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20 NEW TRIANGLE INEQUALITIES

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If n_a-Nagel cevian from A in ABC triangle then will prove:

$$n_a \geq \frac{b^2-bc+c^2}{2R} \text{ (and analogs)}, \quad bc=2Rh_a \text{ (and analogs)}$$

$$\left(\frac{n_a}{h_a}\right)^2=1+\frac{b^2-2bc+c^2}{4r^2}=1+\frac{(b-c)^2}{4r^2} \text{ (and analogs)}$$

We write:

$$n_a \geq \frac{b^2-bc+c^2}{2R} \rightarrow \frac{n_a}{h_a} \geq \frac{b^2-bc+c^2}{2Rh_a}=\frac{b}{c}+\frac{c}{b}-1$$

$$\left(\frac{n_a}{h_a}\right)^2 \geq \left(\frac{b}{c}+\frac{c}{b}-1\right)^2 \rightarrow 1+\frac{(b-c)^2}{4r^2} \geq \left(\frac{b}{c}+\frac{c}{b}-1\right)^2 \rightarrow \frac{(b-c)^2}{4r^2} \geq \left(\frac{b}{c}+\frac{c}{b}-1\right)^2-1$$

$$x^2-y^2=(x-y)(x+y), \frac{(b-c)^2}{4r^2} \geq \left(\frac{b}{c}+\frac{c}{b}-1-1\right)\left(\frac{b}{c}+\frac{c}{b}-1+1\right)$$

$$\frac{(b-c)^2}{4r^2} \geq \left(\frac{b}{c}+\frac{c}{b}-2\right)\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{(b-c)^2}{bc}\left(\frac{b}{c}+\frac{c}{b}\right)$$

If b=c then we obtain equality.

If b≠c then $\frac{(b-c)^2}{4r^2} > \frac{(b-c)^2}{bc}\left(\frac{b}{c}+\frac{c}{b}\right) \rightarrow \frac{1}{4r^2} > \frac{1}{bc}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2Rh_a}\left(\frac{b}{c}+\frac{c}{b}\right)$

$\frac{R}{r}\frac{1}{2r} > \frac{1}{h_a}\left(\frac{b}{c}+\frac{c}{b}\right), \frac{R}{r} \geq \frac{b}{c}+\frac{c}{b}$ (and analogs)(Băndilă)

We will prove : $h_a > 2r$

$ah_a=2S=2pr$; $2p=a+b+c \rightarrow h_a=\left(1+\frac{b+c}{a}\right)r \rightarrow \left(1+\frac{b+c}{a}\right)r > 2r \rightarrow 1+\frac{b+c}{a} > 2$

$\frac{b+c}{a} > 1 \rightarrow b+c>a$ -true.

In conclusion $n_a \geq \frac{b^2-bc+c^2}{2R}$ $(and\ analogs)$

From $n_a \geq \frac{b^2-bc+c^2}{2R}$ and $bc=2Rh_a$ (and analogs) obtain:

$$\mathbf{1+\frac{n_a}{h_a} \geq \frac{b}{c}+\frac{c}{b}} \textbf{ (and analogs) (1)}$$

We also proved $\frac{R}{r} \geq 1+\frac{n_a}{h_a}$ [2] ,obtain from (1):

$\frac{R}{r} \geq 1+\frac{n_a}{h_a} \geq \frac{b}{c}+\frac{c}{b}$ **(Refinement for Băndilă's inequality)**

$$3+\frac{n_a}{h_a} \geq 2+\frac{b}{c}+\frac{c}{b}=\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)^2 \rightarrow \sqrt{3+\frac{n_a}{h_a}} \geq \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}} \text{ (and analogs)} \quad (2)$$

From $1+\frac{n_a}{h_a} \geq \frac{b}{c}+\frac{c}{b} \rightarrow \sqrt{\frac{n_a}{h_a}-1} \geq \left|\sqrt{\frac{b}{c}}-\sqrt{\frac{c}{b}}\right|$ **(and analogs) (3)**

From $n_a \geq \frac{b^2-bc+c^2}{2R}$ and $bc=2Rh_a$ (and analogs) obtain:

$$n_a+h_a \geq \frac{b^2+c^2}{2R} \text{ (and analogs)} \quad (4)$$

In ABC triangle we know that:

$\cos\frac{B-C}{2}=\frac{b+c}{a}\sin\frac{A}{2}$ (and analogs); $a=2R\sin A$ (and analogs);

$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$ (and analogs) $\rightarrow a=4R\sin\frac{A}{2}\cos\frac{A}{2}$ (and analogs)

From $\cos\frac{B-C}{2}=\frac{b+c}{a}\sin\frac{A}{2}$ and $a=4R\sin\frac{A}{2}\cos\frac{A}{2} \rightarrow b+c=4R\cos\frac{A}{2}\cos\frac{B-C}{2} \leq 4R\cos\frac{A}{2}$

