)\left(\frac{r}{R}\right)+\frac{51}{8}$$

$$\cos A\cos B\cos C=\frac{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}{2bc\cdot 2ca\cdot 2ab} \to \quad (i)$$

$$Numerator=(\textstyle\sum a^2-2a^2)(\sum a^2-2b^2)(\sum a^2-2c^2)$$

$$=\left(\sum a^2\right)^3-2\left(\sum a^2\right)^2\left(\sum a^2\right)+4\left(\sum a^2\right)\left(\sum a^2b^2\right)-8a^2b^2c^2$$

$$=-\left(\sum a^2\right)^3+4\left(\sum a^2\right)\left\{\left(\sum ab\right)^2-2abc(2s)\right\}-128R^2r^2s^2$$

$$=\left(\sum a^2\right)\left\{4\left(\sum ab\right)^2-\left(\sum a^2\right)^2-16sabc\right\}-128R^2r^2s^2$$

$$=4\left(\sum a^2\right)\{(s^2+4Rr+r^2)^2-s(s^2-4Rr-r^2)^2-16Rrs^2\}-128R^2r^2s^2$$

$$= 4\left(\sum a^2\right)\{2s^2(8Rr + 2r^2) - 16Rrs^2\} - 128R^2r^2s^2$$

$$= 32r^2s^2(s^2 - 4Rr - r^2) - 128R^2r^2s^2 = 32r^2s^2(s^2 - 4R^2 - 4Rr - r^2) \rightarrow \quad (ii)$$

$$(i), (ii) \Rightarrow \prod \cos A \overset{(iii)}{=} \frac{s^2-4R^2-4Rr-r^2}{4R^2}$$

$$(2) \Leftrightarrow \sum(1 - \sin^2 A)^2 \leq \frac{51R^2-123Rr+48r^2}{8R^2}$$

$$\Leftrightarrow \sum(1 - 2\sin^2 A + \sin^4 A) \leq \frac{51R^2 - 123Rr + 48r^2}{8R^2}$$

$$\Leftrightarrow \sum(\cos 2A) + \sum(\sin^4 A) \leq \frac{51R^2 - 123Rr + 48r^2}{8R^2}$$

$$\Leftrightarrow 1 - 4\left(\prod \cos A\right) + \sum(\sin^4 A) \leq \frac{51R^2 - 123Rr + 48r^2}{8R^2}$$

$$\Leftrightarrow -1 - \frac{s^2 - (2R + r)^2}{R^2} + \sum(\sin^4 A) \overset{by\ (iii)}{\leq} \frac{51R^2 - 123Rr + 48r^2}{8R^2}$$

$$\Leftrightarrow \sum(\sin^4 A) \leq \frac{27R^2 - 155Rr + 40r^2 + 8s^2}{8R^2} \Leftrightarrow \frac{\sum a^4}{16R^4} \leq \frac{27R^2 - 155Rr + 40r^2 + 8s^2}{8R^2}$$

$$\Leftrightarrow \sum a^4 \overset{(a)}{\leq} 54R^4 - 310R^3r + 80R^2r^2 + 16R^2s^2$$

$$Now, \sum a^4 = \left(\sum a^2\right)^2 - 2\left\{\left(\sum ab\right)^2 - 2abc(2s)\right\}$$

$$= 4(s^2 - 4Rr - r^2)^2 - 2(s^2 + 4Rr + r^2)^2 + 32Rrs^2$$

$$= 2\left(s^4 + r^2(4R + r)^2 - 2s^2(4Rr + r^2)\right) + 2(2s^2)(-8Rr - 2r^2) + 32Rrs^2$$

$$= 2s^4 + 2r^2(4R + r)^2 - 4s^2(4Rr + r^2) - 8s^2r^2$$

$$= 2s^4 + 2r^2(4R + r)^2 - 4s^2(4Rr + 3r^2) \overset{?}{\leq} 54R^4 - 310R^3r + 80R^2r^2 + 16R^2s^2$$

$$\Leftrightarrow s^4 \overset{?}{\leq} 27R^4 - 155R^3r + 40R^2r^2 - r^2(4R + r)^2 + 2s^2(4R^2 + 4Rr + 3r^2)$$

$$Now,\ LHS\ of\ (b) \overset{Gerretsen}{\leq} s^2(4R^2 + 4Rr + 3r^2) \overset{?}{\leq}$$

$$27R^4 - 155R^3r + 40R^2r^2 - r^2(4R + r)^2 + 2s^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow s^2(4R^2 + 4Rr + 3r^2) + 27R^4 - 155R^3r + 40R^2r^2 - r^2(4R + r)^2 \underset{(c)}{\overset{?}{\geq}} 0$$

$$Now,\ LHS\ of\ (c) \overset{Gerretsen}{\geq} 27R^4 - 155R^3r + 40R^2r^2 - r^2(4R + r)^2 +$$

$$+(16Rr - 5r^2)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 27t^4 - 91t^3 + 68t^2 + 20t - 16 \overset{?}{\geq} 0 \ \left(where\ t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(27t^2 + 17t + 28) + 64\} \overset{?}{\geq} 0 \rightarrow true \because t \overset{Euler}{\geq} 2$$

$$\Rightarrow (a)\ is\ true \Rightarrow (2)\ is\ true$$

$$Also, \sum \cos^4 A \overset{Chebyshev}{\geq} \left(\frac{1}{3}\right)\left(\sum \cos^2 A\right)^2 \overset{?}{\geq} \frac{3}{16} \Leftrightarrow \sum \cos^2 A \overset{?}{\geq} \frac{3}{4} \Leftrightarrow \sum \sin^2 A \overset{?}{\leq} \frac{9}{4}$$

$$\Leftrightarrow \sum a^2 \overset{?}{\leq} 9R^2 \rightarrow true\ by\ Leibnitz \Rightarrow (1)\ is\ true$$

PROBLEM 3.109-***Solution by Catinca Alexandru-Romania***

$$\frac{(a + b + c + d)^3}{\sum abc} \geq 16 \Leftrightarrow (a + b + c + d)^3 \geq 16 \cdot 4[1,1,1,0];$$

$$\Leftrightarrow \sum a^3 + 3\sum_{sym} a^2 b + 6\sum abc \geq 16[1,1,1,0]4;$$

$$\Leftrightarrow [3,0,0,0]\cdot 4 + 3\cdot[2,1,0,0]\cdot 12 + 6[1,1,1,0]\cdot 4 \geq 16\cdot 4[1,1,1,0]$$

$$\Leftrightarrow 4[3,0,0,0] + 36[2,1,0,0] \geq 40[1,1,1,0] \quad (1)$$

$$4[3,0,0,0] \geq 4[1,1,1,0] \text{ as } (3,0,0,0) > (1,1,1,0) \text{ Muirhead}$$

$$36[2,1,0,0] \geq 36[1,1,1,0] \text{ as } (2,1,0,0) > (1,1,1,0) \text{ Muirhead}$$

$$\overline{\qquad\qquad\qquad\qquad\qquad} +$$

$$4[3,0,0,0] + 36[2,1,0,0] \geq 40[1,1,1,0] \Rightarrow (1) \text{ is True} \Rightarrow \frac{(\sum a)^3}{\sum abc} \geq 16$$

PROBLEM 3.110-***Solution by Marian Ursărescu-Romania***

$$\text{In any } \Delta\, ABC \text{ we have: } m_a \geq \frac{b^2+c^2}{4R} \text{ (1), because:}$$

$$m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} = \frac{2(b^2+c^2) - (b^2+c^2-2bc\sin A)}{4} =$$

$$\frac{b^2+c^2+2bc\cos A}{4} = \frac{b^2+c^2-2bc\cos(B+C)}{4} =$$

$$= \frac{(b\cos B - c\cos C)^2 + (b\sin B + c\sin C)^2}{4} \geq \frac{(b\sin B + c\sin C)^2}{4} \Rightarrow 1 \text{ is true}$$

$$\text{From (1)} \Rightarrow \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{R}{2r^2}$$

$$\Rightarrow m_a \geq \frac{b^2+c^2}{4R} \geq \frac{2bc}{4R} = \frac{bc}{2R} \Rightarrow \frac{1}{m_a} \leq \frac{2R}{bc} \Rightarrow \sum \frac{1}{m_a} \leq 2R\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right)$$

$$\text{Now we show this} \Rightarrow 2R\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) \leq \frac{R}{2r^2} \Leftrightarrow \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \leq \frac{1}{4r^2} \text{ which its true.}$$

$$\text{P.S. } \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \leq \frac{1}{4r^2} \text{ its true because } \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = \frac{1}{2Rr} \leq \frac{1}{4r^2} \Leftrightarrow R \geq 2r \text{ true.}$$

PROBLEM 3.111-***Solution by Marian Ursărescu – Romania***

From Hölder's inequality we have:

$$\left((\sin A + \sin B\cos C)^{\frac{1}{n}}\right)^n + \left((\sin A\cos B\sin C)^{\frac{1}{n}}\right)^n + \left((\cos A\sin B\sin C)^{\frac{1}{n}}\right)^n \geq$$

$$\geq \frac{\left((\sin A + \sin B\cos C)^{\frac{1}{n}} + (\sin A\cos B\sin C)^{\frac{1}{n}} + (\cos A\sin B\sin C)^{\frac{1}{n}}\right)^n}{3^{n-1}} \Leftrightarrow$$

$$\Leftrightarrow \left(\sum (\sin A\sin B\sin C)^{\frac{1}{n}}\right)^n \leq 3^{n-1}\cdot \sum \sin A\sin B\cos C \Rightarrow$$

$$\text{We must show: } \sum \sin A\sin B\cos C \leq \frac{9}{8} \quad (1)$$

$$\text{Now: } \cos 2A + \cos 2B - \cos 2C = 2\cos(A+B)\cos(A-B) - 2\cos^2 C + 1$$

$$= -2\cos C\,(\cos(A-B) + \cos C) + 1 = 1 - 2\cos C\cdot 2\cos\left(\frac{A-B+C}{2}\right)\cos\left(\frac{A-B-C}{2}\right)$$

$$= 1 - 4\sin A\sin B\cos C \Rightarrow \sin A\sin B\sin C = \frac{1}{4}(1 - \cos 2A - \cos 2B + \cos 2C) \quad (2)$$

$$\text{From (1)+(2) we must show: } \frac{3-(\cos 2A + \cos 2B + \cos 2C)}{4} \leq \frac{9}{8} \quad (3)$$

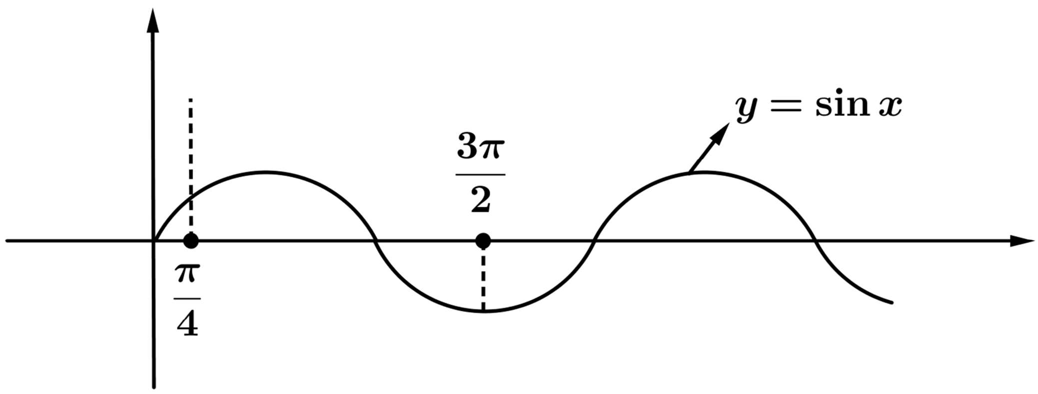
$$\text{But } \cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C \quad (4)$$

From (3)+(4) we must show: $1+\cos A\cos B\cos C\le\frac{9}{8}\Leftrightarrow\cos A\cos B\cos C\le\frac{1}{8}$*, which its true.*

PROBLEM 3.112-*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{x^2}{y}+\frac{y^2}{x}\overset{(1)}{=}\sqrt[8]{128(x^8+y^8)},\quad 4x^3-3y\overset{(2)}{=}\sqrt{\frac{1+\sqrt{1-xy}}{2}}$$

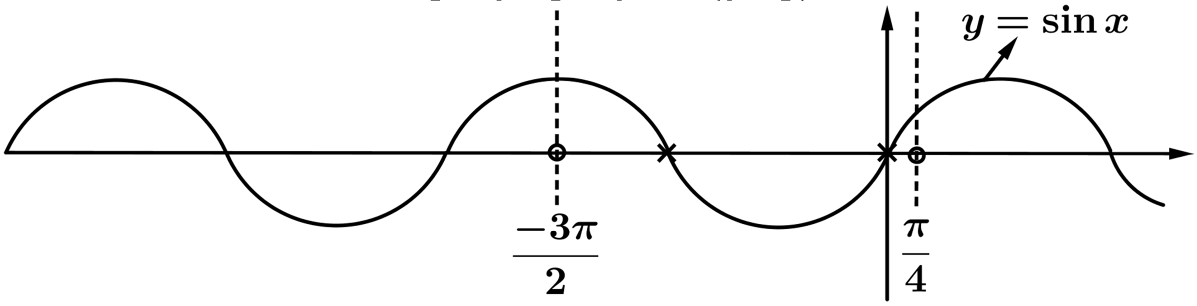
$$(1)\Leftrightarrow(x^3+y^3)^8-128x^8y^8(x^8+y^8)=0\Leftrightarrow t^{24}+8t^{21}+28t^{18}-128t^{16}+56t^{15}+$$
$$+70t^{12}+56t^9-128t^8+28t^6+8t^3+1=0\ \left(t=\frac{x}{y}\right)\Leftrightarrow$$
$$\Leftrightarrow(t-1)^2(t^{22}+2t^{21}+3t^{20}+12t^{19}+21t^{18}+30t^{17}+67t^{16}+104t^{15}+13t^{14}-$$
$$-22t^{13}-57t^{12}-92t^{11}-57t^{10}-22t^9+13t^8+104t^7+67t^6+30t^5+21t^4+$$
$$+12t^3+3t^2+2t+1)=0\Leftrightarrow(t-1)^2\cdot p=0\quad (a)\ (say)$$
$$Now,\ 12t^9-22t^{13}+104t^7=2t^7(6t^{12}-11t^6+52)=$$
$$=2t^7(6a_1^2-11a_1+52)[(a_1=t^6)]>0\ (i)\because discriminant\ \Delta=121-4\cdot6\cdot52<0$$
$$Also,\ 21t^{18}-57t^{12}+67t^6=t^6(21t^{12}-57t^6+67)=t^6(21a_1^2-57a_1+67)>0\ (ii)$$
$$\because\Delta=57^2-84\cdot67<0$$
$$Again,\ 67t^{16}-57t^{10}+21t^4=t^4(67t^{12}-57t^6+21)=$$
$$=t^4(67a_1^2-57a_1+21)>0\ (iii)\because\Delta=57^2-84\cdot67<0$$
$$Moreover,\ 104t^{15}-22t^9+12t^3\ (iv)=2t^3(52t^{12}-11t^6+6)=$$
$$=2t^3(52a_1^2-11a_1+6)>0\ (v)\because\Delta=11^2-24\cdot52<0$$
$$Now,\ 30t^{17}+30t^5\underset{(vi)}{\overset{A-G}{\ge}}60t^{11},13t^4+13t^8\underset{(vii)}{\overset{A-G}{\ge}}26t^{11}$$
$$3t^{20}+3t^2\underset{(viii)}{\overset{A-G}{\ge}}6^{11}\ \&\ of\ course,\ t^{22}+2t^{21}+2t+1\overset{(ix)}{>}0\ as\ t>0\left(t=\frac{x}{y}>0\right)$$
$$(i)+(ii)+(iii)+(iv)+(v)+(vi)+(vii)+(viii)+(ix)\Rightarrow P>0\therefore a\Rightarrow t=1\Rightarrow x=y$$
$$From\ (2),\ we\ have\ 1-xy=1-x^2\ge0\Rightarrow x\le1\Rightarrow0<x\le1$$
$$\therefore 0<x,y\le1.\ Let\ x=y=\cos\theta\quad\left(0<\theta<\frac{\pi}{2}\right)$$
$$\therefore (2)\ becomes\ 4\cos^3\theta-3\cos\theta=\sqrt{\frac{1+\sin\theta}{2}}\Rightarrow\cos\theta=\sqrt{\frac{1+\sin\theta}{2}}\ (2a)\Rightarrow$$
$$\Rightarrow 2\cos^2(3\theta)=1+\sin\theta\Rightarrow1+\cos6\theta=1+\sin\theta\Rightarrow\cos6\theta=\cos\left(\frac{\pi}{2}-\theta\right)\Rightarrow$$
$$\Rightarrow 2\sin\left(\frac{\pi}{4}+\frac{5\theta}{2}\right)\sin\left(\frac{\pi}{4}-\frac{7\theta}{2}\right)=0.\ Now,\ \frac{\pi}{4}<\frac{\pi}{4}+\frac{5\theta}{2}<\frac{3\pi}{2}$$



$\therefore \sin\left(\frac{\pi}{4}+\frac{5\theta}{2}\right)=0 \Rightarrow \frac{\pi}{4}+\frac{5\theta}{2}=\pi \Rightarrow \theta=\frac{3\pi}{10}$. *From (2a),* $\cos 3\theta>0$ *(*), but* $\cos\frac{9\pi}{10}<0$

$\therefore x=y=\cos\frac{3\pi}{10}$ *is not an acceptable solution.*

Also, $-\frac{3\pi}{2}<\frac{\pi}{4}-\frac{7\theta}{2}<\frac{\pi}{4} \therefore \sin\left(\frac{\pi}{4}-\frac{7\theta}{2}\right)=0$



$$\Rightarrow \frac{\pi}{4}-\frac{7\theta}{2}=0,-\pi \Rightarrow \theta=\frac{\pi}{14},\frac{5\pi}{14}$$

But $\cos\frac{15\pi}{14}<0, \therefore x=y=\cos\frac{5\pi}{14}$ *is unacceptable*

$\therefore$ *only possible solution is:* $x=y=\cos\frac{\pi}{14}$

PROBLEM 3.113-*Solution by proposer*

By the AM-GM inequality, we have:

$$\sqrt{\frac{3x^2+yz}{y^2+z^2}}=\sqrt{\frac{3x^2+\frac{y^2z^2}{yz}}{y^2+z^2}}\ge\sqrt{\frac{3x^2+\frac{2y^2z^2}{y^2+z^2}}{y^2+z^2}}=\frac{\sqrt{2y^2z^2+3x^2y^2+3z^2x^2}}{y^2+z^2}$$

Similarly, we have: $LHS\ge\sum\frac{\sqrt{2x^2y^2+3y^2z^2+3z^2x^2}}{x^2+y^2}$

Put $x^2=a, y^2=b, z^2=c \quad (a,b,c>0)$. *Thus, we need to prove:*

$$\sum\frac{\sqrt{2ab+3bc+3ca}}{a+b}\ge\sqrt{\frac{103}{6}+\frac{20abc}{3(a+b)(b+c)(c+a)}}$$

$$\Leftrightarrow\sum\frac{2ab+3bc+3ca}{(a+b)^2}+2\sum\frac{\sqrt{(2bc+3ca+3ab)(2ca+3ab+3bc)}}{(b+c)(c+a)}\ge$$

$$\ge\frac{103}{6}+\frac{20abc}{3(a+b)(b+c)(c+a)}$$

By the Cauchy – Schwarz inequality, we have:

$$\sum\frac{\sqrt{(2bc+3ca+3ab)(2ca+3ab+3bc)}}{(b+c)(c+a)}\ge\sum\frac{3ab+c\sqrt{(2a+3b)(3a+b)}}{(b+c)(c+a)}$$

$$=\sum\frac{3ab+c\cdot\frac{(2a+3b)(3a+2b)}{\sqrt{(2a+3b)(3a+2b)}}}{(b+c)(c+a)}\ge\sum\frac{3ab+c\cdot\frac{2(6a^2+13ab+6b^2)}{5(a+b)}}{(b+c)(c+a)}$$

$$\ge\sum\frac{3ab+c\cdot\frac{7a^2+16ab+7b^2}{3(a+b)}}{(b+c)(c+a)}=\frac{16(a+b+c)(ab+bc+ca)}{3(a+b)(b+c)(c+a)}$$

By the Iran 1996 inequality, we have: $(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right]\ge\frac{9}{4}$

Thus, we need to prove: $\frac{9}{2}+\sum\frac{a}{b+c}+\frac{32(a+b+c)(ab+bc+ca)}{3(a+b)(b+c)(c+a)}\ge\frac{103}{6}+\frac{20abc}{3(a+b)(b+c)(c+a)}$

$$\Leftrightarrow\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\ge 2$$

$$\Leftrightarrow a^3+b^3+c^3+3abc\ge ab(a+b)+bc(b+c)+ca(c+a)$$

True by Schur inequality. The equality holds for $x=y=z$.

PROBLEM 3.114-*Solution by proposer*

We use the substitution: $a=y+z, b=z+x, c=x+y$, *for three positive real numbers* $x,y,z>0$. *Then* $x=s-a, y=s-b, z=s-c$, *with* s *the semiperimeter of the triangle. Hence:* $xy+yz+zx=-s^2+ab+ab+bc+ca=r(4R+r)$, *be the well-known relations [1],* $ab+bc+ca=s^2+4Rr+r^2$. *The inequality is transformed to* $(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\ge\frac{9}{4}$, *and this is a famous inequality, [2].*

Remark 1. By Euler's inequality $R\ge 2r$, *we have* $(9R+2r)(R-2r)\ge 0$. *Hence* $\frac{9}{4r(4R+r)}\ge\frac{1}{R^2}$. *Thus the inequality is a sharpening of the well-known inequality*

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{1}{R^2}$$

References

[1] O. Bottema, R.Z. Djordjevic, R. R. Janic, D. S. Mitrinovic, P.M. Vasic, *Geometric inequalities., Groningen, Wolters-Noordhoff, 1969.*

[2] CRUX Mathematicorum, 1994, No. 4, p. 108, Problem 1940

PROBLEM 3.115-*Solution by Feti Sinani-Podujeve-Kosovo*

$$\int_0^{+\infty}\frac{\ln(x)\sin x}{x^{\frac{1}{2}}}dx=Im\int_0^{+\infty}e^{xi}x^{-\frac{1}{2}}\ln(x)\,dx=\left[\frac{x}{i}=t\right]=Im\int_0^{+\infty}e^{-x}x^{-\frac{1}{2}}\ln(xi)\,i^{\frac{1}{2}}dx=$$

$$=Im\left(\int_0^{+\infty}e^{-x}x^{-\frac{1}{2}}\ln(x)\left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}}dx+Im\int_0^{+\infty}e^{-x}x^{-\frac{1}{2}}\ln\left(e^{i\frac{\pi}{2}}\right)\left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}}dx\right)=$$

$$=Im\left(\Gamma\left(\frac{1}{2}\right)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)+\Gamma\left(\frac{1}{2}\right)\frac{\pi}{2}i\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)=$$

$$=\frac{\sqrt{2}}{2}\left(\Gamma\left(\frac{1}{2}\right)+\frac{\pi}{2}\Gamma\left(\frac{1}{2}\right)\right)=\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{2}}\left(\Psi\left(\frac{1}{2}\right)+\frac{\pi}{2}\right)=\sqrt{\frac{\pi}{2}}\left(-\gamma-2\ln 2+\frac{\pi}{2}\right)$$

PROBLEM 3.116-*Solution by proposers*

Apply the Stirling's second sum, we get,

$$\int_0^1 x^{n-1}\ln^k(1-x)\,dx=(-1)^k\frac{\frac{2}{k}(n)}{n}\quad (1)$$

where,

$$\frac{2}{k}(n)=\sum_{j=0}^{k-1}\binom{k-1}{j}(k-j-1)!\,H_n^{(k-j)}\frac{2}{j}(n)$$

Putting $k=4$, we get,

$$\tfrac{2}{4}(n)=H_n^4+6H_n^2H_n^{(2)}+3\left(H_n^{(2)}\right)^2+8H_nH_n^{(3)}+6H_n^{(4)}\quad (2)$$

then using (1) & (2), and then summing both sides, we get

$$-S=\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{n-1}\ln^4(1-x)\,dx\Rightarrow\int_0^1\sum_{n=1}^{\infty}(-x)^n\frac{\ln^4(1-x)}{x}dx\Rightarrow\int_0^1\frac{\ln^4(1-x)}{1+x}dx$$

Replace, $x\to 1-x$

$$\Rightarrow\int_0^1\frac{\ln^4(x)}{2-x}dx\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}\int_0^1\frac{x^n}{2^n}\ln^4(x)\,dx\Rightarrow\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 x^{n-1}\ln^4(x)\,dx$$

$$\Rightarrow\sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{\partial^4}{\partial n^4}\left[\int_0^1 x^{n-1}\,dx\right]\Rightarrow$$

$$\Rightarrow\sum_{n=0}^{\infty}\frac{1}{2^n}\left[\frac{x^n\ln^4(x)}{n}-\frac{4x^n}{n^2}\ln^3(x)+12\frac{x^n}{n^3}\ln^2(x)-24\frac{x^n}{n^4}\ln(x)+24\frac{x^n}{n^5}\right]_0^1\Rightarrow$$

$$\Rightarrow 24\sum_{n=1}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^n}{n^5}\right)\Rightarrow 24Li_5\left(\frac{1}{2}\right)\ \textit{(OR)}\ -S=24Li_5\left(\frac{1}{2}\right)\ \textit{(OR)}\ S=-24Li_5\left(\frac{1}{2}\right)\ \textit{(Answer)}$$

PROBLEM 3.117-*Solution by proposer*

** By AM-GM inequality we have:*

$$a^3+a^3+a^3+\sqrt[4]{a}+\sqrt[4]{a}+\sqrt[4]{a}+\sqrt[4]{a}+1+1+1\ge 10\cdot\sqrt[10]{a^3\cdot a^3\cdot a^3\cdot\left(\sqrt[4]{a}\right)^4\cdot 1\cdot 1\cdot 1}=$$

$$=10\cdot\sqrt[10]{a^{10}}=10a\Rightarrow 3a^3+4\cdot\sqrt[4]{a}+3\ge 10a\Rightarrow 4\cdot\sqrt[4]{a}\ge 10a-3a^3-3\qquad (1)$$

- Other, because $a,b,c>0;a+b+c=3\Rightarrow a<3\Rightarrow(a-3)(a-1)^2\le 0\Leftrightarrow(a-3)(a^2-2a+1)\le 0$

$$\Leftrightarrow a^3-5a^2+7a-3\le 0\Leftrightarrow a^3\le 5a^2-7a+3\qquad (2)$$

- Let (1), (2): $\Rightarrow 4\sqrt[4]{a}\ge 10a-3(5a^2-7a+3)-3=31a-15a^2-12\Rightarrow$

$$\sqrt[4]{a}\ge\frac{31a-15a^2-12}{4}$$

$+$ *Similar:* $\sqrt[4]{b}\ge\frac{31b-15b^2-12}{4};\ \sqrt[4]{c}\ge\frac{31c-15c^2-12}{4}$

$-$ *Therefore:* $\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}\ge\frac{31(a+b+c)-15(a^2+b^2+c^2)-36}{4}=\frac{31\cdot 3-15(a^2+b^2+c^2)-36}{4}$

$$\Rightarrow\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}}{30}\ge\frac{57-15[(a+b+c)^2-2(ab+bc+ca)]}{120}=\frac{30(ab+bc+ca)-78}{120}=\frac{ab+bc+ca}{4}-\frac{13}{20}$$

$$\Rightarrow\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}}{30}+\frac{11}{40}\ge\frac{ab+bc+ca}{4}-\frac{3}{8}\qquad (3)$$

$+$ *Other:* $\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}+a^2+a^2\ge 5\cdot\sqrt[5]{\left(\sqrt[3]{a}\right)^3\cdot a^2\cdot a^2}=5\cdot\sqrt[5]{a^5}=5a$

$$\Leftrightarrow 3\cdot\sqrt[3]{a}+2a^2\geq 5a\Leftrightarrow \sqrt[3]{a}\geq\frac{5a-2a^2}{3}.\ \textit{Similar: } \sqrt[3]{b}\geq\frac{5b-2b^2}{3};\ \sqrt[3]{c}\geq\frac{5c-2c^2}{3}$$

- *Therefore:* $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\geq\frac{5(a+b+c)-2(a^2+b^2+c^2)}{3}=\frac{5\cdot 3-2[(a+b+c)^2-2(ab+bc+ca)]}{3}$

$$\Rightarrow \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\geq\frac{15-2\cdot 3^2+4(ab+bc+ca)}{3}\Leftrightarrow \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\geq\frac{4(ab+bc+ca)-3}{3}\quad (4)$$

- *Because* $a,b,c\in\left[\frac{1}{2};3\right)\Rightarrow\left(a-\frac{1}{2}\right)\left(b-\frac{1}{2}\right)+\left(b-\frac{1}{2}\right)\left(c-\frac{1}{2}\right)+\left(c-\frac{1}{2}\right)\left(a-\frac{1}{2}\right)\geq 0$

$$\Leftrightarrow ab+bc+ca\geq a+b+c-\frac{3}{4}=3-\frac{3}{4}=\frac{9}{4}\Rightarrow ab+bc+ca-2>0\quad (5)$$

\+ *Let (4), (5):* $\Rightarrow\frac{3(ab+bc+ca-2)}{2\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)}\leq\frac{3(ab+bc+ca-2)}{\frac{8(ab+bc+ca)}{3}}=\frac{9(ab+bc+ca-2)}{8(ab+bc+ca)}\quad (6)$

- *Let (3), (6), (*). We need to prove:* $\frac{ab+bc+ca}{4}-\frac{3}{8}\geq\frac{9(ab+bc+ca-2)}{8(ab+bc+ca)}\Leftrightarrow$

$\Leftrightarrow\frac{ab+bc+ca}{4}+\frac{9}{4(ab+bc+ca)}\geq\frac{3}{2}$ *(True because by AM-GM inequality) and we get the result.*

\+ *Equality occurs if:* $a=b=c=1$.

PROBLEM 3.118-*Solution by Togrul Ehmedov-Baku-Azerbaidian*

$$1+x^2=y^2\Rightarrow x=\sqrt{y^2-1}\Rightarrow dx=\frac{y}{\sqrt{y^2-1}}dy$$

$$I=\int_0^\infty x^p\sqrt{\frac{1+\sqrt{1+x^2}}{1+x^2}}dx=\int_0^\infty (y^2-1)^{\frac{p}{2}}\frac{dy}{\sqrt{y-1}}=\int_0^\infty\frac{(y-1)^{\frac{p}{2}}(y+1)^{\frac{p}{2}}}{(y-1)^{\frac{1}{2}}}dy$$

$$=\int_0^\infty (y-1)^{\frac{p-1}{2}}(y+1)^{\frac{p}{2}}dy$$

$$\frac{1}{y-1}=t\Rightarrow y=\frac{1}{t}-1\Rightarrow dy=-\frac{1}{t^2}dt$$

$$I=\int_0^\infty t^{\frac{1-p}{2}}\left(\frac{1+2t}{t}\right)^{\frac{p}{2}}\frac{dt}{t^2}=\int_0^\infty t^{\frac{-2p-3}{2}}(1+2t)^{\frac{p}{2}}dt$$

$$2t=z\Rightarrow\frac{z}{2}\Rightarrow dt=\frac{dz}{2}$$

$$I=\int_0^\infty 2^{\frac{2p+1}{2}}z^{\frac{-2p-3}{z}}(1+z)^{\frac{p}{2}}dx=2^{p+\frac{1}{2}}\int_0^\infty\frac{z^{\frac{-2p-3}{2}}}{(1+z)^{-\frac{p}{2}}}dz=2^{p+\frac{1}{2}}B\left(\frac{p+1}{2},-p-\frac{1}{2}\right)$$

Note: $B(x,y)=\int_0^\infty\frac{t^{x-1}}{(1+t)^{x+y}}dt, Re(x)>0; Re(y)>0$

PROBLEM 3.119-*Solution by Ali Shather-Iraq and Shivam Sharma-India*

$$S=\sum_{k=1}^\infty\frac{H_n^{(3)}}{n^8}=\sum_{n=1}^\infty\frac{1}{n^8}\left(\zeta(3)-\sum_{k=1}^\infty\frac{1}{(n+k)^3}\right)=\zeta(3)\zeta(8)+\sum_{k=1}^\infty\sum_{n=1}^\infty\frac{-1}{n^8(n+k)^3}$$

$$S=\zeta(3)\zeta(8)+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\begin{array}{c}\frac{36}{k^{10}n}-\frac{36}{k^{10}(k+n)}-\frac{28}{k^9n^2}-\frac{8}{k^9(k+n)^2}+\frac{21}{k^8n^5}-\\-\frac{1}{k^5(k+n)^3}-\frac{15}{k^7n^4}+\frac{10}{k^6n^5}-\frac{6}{k^5n^6}+\frac{3}{k^4n^7}-\frac{1}{k^3n^8}\end{array}\right]$$

$$S=\zeta(3)\zeta(8)+\sum_{k=1}^{n}\left[\begin{array}{c}\frac{36H_k}{k^{10}}-\frac{28\zeta(2)}{k^9}-\frac{8}{k^9}\left(\zeta(2)-H_k^{(2)}\right)+\frac{21\zeta(3)}{k^8}-\frac{35}{k^8}\left(\zeta(3)-H_k^{(3)}\right)-\\-\frac{15\zeta(4)}{k^7}+\frac{10\zeta(5)}{k^6}-\frac{6\zeta(6)}{k^5}+\frac{3\zeta(7)}{k^4}-\frac{\zeta(8)}{k^3}\end{array}\right]$$

$$S=\zeta(3)\zeta(8)+\left[\begin{array}{c}36\big(6\zeta(11)-\zeta(2)\zeta(9)-\zeta(3)\zeta(8)-\zeta(4)\zeta(7)-\zeta(5)\zeta(6)\big)-\\-28\zeta(2)\zeta(9)-8\zeta(2)\zeta(9)+8\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^8}+21\zeta(3)\zeta(8)-\zeta(3)\zeta(8)+\\+S-15\zeta(4)\zeta(7)+10\zeta(5)\zeta(6)-6\zeta(6)\zeta(5)+3\zeta(7)\zeta(4)-\zeta(8)\zeta(3)\end{array}\right]$$

$$-8\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^9}=216\zeta(11)-72\zeta(2)\zeta(9)-16\zeta(3)\zeta(8)-48\zeta(4)\zeta(7)-32\zeta(5)\zeta(6)$$

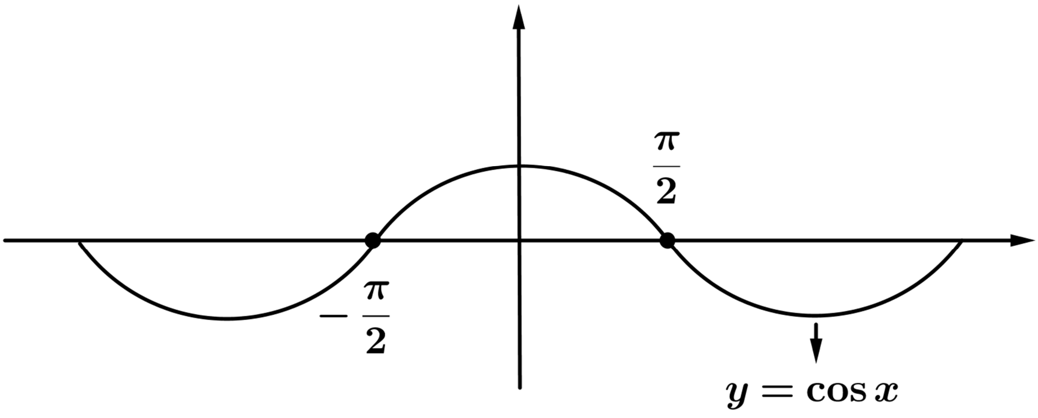
Or

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^9}=9\zeta(2)\zeta(9)+2\zeta(3)\zeta(8)+6\zeta(4)\zeta(7)+4\zeta(5)\zeta(6)-27\zeta(11)$$

PROBLEM 3.120-*Solution by Soumava Chakraborty-Kolkata-India*

$\Delta\,ABC$ *is acute-angled,* $\therefore\ 0<A,B,C<\frac{\pi}{2}\ \therefore\ 0<A<\frac{\pi}{2}\ \&\ -\frac{\pi}{2}<-B<0$

Adding these last 2, $-\frac{\pi}{2}<A-B<\frac{\pi}{2}$



From the graph, we see $0<\cos(A-B)\le 1$ *(1)*

Similarly, $0<\cos(B-C)\le 1$ *(2)*

$0<\cos(C-A)\le 1$ *(3)*

$$\therefore\sum\sin C\,(1+\cos 2(A-B))=\sum\sin C\cdot 2\cos^2(A-B)$$

$$\overset{by\ (1),(2),(3)}{\le}\sum 2\sin C=2\sum\sin A\overset{?}{<}\sqrt{2}\sum(\sin A+\cos A)$$

$$\Leftrightarrow\sqrt{2}\sum\cos A\overset{?}{>}(2-\sqrt{2})\sum\sin A\Leftrightarrow\sqrt{2}\left(\frac{R+r}{R}\right)\overset{?}{>}(2-\sqrt{2})\left(\frac{s}{R}\right)$$

$$\Leftrightarrow 2(R+r)^2 \overset{?}{>} \left(6-4\sqrt{2}\right)s^2 \quad (a)$$

$$\because 128 > 121 \therefore 8\sqrt{2} > 11 \Rightarrow 4\sqrt{2} > \frac{11}{2} = 6 - \frac{1}{2} \Rightarrow \frac{1}{2} > 6 - 4\sqrt{2} \quad (4)$$

(4) $\Rightarrow$ *in order to prove (a), it suffices to show:* $2(R+r) > \frac{1}{2}s^2 \Leftrightarrow$

$$\Leftrightarrow s^2 < 4(R+r)^2 \quad (b)$$

Now, LHS of (b) $\overset{Gerretsen}{\leq} 4R^2 + 4Rr + 3r^2 \overset{?}{<} 4R^2 + 8Rr + 4r^2 \Leftrightarrow$

$$\Leftrightarrow 4Rr + r^2 \overset{?}{>} 0 \to \text{true} \Rightarrow (b) \text{ is true} \Rightarrow (a) \text{ is true}$$

PROBLEM 3.121-*Solution by Khalef Ruhemi-Jarash-Jordan*

$$I = \int_0^\infty x^{p-1}\left(\frac{2}{1+\sqrt{1+4x}}\right)^n dx. \text{ Let } \frac{2}{\sqrt{1+4x}+1} = y \Rightarrow x = \frac{1}{y^2} - \frac{1}{y} = \frac{1-y}{y^2}$$

$$dx = -\frac{2}{y^3} + \frac{1}{y^2}$$

$$\therefore I = -\int_0^1 (1-y)^{p-1} y^{2-2p} y^n (y-2)^{-3} y \cdot dy$$

$$\therefore I = -\int_0^1 y^{n-2p-1}(1-y)^{p-1}(y-2)dy = \int_0^1 x^{n-2p-1}(1-x)^{p-1}\big(1+(1-x)\big)dx =$$

$$= \int_0^1 x^{n-2p-1}(1-x)^{p-1}dx + \int_0^1 x^{n-2p-1}(1-x)^p dx = \int_0^1 x^{n-2p-1}(1-x)^{p-1}dx +$$

$$+ \left.\frac{x^{n-2p}(1-x)^p}{n-2p}\right|_0^1 + \frac{p}{n-2p}\int_0^1 x^{n-2p}(1-x)^{p-1} = \frac{\Gamma(n-2p)\Gamma(p)}{\Gamma(n-p)} +$$

$$+\left(\frac{p}{n-2p}\right)\frac{\Gamma(n-2p+1)\Gamma(p)}{\Gamma(n-p+1)} =$$

$$\beta(n-2p,p) + \left(\frac{p}{n-2p}\right)\left(\frac{(n-2p)\Gamma(n-2p)\Gamma(p)}{(n-p)\Gamma(n-p)}\right) =$$

$$= \mathrm{B}(n-2p,p) + \left(\frac{p}{n-p}\right)\mathrm{B}(n-2p,p) = \left(\frac{n}{n-p}\right)\mathrm{B}(n-2p,p) \to$$

$$I = \left(\frac{n}{n-p}\right)\mathrm{B}(n-2p,p)$$

PROBLEM 3.122-*Solution by proposer*

$$a_n = \sum \arctan\frac{1}{k^2+k+1} = \sum \arctan\frac{\frac{1}{k}-\frac{1}{k+1}}{1+\frac{1}{k(k+1)}} =$$

$$= \sum\left(\arctan\frac{1}{k} - \arctan\frac{1}{k+1}\right) = \frac{\pi}{4} - \arctan\frac{1}{n+1} \to \frac{\pi}{4}$$

$$\lim_{n\to\infty} \frac{a_n^{\frac{\pi}{4}} - \frac{\pi}{4}^{a_n}}{\frac{\pi}{4} - a_n} \cdot 4\left(\frac{\pi}{4} - a_n\right) \quad (1)$$

$$a_n = x \to \frac{\pi}{4} \Rightarrow \lim_{x\to\frac{\pi}{4}} \frac{x^{\frac{\pi}{4}} - \frac{\pi}{4}^x}{\frac{\pi}{4} - x} = \left(\frac{\pi}{4}\right)^{\frac{\pi}{4}} \left(\ln\frac{\pi}{4} - 1\right) \quad (2)$$

$$\lim_{x\to a} \frac{x^a - a^x}{a - x} = a^a(\ln a - 1)$$

$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - a_n\right) = \lim_{n\to\infty} \frac{\frac{\pi}{4} - a_n}{\frac{1}{n}} \overset{C.S.}{=} \lim_{n\to\infty} \frac{\frac{\pi}{4} - a_{n+1} - \frac{\pi}{4} + a_n}{\frac{1}{n+1} - \frac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{-\arctan\frac{1}{(n+1)^2+n+1+1}}{-\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{\arctan\frac{1}{(n+1)^2+n+2}}{\frac{1}{(n+1)^2+(n+2)} \cdot \frac{(n+1)^2+n+2}{n(n+1)}} = 1 \quad (3)$$