Use $b+c \leq 4R\cos\frac{A}{2}$ (and analogs) and $\cos^2\frac{A}{2}=\frac{r_b+r_c}{4R}$ (and analogs) and $bc=2Rh_a$ (and analogs) we obtain:

$$\frac{r_b+r_c}{4R} \geq \frac{4Rh_a}{4R\,4R}+\frac{b^2+c^2}{4R\,4R} \rightarrow r_b+r_c-h_a \geq \frac{b^2+c^2}{4R} \text{ (and analogs)} \quad (5)$$

From (4) and (5) $\rightarrow n_a+r_b+r_c \geq \frac{3(b^2+c^2)}{4R}$ **(and analogs) (6)**

$m_a \geq \frac{b^2+c^2}{4R}$ (and analogs)(Tereshin) and (6) we obtain:

$$n_a+r_b+r_c+m_a \geq \frac{b^2+c^2}{R} \text{ (and analogs)} \quad (7)$$

From (4) and Tereshin we obtain: $2m_a+n_a+h_a \geq \frac{b^2+c^2}{2R}+\frac{b^2+c^2}{2R}=\frac{b^2+c^2}{R}$

$$\mathbf{2m_a + n_a + h_a \geq \frac{b^2+c^2}{R}} \text{ (and analogs) } (8)$$

$$\mathbf{m_a + n_a + h_a \geq \frac{3(b^2+c^2)}{4R}} \text{ (and analogs) } (9)$$

From (1) $\rightarrow 3+\frac{n_a}{h_a} \geq \frac{b}{c} + \frac{c}{b} + 2 = \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right)^2$

$$\sqrt{3 + \frac{n_a}{h_a}} \geq \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} \text{ (and analogs) } (10)$$

From $l_a = 2\frac{\sqrt{bc}}{b+c}\sqrt{r_b r_c}$ (and analogs) $\rightarrow \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} = 2\frac{\sqrt{r_b r_c}}{l_a}$ and from (10) we obtain:

$$l_a\sqrt{3 + \frac{n_a}{h_a}} \geq 2\sqrt{r_b r_c} \text{ (and analogs) } (11)$$

From (11) and $h_a = 2\frac{r_b r_c}{r_b + r_c}$(and analogs) after simple manipulations:

$$\frac{l_a}{h_a} \geq \sqrt{\frac{2(r_b+r_c)}{3h_a+n_a}} \text{ (and analogs) } (12)$$

From [3]: $\mathbf{m_a} \geq \sqrt{r_b r_c}\sqrt[4]{\frac{p_a}{l_a}}$(and analogs)

$\mathbf{m_a} \geq \sqrt{r_b r_c}\sqrt{\frac{l_a+p_a}{2l_a}}$(and analogs) and (11) $\rightarrow$

$$m_a l_a\sqrt{3 + \frac{n_a}{h_a}} \geq 2r_b r_c\sqrt[4]{\frac{p_a}{l_a}} \quad (13)$$

$$m_a l_a\sqrt{\frac{l_a}{h_a}} \geq r_b r_c\sqrt{\frac{2(l_a+p_a)}{3h_a+n_a}} \text{ (and analogs) } (14)$$

From (6) and $bc=2Rh_a$ (and analogs) :

$$\frac{n_a+r_b+r_c}{h_a} \geq \frac{3}{2}\left(\frac{b}{c}+\frac{c}{b}\right) \text{ (and analogs)} \quad (15)$$

From (7) and bc=2Rh_a (and analogs):

$$\frac{n_a+r_b+r_c+m_a}{2h_a} \geq \frac{b}{c}+\frac{c}{b} \text{(and analogs)} \quad (16)$$

From (15),(16) we get:

$$\sqrt{2+\frac{2(n_a+r_b+r_c)}{3h_a}} \geq \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}=2\frac{\sqrt{r_b r_c}}{l_a} \text{ (and analogs)} \quad (17)$$

$$\sqrt{2+\frac{n_a+r_b+r_c+m_a}{2h_a}} \geq \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}=2\frac{\sqrt{r_b r_c}}{l_a} \text{(and analogs)} \quad (18)$$

From (8) and bc=2Rh_a (and analogs) :

$$\frac{2m_a+n_a+h_a}{2h_a} \geq \frac{b}{c}+\frac{c}{b} \text{(and analogs)}(19)$$

From(19) :

$$\sqrt{2+\frac{h_a+n_a+2m_a}{2h_a}} \geq \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}=2\frac{\sqrt{r_b r_c}}{l_a} \text{(and analogs)} \quad (20)$$

REFERENCE:

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