From (1)+(2)+(3)

$$\Rightarrow \lim_{n\to\infty} n\left(a_n^{\frac{\pi}{4}} - \frac{\pi^{a_n}}{4}\right) = \left(\frac{\pi}{4}\right)^{\frac{\pi}{4}} \left(\ln\frac{\pi}{4} - 1\right)$$

PROBLEM 3.123-*Solution by Shivam Sharma-New Delhi-India*

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{\sqrt{n^4+k}+k}{n^2}\right)^2 \Rightarrow \lim_{n\to\infty} \prod_{k=1}^{n} \left(\sqrt{1+\frac{k^2}{n^4}} + \frac{k}{n^2}\right)^2 \Rightarrow \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k^2}{2n^4} + \frac{k}{n^2} + \cdots\right)^2$$

$$\ln(\Omega) = 2\lim_{n\to\infty} \ln\left(\prod_{k=1}^{n} \left(1 + \frac{k}{n^2}\right)\right)$$

$$\ln(\Omega) = 2\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n^2}\right)$$

As we know, $x - \frac{x^2}{2} \le \ln(1+x) \le x$*. Using this, we get,*

$$\sum_{k=1}^{n} \left(\frac{k}{n^2} - \frac{k^2}{n^4}\right) \le \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n^2}\right) \le \sum_{k=1}^{n} \left(\frac{k}{n^2}\right)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right) - \lim_{n\to\infty} \frac{1}{2n^2} \left(\sum_{k=1}^{n} \frac{k^2}{n^2}\right) \le \lim_{n\to\infty} \sum_{k} \ln\left(1 + \frac{k}{n^2}\right) \le \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)$$

$$2\left[\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right) - \lim_{n\to\infty} \frac{1}{2n^2} \left(\sum_{k=1}^{n} \frac{k^2}{n^2}\right)\right] \le 2\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n^2}\right) \le 2\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)$$

$$2\left[\int_0^1 x\,dx - \lim_{n\to\infty} \left(\frac{1}{2n^2}\right) \int_0^1 x^2\,dx\right] \le 2\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n^2}\right) \le 2\int_0^1 x\,dx$$

$$2\left[\frac{x^2}{2}\right]_0^1 - 0 \le 2\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n^2}\right) \le 2\left[\frac{x^2}{2}\right]_0^1$$

$$1 \le 2\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n^2}\right) \le 1, 1 \le \ln(\Omega) \le 1$$

By squeeze theorem, we get $\Omega = e$ *(Answer)*

PROBLEM 3.124-***Solution by Ravi Prakash-New Delhi-India***

$$\int_{e^x}^{e^{2x}} f(t)\,dt = \int_1^{e^x} f(t)\,dt$$

Differentiating w.r.t. x we get: $2e^{2x}f(e^{2x}) - e^x f(e^x) = e^x f(e^x) \Rightarrow 2e^{2x}f(e^{2x}) =$
$= 2e^x f(e^x) \Rightarrow e^x f(e^{2x}) = f(e^x)$. *Put* $x = \ln t, t > 0$ *to obtain:*

$$tf(t^2) = f(t)\ \ \forall t > 0 \quad (1)$$

Taking $\lim t \to 0+$, *we get:* $0 = f(0)$. *For* $0 < t < 1$, $tf(t) = t^2 f(t^2) = t^{2^2} f\left(t^{2^2}\right) =$
$= t^{2^3} f\left(t^{2^3}\right) = \cdots t^{2^n} f\left(t^{2^n}\right)\ \forall n \in \mathbb{N}$. *Taking limit as* $n \to \infty$, *we get*

$$tf(t) = \lim_{n\to\infty} t^{2^n} f\left(t^{2^n}\right), 0 < t < 1$$

$= 0f(0) = 0 \Rightarrow f(t) = 0$ *for* $0 < t < 1$. *As* f *is continuous* $f(1) = \lim_{t\to 1-} f(t) = 0$.
$\therefore$ *for* $0 \le t \le 1, f(t) = 0$. *Let* $t > 1, t^2 f(t^2) = tf(t) = t^{\frac{1}{2}} f\left(t^{\frac{1}{2}}\right) = t^{\frac{1}{4}} f\left(t^{\frac{1}{4}}\right) = \cdots =$
$= t^{\frac{1}{2^n}} f\left(t^{\frac{1}{2^n}}\right)\ \forall n \in \mathbb{N}$. *Taking limit as* $n \to \infty$, *we get, for* $t > 1$

$$tf(t) = \lim_{n\to\infty} t^{\frac{1}{2^n}} f\left(t^{\frac{1}{2^n}}\right) = (1)f(1)\ [\because f \text{ is continuous}] \Rightarrow tf(t) = 0\ \forall t > 1$$

$\Rightarrow f(t) = 0\ \forall t > 1$. *Thus,* $f(x) = 0, \forall x \ge 0$. *Let* g *be any continuous function on* $(-\infty, 0]$ *such that* $g(0) = 0$, *then:* $f(x) = \begin{cases} g(x), \forall x < 0 \\ 0, \forall x \ge 0 \end{cases}$

PROBLEM 3.125-***Solution by proposer***

$$\text{If } x > 0 \text{ then: } \frac{2}{x+1} \le \ln\left(1 + \frac{1}{x}\right) \le \frac{1}{\sqrt{x^2+x}} \quad (1)$$

Let be $f(x) = \frac{1}{\sqrt{x^2+x}} - \ln\left(1 + \frac{1}{x}\right)$ *then:* $f'(x) = \frac{2\sqrt{x^2+1} - (2x+1)}{2x(x+1)\sqrt{x^2+x}} < 0$
therefore $x < \infty \Rightarrow f(x) > f(\infty) = 0$. *Let be* $g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{2}{2x+1}$ *then:*
$g'(x) = \frac{-1}{x(x+1)} - \frac{2}{2x+1} < 0$ *therefore* $x < \infty \Rightarrow g(x) > g(\infty) = 0$
Using (1) we have: $\sum_{k=1}^n \ln^2\left(1 + \frac{1}{k}\right) \le \sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1}$ *and if* $x = \frac{k^2+k-1}{2}$
then $\ln\left(1 + \frac{2}{k^2+k-1}\right) \ge \frac{2}{k(k+1)} \Rightarrow \sum_{k=1}^n \left(1 + \frac{2}{k^2+k-1}\right) \ge 2\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{2n}{n+1}$

PROBLEM 3.126-***Solution by Khalef Ruhemi-Jarash-Jordan***

$I := \int_0^1 \int_0^1 \frac{\sqrt{x+y} + \sqrt{xy}}{x\sqrt{y} + y\sqrt{x}} dx\,dy \to$ *Let* $x = t^2 \Rightarrow dx = 2t\,dt; y = v^2 \Rightarrow dy = 2vdv$

$$\therefore I = 4\int_0^1\int_0^1 \frac{\left(\sqrt{t^2+v^2} + tv\right)tv \cdot dt\,dv}{t^2v + v^2t} = 4\int_0^1 \left(\frac{\sqrt{x^2+y^2} + xy}{x+y} \cdot dx\right) dy$$

Let $x = yt \Rightarrow dx = y\,dt$

$$= 4\int_0^1\left(\int_0^{\frac{1}{y}}\frac{y\sqrt{1+x^2}+y^2x}{1+x}dx\right)dy = 4\int_0^1\int_0^1\frac{y\sqrt{1+x^2}+y^2x}{1+x}dy\,dx +$$

$$+4\int_0^{\infty}\int_0^{\frac{1}{x}}\frac{y\sqrt{1+x^2}+y^2x}{1+x}dy\,dx = 4\int_0^1\frac{\frac{1}{2}\sqrt{1+x^2}+\frac{1}{3}x}{1+x}dx + 4\int_0^{\infty}\frac{\frac{\sqrt{1+x^2}}{2x^2}+\frac{1}{3x^2}}{1+x}dx$$

$$Let\,\frac{1}{x} = y \Rightarrow \frac{1}{y};\, dx = -\frac{dy}{y^2}$$

$$= 4\int_0^1\frac{\frac{1}{2}\sqrt{1+x^2}+\frac{1}{3}x}{1+x}dx + 4\int_0^1\frac{\frac{1}{2}\sqrt{1+x^2}+\frac{1}{3}x}{1+x}dx$$

$$= 8\int_0^1\frac{\frac{1}{2}\sqrt{1+x^2}+\frac{1}{3}x}{1+x}dx = 4\int_0^1\frac{\sqrt{1+x^2}}{1+x}dx + \frac{8}{3}\int_0^1\frac{1+x-1}{1+x}dx$$

$$= \left(\frac{8}{3}\right)\int_0^1\left(1-\frac{1}{1+x}\right)dx + 4\int_0^1\frac{x}{\sqrt{1+x^2}}dx - \underbrace{4\int_0^1\frac{dx}{\sqrt{1+x^2}} + 8\int_0^1\frac{dx}{(1+x)\sqrt{1+x^2}}}_{let\,x=\tan(\theta);dx=\sec^2(\theta)d\theta}$$

$$= \left(\frac{8}{3}\right)(x-\ln(1+x))\Big|_0^1 + \left(4\sqrt{1+x^2}\right)\Big|_0^1 - 4\int_0^{\frac{\pi}{4}}\sec(\theta)\,d\theta + \frac{8}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\sec\left(\frac{\pi}{4}-\theta\right)d\theta$$

$$= \left(\frac{8}{3}\right)(1-\ln(2)) + 4\sqrt{2} - 4 - \left(4\ln(\tan(\theta)+\sec(\theta))\Big|_0^{\frac{\pi}{4}}\right) -$$

$$-\frac{8}{\sqrt{2}}\left(\ln\left(\tan\left(\frac{\pi}{4}-\theta\right)+\sec\left(\frac{\pi}{4}-\theta\right)\right)\Big|_0^{\frac{\pi}{4}}\right) = \frac{8}{3}-\frac{8}{3}\ln(2) + 4\sqrt{2} - 4 - 4\ln\left(1+\sqrt{2}\right) +$$

$$+\frac{8}{\sqrt{2}}\ln\left(1+\sqrt{2}\right) = -\frac{4}{3}-\frac{8}{3}\ln(2) + 4\sqrt{2} + 4\sqrt{2}\ln\left(1+\sqrt{2}\right) - 4\ln\left(1+\sqrt{2}\right) =$$

$$= -\frac{4}{3}\left(1-3\sqrt{2}+2\ln(2)\right) + 4\ln\left(\sqrt{2}-1\right) + 4\sqrt{2}\ln\left(\sqrt{2}+1\right) =$$

$$= -\frac{4}{3}\left(1-3\sqrt{2}+2\ln(2)\right) + \ln\left(\left(\sqrt{2}-1\right)^4\right) + \sqrt{2}\ln\left(\left(\sqrt{2}+1\right)^4\right) =$$

$$= -\frac{4}{3}\left(1-3\sqrt{2}+2\ln(2)\right) + \ln\left(17-12\sqrt{2}\right) + \sqrt{2}\ln\left(17+12\sqrt{2}\right) = I$$

$$\therefore \int_0^1\int_0^1\frac{\sqrt{x+y}+\sqrt{xy}}{x\sqrt{y}+y\sqrt{x}}dx\,dy = -\frac{4}{3}\left(1-3\sqrt{2}+2\ln(2)\right) + \ln\left(17-12\sqrt{2}\right) +$$

$$+\sqrt{2}\ln\left(17+12\sqrt{2}\right)$$

PROBLEM 3.127- ***Solution by proposer***

$$If\,x_1+x_2+x_3+x_4+x_5 = 0\,then = \left|\cos\left(\sum_{k=1}^5 x_k\right)\right| \le |\cos x_1| + \left|\sin\left(\sum_{k=2}^5 x_k\right)\right| \le$$
$$\le |\cos x_1| + |\cos x_2| + |\cos(x_3+x_4+x_5)| \le |\cos x_1| + |\cos x_2| +$$

$$+|\cos x_3| + |\cos x_4| + |\cos x_5|$$

If $x_1 = x, x_2 = y, x_3 = z, x_4 = t, x_5 = -x - y - z - t$ *then:*

$$|\cos x| + |\cos y| + |\cos z| + |\cos t| + |\cos(x + y + z + t)| \geq 1 \quad (1)$$

In (1) we take $x = \frac{\pi}{2} - y; y = \frac{\pi}{2} - y_2, z = \frac{\pi}{2} - y_3, t = \frac{\pi}{2} - y_4 \Rightarrow$

$$\Rightarrow |\sin y_1| + |\sin y_2| + |\sin y_3| + |\sin y_4| + |\cos(y_1 + y_2 + y_3 + y_4)| \geq 1 \text{ or}$$

$$|\sin x| + |\sin y| + |\sin z| + |\sin t| + |\sin(x + y + z + t)| \geq 1 \quad (2)$$

Adding (1) and (2) we obtain the result.

PROBLEM 3.128- ***Solution by proposer***

In inequality $\sum_{k=1}^{\infty} \frac{x_k^2}{a_k} \geq \frac{\left(\sum_{k=1}^{\infty} x_k\right)^2}{\sum_{k=1}^{\infty} a_k}$ *we take* $x_k = \frac{1}{k^\alpha}, a_k = \frac{1}{k\hat{}\beta}$ *and we obtain*

$$\sum_{k=1}^{\infty} \frac{1}{k^{2\alpha-\beta}} \geq \frac{\left(\sum_{k=1}^{\infty} \frac{1}{k^\alpha}\right)^2}{\sum_{k=1}^{\infty} \frac{1}{k^\beta}} \text{ or } \zeta(2\alpha - \beta) \geq \frac{\zeta^2(\alpha)}{\zeta(\beta)} \text{ or } \zeta(2\alpha - \beta)\zeta(\beta) \geq \zeta^2(\alpha)$$

PROBLEM 3.129-*Solution by Marian Ursărescu – Romania*

We use Cauchy – D'Alembert, Cesaró – Stolz and $f^9 = e^{9\ln 7}$ *and* $\frac{a^{x_n}-1}{x_n} \to \ln a$*, with* $x_n \to 0$.

$$L = \lim_{n\to\infty} b_{n+1}^u \sqrt[n+1]{(P_{n+1})!^\gamma} - b_n^u \sqrt[n]{(P_{n!})^\gamma} = \lim_{n\to\infty} b_n^u \sqrt[n]{(P_n)!^n} \left(\frac{b_{n+1}^u \sqrt[n+1]{(P_{n+1})!^v}}{b_n^u \cdot \sqrt[n]{(P_n)!^v}} - 1 \right) =$$

$$= \lim_{n\to\infty} \frac{b_n^n \cdot \sqrt[n]{(P_n)^v}}{n} \cdot n \left(\frac{b_{n+1}^u \cdot \sqrt[n+1]{(P_{n+1})!^v}}{b_n^u \cdot \sqrt[n]{(P_n)!^v}} - 1 \right) \quad (1)$$

$$\lim_{n\to\infty} \frac{b_n^u \cdot \sqrt[n+1]{(P_n)!^v}}{n} = \lim_{n\to\infty} \left(\frac{b_n}{n}\right)^u \cdot \left(\frac{\sqrt[n]{(P_n)!}}{n}\right)^v \quad (2)$$

$$\lim_{n\to\infty} \frac{b_n}{n} = \lim_{n\to\infty} \frac{b_{n+1} - b_n}{n+1-n} = b \quad (3)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{(P_n)!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(P_n)!}{n^n}} = \lim_{n\to\infty} \frac{(P_{n+1})!}{(n+1)^{n+1}} \cdot \frac{n^n}{(P_n)!} = \lim_{n\to\infty} \frac{P_{n+1}}{n+1} \cdot \left(\frac{n}{n+1}\right)^n = \frac{1}{e} \lim_{n\to\infty} \frac{P_n}{n} =$$

$$= \frac{1}{e} \lim_{n\to\infty} \frac{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}}{n} = \frac{1}{e} \lim_{n\to\infty} \sqrt{\frac{a_1^2 + \dots + a_n^2}{n^3}} = \frac{1}{e} \sqrt{\lim_{n\to\infty} \frac{a_1^2 + \dots + a_n^2}{n^3}} =$$

$$= \frac{1}{e} \sqrt{\lim_{n\to\infty} \frac{a_{n+1}^2}{3n^2 + 3n + 1}} = \frac{1}{e} \sqrt{\lim_{n\to\infty} \frac{a_{n+1}^2}{(n+1)^2} \cdot \frac{(n+1)^2}{3n^2 + 3n}} = \frac{1}{e\sqrt{3}} \lim_{n\to\infty} \frac{a_n}{n} =$$

$$= \frac{1}{e\sqrt{3}} \lim_{n\to\infty} (a_{n+1} - a_n) = \frac{a}{e\sqrt{3}} \quad (4)$$

From (2) + (3) + (4) $\Rightarrow \lim_{n\to\infty} \frac{b_n^u \cdot \sqrt[n]{(P_n)!^v}}{n} = \frac{b^n \cdot a^v}{e^v \cdot \sqrt{e}^v} \quad (5)$

Now let $\lim_{n\to\infty} n \left(\frac{b_{n+1}^u \sqrt[n+1]{P_{n+1}!^v}}{b_n^u \cdot \sqrt[n]{P_{n!}^v}} - 1 \right) = \lim_{n\to\infty} n \left(\frac{\ln \frac{b_{n+1}^u \cdot \sqrt[n+1]{P_{n+1}!^v}}{b_n^u \cdot \sqrt[n]{P_{n!}^v}} - 1}{\ln\left(\frac{b_{n+1}^u \cdot \sqrt[n]{P_{n+1}!^v}}{b_n^u \cdot \sqrt[n]{P_{n!}^v}} \right)} \right) \ln \left(\frac{b_{n+1}^u \sqrt[n+1]{P_{n+1}^v}}{b_n^u \cdot \sqrt[n]{P_{n!}^v}} \right)$

$$=\lim_{n\to\infty} n\ln\left(\frac{b_{n+1}^{u}\sqrt[n+1]{P_{n+1}^{v}!}}{b_n^n\cdot\sqrt[n]{P_{n!}^{v}}}\right)=\lim_{n\to\infty}\ln\left(\frac{b_{n+1}^{u}\cdot\sqrt[n+1]{P_{n+1}!^{v}}}{b_n^u\cdot\sqrt[n]{P_n^{\overline{n}}}}\right)^n \quad (6)$$

$$\lim_{n\to\infty}\frac{b_{n+1}^n}{b_n^n}\cdot\lim_{n\to\infty}\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{P_{n+1}!}}{\sqrt[n]{P_n!}}\right)^v=$$

$$=\lim_{n\to\infty}\left(\frac{b_{n+1}}{n+1}\cdot\frac{n}{b_n}\cdot\frac{n+1}{n}\right)^n\cdot\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{P_{n+1}!}}{n+1}\cdot\frac{n}{\sqrt[n]{P_n}}\cdot\frac{n+1}{n}\right)$$

$$s\underset{(4)}{\overset{(3)}{=}}1\cdot\frac{a\sqrt{3}}{3e}\cdot\frac{3e}{a\sqrt{3}}=1 \quad (7)$$

From (6)+(7) we have :

$$\ln\lim_{n\to\infty}\left(\frac{b_{n+1}^n}{b_n^n}\cdot\frac{\sqrt[n+1]{P_{n+1}!^{u}}}{\sqrt[n]{P_n^u}}\right)^n=\ln\left(\lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\right)^{nu}\cdot\lim_{n\to\infty}\left(\frac{P_{n+1}!}{P_n!}\cdot\frac{1}{\sqrt[n]{P_{n+1}!}}\right)^v\right)$$

$$=\ln\left(\lim_{n\to\infty}\left[\left(1+\frac{b_{n+1}-b_n}{b_n}\right)^{\frac{b_n}{b_{n+1}-b_n}}\right]^{\frac{b_n}{b_{n+1}-b_n}nu}\cdot\left(\frac{a\sqrt{3}}{3}\cdot\frac{3e}{a\sqrt{3}}\right)\right)=$$

$$=\ln(e^u\cdot e^v)=\ln(e^{u+v})=\ln e=1 \quad (8)$$

$$\text{From } (1)+(5)+(8)\Rightarrow L=\frac{a^v b^n}{e^v\cdot\sqrt{3}^v}$$

PROBLEM 3.130-*Solution by Marian Ursărescu – Romania*

$$\text{We have: } a^{\sin x}\cdot b^{\cos x}\cdot c^{\sin x}\cdot d^{\cos x}\le 2^{\sqrt{2(\log_2^2 a+\log_2^2 b+\log_2^2 c+\log_2^2 d)}} \quad (1)$$

$$\text{But } a^{\sin x}\cdot b^{\cos x}=\left(2^{\log_2 a}\right)^{\sin x}\cdot\left(b^{\log_2 b}\right)^{\cos x}=$$

$$=2^{\log_2 a\cdot\sin x}\cdot 2^{\log_2 b\cdot\cos x}=2^{\log_2 a\cdot\sin x+\log_2 b\cos x} \quad (2)$$

From Cauchy's inequality $\Rightarrow$

$$(\log_2 a\cdot\sin x+\log_2 b\cos x)^2\le(\log_2^2 a+\log_2^2 b)(\sin^2 x+\cos^2 x) \quad (3)$$

$$\text{From } (2)+(3)\Rightarrow a^{\sin x}\cdot b^{\cos x}\le 2^{\sqrt{\log_2^2 a+\log_2^2 b}} \quad (4)$$

$$\text{Similarly}\Rightarrow c^{\sin x}d^{\cos x}\le 2^{\sqrt{\log_2^2 c+\log_2^2 d}} \quad (5)$$

$$\text{From } (4)+(5)\Rightarrow a^{\sin x}\cdot b^{\cos x}\cdot c^{\sin x}\cdot d^{\cos x}\le 2^{\sqrt{\log_2^2 a+\log_2^2 b}+\sqrt{\log_2^2 c+\log_2^2 d}} \quad (6)$$

$$\text{From Cauchy's inequality}\Rightarrow 2(\alpha^2+\beta^2)\ge(\alpha+\beta)^2\Rightarrow$$

$$\Rightarrow\sqrt{\log_2^2 a+\log_2^2 b}+\sqrt{\log_2^2 c+\log_2^2 d}\le\sqrt{2(\log_2^2 a+\log_2^2 b+\log_2^2 c+\log_2^2 d)} \quad (7)$$

$$\text{From } (6)+(7)\Rightarrow (ac)^{\sin x}\cdot(bd)^{\cos x}\le 2^{\sqrt{2(\log_2^2 a+\log_2^2 b+\log_2^2 c+\log_2^2 d)}}$$

PROBLEM 3.131-*Solution by proposer*

By AM-GM:

$$2^{\sin A}+2^{\cos A}\ge 2\sqrt{2^{\sin A}\cdot 2^{\cos A}}=2\sqrt{2^{\sin A+\cos A}}=2\sqrt{2^{\sqrt{2}\cos\left(\frac{\pi}{4}-A\right)}}=$$

$$= 2 \cdot 2^{\frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4}-A\right)} > 2 \cdot 2^{-\frac{\sqrt{2}}{2}} = \frac{2}{2^{\frac{\sqrt{2}}{2}}} = \frac{2}{(\sqrt{2})^{\sqrt{2}}}$$

Analogous: $2^{\sin B} + 2^{\cos B} > \frac{2}{(\sqrt{2})^{\sqrt{2}}}$; $2^{\sin C} + 2^{\cos C} > \frac{2}{(\sqrt{2})^{\sqrt{2}}}$

By adding: $2^{\sin A} + 2^{\sin B} + 2^{\sin C} + 2^{\cos A} + 2^{\cos B} + 2^{\cos C} > \frac{6}{(\sqrt{2})^{\sqrt{2}}}$

PROBLEM 3.132-*Solution by Rovsen Pirguliyev-Sumgait-Azerbaidian*

If $x, y > 0; x + y = 1, n, m \in \mathbb{N}$ *then* $x^n y^m \le \frac{n^n \cdot m^m}{(n+m)^{n+m}}$, *if* $x = \sin^2 x, y = \cos^2 x$ *then:*

$$(\sin^2 x)^n \cdot (\cos^2 x)^m \le \frac{n^n \cdot m^m}{(n+m)^{m+n}} \quad (1)$$

Using (1) we have: $LHS \le \frac{n^n m^m}{(n+m)^{m+n}} \cdot a \cdot \frac{n^n \cdot m^m}{(n+m)^{m+n}} = \frac{n^{2n} \cdot m^{2m} \cdot ab}{(n+m)^{2(m+n)}}$ *(2)*

$$n + m \ge 2\sqrt{nm} \Rightarrow \frac{1}{n+m} \le \frac{1}{2\sqrt{nm}},$$

$$(2) \Rightarrow \frac{n^{2n} \cdot m^{2m} \cdot ab}{(n+m)^{2(m+n)}} \le \left(\frac{1}{2\sqrt{nm}}\right)^{2(m+n)} \cdot n^{2n} \cdot m^{2m} \cdot ab = \frac{ab}{4^{m+n}} \cdot \left(\frac{m}{n}\right)^{m-n}$$

PROBLEM 3.133-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\int_a^b \int_a^b \left(\frac{\sin^2(x+y) + \sin^2(x-y) - 1}{1 + 2\sin x \cos y}\right) dx\, dy = 2\int_a^b (a+b-x)\frac{\sin^2(2x) - 1}{1 + 2\sin x \cos x} dx =$$

$$= 2\int_a^b (a+b-x)(\sin 2x - 1)dx = 2(a+b)\int_a^b (\sin 2x - 1)\, dx - 2\int_a^b x(\sin 2x - 1)dx$$

$$= 2(a+b)\left(\frac{\cos 2b - \cos 2a}{2} + a - b\right) - 2\int_b^a x(\sin 2x - 1)dx$$

$$= 2(a+b)(\sin^2 a - \sin^2 b + a - b) - 2\int_b^a (a+b-z)(\sin(2a+2b-2z) - 1)dz$$

$$\left[\begin{matrix} \therefore\ let\ z = a+b-x, dx = -dz; when\ x = b, z = a \\ when\ x = a, z = b \end{matrix}\right]$$

$$\ge 2(a+b)(\sin^2 a - \sin^2 b + a - b) - 2\int_b^a (a+b-z)(\sin(\pi - 2z) - 1)dz$$

$$\Rightarrow 2\int_b^a (a+b-x)(\sin 2x - 1)dx \ge 2(a+b)(\sin^2 a - \sin^2 b + a - b)$$

$$\Rightarrow \int_a^a (a+b-x)(\sin 2x - 1)dx \ge (a+b)(\sin^2 a - \sin^2 b + a - b)$$

We need need to prove, $(a+b)(\sin^2 a - \sin^2 b + a - b) \ge$

$$\geq (a-b)(\sin^2 a - \sin^2 b + a - b)$$
$$\Leftrightarrow b(\sin^2 a - \sin^2 b + a - b) \geq 0, \textit{ which is true (Hence Proved)}$$

PROBLEM 3.134-***Solution by Marian Ursărescu-Romania***

$$\frac{a^2(m_a+m_b)}{h_c}+\frac{b^2(m_b+m_c)}{h_a}+\frac{c^2(m_c+m_a)}{h_b}\geq 8\sqrt{3}S$$

We must show this: $\frac{a^2m_a}{h_c}+\frac{b^2m_b}{h_a}+\frac{c^2m_c}{h_b}+\frac{a^2m_b}{h_c}+\frac{b^2m_c}{h_a}+\frac{c^2m_a}{h_b}\geq 8\sqrt{3}S$. *First we show this:*

$$\frac{a^2m_a}{h_c}+\frac{b^2m_b}{h_a}+\frac{c^2m_c}{h_b}\geq 4\sqrt{3}S \quad (1)$$

We know: $m_a \geq \frac{b^2+c^2}{4R} \Rightarrow m_a \geq \frac{bc}{2R} \Rightarrow \frac{a^2m_a}{h_c}+\frac{b^2m_b}{h_a}+\frac{c^2m_c}{h_b}\geq \frac{a^2bc}{2Rh_c}+\frac{ab^2c}{2Rh_a}+\frac{c^2ab}{2Rh_b}=$

$$=\frac{abc}{2R}\left(\frac{a}{h_c}+\frac{b}{h_a}+\frac{c}{h_b}\right)=2S\left(\frac{ac}{ch_c}+\frac{ab}{ah_a}+\frac{h_c}{bh_b}\right)=(ab+bc+ac) \quad (2)$$

From (1)+(2) we must show: $ab+bc+ac\geq 4\sqrt{3}S \quad (3)$

Now $ab+bc+ac=s^2+r^2+4Rr \Rightarrow (3) \Leftrightarrow s^2+r^2+4Rr\geq 4\sqrt{3}sr \quad (4)$

Now use Doucet inequality: $9r(4R+r)\leq 3s^2\leq (4R+r)^2$

From left side of Doucet and (4) we must show this:

$16Rr+4r^2\geq 4\sqrt{3}sr \Leftrightarrow 4r(4R+r)\geq 4\sqrt{3}sr \Leftrightarrow 4R+r\geq \sqrt{3}s \Leftrightarrow (4R+r)^2\geq 3s^2$ *true* $\Rightarrow$ *then (3) its true. Now, we show this:*

$$\left.\begin{matrix}\frac{a^2m_b}{h_c}+\frac{b^2m_c}{h_a}+\frac{c^2m_a}{h_b}\geq 4\sqrt{3}S\ (5)\\ \text{but } m_a\geq h_a, m_b\geq h_b, m_c\geq h_c\end{matrix}\right\}\Rightarrow (5) \Rightarrow \textit{we must show: } \frac{a^2h_b}{h_c}+\frac{b^2h_c}{h_a}+\frac{c^2h_a}{h_b}\geq 4\sqrt{3}S \quad (6)$$

$$\textit{But } \frac{a^2h_b}{h_c}+\frac{b^2h_c}{h_a}+\frac{c^2h_a}{h_b}=\frac{a^2c}{b}+\frac{b^2a}{c}+\frac{c^2b}{a}=\frac{a^3c^2+b^3a^2+c^3b^2}{abc} \quad (7)$$

Now, we show this: $\frac{a^2c}{b}+\frac{b^2a}{c}+\frac{c^2b}{a}\geq ab+bc+ac \Leftrightarrow$

$$a^3c^2+b^3a^2+c^3b^2\geq abc(ab+bc+ac) \quad (8)$$

Now, use inequality of generalized enviroments $\Rightarrow$

$$\left.\begin{matrix}\frac{4}{7}a^3c^2+\frac{2}{7}b^3a^2+\frac{1}{7}c^3b^2\geq a^2b^2c\\ \frac{4}{7}b^3a^2+\frac{2}{7}c^3b^2+\frac{1}{7}a^3c^2\geq a^2bc^2\\ \frac{4}{7}c^3b^2+\frac{2}{7}a^3c^2+\frac{1}{7}b^2a^2\geq ab^2c^2\end{matrix}\right\}\Rightarrow a^3c^2+b^3a^2+c^2b^2\geq abc(ab+ac+bc)\Rightarrow$$

$\Rightarrow$ *(8) its true. From (7) + (8) we must show:* $ab+bc+ac\geq 4\sqrt{3}S$, *but this inequality its true from (3). From (1) + (5)* $\Rightarrow$ *inequality its true.*

PROBLEM 3.135-***Solution by Marian Ursărescu-Romania***

From Bergström inequality $\Rightarrow$

$$\frac{c^2}{xb+yc}+\frac{a^2}{xc+ya}+\frac{b^2}{xa+yb}\geq \frac{(a+b+c)^2}{(a+b+c)(x+y)}=\frac{a+b+c}{x+y} \quad (1)$$

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}=\frac{a^2}{xab+yac}+\frac{b^2}{xbc+yab}+\frac{c^2}{xac+ybc}\geq \frac{(a+b+c)^2}{(ab+ac+bc)(x+y)} \quad (2)$$

From (2)+(3)$\Rightarrow$ *But* $(a+b+c)^2\geq 3(ab+ac+bc) \quad (3)$

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}\geq\frac{3}{x+y}\quad (4)$$

From (1)+(4)$\Rightarrow \frac{a+c^2}{xb+yc}+\frac{b+a^2}{xc+ya}+\frac{c+b^2}{xa+yb}\geq\frac{3}{x+y}+\frac{a+b+c}{x+y}$

PROBLEM 3.136-*Solution by proposer*

$$S=\sum_{k=0}^{n}T_{4k}(x)$$

Let, $x=\cos(\theta)$*. Then, by definition, we have,*

$$\Rightarrow\sum_{k=0}^{n}\cos(4k\cos^{-1}(x))\Rightarrow\sum_{k=0}^{n}\cos(4k\cos^{-1}(\cos\theta))\Rightarrow\sum_{k=0}^{n}\cos(4k\theta)\Rightarrow$$

$$\Rightarrow \text{Real Part}\sum_{k=0}^{n}e^{4ki\theta}\Rightarrow 1+e^{4i\theta}+e^{8i\theta}+\cdots+(n+1)\text{ terms}\Rightarrow\frac{1-\left(e^{4i\theta}\right)^{n+1}}{1-e^{4i\theta}}\Rightarrow$$

$$\Rightarrow\frac{\left(1-e^{i(4n+4)\theta}\right)\left(1-e^{-4i\theta}\right)}{\left(1-e^{4i\theta}\right)\left(1-e^{-4i\theta}\right)}\Rightarrow\frac{\left(1-e^{i(4n+4)\theta}\right)\left(1-e^{-4i\theta}\right)}{2\left(e^{4i\theta}+e^{-4i\theta}\right)}\Rightarrow$$

$$\Rightarrow\frac{(1-\cos(4n+4)\theta)-i\sin\left((4n+4)\theta\right)(1-(\cos(4\theta))+i\sin(4\theta))}{2-2\cos(4\theta)}\Rightarrow$$

$$\Rightarrow\frac{1-\cos\left((4n+4)\theta\right)-\cos(4\theta)+\cos\left((4n+4)\theta\right)\cos(4\theta)+\sin\left((4n+4)\theta\right)\sin(4\theta)}{2(1-\cos(4\theta))}$$

$$\Rightarrow\frac{(1-\cos(4\theta))-\cos\left((4n+4)\theta\right)(1-\cos(4\theta))+\sin\left((4n+4)\theta\right)\sin(4\theta)}{2(1-\cos(4\theta))}$$

$$\Rightarrow\frac{1}{2}\left[1-\cos\left((4n+4)\theta\right)+\frac{\sin\left((4n+4)\theta\right)\sin(4\theta)}{1-\cos(4\theta)}\right]$$

$$\Rightarrow\frac{1}{2}\left[1-\cos\left((4n+4)\theta\right)+\frac{\sin\left((4n+4)\theta\right)2\sin(2\theta)\cos(2\theta)}{2\sin^2(2\theta)}\right]$$

$$\Rightarrow\frac{1}{2}\left[\frac{1+\sin\left((4n+4)\theta\right)\cos(2\theta)-\cos\left((4n+4)\theta\right)\sin(2\theta)}{\sin(2\theta)}\right]$$

$$\Rightarrow\frac{1}{2}\left[1+\frac{\sin\left(\left((4n+4)\theta\right)-2\theta\right)}{\sin(2\theta)}\right]\Rightarrow\frac{1}{2}\left[1+\frac{\sin\left((4n+2)\theta\right)}{\sin(2\theta)}\right]\Rightarrow$$

$$\Rightarrow\frac{1}{2}\left[1+\frac{\sin\left((4n+2)\theta\right)}{2\left[\cos(\theta)\sqrt{1-\cos^2\theta}\right]}\right]$$

As, $x=\cos\theta\Rightarrow U_{4n+2}(x)=\sin\left((4n+2)\cos^{-1}(\cos\theta)\right)$*, so,*

$U_{4n+2}(x)=\sin\left((4n+2)\theta\right)$*. Using above, we get,* $S=\frac{1}{4}\left[\frac{2+U_{4n+2}(x)}{x\sqrt{1-x^2}}\right]$ *(Answer)*

PROBLEM 3.137-*Solution by Shafiqur Rahman-Bangladesh*

$$\lim_{n\to\infty}\frac{f(x)}{x}=\lim_{n\to\infty}\left(f(x+1)-f(x)\right)=a;\ \lim_{n\to\infty}\frac{\left(g(x)\right)^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{g(x+1)x^x}{g(x)\cdot(x+1)^{x+1}}=\frac{b}{e}$$

$$Now, \lim_{n\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}}-(g(x))^{\frac{\sin^2 t}{x}}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{f(x)}{x}\right)^{\cos^2 t}\cdot x^{\cos^2 t}\left(\frac{\left(g(x)^{\frac{1}{x}}\right)}{x}\right)^{\sin^2 t}\cdot x^{\sin^2 t}\left((g(x))^{-\frac{\sin^2 t}{x(x+1)}}-1\right)=$$

$$=a^{\cos^2 t}\cdot\left(\frac{b}{e}\right)^{\sin^2 t}\cdot\lim_{n\to\infty}\left(\frac{e^{-\frac{\sin^2 t}{x(x+1)}\ln(g(x))}-1}{-\frac{\sin^2 t}{x(x+1)}\ln(g(x))}\cdot\ln(g(x))^{-\frac{\sin^2 t}{x+1}}\right)=a^{\cos^2 t}\cdot\left(\frac{b}{e}\right)^{\sin^2 t}\cdot 1\cdot\ln 0$$

$$\therefore \lim_{n\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}}-(g(x))^{\frac{\sin^2 t}{x}}\right)=-\infty$$

Note: $\lim_{n\to\infty}\frac{\sin^2 t}{x(x+1)}\ln(g(x))=\lim_{n\to\infty}\frac{\sin^2 t}{x+1}\ln\left(\frac{bx}{e}\right)=0$ *and* $\lim_{n\to\infty}(g(x))^{-\frac{\sin^2 t}{x+1}}=$

$$\lim_{n\to\infty}\left(\frac{bx}{e}\right)^{-\frac{x\sin^2 t}{x+1}}=0$$

$$\lim_{n\to\infty}\frac{f(x)}{x}=\lim_{n\to\infty}(f(x+1)-f(x))=a;\ \lim_{n\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{g(x+1)x^x}{g(x)\cdot(x+1)^{x+1}}=\frac{b}{e}$$

$$Now, \lim_{n\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}}-(g(x))^{\frac{\sin^2 t}{x}}\right)=\lim_{n\to\infty}\left(\frac{f(x)}{x}\right)^{\cos^2 t}\cdot x^{\cos^2 t}\cdot$$

$$\left(\frac{(g(x))^{\frac{1}{x}}}{x}\right)^{\sin^2 t}\cdot$$

$$\cdot x^{\sin^2 t}\left((g(x))^{-\frac{\sin^2 t}{x(x+1)}}-1\right)=a^{\cos^2 t}\left(\frac{b}{e}\right)^{\sin^2 t}\cdot x\cdot\lim_{n\to\infty}\left(\left(\frac{bx}{e}\right)^{-\frac{\sin^2 t}{x+1}}-1\right)=$$

$$=a^{\cos^2 t}\cdot\left(\frac{b}{e}\right)^{\sin^2 t}\lim_{n\to\infty}\left(-\frac{x\sin^2 t}{x+1}\ln\left(\frac{bx}{e}\right)+0\left(\frac{\ln^2\left(\frac{bx}{e}\right)}{x+1}\right)\right)$$

$$=-a^{\cos^2 t}\cdot\left(\frac{b}{e}\right)^{\sin^2 t}\lim_{n\to\infty}\frac{x\sin^2 t}{x+1}\ln\left(\frac{bx}{e}\right)$$

$$\therefore \lim_{n\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}}-(g(x))^{\frac{\sin^2 t}{x}}\right)=-\infty$$

$$\lim_{n\to\infty}\frac{f(x)}{x}=\lim_{n\to\infty}(f(x+1)-f(x))=a;\ \lim_{n\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{g(x+1)\cdot x^x}{g(x)(x+1)^{x+1}}=\frac{b}{e}$$

$$Now, \lim_{n\to\infty}(g(x))^{\cos^2 t}\left(g(x+1)^{\frac{\sin^2 t}{x+1}}-(g(x))^{\frac{\sin^2 t}{x}}\right)=$$

$$= \lim_{n\to\infty}\left(\frac{f(x)}{x}\right)^{\cos^2 t}\cdot\left((x+1)^{\sin^2 t}\left(\frac{g(x+1)}{(x+1)^{x+1}}\right)^{\frac{\sin^2 t}{x+1}} - x^{\sin^2 t}\left(\frac{g(x)}{x^x}\right)^{\frac{\sin^2 t}{x}}\right) =$$

$$= a^{\cos^2 t}\cdot\sin^2 t\cdot\lim_{n\to\infty}\left(\frac{\frac{g(x+1)}{(x+1)^{x+1}}}{\frac{g(x)}{x^x}}\right)^{\sin^2 t}$$

$$\therefore \lim_{n\to\infty}\left(f(x)\right)^{\cos^2 t}\left(\left(g(x)\right)^{\frac{\sin^2 t}{x+1}} - \left(g(x)\right)^{\frac{\sin^2 t}{x}}\right) = a^{\cos^2 t}\cdot\sin^2 t\left(\frac{b}{e}\right)^{\sin^2 t}$$

PROBLEM 3.138-*Solution by proposers*

By Cesaro – Stolz theorem we have:

$$\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{\substack{x\to\infty\\ n\to\mathbb{N}^*}}\frac{f(n)}{n} \overset{C-S}{=} \lim_{x\to\infty}\frac{f(n+1)-f(n)}{(n+1)-n} = \lim_{x\to\infty}\left(f(n+1)-f(n)\right) = a,$$

and by Cauchy-D'Alembert theorem we deduce that:

$$\lim_{x\to\infty}\frac{\left(g(x)\right)^{\frac{1}{x}}}{x} = \lim_{\substack{x\to\infty\\ n\in\mathbb{N}^*}}\frac{\left(g(n)\right)^{\frac{1}{n}}}{n} = \lim_{x\to\infty}\sqrt[n]{\frac{g(n)}{n^2}} \overset{C-D'A}{=} \lim_{n\to\infty}\left(\frac{g(n+1)}{(n+1)^{n+1}}\cdot\frac{n^n}{g(n)}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{g(n+1)}{ng(n)}\left(\frac{n}{n+1}\right)^{n+1}\right) = \frac{b}{e}. \text{ So, } \lim_{x\to\infty}\left(f(x)\right)^{\sin^2 t}\left(\left(g(x)\right)^{\frac{\cos^2 t}{x+1}} - \left(g(x)\right)^{\frac{\cos^2 t}{x}}\right) =$$

$$= \lim_{x\to\infty}\left(\frac{f(x)}{x}\right)^{\sin^2 t}\lim_{x\to\infty}\left(\frac{\left(g(x)\right)^{\frac{1}{x}}}{x}\right)^{\cos^2 t}\left(u(x)-1\right)x^{\sin^2 t+\cos^2 t} =$$

$$= a^{\sin^2 t}\cdot\frac{b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\lim_{x\to\infty}\left(\frac{u(x)-1}{\ln u(x)}\cdot\ln\left(u(x)\right)^x\right),$$

where $u(x) = \left(\frac{\left(g(x+1)\right)^{\frac{1}{x+1}}}{\left(g(x)\right)^{\frac{1}{x}}}\right)^{\cos^2 t}$ *with* $\lim_{x\to\infty} u(x) = 1$*, then* $\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)} = 1$*. We have:*

$$\lim_{x\to\infty}\left(u(x)\right)^x = \lim_{x\to\infty}\left(\frac{\left(g(x+1)\right)^{\frac{x}{x+1}}}{g(x)}\right)^{\cos^2 t} = \lim_{x\to\infty}\left(\frac{g(x+1)}{g(x)}\cdot\frac{1}{\left(g(x+1)\right)^{\frac{1}{x+1}}}\right)^{\cos^2 t} =$$

$$= \lim_{x\to\infty}\left(\frac{g(x+1)}{xg(x)}\cdot\frac{x+1}{\left(g(x+1)\right)^{\frac{1}{x+1}}}\cdot\frac{x}{x+1}\right)^{\cos^2 t} = \left(b\cdot\frac{e}{b}\cdot 1\right)^{\cos^2 t} = e^{\cos^2 t}$$

Therefore: $\lim_{x\to\infty}\left(f(x)\right)^{\sin^2 t}\left(\left(g(x)\right)^{\frac{\cos^2 t}{x+1}} - \left(g(x)\right)^{\frac{\cos^2 t}{x}}\right) = \frac{a^{\sin^2 t}\cdot b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\ln e^{\cos^2 t} =$

$= \frac{a^{\sin^2 t}\cdot b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\cos^2 t$*, and we are done.*

PROBLEM 3.139-*Solution by proposers*

We have: $\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'A}{=}$

$=\lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$. *Let*

$f(x)=x^{\cosh^2 t}\left((\Gamma(x+1))^{-\frac{\sinh^2 t}{x}}-(\Gamma(x+2))^{-\frac{\sinh^2 t}{x+1}}\right)=-x^{\cosh^2 t}(\Gamma(x+1))^{-\frac{\sinh^2 t}{x}}(u(x)-1),$

where $u:\mathbb{R}_+^*\to\mathbb{R}, u(x)=\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{-\sinh^2 t}$. *We deduce that:*

$$\lim_{n\to\infty}u(x)=\lim_{n\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1}\cdot\frac{x}{(\Gamma(x+1))^{\frac{1}{x}}}\cdot\frac{x+1}{x}\right)^{-\sinh^2 t}=\left(\frac{1}{e}\cdot e\cdot 1\right)^{-\sinh^2 t}=1$$

We have, $\lim_{n\to\infty}\frac{u(x)-1}{\ln u(x)}=1$. *Also, we have:*

$$\lim_{x\to\infty}(u(x))^x=\lim_{x\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{-x\sin^2 t}=\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{-\sin^2 t}=$$

$$=\lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{-\sin^2 t}=e^{-\sin^2 t}.$$ *Therefore:*

$$\lim_{x\to\infty}f(x)=-\lim_{x\to\infty}\left(x^{ch^2 t}\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\cdot x\right)^{-\sin^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$=-\lim_{x\to\infty}\left(\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\right)^{-\sin^2 t}\cdot x^{\cosh^2 t-\sinh^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$=-\left(\frac{1}{e}\right)^{-\sinh^2 t}\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}\cdot\ln\left(\lim_{x\to\infty}(u(x))^x\right)=-e^{\sinh^2 t}\cdot 1\cdot\ln\ e^{-\sinh^2 t}$$

$$=e^{\sinh^2 t}\cdot\sinh^2 t$$

PROBLEM 3.140-*Solution by proposers*

We have: $\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{x\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'A}{=}$

$=\lim_{x\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n!}{n!}\right)=\lim_{x\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$. *We denote:*

$$f(x)=x^{\sin^2 t}\left(\left(\Gamma(x+2)\right)^{\frac{\cos^2 t}{x+1}}-\left(\Gamma(x+1)\right)^{\frac{\cos^2 t}{x}}\right)=x^{\sin^2 t}\left(\Gamma(x+1)\right)^{\frac{\cos^2 t}{x}}(u(x)-1),\text{ where}$$

$$u:\mathbb{R}_+^*\to\mathbb{R},u(x)=\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{\cos^2 t}.\text{ We deduce that:}$$

$$\lim_{x\to\infty}u(x)=\lim_{x\to\infty}\left(\frac{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}{x+1}\cdot\frac{x}{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}\cdot\frac{x+1}{x}\right)^{\cos^2 t}=\left(\frac{1}{e}\cdot e\cdot 1\right)^{\cos^2 t}=1.$$

We have, $\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}=1$ *and also we have:*

$$\lim_{x\to\infty}\left(u(x)\right)^x=\lim_{x\to\infty}\left(\frac{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}\right)^{x\cos^2 t}=\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}}\right)^{\cos^2 t}=$$

$$=\lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{\cos^2 t}=e^{\cos^2 t}.\text{ Therefore:}$$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\left(x^{\sin^2 t}\left(\frac{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}{x}\cdot x\right)\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$=-\lim_{x\to\infty}\left(\left(\frac{\left(\Gamma(x+1)\right)^{\frac{1}{x}}}{x}\right)^{\cos^2 t}\cdot x^{\sin^2 t+\cos^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$=e^{\cos^2 t}\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}\cdot\ln\left(\lim_{x\to\infty}\left(u(x)\right)^x\right)=e^{\cos^2 t}\cdot 1\cdot\ln e^{\cos^2 t}=e^{\cos^2 t}\cdot\cos^2 t$$

PROBLEM 3.141-*Solution by Marian Ursărescu – Romania*

$$L=\lim_{n\to\infty}n^{\sin^2 x}\left[\left(\frac{(n+3)^{n+2}}{(n+2)^{n+1}}\right)^{\cos^2 x}-\left(\frac{(n+2)^{n+1}}{(n+1)^n}\right)^{\cos^2 x}\right]$$

Let $f:[n,n+1]\to\mathbb{R},f(t)=\left(\frac{(t+2)^{t+1}}{(t+1)^t}\right)^{\cos^2 x}$

From Lagrange's theorem we have: $\exists c\in(n,n+1)$ *such that* $f(n+1)-f(n)=f'(c)\Rightarrow$

$$L=\lim_{n\to\infty}n^{\sin^2 x}\left(f(n+1)-f(n)\right)=\lim_{n\to\infty}n^{\sin^2 x}f'(c)\quad (1)$$

$$f(t)=\left((t+2)\left(\frac{t+2}{t+1}\right)^t\right)^{\cos^2 x}=(t+2)^{\cos^2 x}\left(1+\frac{1}{t+1}\right)^{t\cos^2 x}$$

$$f'(t)=\cos^2 x\,(t+2)^{\cos^2 x-1}\left(1+\frac{1}{t+1}\right)^{t\cos^2 x}+(t+2)^{\cos^2 x}\,.$$

$$\cdot\left(\left(1+\left(\frac{1}{t+1}\right)\right)^{t\cos^2 x}\left(\cos^2 x\cdot\ln\left(1+\frac{1}{t+1}\right)\right)+\tan^2 x\cdot\frac{-\frac{1}{t+1}}{1+\frac{1}{t+1}}\right)\Rightarrow$$

$$f'(t)=\cos^2 x\left(1+\frac{1}{t+1}\right)^{t\cos^2 x}\left[(t+2)^{-\sin^2 x}+(t+2)^{\cos^2 x}\left(\ln\left(1+\frac{1}{t+1}\right)-\frac{t}{(t+1)(t+2)}\right)\right]\quad (2)$$

From (1)+(2) and because $c\in(n,n+1)$ we must calculate:

$$L=\lim_{n\to\infty} n^{\sin^2 x}\cdot\cos^2 x\left(1+\frac{1}{n+1}\right)^{\cos^2 x}\left[(n+2)^{-\sin^2 x}+(n+2)^{\cos^2 x}\left(\ln\left(1+\frac{1}{n+1}\right)-\frac{n}{(n+1)(n+2)}\right)\right]\quad (3)$$

$$\lim_{n\to\infty}\cos^2 x\left(1+\frac{1}{n+1}\right)^{n\cos^2 x}=\cos^2 x\cdot e^{\cos^2 x}\quad (4)$$

$$\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^{\sin^2 x}+(n+2)^{\cos^2 x-1}\cdot n\cdot n^{\sin^2 x}\left(\ln\left(1+\frac{1}{n+1}\right)-\frac{n}{(n+1)(n+2)}\right)$$

$$=1+\lim_{n\to\infty}\left(\frac{n}{n+2}\right)^{\sin^2 x}\cdot\left(\frac{\ln\left(1+\frac{1}{n+1}\right)}{\frac{1}{n}}-\frac{n}{(n+1)(n+2)}\right)=$$

$$1+1(1-1)=1\quad (5)$$

From (3)+(4)+(5) $\Rightarrow L=\cos^2 x\, e^{\cos^2 x}$

PROBLEM 3.142-*Solution by Shafiqur Rahman-Bangladesh*

$$-\ln(mn+x_n)+\sum_{k=1}^{mn}\frac{1}{k}=\gamma\Rightarrow\lim_{n\to\infty}x_n=\lim_{x\to\infty}\left(e^{\left(\sum_{k=1}^{mn}\frac{1}{k}-\gamma\right)}-mn\right)=$$

$$=\lim_{n\to\infty}\left(e^{\left(\sum_{k=1}^{mn}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k}+\ln n\right)}-mn\right)=\lim_{n\to\infty}n\left(e^{\left(\sum_{k=n+1}^{mn}\frac{1}{k}\right)}-m\right)=$$

$$=\lim_{n\to\infty}\left(e^{\int_1^m\frac{dx}{x}}-m\right)=\lim_{n\to\infty}n\left(e^{\ln m}-m\right)\therefore\lim_{n\to\infty}x_n=0$$

PROBLEM 3.143-*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\lim_{n\to\infty}n\big(\gamma_n(a,b)-\gamma(a,b)\big)=\lim_{n\to\infty}\frac{\gamma_n(a,b)-\gamma(a,b)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\gamma_{n+1}(a+b)-\gamma_n(a,b)}{\frac{1}{n+1}-\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}\frac{1}{k+b}-\ln(n+1+a)-\sum_{k=1}^{n}\frac{1}{k+b}+\ln(n+a)}{\frac{1}{n+1}-\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n+a}\right)-\frac{1}{n+1+b}}{\frac{1}{(n+1)}}=\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)-\frac{1}{n+1}}{\frac{1}{n(n+1)}}=$$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\ln\left(1+\frac{1}{n}\right)-\frac{1}{n}}{\frac{1}{n^2}}=$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{(1+x)\ln(1+x)-x}{x^2} \overset{L'HOSPITAL}{=} \lim_{\substack{x\to 0\\ x>0}} \frac{\ln(1+x)+1-1}{2x} = \frac{1}{2}\lim_{\substack{x\to 0\\ x>0}} \ln \sqrt[x]{1+x} =$$

$$= \frac{\ln e}{2} = \frac{1}{2}$$

$$\lim_{n\to\infty}\left(\ln\frac{e}{n+a}+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^n = e^{b-a+\frac{1}{2}}$$

PROBLEM 3.144-*Solution by proposer*

If $a,b \geq 0$ *then:* $(a^2+1)(b^2+1) = (a^2b^2+1)+a^2+b^2 \geq a^2+2ab+b^2 = (a+b)^2$

$\prod(a^2+1)^2 = \prod(a^2+1)(b^2+1) \geq \prod(a+b)^2$ *therefore* $\prod(a^2+1) \geq \prod(a+b)$

We take $a = \sinh x, b = \sinh y, c = \sinh z \Rightarrow \prod \cosh^2 x = \prod(1+\sinh^2 x) \geq$

$$\geq \prod(\sinh x + \sinh y) = \prod 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2} =$$

$$= 2\prod \sinh\frac{x+y}{2}\cdot\left(4\prod\sinh\frac{x-y}{2}\right) = 2\prod\sinh\frac{x+y}{2}\left(1+\sum\cosh(x-y)\right)$$

PROBLEM 3.145-*Solution by Ruanghaw Chaokha-Chiangrai-Thailand*

$$\lim_{n\to\infty}(x_{n+1}-x_n) = x \Rightarrow L = \lim_{n\to\infty}\left(\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}} - \frac{nx_n}{\sqrt[n]{(2n-1)!!}}\right) = ??$$

Stolz-Cesaro; $\lim_{n\to\infty}(a_n - a_{n-1}) = \lim_{n\to\infty}\frac{a_n}{n}$; $a_n = \frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}}$

$$L = \lim_{n\to\infty}\frac{(n+1)x_{n+1}}{n\cdot\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty}\frac{x_{n+1}}{\sqrt[n+1]{(2n+1)!!}} \overset{n\to n-1}{=} \lim_{n\to\infty}\frac{x_n}{\sqrt[n]{(2n-1)!!}}$$

Again, Stolz – Cesaro; $\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}$; $y_n = \sqrt[n]{(2n-1)!!}$

$$L = \lim_{n\to\infty}\frac{x_n-x_{n-1}}{\sqrt[n]{(2n-1)!!}-\sqrt[n-1]{(2n-3)!!}} = \frac{\lim_{n\to\infty}(x_n-x_{n-1})}{\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}-\sqrt[n-1]{(2n-3)!!}\right)} = \frac{x}{K} \to *$$

$$a_n = \sqrt[n]{(2n-1)!!} \Rightarrow K = \lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}$$

And Stolz-Cesaro again; $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_n}{a_{n-1}}$; $a_n = \frac{(2n-1)!!}{n^n}$

$$K = \lim_{n\to\infty}\frac{(2n-1)!!}{n^n}\cdot\frac{(n-1)^{n-1}}{(2n-3)!!} = \lim_{n\to\infty}\frac{(2n-1)}{(n-1)}\cdot\left(\frac{n-1}{n}\right)^n =$$

$$= \lim_{n\to\infty}\frac{(2n-1)}{(n-1)}\cdot\left(1-\frac{1}{n}\right)^n = 2e^{-1}; *; \therefore L = \frac{xe}{2}$$

PROBLEM 3.146-*Solution by Remus Florin Stanca-Romania*

$$l = \lim_{n\to\infty} n^{\sin^2 t}\sqrt[n]{\left(f(1)\cdot\ldots\cdot f(n)\right)^{\cos^2 t}}\left(\left(\frac{n+1}{n}\right)^{\sin^2 t}\cdot\left(\frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}\right)-1\right)$$

$$= \lim_{n\to\infty} n \cdot \sqrt[n]{\left(\frac{f(1)\cdot\ldots\cdot f(n)}{n^n}\right)^{\cos^2 t}} \cdot \left(\left(\left(\frac{n+1}{n}\right)^{\tan^2 t} \cdot \frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}\right)^{\cos^2 t} - 1\right)$$

We know that $\lim_{n\to\infty}\left(\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)} - \sqrt[n]{f(1)\cdot\ldots\cdot f(n)}\right) = \frac{a}{e}$ *and* $\lim_{x\to 1}\frac{x^a-1}{x-1} = a$

$$\lim_{n\to\infty}\sqrt[n]{\frac{f(1)\cdot\ldots\cdot f(n)}{n^n}} = \lim_{n\to\infty} e^{\frac{\ln\frac{f(1)\cdot\ldots\cdot f(n)}{n^n}}{n}} = \lim_{n\to\infty} e^{\ln\frac{f(n+1)}{(n+)}} = \frac{a}{e}$$

$$\lim_{n\to\infty} n\cdot\left(\left(\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{\prod_{k=1}^{n+1}f(k)}}{\sqrt[n]{\prod_{k=1}^{n}f(k)}}\right)^{\tan^2 t} - 1\right) =$$

$$\tan^2 t\cdot\lim_{n\to\infty} n\cdot\left(\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}} - 1\right) =$$

$$\tan^2 t\cdot\lim_{n\to\infty}\frac{(n+1)\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)} - n\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}} =$$

$$t = \tan^2 t\cdot\frac{e}{a}\cdot\left(\frac{a}{e}+\frac{a}{e}\right) = 2\tan^2 t$$

$$\lim_{n\to\infty} n\left(\left(\frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}\right)^{\tan^2 t - 1} - 1\right) =$$

$$(\tan^2 t - 1)\lim_{n\to\infty} n\cdot\left(\frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}} - 1\right) = (\tan^2 2 - 1)\cdot\frac{e}{a}\cdot\frac{a}{e} = \tan^2 t - 1$$

$$\Rightarrow l_1 = \tan^2 t + 1 \Rightarrow l = \left(\frac{a}{e}\right)^{\cos^2 t}\cdot\cos^2 t\cdot(\tan^2 t + 1) = \left(\frac{a}{e}\right)^{\cos^2 t} \Rightarrow l = \left(\frac{a}{e}\right)^{\cos^2 t}$$

PROBLEM 3.147-*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}} = \sum\frac{r_a^3}{s\prod\tan\frac{A}{2}} = \frac{\sum r_a^3}{s\left(\frac{r}{4R}\right)\left(\frac{4R}{s}\right)} = \frac{\sum r_a^3}{r} \geq \frac{9\sum a^2}{4} \Leftrightarrow 4\sum r_a^3 \overset{(1)}{\geq} 9r\sum a^2$$

Now, $\sum r_a^3 = (\sum r_a)^3 - 3\prod(r_a + r_b) = (4R+r)^3 - 3\left(2r_ar_br_c + \sum r_ar_b\left(\sum r_a - r_c\right)\right) =$

$$= (4R+r)^3 - 3\left((4R+r)s^2 - rs^2\right) \overset{(2)}{=} (4R+r)^3 - 12Rs^2$$

Now, RHS of (1) $\overset{Leibniz}{\leq} 81R^2r \overset{?}{\leq} 4\sum r_a^3 \Leftrightarrow 4(4R+r)^3 - 48Rs^2 \overset{?}{\geq} 81R^2r$ *(by (2))*

$\Leftrightarrow 4(4R+r)^3 - 81R^2r \underset{(3)}{\overset{?}{\geq}} 48s^2$*. Now, RHS of (3)* $\overset{Gerretsen}{\leq} 48R(4R^2 + 4Rr + 3r^2)$

$$\overset{?}{\leq} 4(4R+r)^3 - 81R^2r \Leftrightarrow 64t^3 - 81t^2 - 96t + 4 \overset{?}{\geq} 0 \quad \left(t = \frac{R}{r}\right)$$

$\Leftrightarrow (t-2)(64t^2 + 47t - 2) \overset{?}{\geq} 0 \to$ *true* $\because t \overset{Euler}{\geq} 2$ *(Proved)*

PROBLEM 3.148-*Solution by Marian Ursărescu-Romania*

We use Vasc's inequality: $\forall a,b,c \in \mathbb{R} \Rightarrow (a^2+b^2+c^2)^2 \geq 3(a^3b + b^3c + c^2a) \Rightarrow$

$$\Rightarrow \sqrt{3(a^3b+b^3c+c^3a)} \le a^2+b^2+c^2 \Rightarrow \textit{inequality becomes:}$$

$$2(a^2+b^2+c^2)+3 \ge 3\sqrt{abc}(a^2+b^2+c^2) \quad (1)$$

$$\left.\begin{array}{l} \sqrt{abc} \le \frac{a+bc}{2} \Rightarrow \sqrt{abc}(a^2+b^2+c^2) \le \frac{(a+bc)}{2}(a^2+b^2+c^2) \\ \sqrt{abc} \le \frac{b+ac}{2} \Rightarrow \sqrt{abc}(a^2+b^2+c^2) \le \frac{(b+ac)}{2}(a^2+b^2+c^2) \end{array}\right\} \Rightarrow$$

$$\sqrt{abc} \le \frac{c+ab}{2} \Rightarrow \sqrt{abc}(a^2+b^2+c^2) \le \frac{(c+ab)}{2}(a^2+b^2+c^2)$$

$$3\sqrt{abc}(a^2+b^2+c^2) \le \frac{(a^2+b^2+c^2)}{3}(3+ab+ac+bc) \quad (2)$$

From (1)+(2) we must show: $2(a^2+b^2+c^2)+3 \ge \frac{a^2+b^2+c^2}{2}(3+ab+ac+bc) \Leftrightarrow$

$$4(a^2+b^2+c^2)+6 \ge 3(a^2+b^2+c^2)+(a^2+b^2+c^2)(ab+ac+bc) \Leftrightarrow$$

$$a^2+b^2+c^2+6 \ge (a^2+b^2+c^2)(ab+ac+bc) \quad (3)$$

Because $a,b,c>0$ *such that* $a+b+c=3 \Rightarrow \exists x,y,z>0$ *such that:* $a=\frac{3x}{x+y+t}$, $b=\frac{3y}{x+y+t}$, $c=\frac{3z}{x+y+t}$. *Inequality (3) becomes:* $9\left(\frac{x^2+y^2+z^2}{(x+y+t)^2}\right)+6 \ge \frac{9(x^2+y^2+z^2)}{(x+y+z)^2}\cdot\frac{9(xy+xz+yt)}{(x+y+z)^2}$

$$\Leftrightarrow \frac{3(x^2+y^2+z^2)}{(x+y+z)^2}+2 \ge \frac{27(x^2+y^2+z^2)(xy+xz+yz)}{(x+y+z)^4} \Leftrightarrow$$

$$\Leftrightarrow 3(x^2+y^2+z^2)(x+y+z)^2+2(x+y+z)^4 \ge 27(x^2+y^2+z^2)(xy+xz+yz) \quad (4)$$

Now, using Cartoaje's theorem: If $f_4(x,y,z)$ *is an symmetric polynomial function of degree* $n=4$ *then:* $f_4(x,y,z)\ge 0, \forall x,y,z\ge 0 \Leftrightarrow x=0$ *and* $y=z$

(if and only if)

In our case let $f_4(x,y,z)=3(x^2+y^2+z^2)(x+y+z)^2+2(x+y+z)^4=$

$$=2t(x^2+y^2+z^2)(xy+zx+yz)$$

$$f_4(0,y,y)=3\cdot 2y^2\cdot 4y^2+2\cdot 2^4y^4-27\cdot 2y^2\cdot y^2=$$

$$24y^4+32y^4-54y^4=2y^4\ge 0 \Rightarrow \textit{inequality (4) its true.}$$

PROBLEM 3.149-*Solution by proposer*

$$S=\sum_{k=-l}^{l}\left[(-1)^k\binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k}\right]$$

Assuming that, $l=\min(l,m,n)$. *This reduces to the series,*

$$\frac{(-1)^l(2m)!\,(2n)!}{(m-l)!\,(m+l)!\,(n-l)!\,(n+l)!}\,3^{f_2}\left(\begin{matrix}-2l,-m-l,-n-l\\ m-l+1,n-l+1\end{matrix};1\right)$$

Now, Applying Dixon's formula, we get $\Rightarrow 3^{f_2}\left(\begin{matrix}-2l-2\varepsilon,-m-l-\varepsilon,-n-l-\varepsilon\\ m-l-\varepsilon+1,n-l-\varepsilon+1\end{matrix};1\right)$

$$=\frac{\Gamma(1-l-\varepsilon)\Gamma(1+m-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1-2l-2\varepsilon)\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}$$

Now, apply Euler's reflection formula, we get,

$$\frac{\sin\pi(2l+2\varepsilon)}{\sin\pi(l+\varepsilon)}\cdot\frac{\Gamma(2l+2\varepsilon)}{\Gamma(l+\varepsilon)}\cdot\frac{\Gamma(1+m-l-\varepsilon)\Gamma(1+n-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}$$

As the limit $\varepsilon\to 0$, *this expression gives* $\Rightarrow 2(-1)^l\frac{(2l-1)!}{(l-1)!}\cdot\frac{(m-l)!(n-l)!(m+n+l)!}{m!n!(m+n)!}$

$$(or)\ S=\frac{(l+m+n)!(2l)!(2m)!(2n)!}{(l+m)!(l+n!)(m+n)!l!m!n!}\ (Answer)$$

PROBLEM 3.150-*Solution by proposer*

$$S=\sum_{k=-l}^{l}\left[(-1)^k\binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k}\right]$$

Assuming that, $l=\min(l,m,n)$. *This reduces to the series,*

$$\frac{(-1)^l(2m)!\,(2n)!}{(m-l)!\,(m+l)!\,(n-l)!\,(n+l)!}\,3^{f_2}\left(\begin{matrix}-2l,-m-l,-n-l\\ m-l+1,n-l+1\end{matrix};1\right)$$

Now, Applying Dixon's formula, we get $\Rightarrow 3^{f_2}\left(\begin{matrix}-2l-2\varepsilon,-m-l-\varepsilon,-n-l-\varepsilon\\ m-l-\varepsilon+1,n-l-\varepsilon+1\end{matrix};1\right)$

$$=\frac{\Gamma(1-l-\varepsilon)\Gamma(1+m-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1-2l-2\varepsilon)\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}$$

Now, apply Euler's reflection formula, we get,

$$\frac{\sin\pi(2l+2\varepsilon)}{\sin\pi\,(l+\varepsilon)}\cdot\frac{\Gamma(2l+2\varepsilon)}{\Gamma(l+\varepsilon)}\cdot\frac{\Gamma(1+m-l-\varepsilon)\Gamma(1+n-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}$$

As the limit $\varepsilon\to 0$, *this expression gives* $\Rightarrow 2(-1)^l\frac{(2l-1)!}{(l-1)!}\cdot\frac{(m-l)!(n-l)!(m+n+l)!}{m!n!(m+n)!}$

$$(or)\ S=\frac{(l+m+n)!(2l)!(2m)!(2n)!}{(l+m)!(l+n!)(m+n)!l!m!n!}\ (Answer)$$

PROBLEM 3.151-*Solution by Michael Sterghiou-Greece*

$$A=a_1+a_2+\dots+a_n+(1-a_1)(1-a_2)\cdot\ldots\cdot(1-a_n)\quad(1)$$

From Weierstrass product inequality we have $\prod_{k=1}^{n}(1-a_k)\ge 1-\sum_{k=1}^{n}a_k$ (*) which gives immediately $\min A=1$ when $a_k=0\ \forall k\in\{1,2,\dots,n\}$. Now,

$\prod_{k=1}^{n}(1-a_k)\le\left[\frac{\sum_{k=1}^{n}(1-a_k)}{n}\right]^n$ from AM-GM given $1-a_k\ge 0\ \ \forall k$ or

$\prod_{k=1}^{n}(1-a_k)\le\left(1-\frac{S_n}{n}\right)^n$ where $S_n=\sum_{k=1}^{n}a_k$ (1) becomes $A\le S_n+\left(1-\frac{S_n}{n}\right)^n$

Let $S_n=x, x\in[0,n]$: $f(x)=x+\left(1-\frac{x}{n}\right)^n$. $f'(x)=1-\left(1-\frac{x}{n}\right)^{n-1}>0$ as $\frac{x}{n}\le 1$,

$0\le\left(1-\frac{x}{n}\right)^{n-1}\le 1$ so $f(x)\uparrow$. This means $f(x)\le f(n)$ or

$S_n+\left(1-\frac{S_n}{n}\right)^n\le n+\left(1-\frac{n}{n}\right)^n=n$. Therefore $A\le n$ and $A_{max}=n$ when $a_k=1$ $\forall k\in\{1,2,\dots,n\}$

* proved easily by induction over n.

PROBLEM 3.152-*Solution by Nassim Nicholas Taleb-USA*

Solution with probabilistic commentary:

There are $\frac{n(n-1)}{2}$ *distinct terms of* $\sum_{j=1}^{n}\sum_{i=1}^{j-1}a_ia_j$, $\sum_{j=1}^{n}\sum_{i=1}^{j-1}1=\frac{n(n-1)}{2}$

Note that we have the lower or upper half of a symmetric square matrix minus the diagonal a_i^2.

The average term $a_ia_j\ge 1$, *allora the mean* $a_i^2\ge 1$. *In probabilistic terms:*

$$\mathbb{E}(a_i a_{j\neq i}) \leq \mathbb{E}(a^2)$$

Proof: (Cauchy – Schwarz): $\left(\frac{2}{n(n-1)}\sum_{j=1}^{n}\sum_{i=1}^{j-1} a_i\, a_j\right)^2 \leq \left(\frac{1}{n}\sum_{j=1}^{n} a_i^2\right)\left(\frac{1}{n}\sum_{j=1}^{n} a_j^2\right) = \left(\frac{1}{n}\sum_{j=1}^{n} a_i^2\right)^2$

hence, since all a_i *are positive:* $\frac{1}{n}\sum_{j=1}^{n} a_i^2 \geq 1$

by Harmonic/Arthmetic Mean inequality, $\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i^2+n-1} \geq \frac{n}{\sum_{i=1}^{n}(a_i^2+n-1)} = \frac{n}{\sum_{i=1}^{n} a_i^2+n^2-n}$,

allora $\frac{n^2}{\sum_{i=1}^{n} a_i^2+n^2-n} \leq 1$ *when* $\frac{1}{n}\sum_{j=1}^{n} a_i^2 \geq 1.$

PROBLEM 3.153-*Solution by Yen Tung Chung-Taichung-Taiwan*

$$\Omega = \int_0^1\int_0^\infty \left(\frac{x^{100}}{100^x}\right)(y\ln y)^{100}dx\,dy = \left(\int_0^1 (y\ln y)^{100}\,dy\right)\left(\int_0^\infty \frac{x^{100}}{100^x}dx\right)$$

$$= \left(\frac{100!}{101^{101}}\right)\left(\frac{100!}{(\ln 100)^{101}}\right) = \frac{(100!)^2}{(101\ln 100)^{101}}$$

where

let $t = -\ln y \Rightarrow y = e^{-t}, dy = -e^{-t}dt$

$$\int_0^1 (y\ln y)^{100}\,dy = \int_\infty^0 (-te^{-t})^{100}\,(-e^{-t}dt) = \int_0^\infty t^{100}\,e^{-101t}dt = \frac{\Gamma(101)}{101^{101}} = \frac{100!}{101^{101}}$$

$$\int_0^\infty \frac{x^{100}}{100^x}dx = \int_0^\infty x^{100}\,e^{-x\ln 100}dx = \frac{\Gamma(101)}{(\ln 100)^{101}} = \frac{100!}{(\ln 100)^{101}}$$

PROBLEM 3.154-*Solution by proposers*

Let be: $v_n = \sqrt[2n]{n!} + \sqrt[2n+2]{(n+1)!}; n \in \mathbb{N}^*$

$$x_n = \frac{1}{\sqrt[n]{a_n}}\sum_{k=1}^{n}\left[\left(\sqrt[2k]{k!} + \sqrt[2k+2]{(k+1)!}\right)^2\right] = \frac{1}{\sqrt[n]{a_n}}\sum_{k=1}^{n}[v_k^2]\,; n \in \mathbb{N}^*$$

$$\Omega = \lim_{n\to\infty}\frac{n^2}{\sqrt[n]{a_n}}\cdot\frac{1}{n^2}\sum_{k=1}^{n}[v_k^2]$$

$$\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{a_n}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{a_n}} \overset{CDA}{=} \lim_{n\to\infty}\frac{(n+1)^{2n+2}}{a_{n+1}}\cdot\frac{a_n}{n^{2n}} = \lim_{n\to\infty}\frac{n^2 a_n}{a_{n+1}}\cdot\left(\frac{n+1}{n}\right)^{2n+2} = \frac{e^2}{a}$$

and $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = e.$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}[v_k^2] \overset{CS}{=} \lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}[v_k^2] - \sum_{k=1}^{n}[v_k^2]}{(n+1)^2-n^2} = \lim_{n\to\infty}\frac{[v_{n+1}^2]}{2n+1}$$

$$[v_n^2] \leq v_n^2 < [v_n^2]+1; n \in \mathbb{N}^*$$

$$\frac{[v_n^2]}{2n+1} \leq \frac{v_n^2}{2n+1} < \frac{[v_n^2]}{2n+1} + \frac{1}{2n+1}; n \in \mathbb{N}^*$$

$$\lim_{n\to\infty}\frac{[v_n^2]}{2n+1}=\lim_{n\to\infty}\frac{v_n^2}{2n+1}=\lim_{n\to\infty}\frac{\left(\sqrt[2n]{n!}+\sqrt[2n+2]{(n+1)!}\right)^2}{2n+1}=$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{2n+1}+2\sqrt{\frac{\sqrt[n]{n!}}{n}}\cdot\sqrt{\frac{\sqrt[n+1]{(n+1)!}}{n+1}}\cdot\frac{\sqrt{n(n+1)}}{2n+1}+\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{2n+1}\right)=$$

$$=\frac{1}{2}\left(\frac{1}{e}+\frac{2}{e}+\frac{1}{e}\right)=\frac{2}{e}\ \textit{then:}$$

$$\lim_{n\to\infty}x_n=\frac{e^2}{a}\cdot\frac{2}{e}=\frac{2e}{a}$$

PROBLEM 3.155-*Solution by Serban George Florin-Romania*

$$(1+x)^\alpha>1+\alpha x,\ \textit{Bernoulli}\Rightarrow x^\alpha>1+\alpha(x-1)\Rightarrow$$

$$\Rightarrow\left(\frac{a^2x^2}{yz}\right)^{m+1}>1+(m+1)\left(\frac{a^2x^2}{yz}-1\right)$$

$$\left(\frac{b^2y^2}{xz}\right)^{m+1}>1+(m+1)\left(\frac{b^2y^2}{xz}-1\right)$$

$$\left(\frac{c^2z^2}{xy}\right)^{m+1}>1+(m+1)\left(\frac{c^2z^2}{xy}-1\right)$$

$$\Rightarrow 3m+\sum\left(\frac{a^2x^2}{yz}\right)^{m+1}>3+(m+1)\left(\sum\frac{a^2x^2}{yz}-3\right)+3m=$$

$$=3+(m+1)\sum\frac{a^2x^2}{yz}-3(m+1)+3m=(m+1)\sum\frac{a^2x^2}{yz}\geq 4(m+1)\sqrt{3}S$$

$$\Rightarrow\sum\frac{a^2x^2}{yz}\geq 4\sqrt{3}S$$

$$\sum\frac{a^2x^2}{yz}\overset{(Ma\geq Mg)}{\geq}3\sqrt[3]{\prod\frac{a^2x^2}{yz}}=3\sqrt[3]{a^2b^2c^2}=4\sqrt{3}S$$

$$3\sqrt[3]{a^2b^2c^2}\geq 4\sqrt{3}S,\ 27(abc)^2\geq\left(4\sqrt{3}S\right)^3\Rightarrow$$

$$\Rightarrow(abc)^2\geq\left(\frac{4\sqrt{3}S}{3}\right)^3,(abc)^2\geq\left(\frac{4S}{\sqrt{3}}\right)^3,\ \textit{true as in the inequality Carlitz.}$$

PROBLEM 3.156-*Solution by Marian Ursărescu – Romania*

From A.M-G.M. inequality $\Rightarrow$

$$\sum_{k=1}^n t_kx_k\left(\sum_{k=1}^n\frac{t_k}{x_k}\right)^{ab+bc}\leq\left(\frac{\sum_{k=1}^n t_kx_k+(ab+bc)\sum_{k=1}^n\frac{t_k}{x_k}}{1+ab+bc}\right)^{1+ab+bc}\quad(1)$$

From (1) we must show this:

$$\left(\frac{\sum_{k=1}^n t_kx_k+(ab+bc)\sum_{k=1}^n\frac{t_k}{x_k}}{1+ab+bc}\right)^{1+ab+bc}\leq\left(\frac{a+b+c}{1+ab+bc}\right)^{1+ab+bc}\left(\sum_{k=1}^n t_k\right)^{1+ab+bc}$$

$$\Leftrightarrow \sum_{k=1}^{n} t_k x_k + (ab+bc)\sum_{k=1}^{n}\frac{t_k}{x_k} \le (a+b+c)\sum_{k=1}^{n} t_k \quad (2)$$

But $x_k \in [b,c] \Rightarrow b \le x_k \le c \Rightarrow (x_k - b)(c - x_k) \ge 0 \Rightarrow x_k^2 - (b+c)x_k + bc \le 0 \Leftrightarrow$

$$\Leftrightarrow x_k^2 + bc \le (b+c)x_k \Rightarrow x_k + \frac{bc}{x_k} \le b + c \Rightarrow$$

$$t_k x_k + bc\frac{t_k}{x_k} \le (b+c)t_k \Rightarrow \sum_{k=1}^{n} t_k x_k + bc\sum_{k=1}^{n}\frac{x_k}{x_k} \le (b+c)\sum_{k=1}^{n} t_k \Rightarrow$$

$$\sum_{k=1}^{n} t_k x_k + (ab+bc)\sum_{k=1}^{n}\frac{t_k}{x_k} \le ab\sum_{k=1}^{n}\frac{t_k}{x_k} + (b+c)\sum_{k=1}^{n} t_k \quad (3)$$

From (2)+(3) we must show:

$$ab\sum_{k=1}^{n}\frac{t_k}{x_k} + (b+c)\sum_{k=1}^{n} t_k \le (a+b+c)\sum_{k=1}^{n} t_k \Leftrightarrow ab\sum_{k=1}^{n}\frac{t_k}{x_k} \le a\sum_{k=1}^{n} t_k \Leftrightarrow$$

$$b\sum_{k=1}^{n}\frac{t_k}{x_k} \le \sum_{k=1}^{n} t_k \quad (4)$$

But $x_k \ge b \Rightarrow \frac{1}{x_k} \le \frac{1}{b} \Rightarrow \frac{t_k}{x_k} \le \frac{t_k}{b} \Rightarrow \sum_{k=1}^{n}\frac{t_k}{x_k} \le \frac{1}{b}\sum_{k=1}^{n} t_k \Rightarrow b\sum_{k=1}^{n}\frac{t_k}{x_k} \le \sum_{k=1}^{n} t_k \Rightarrow$

(4) is true.

PROBLEM 3.157-*Solution by Marian Ursărescu-Romania*

We must show:

$$\frac{h_a^{m+1}}{h_a^m h_b^m h_c^m (xh_b+yh_c)^m} + \frac{h_b^{m+1}}{(h_ah_bh_c)^m (xh_c+yh_a)^m} + \frac{h_c^{m+1}}{(h_ah_bh_c)^m (xh_a+yh_b)^m} \ge \frac{9}{(x+y)^m s^{2m} r^{m-1}} \quad (1)$$

From Hölder's inequality we have:

$$\frac{1}{(h_ah_bh_c)^m}\cdot\sum\frac{h_a^{m+1}}{(xh_b+yh_c)^m} \ge \frac{1}{(h_ah_bh_c)^m}\cdot\frac{(h_a+h_b+h_c)^{m+1}}{(x+y)^m(h_a+h_b+h_c)^m} \Leftrightarrow$$

$$\sum\frac{h_a}{h_b^m h_c^m (xh_c+yh_c)^m} \ge \frac{(h_a+h_b+h_c)}{(h_ah_bh_c)^m (x+y)^m} \quad (2)$$

From (2) we must show: $\frac{h_a+h_b+h_c}{(h_ah_bh_c)^m (x+y)^m} \ge \frac{9}{(x+y)^m s^{2m} r^{m-1}} \Leftrightarrow$

$$\frac{h_a+h_b+h_c}{(h_ah_bh_c)^m} \ge \frac{9}{s^{2m}\cdot r^{m-1}} \quad (3)$$

Because $h_a + h_b + h_c \ge 9r$ *(4). From (3) + (4)* $\Rightarrow$

$$\frac{9r}{(h_ah_bh_c)^m} \ge \frac{9}{s^{2m} r^{m-1}} \Leftrightarrow \frac{1}{(h_ah_bh_c)^m} \ge \frac{1}{s^{2m} r^m}$$

$$\Leftrightarrow \frac{1}{h_ah_bh_c} \ge \frac{1}{s^2 r} \Leftrightarrow h_ah_bh_c \le s^2 r \quad (5)$$

But $h_ah_bh_c = \frac{2s^2r^2}{R}$ *(6). From (5)+(6) we must show:*

$$\frac{2s^2r^2}{R} \le s^2 r \Leftrightarrow 2r \le R \quad \text{true (Euler)}$$

PROBLEM 3.158-*Solution by proposer*

We know, $\frac{1}{2}\int_a^b g(x)\cot\left(\frac{ax}{2}\right)dx = \sum_{n=1}^{\infty}\int_a^b g(x)\sin(anx)\,dx$

put, $g(x) = x^2\log(x)$, $a = \pi$ *and* $(a,b) = (0,1)$. *Now,*

$$\frac{1}{2}\int_0^1 x^2\log(x)\cot\left(\frac{\pi x}{2}\right)dx = \sum_{n=1}^{\infty}\int_0^1 x^2\log(x)\sin(\pi n x)\,dx \quad (1)$$

We have that, $\int x^2\log(x)\sin(ax)\,dx = \frac{3\cos ax}{a^3}+\frac{x\sin ax}{a^2}+\left(\frac{2\cos ax+2ax\sin ax-a^2x^2\cos ax}{a^3}\right)$

$$\log(x)-\frac{2\,Ci(ax)}{a^3}\therefore\int_0^1 x^2\log(x)\sin(\pi nx)\,dx=$$

$$=\frac{(-1)^n3}{n^3\pi^3}-\frac{3}{n^3\pi^3}-\frac{2Ci(n\pi)}{n^3\pi^3}+\frac{2(\gamma+\log n\pi)}{n^3\pi^3}$$

From (1) we get,

$$\frac{1}{2}\int_0^1 x^2\log(x)\cot\left(\frac{\pi x}{2}\right)dx=\sum_{n=1}^{\infty}\left(\frac{(-1)^n3}{n^3\pi^3}-\frac{3}{n^3\pi^3}-\frac{2Ci(n\pi)}{n^3\pi^3}+\frac{2(\gamma+\log(n\pi))}{n^3\pi^3}\right)=$$

$$=-\frac{3}{\pi^3}[\xi_a(3)+\xi(3)]-\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{Ci(n\pi)}{n^3}+\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(\gamma+\log n\pi)}{n^3}=$$

$$=-\frac{3}{\pi^3}[\xi_a(3)+\xi(3)]-\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{Ci(n\pi)}{n^3}+\frac{2}{\pi^3}(-\xi'(3)+\gamma\xi(3)+\xi(3)\log(\pi))$$

$$\therefore\int_0^1 x^2\log(x)\cot\left(\frac{\pi x}{2}\right)dx=\frac{-6}{\pi^3}[\xi_a(3)+\xi(3)]-\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{Ci(n\pi)}{n^3}+$$

$$+\frac{4}{\pi^3}\xi(3)[\gamma+\log\pi]-\frac{4}{\pi^3}-\xi'(3)$$

PROBLEM 3.159-*Solution by proposer*

$$\int_{-1}^{+1}[(x+1)(x+1)]^{\frac{3}{2}}(1-x)^{\frac{1}{2}}\,dx$$

$$\int_{-1}^{+1}(1+x)^{\frac{3}{2}}(1+x)^{\frac{3}{2}}(1-x)^{\frac{1}{2}}dx\Rightarrow\int_{-1}^{+1}((1+x)^2)^{\frac{3}{2}}(1-x)^{\frac{1}{2}}dx$$

(ex $x=\cos 2t\quad dx=-2\sin 2\theta\,d\theta$

$cos^{-1}(x)=2\theta$

$\cos^{-1}(1)=2\theta;0=2\theta\therefore\theta=0$

$\cos^{-1}(-1)=2\theta;\pi=2\theta;0=\frac{\pi}{2}$

$$\int_{\frac{\pi}{2}}^{0}(1+\cos 2\theta)^3\left(1-(\cos 2\theta)\right)^{\frac{1}{2}}(-2\sin 2\theta)d\theta$$

$$=-2\int_{\frac{\pi}{2}}^{0}(1+2\cos^2 t-1)^3\left(1-(2\cos^2 t-1)\right)^{\frac{1}{2}}\sin 2t\,dt$$

$$2\int_0^{\frac{\pi}{2}}(1-1+2\cos^2 t)^3(1+1-2\cos^2 t)^{\frac{1}{2}}\sin 2t\,dt=2\int_0^{\frac{\pi}{2}}2^3(\cos t)^6\left(2(1-\cos^2 t)^{\frac{1}{2}}\right)\sin 2t\,dt$$

$$2\cdot 2^3\int_0^{\frac{\pi}{2}}\cos^6 t\, 2^{\frac{1}{2}}(\sin^2 t)^{\frac{1}{2}}\sin 2t\, dt$$

$$2^{\frac{1}{2}}2^4\int_0^{\frac{\pi}{2}}\cos^6 t\sin t\, 2\sin t\cos t\, dt = 2^{4+\frac{1}{2}+1}\int_0^{\frac{\pi}{2}}\cos^2 t\sin^2 t\, dt$$

$$2^{\frac{11}{2}}\int_0^{\frac{\pi}{2}}\cos^2 t\sin^2 t\, dt \Rightarrow 2^{\frac{9}{2}}\left[2\int_0^{\frac{\pi}{2}}\cos^2 t\sin^2\theta\, d\theta\right] or\ 2^4\sqrt{2}\left[2\int_0^{\frac{\pi}{2}}\cos^2 t\sin^2 t\, dt\right]$$

$$16\sqrt{2}\left[2\int_0^{\frac{\pi}{2}}\cos^2 t\sin^2 t\, dt\right], \beta(m,n) = 2\int_0^{\frac{\pi}{2}}\cos^{2n-1} t\sin^{2n-1}\theta\, dt$$

$$\therefore 2m-1 = 1\ and\ 2n-1 = 2,\ 2m-2 \to +1\ \ 2n = 1+1$$

$$2m = 0\quad 2n = 3,\ m = 4\ \ 2n = 3.\ n = \frac{3}{2}$$

$$\therefore 10\sqrt{2}\left[2\int_0^{\frac{\pi}{2}}\cos^{2(4)-1} t\sin^{2\left(\frac{3}{2}\right)-1} t\, dt\right]$$

$$= 10\sqrt{2}\left[\beta\left(4,\frac{3}{2}\right)\right] \Rightarrow 10\sqrt{2}\left(\frac{\sqrt{4}\sqrt{\frac{3}{2}}}{\sqrt{4+\frac{3}{2}}}\right) = \sqrt{2}(10)\left[\frac{\sqrt{4}\sqrt{\frac{3}{2}}}{\sqrt{\frac{11}{2}}}\right]$$

$$\sqrt{2}\left[\frac{16\ \times\ 6\ \times\ \sqrt{\frac{\pi}{2}}}{\frac{945\sqrt{\pi}}{32}}\right] = \sqrt{2}\left[\frac{3072\ \times\ \sqrt{\frac{\pi}{2}}}{945\sqrt{\pi}}\right] = \sqrt{2}\left[\frac{1536}{945}\right] = \frac{512\sqrt{2}}{315} \approx 2\cdot 2987$$

PROBLEM 3.160-*Solution by Tran Hong-Vietnam*

$$x,y,z \ge 0 \Rightarrow [x],[y],[z] \ge 0\ and\ 0 \le \{x\},\{y\},\{z\} < 1$$

$\Rightarrow [x]+\{y\}+1, [y]+\{z\}+1, [z]+\{x\}+1 > 0$. *Using Cauchy's inequality we have:*

$$LHS \ge \frac{3}{\sqrt[3]{([x]+[y]+1)^\alpha([y]+\{z\}+1)^\alpha([z]+\{x\}+1)^\alpha}} =$$

$$= \frac{3}{\sqrt[3]{\big(([x]+\{y\}+1)([y]+\{x\}+1)([z]+\{x\}+1)\big)^\alpha}} \ge \frac{3}{\sqrt[3]{\left(\frac{3}{x+y+3}\right)^{3\alpha}}} = \frac{(x+y+3)^\alpha}{3^{\alpha-1}}$$

We need to prove that: $\frac{(x+y+z+3)^\alpha}{3^{\alpha-1}} \ge \frac{3^{\alpha+1}}{(x+y+z+3)^\alpha} \Leftrightarrow (x+y+3)^{2\alpha} \ge 3^{2\alpha-2} \Leftrightarrow$

$\Leftrightarrow (x+y+z+3)^\alpha \ge 3^{\alpha-1}$. *Which is true because* $x+y+z+3 \ge 3 \Leftrightarrow$

$\Leftrightarrow (x+y+3)^\alpha \ge 3^\alpha \ge 3^{\alpha-1}$ *for* $x,y,z \ge 0$ *and* $\alpha \ge 1$. *Proved.*

PROBLEM 3.161-*Solution by Ravi Prakash-New Delhi-India*

$$\Omega = (I_1)(I_2)\ where\ I_1 = \int_0^\infty \frac{dx}{x^4+2x^2+1} = \int_0^\infty \frac{dx}{(x^2+1)^2}.\ Put\ x = \tan\theta$$

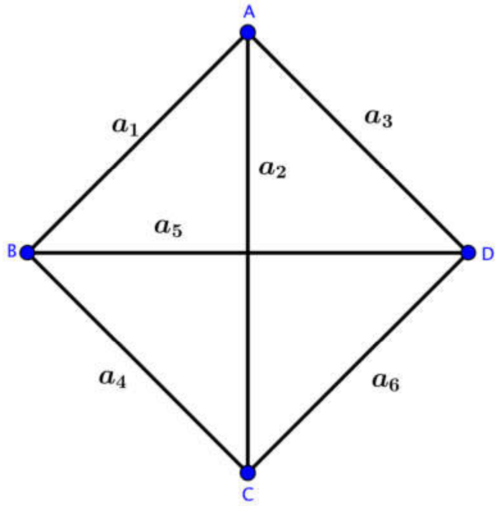
$$I_1 = \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\sec^4\theta} d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \frac{\pi}{4} \text{ and } I_2 = \int_0^{\infty} \frac{dy}{(y^2+25)^4}$$

Put $y = 5\tan\varphi$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{5\sec^2\varphi}{5^8\sec^8\varphi} d\varphi = \frac{1}{5^7}\int_0^{\frac{\pi}{2}} \cos^6\varphi\, d\varphi = \frac{1}{5^7}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1}{5^6}\cdot\frac{\pi}{32}$$

$$\therefore \Omega = \frac{\pi^2}{5^6 2^7} = \frac{\pi^2}{2000000}$$

PROBLEM 3.162-*Solution by Marian Ursărescu – Romania*



$(a_i + a_j)^2 \geq 3a_ia_j \Rightarrow$ *we must show:*

$$3\sum_{1\leq i<j\leq 6} a_ia_j \geq 12\sqrt{3}S[ABCD] \Leftrightarrow \sum_{1\leq i<j\leq 6} a_ia_j \geq 4\sqrt{3}S[ABCD] \quad (1)$$

Now, using Gordon's inequality: in any ΔABC *we have:*

$$ab + bc + ac \geq 4\sqrt{3} \Rightarrow$$

$$\left.\begin{array}{l} a_1a_2 + a_1a_4 + a_2a_4 \geq 4\sqrt{3}S_{ABC} \\ a_2a_6 + a_2a_3 + a_3a_6 \geq 4\sqrt{3}S_{ACD} \\ a_1a_5 + a_1a_3 + a_3a_5 \geq 4\sqrt{3}S_{ABD} \\ a_4a_5 + a_4a_6 + a_5a_6 \geq 4\sqrt{3}S_{BCD} \end{array}\right\} \Rightarrow \sum_{1\leq i<j\leq 6} a_ia_j \geq 4\sqrt{3}S[ABCD]$$

PROBLEM 3.163-*Solution by proposer*

** By Cauchy Schwarz inequality we have:*

$$P = \frac{a}{b(b+2c+1)(a+3c)^2} + \frac{b}{c(c+2a+1)(b+3a)^2} + \frac{c}{a(a+2b+1)(c+3b)^2}$$

$$= \frac{\left(\frac{a}{a+3c}\right)^2}{ab(b+2c+1)} + \frac{\left(\frac{b}{b+3a}\right)^2}{bc(c+2a+1)} + \frac{\left(\frac{c}{c+3b}\right)^2}{ca(a+2b+1)} \geq$$

$$\geq \frac{\left(\frac{a}{a+3}+\frac{b}{b+3a}+\frac{c}{c+3b}\right)^2}{ab(b+2c+1)+bc(c+2a+1)+ca(a+2b+1)}$$

$$\Leftrightarrow P \geq \frac{\left(\frac{a}{a+3c}+\frac{b}{b+3a}+\frac{c}{c+3b}\right)^2}{(ab^2+bc^2+ca^2)+6abc+(ab+bc+ca)} \quad (1)$$

- By inequality: $x^2+y^2 \geq \frac{(x+y)^2}{2}$ *(equality occurs if* $x=y$*) we have:*

$$8=(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq \frac{(a+b)^2}{2}\cdot\frac{(b+c)^2}{2}\cdot\frac{(c+a)^2}{2} \Leftrightarrow$$

$$\Leftrightarrow 64 \geq (a+b)^2(b+c)^2(c+a)^2$$

$$\Leftrightarrow 8 \geq (a+b)(b+c)(c+a) \quad (2)$$

- Other, by AM-GM inequality for 2 positive real numbers:

$$(a+b)(b+c)(c+a) \geq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}=8\sqrt{(abc)^2}=8abc \Leftrightarrow$$

$$abc \leq \frac{(a+b)(b+c)(c+a)}{8} \quad (3)$$

- Hence (3):

$$\Rightarrow (a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc \leq$$

$$\leq (a+b)(b+c)(c+a)+\frac{(a+b)(b+c)(c+a)}{8}=\frac{9(a+b)(b+c)(c+a)}{8}$$

$$\Leftrightarrow (a+b+c)(ab+bc+ca) \leq \frac{9(a+b)(b+c)(c+a)}{8} \quad (4)$$

- Let (2), (4): $\Rightarrow 8 \geq \frac{8(a+b+c)(ab+bc+ca)}{9} \Leftrightarrow 9 \geq (a+b+c)(ab+bc+ca)$

$$\Leftrightarrow 9 \geq (ab^2+bc^2+ca^2)+(a^2b+b^2c+c^2a)+3abc \quad (5)$$

- By AM-GM inequality we have:

$$a^2b+b^2c+c^2a \geq 3\sqrt[3]{(a^2b)\cdot(b^2c)\cdot(c^2a)}=3\sqrt[3]{(abc)^3}=3abc \Leftrightarrow a^2b+b^2c+c^2a \geq 3abc \quad (6)$$

- Let (5), (6): $\Rightarrow 9 \geq (ab^2+bc^2+ca^2)+3abc+3abc \Leftrightarrow (ab^2+bc^2+ca^2)+6abc \leq 9$ *(7)*

- Other: $a^2+b^2+c^2=\frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2} \geq \frac{2ab}{2}+\frac{2bc}{2}+\frac{2ca}{2}=ab+bc+ca$

$$\Leftrightarrow a^2+b^2+c^2 \geq ab+bc+ca \Leftrightarrow a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca)$$

$$\Leftrightarrow (a+b+c)^2 \geq 3(ab+bc+ca) \Leftrightarrow a+b+c \geq \sqrt{3(ab+bc+ca)} \quad (8)$$

- Let (4), (8): $\Rightarrow (a+b)(b+c)(c+a) \geq \frac{8\sqrt{3(ab+bc+ca)}\cdot(ab+bc+ca)}{9}$ *(9)*

- Let (2), (9):

$$\Rightarrow 8 \geq \frac{8\sqrt{3(ab+bc+ca)}\cdot(ab+bc+ca)}{9} \Leftrightarrow 9 \geq \sqrt{3(ab+bc+ca)^3} \Leftrightarrow$$

$$81 \geq 3(ab+bc+ca)^3$$

$$\Leftrightarrow 27 \geq (ab+bc+ca)^3 \Leftrightarrow 3 \geq (ab+bc+ca) \Leftrightarrow (ab+bc+ca) \leq 3 \quad (10)$$

- Let (7), (10): $\Rightarrow (ab^2+bc^2+ca^2)+6abc+(ab+bc+ca) \leq 9+3$

$$\Leftrightarrow (ab^2+bc^2+ca^2)+6abc+(ab+bc+ca) \leq 12 \quad (11)$$

- Let (1), (11): $\Rightarrow P \geq \frac{\left(\frac{a}{a+3c}+\frac{b}{b+3a}+\frac{c}{c+3b}\right)^2}{12}$ *(12)*

- By Cauchy Schwarz inequality we have:

$$\frac{a}{a+3c}+\frac{b}{b+3a}+\frac{c}{c+3b}=\frac{a^2}{a^2+3ca}+\frac{b^2}{b^2+3ab}+\frac{c^2}{c^2+3bc}$$
$$\geq\frac{(a+b+c)^2}{(a^2+3ca)+(b^2+3ab)+(c^2+3bc)}$$
$$\Leftrightarrow\frac{a}{a+3c}+\frac{b}{b+3a}+\frac{c}{c+3b}\geq\frac{(a+b+c)^2}{(a^2+b^2+c^2+2ab+2bc+2ca)+(ab+bc+ca)}$$
$$\Leftrightarrow\frac{a}{a+3}+\frac{b}{b+3a}+\frac{c}{c+3b}\geq\frac{(a+b+c)^2}{(a+b+c)^2+(ab+bc+ca)}\quad (13)$$

- Let (8): $(a+b+c)^2\geq 3(ab+bc+ca)\Leftrightarrow ab+bc+ca\leq\frac{(a+b+c)^2}{3}\quad (14)$

- Let (13), (14): $\Rightarrow\frac{a}{a+3c}+\frac{b}{b+3a}+\frac{c}{c+3b}\geq\frac{(a+b+c)^2}{(a+b+c)^2+\frac{(a+b+c)^2}{3}}=\frac{3(a+b+c)^2}{4(a+b+c)^2}=\frac{3}{4}$

$$\Leftrightarrow\frac{a}{a+3c}+\frac{b}{b+3a}+\frac{c}{c+3b}\geq\frac{3}{4}\quad (15)$$

- Let (12), (15): $\Rightarrow P\geq\frac{\left(\frac{3}{4}\right)^2}{12}=\frac{9}{16\cdot 12}=\frac{3}{64}\Rightarrow P\geq\frac{3}{64}\Rightarrow P_{\min}=\frac{3}{64}$

+ Equality occurs if: $\begin{cases}a,b,c>0;(a^2+b^2)(b^2+c^2)(c^2+a^2)=8\\ a=b;b=c;c=a\\ \frac{a}{(a+3c)\cdot ab(b+2c+1)}=\frac{b}{(b+3a)\cdot bc(c+2a+1)}=\frac{c}{(c+3b)\cdot ca(a+2b+1)}\\ \frac{a}{a(a+3c)}=\frac{b}{b(b+3a)}=\frac{c}{c(c+3b)}\end{cases}$

$$\Leftrightarrow\begin{cases}a=b=c>0\\(a^2+b^2)(b^2+c^2)(c^2+a^2)=8\end{cases}\Leftrightarrow\begin{cases}a=b=c>0\\2a^2\cdot 2a^2\cdot 2a^2=8\end{cases}\Leftrightarrow\begin{cases}a=b=c>0\\a^6=1\end{cases}\Leftrightarrow a=b=c=1.$$

Therefore, minimum of $P:\frac{3}{64}$ *then* $a=b=c=1$**.**

PROBLEM 3.164-*Solution by Amit Dutta-Jamshedpur-India*

Using Cauchy – Schwarz's Inequality:

$$\left(\sqrt{2(x^4+y^4)}+2xy\right)^2\leq 2\left(2(x^4+y^4+2x^2y^2)\right)\leq 4(x^2+y^2)^2\Rightarrow$$
$$\Rightarrow\left(\sqrt{2(x^4+y^4)}+2xy\right)\leq 2(x^2+y^2)$$
$$\sqrt{2(x^4+y^4)}\leq 2(x^2-xy+y^2)$$

Putting $y=1:\sqrt{2(x^4+1)}\leq 2(x^2-x+1)\quad$ *(1)*

$$\begin{Bmatrix}\text{Equality holds when}\\ \frac{1}{\sqrt{2(x^4+y^4)}}=\frac{1}{2xy}\ or\ \frac{1}{\sqrt{2(x^4+1)}}=\frac{1}{2x}\\ \Rightarrow x=1\quad (a)\end{Bmatrix}$$

$$\because GM\leq AM$$

$$2\sqrt{3x-2x^4}\leq 2\sqrt{x(3-2x^3)}\leq(x+3-2x^3)\quad (2)$$

$$\begin{Bmatrix}\text{Equality holds when}\\ x=3-2x^3\Rightarrow x=1\\ (b)\end{Bmatrix}$$

Adding (1) & (2):

$$\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}\le 2(x^2-x+1)+x+3-2x^3\Rightarrow$$
$$\Rightarrow 7-3x\le 2x^2-x+5-2x^3\Rightarrow 2x^3-2x^2-2x+2\le 0\Rightarrow x^3-x^2-x+1\le 0\Rightarrow$$
$$\Rightarrow x^2(x-1)-1(x-1)\le 0\Rightarrow (x^2-1)(x-1)\le 0\Rightarrow (x+1)(x-1)^2\le 0$$
$\because (x+1)>0$ *{$\because x>0$ from domain}* $\Rightarrow (x-1)^2\le 0\Rightarrow (x-1)^2=0\Rightarrow x=1$ *(c)*

From (a), (b) & (c) we can conclude that the only real solution is $x=1$.

PROBLEM 3.165-*Solution by proposer*

** Hence (1), by AM-GM inequality for three positive real numbers we have:*

$$3=\frac{1}{\sqrt{a^3}}+\frac{1}{\sqrt{b^3}}+\frac{1}{\sqrt{c^3}}\ge 3\cdot\sqrt[3]{\frac{1}{\sqrt{a^3}\cdot\sqrt{b^3}\cdot\sqrt{c^3}}}=\frac{3}{\sqrt[6]{(abc)^3}}=\frac{3}{\sqrt{abc}}\Leftrightarrow 3\ge\frac{3}{\sqrt{abc}}\Leftrightarrow$$
$$\Leftrightarrow\sqrt{abc}\ge 1\Leftrightarrow abc\ge 1$$

Hence (2): $\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{a^3c+b^3a+c^3b}{abc}\le\frac{a^3c+b^3a+c^3b}{1}=a^3c+b^3a+c^3b$ *(3)*

- By AM-GM inequality, we have:

$$a^3c+b^3a+c^3b=a^3ac+b^2ba+c^2cb\le\frac{a^3+(ac)^2}{2}+\frac{b^4+(ba)^2}{2}+\frac{c^4+(cb)^2}{2}$$
$$\Leftrightarrow a^3c+b^3a+c^3b\le\frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2}\quad (4)$$

- Hence (3), (4): $\Rightarrow\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\le\frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2}$ *(5)*

- Other, by AM-GM inequality:

$$\frac{a^6+a^6+1}{2}+\frac{b^6+b^6+1}{2}+\frac{c^6+c^6+1}{2}\ge\frac{3\sqrt[3]{a^6\cdot a^6\cdot 1}}{2}+\frac{3\sqrt[3]{b^6\cdot b^6\cdot 1}}{2}+\frac{3\sqrt[3]{c^6\cdot c^6\cdot 1}}{2}$$
$$=\frac{3(a^4+b^4+c^4)}{2}$$
$$\Leftrightarrow a^6+b^6+c^6+\frac{3}{2}\ge\frac{3(a^4+b^4+c^4)}{2}\Leftrightarrow 2(a^6+b^6+c^6)+3\ge 3(a^4+b^4+c^4)\quad (6)$$
$$(a^3b^3+a^3b^3+1)+(b^3c^3+b^3c^3+1)+(c^3a^3+c^3a^3+1)\ge$$
$$\ge 3\sqrt[3]{(a^3b^3)(a^3b^3)\cdot 1}+3\sqrt[3]{(b^3c^3)(b^3c^3)\cdot 1}+3\sqrt[3]{(c^3a^3)(c^3a^3)\cdot 1}$$
$$=3(a^2b^2+b^2c^2+c^2a^2)$$
$$\Leftrightarrow 2(a^3b^3+b^3c^3+c^3a^3)+3\ge 3(a^2b^2+b^2c^2+c^2a^2)$$
$$\Leftrightarrow 4(a^3b^3+b^3c^3+c^3a^3)+6\ge 6(a^2b^2+b^2c^2+c^2a^2)\quad (7)$$

- Let (6), (7): $\Rightarrow 2(a^6+b^6+c^6)+3+4(a^3b^3+b^3c^3+c^3a^3)+6\ge$
$$\ge 3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)$$
$$\Leftrightarrow 2(a^6+b^6+c^6+2a^3b^3+2b^3c^3+2c^3a^3)+9\ge$$
$$\ge 3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)$$
$$\Leftrightarrow 2(a^3+b^3+c^3)^2\ge 3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)-9\quad (8)$$

- By AM-GM inequality and (2). We have:

$$3(a^2b^2+b^2c^2+c^2a^2)\ge 3\cdot 3\cdot\sqrt[3]{(a^2b^2)(b^2c^2)(c^2a^2)}=9\sqrt[3]{(abc)^4}\ge 9\sqrt[3]{1^4}=9$$
$$\Leftrightarrow 6(a^2b^2+b^2c^2+c^2a^2)-9\ge 3(a^2b^2+b^2c^2+c^2a^2)\quad (9)$$

- Let (8),(9): $\Rightarrow 2(a^3+b^3+c^3)^2\ge 3(a^4+b^4+c^4)+3(a^2b^2+b^2c^2+c^2a^2)$
$$\Leftrightarrow 2(a^3+b^3+c^3)^2\ge 3(a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2)$$
$$\Leftrightarrow\frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2}\le\frac{(a^3+b^3+c^3)^2}{3}\quad (10)$$

- (5), (10):$\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\le\frac{(a^3+b^3+c^3)^2}{3}$ (11)

- (1), (11): $\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{(a^3+b^3+c^3)^2}{3}$ occurs if: $\begin{cases} a=b=c>0 \\ \frac{1}{\sqrt{a^3}}+\frac{1}{\sqrt{b^3}}+\frac{1}{\sqrt{c^3}}=3 \end{cases}\Leftrightarrow a=b=c=1.$

Solution of equation is: $(a,b,c)=(1,1,1)$.

PROBLEM 3.166-*Solution by Amit Dutta-Jamshedpur-India*

$$\text{Domain}\to\begin{cases} x^3-2x^2+2x>0 \\ 4x-3x^4>0 \end{cases}$$

$$x^3-2x^2+2x=x(x^2-2x+2)=x[(x-1)^2+1]$$

$$\because x^3-2x^2+2x>0\Rightarrow x[(x-1)^2+1]>0\Rightarrow x>0$$

$$GM\le AM\ \sqrt{x^2-2x^2+2x}\le\frac{(x^2-2x^2+2x)+1}{2}$$

$$\sqrt{x^3-2x^2+2x}\le\left(\frac{x^2-2x^2+2x+1}{2}\right)\quad (a)$$

Equality holds when $x^2-2x^2+2x=1$ (1)

Again, using $GM\le AM$

$$3\sqrt[3]{x^2-x+1}\le(x^2-x+1)+1+1\le(x^2-x+3)\quad (2)$$

Equality holds when $x^2-x+1=1$ (2)

Again, using $GM\le AM$

$$2\sqrt[4]{4x-3x^4}\le 2\left\{\frac{(4x-3x^4)+1+1+1}{4}\right\}\le\left(\frac{4x-3x^4+3}{2}\right)\quad (3)$$

Equality holds when $4x-3x^4=1$ (3)

Adding (1), (2), (3):

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}\le$$

$$\le\left(\frac{x^3-2x^2+2x+1}{2}\right)+(x^2-x+3)+\left(\frac{4x-3x^4+3}{2}\right)$$

$$\Rightarrow\frac{x^4-3x^3}{2}+7\le\frac{-3x^4+4x+10+x^3}{2}$$

$$\Rightarrow x^4-3x^3+14\le-3x^4+4x+10+x^3\Rightarrow 4x^4-4x^3-4x+4\le 0$$

$$\Rightarrow x^4-x^3-x+1\le 0\Rightarrow x^3(x-1)-1(x-1)\le 0$$

$$\Rightarrow(x^3-1)(x-1)\le 0\Rightarrow(x-1)(x^2+x+1)(x-1)\le 0$$

$$\Rightarrow(x-1)^2(x^2+x+1)\le 0$$

$$\because x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0\Rightarrow(x-1)^2\le 0$$

$$(x-1)^2=0\Rightarrow x=1\quad (4)$$

From (1), (2), (3) & (4): The only real solution is $x=1$.

PROBLEM 3.167-*Solution by Tran Hong-Vietnam*

$$3a^4-4a+2b^2+11=\{a^4-4a^3+6a^2-4a+1\}+\{2a^4+4a^3-6a^2+10+2b^2\}$$

$$=(a-1)^4+2(a^4+2a^3-3a^2+b^2+5)$$

$$\ge 2(a^4+2a^3-4a^2+a^2+b^2+5)\overset{(*)}{\ge}4(a+ab+1)$$

$$(*)\Leftrightarrow a^4+2a^3-4a^2+2ab+5\ge 2(a+ab+1)$$

$$(*) \Leftrightarrow a^4+2a^3-4a^2+5 \geq 2(a+1)$$

$$\Leftrightarrow a^4+2a^3-4a^2-2a+3 \geq 0 \Leftrightarrow (a-1)^2(a+1)(a+3) \geq 0 \ \ (\text{true with } a>0)$$

Hence: $3a^4-4a+2b^2+11 \geq 4(a+ab+1)$*, etc. Now,*

Let $f(t)=\sqrt[3]{t}, t>0 \Rightarrow f'(t)=-\frac{2}{9}t^{-\frac{5}{3}}<0\ (\forall t>0)$

Using Jensen's inequality, we have: $P \leq 3\sqrt[3]{\frac{Q}{3}}$

$$\therefore Q=\frac{1}{3a^4-4a+2b^2+11}+\frac{1}{3b^4-4b+2c^2+11}+\frac{1}{3c^4-4c+2a^2+11}$$

$$\leq \frac{1}{4}\left(\frac{1}{a+ab+1}+\frac{1}{b+bc+1}+\frac{1}{c+ca+1}\right)$$

$$=\frac{1}{4}\left(\frac{1}{a+ab+1}+\frac{a}{a+ab+1}+\frac{ab}{a+ab+1}\right)=\frac{1}{4}\left(\frac{1+a+ab}{1+a+ab}\right)=\frac{1}{4}$$

$$\Rightarrow P \leq 3\sqrt[3]{\frac{1}{4\cdot 3}}=\frac{3}{\sqrt[3]{12}}=\sqrt[3]{\frac{9}{4}}.\ \text{Equality} \Leftrightarrow a=b=c=1.$$

PROBLEM 3.168-*Solution by Remus Florin Stanca-Romania*

Let be $a>0$ *and* $f:(-\infty;-a-1)\cup(-a;+\infty)\to\mathbb{R}$

$f(x)=\frac{1}{x^2+(2a+1)x+a^2+a}$. *Find:*

$$\Omega=\lim_{n\to\infty}\sqrt[n^2]{\left|\lim_{p\to\infty}\sum_{k=1}^{p} f^{(n)}(k)\right|}$$

$$f(x)=\frac{1}{x^2+2\cdot x\cdot\frac{2a+1}{2}+\frac{4a^2+4a+1}{4}+a^2+a-\frac{4a^2+4a+1}{4}}$$

$$=\frac{1}{\left(x+\frac{2a+1}{2}\right)^2-\frac{1}{4}}=\frac{1}{(x+a)(x+a+1)}=\frac{x+a+1-(x+a)}{(x+a)(x+a+1)}=\frac{1}{x+a}-\frac{1}{x+a+1}$$

$$\succ f^{(n)}(x)=n!\cdot(-1)^n\cdot\frac{1}{(x+a)^{n+1}}+(-1)^{n+1}\cdot n!\ \frac{1}{(x+a+1)^{n+1}}$$

$$\Rightarrow\sum_{k=1}^{p} f^{(n)}(k)=n!\cdot(-1)^n\cdot\left(\frac{1}{(a+1)^{n+1}}-\frac{1}{(p+a+1)^{n+1}}\right)\Rightarrow\left|\lim_{p\to\infty}\sum_{k=1}^{p} f^{(n)}(k)\right|=$$

$$=\frac{n!}{(a+1)^{n+1}}\Rightarrow\lim_{n\to\infty}\left(\frac{n!}{(a+1)^{n+1}}\right)^{\frac{1}{n^2}}=\lim_{n\to\infty}e^{\frac{\ln\left(\frac{n!}{(a+1)^{n+1}}\right)}{n^2}}=\lim_{n\to\infty}e^{\frac{\ln\frac{n+1}{a+1}}{2n+1}}=$$

$$=\lim_{n\to\infty}e^{\frac{\ln\frac{n+2}{n+1}}{2}}=e^0=1\succ\Omega=1$$

PROBLEM 3.169-*Solution by Remus Florin Stanca-Romania*

$$\lim_{n\to\infty} n^{p+1}x_n^{p^2+p+1}=\lim_{n\to\infty} n^{p+1}x_n^{(p+1)^2}\cdot\frac{1}{x_n^p}==\lim_{n\to\infty}\frac{n^{p+1}x_n^{(p+1)^2}x_n}{x_n^{p+1}}\qquad (1)$$

$$x_1^p+\cdots+x_n^p=\frac{1}{\sqrt[p+1]{x_{n+1}}}\succ\frac{1}{\sqrt[p+1]{x_n}}+x_n^p=\frac{1}{\sqrt[p+1]{x_{n+1}}}$$

we prove by using the Mathematical induction that $x_n>0;\forall n\in\mathbb{N}$*:*

1. we prove that $P(0)$: *"*$x_0>0$*" is true (true).*

2. we suppose that $P(n)$: *"*$x_n>0$*" is true*

3. we prove that $P(n+1)$: *"*$x_{n+1}>0$*" is true by using* $P(n)$*:*

$$\frac{1}{\sqrt[p+1]{x_{n+1}}}=x_n^p+\frac{1}{\sqrt[p+1]{x_n}};x_n>0\Rightarrow\frac{1}{\sqrt[p+1]{x_{n+1}}}>0\Rightarrow x_{n+1}>0\Rightarrow true\Rightarrow x_n>0;\forall n\in\mathbb{N}$$

$$\frac{1}{\sqrt[p+1]{x_n}}+x_n^p=\frac{1}{\sqrt[p+1]{x_{n+1}}}\succ\frac{1}{\sqrt[p+1]{x_{n+1}}}-\frac{1}{\sqrt[p+1]{x_n}}=x_n^p>0\succ\sqrt[p+1]{x_{n+1}}<\sqrt[p+1]{x_n}$$

$\succ x_{n+1}<x_n\succ(x_n)_{n\in\mathbb{N}}$ *is a decreasing sequence,* $x_n>0\succ|l\in\mathbb{R}$ *such that:*

$$\lim_{n\to\infty}x_n=l=\frac{1}{\sqrt[p+1]{l}}=l^p+\frac{1}{\sqrt[p+1]{l}}\Rightarrow l=0\Rightarrow\lim_{n\to\infty}x_n=0$$

$$\text{(1)}\Rightarrow\lim_{n\to\infty}n^{p+1}x_n^{p^2+p+1}=\lim_{n\to\infty}\left(\frac{x_n^{p+1}\cdot n\cdot\sqrt[p+1]{x_n}}{x_n}\right)^{p+1}$$

$$=\lim_{n\to\infty}\left(x_n^p\cdot n\cdot\sqrt[p+1]{x_n}\right)^{p+1}=L^{p+1}$$

$$x_n^p=\frac{1}{\sqrt[p+1]{x_{n+1}}}-\frac{1}{\sqrt[p+1]{x_n}}\Rightarrow L=\lim_{n\to\infty}\left(\left(\frac{1}{\sqrt[p+1]{x_{n+1}}}-\frac{1}{\sqrt[p+1]{x_{n+1}}}\right)\cdot n\cdot\sqrt[p+1]{x_n}\right)=$$

$$\lim_{n\to\infty}\left(n\cdot\left(\sqrt[p+1]{\frac{x_n}{x_{n+1}}}-1\right)\right)\quad\text{(2)}$$

$$x_n^p=\frac{1}{\sqrt[p+1]{x_{n+1}}}-\frac{1}{\sqrt[p+1]{x_n}}\Rightarrow\lim_{n\to\infty}\sqrt[p+1]{\frac{x_{n+1}}{x_n}}=1$$

$$\overset{(2)}{\Rightarrow}L=\lim_{n\to\infty}n\cdot\frac{\frac{x_n}{x_{n+1}}-1}{\left(\sqrt[p+1]{\frac{x_n}{x_{n+1}}}\right)^0+\cdots+\left(\sqrt[p+1]{\frac{x_n}{x_{n+1}}}\right)^p}=\frac{1}{p+1}\cdot\lim_{n\to\infty}n\left(\frac{x_n}{x_{n+1}}-1\right)=$$

$$=\frac{1}{p+1}\cdot\lim_{n\to\infty}\left(\left(x_n^{\frac{p^2+p+1}{p+1}}\right)^{p+1}-1\right)$$

$$=\frac{1}{p+1}\cdot\lim_{n\to\infty}n\cdot x_n^{\frac{p^2+p+1}{p+1}}(p+1)=\lim_{n\to\infty}\frac{n}{x_n^{-\frac{p^2+p+1}{p+1}}}\overset{Stolz\ Cesaro}{=}$$

$$=\lim_{n\to\infty}\frac{1}{x_{n+1}^{-\frac{p^2+p+1}{p+1}}-x_n^{-\frac{p^2+p+1}{p+1}}}$$

$$\lim_{n\to\infty}x_{n+1}^{-\frac{p^2+p+1}{p+1}}-x_n^{-\frac{p^2+p+1}{p+1}}=\lim_{n\to\infty}\left(x_n^p+\frac{1}{\sqrt[p+1]{x_n}}\right)^{p^2+p+1}-x_n^{-\frac{p^2+p+1}{p+1}}=$$

$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{x_n}}\right)^{p^2+p+1}\cdot\left(\left(\frac{x_n^p+\frac{1}{\sqrt[p+1]{x_n}}}{\frac{1}{\sqrt[p+1]{x_n}}}\right)^{p^2+p+1}-1\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{x_n}}\right)^{p^2+p+1}\cdot\frac{\left(x_n^{p+\frac{1}{p+1}}+1\right)^{p^2+p+1}-1}{x_n^{p+\frac{1}{p+1}}}\cdot x_n^{p+\frac{1}{p+1}}=$$

$$= (p^2+p+1)\cdot\lim_{n\to\infty}x_n^{\frac{p^2+p+1}{p+1}}\cdot x_n^{-\frac{p^2+p+1}{p+1}}=p^2+p+1$$

$$\Rightarrow L=\frac{1}{p^2+p+1}\Rightarrow\lim_{n\to\infty}n^{p+1}\cdot x_n^{p^2+p+1}=\frac{1}{(p^2+p+1)^{p+1}}$$

PROBLEM 3.170-*Solution by Avishek Mitra-India*

$$\Omega=\lim_{n\to\infty}\int_0^1\tan^{-1}(nx)\frac{\ln(1+x)}{(1+x^2)}dx=\frac{\pi}{2}\int_0^1\frac{\ln(1+x)}{(1+x^2)}dx$$

$$Let\ I=\int_0^1\frac{\ln(1+x)dx}{(1+x^2)}=[\ln(1+x)\cdot\tan^{-1}x]_0^1-\int_0^1\frac{\tan^{-1}xdx}{(1+x)}$$

$$[let\ x=\tan z\Rightarrow dx=\sec^2 z\,dz]$$

$$=\frac{\pi}{4}\ln 2-\int_0^{\frac{\pi}{4}}\frac{z\cdot\sec^2 z\,dz}{(1+\tan z)}=\frac{\pi}{4}\ln 2-\int_0^{\frac{\pi}{4}}\frac{zdz}{\cos z(\sin z+\cos z)}$$

$$Let\ I_1=\int_0^{\frac{\pi}{4}}\frac{zdz}{\cos z(\sin z+\cos z)}=\int_0^{\frac{\pi}{4}}\frac{\left(\frac{\pi}{4}-z\right)}{\cos\left(\frac{\pi}{4}-z\right)\left[\sin\left(\frac{\pi}{4}-z\right)+\cos\left(\frac{\pi}{4}-z\right)\right]}dz$$

$$=\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{dz}{\cos z\,(\sin z+\cos z)}-I_1\Rightarrow$$

$$\Rightarrow 2I_1=\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{\sec^2 z\,dz}{(1+\tan z)}=\frac{\pi}{4}[\ln(1+\tan z)]_0^{\frac{\pi}{4}}\ln 2\Rightarrow I_1=\frac{\pi}{8}\ln 2$$

$$Hence\ I=\frac{\pi}{4}\ln 2-\frac{\pi}{8}\ln 2=\frac{\pi}{8}\ln 2$$

$$Hence\ \Omega=\frac{\pi}{2}\cdot\frac{\pi}{8}\ln 2=\frac{\pi^2}{16}\ln 2\ \ (answer)$$

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