A Sierpinski type inequality for geometric , arithmetic and quadratic means

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The purpose of this note is to establish a Sierpinski type inequality for the means G_n , A_n , Q_n *of* n *positive real numbers. They are also highlighted refinements of the inequalities of classical means , as well as new types of means .*

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If $a_1, a_2, \ldots, a_n$ are strictly positive real numbers – the following numerical means are well known in literature and mathematical practice :

$$A_n[a] := \frac{a_1 + a_2 + \ldots + a_n}{n} \qquad (\textit{arithmetic mean}), \tag{1}$$

$$G_n[a] := \sqrt[n]{a_1 \cdot a_2 \cdot \ \ldots \cdot a_n} \qquad (\textit{geometric mean}), \tag{2}$$

$$H_n[a] := \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} \qquad (\textit{harmonic mean}), \tag{3}$$

$$Q_n[a] := \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} \qquad (\textit{quadratic mean}), \tag{4}$$

associated with the vector $a = (a_1, a_2, \ldots, a_n) \in \mathbf{R}_+^n$, as well as the inequality between these means ,

$$H_n[a] \leqslant G_n[a] \leqslant A_n[a] \leqslant Q_n[a] \ . \tag{5}$$

Regarding the first three means above - it is also known and is already classic, the next double *inequality of Sierpinski*, published in 1909 in the work [7.] and resumed, demonstrated, commented or annotated in [1.], [2.], [3.], [5.], [6.] :

$$A_n[a] \cdot H_n^{n-1}[a] \le G_n^n[a] \le A_n^{n-1}[a] \cdot H_n[a] \ . \tag{6}$$

In the following we will demonstrate a *Sierpinski type inequality,* in which this time appear the means : *arithmetic, geometric, quadratic* , and the distribution of powers is now different .

In the demonstration we will use *Liouville's method* – reconsidered , named , studied and applied on several examples in [4.].

More precisely we will have the following,

1. *Proposition*

For any $n \in \mathbb{N}$, $n \geqslant 2$, $a_k > 0$, $k = \overline{1, n}$, and the vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n_+$,

there is the inequality, $$\mathrm{A}_n^{n+2}[a] \geq \mathrm{G}_n^n[a] \cdot \mathrm{Q}_n^2[a] \ , \qquad \mathbf{(7)}$$

with equality if and only if $a_1 = a_2 = \ \ldots \ = a_n$.

Proof

We will prove by mathematical induction. For $n = 2$, the inequality to be demonstrated is written,

$$\left(\frac{a_1 + a_2}{2}\right)^4 \geq a_1 a_2 \cdot \frac{a_1^2 + a_2^2}{2} \quad , \qquad \mathbf{(8)}$$

By dividing with a_2^4 and notation $a_1 / a_2 =: t$, we get , $\left(\frac{t+1}{2}\right)^4 \geq t \cdot \frac{t^2+1}{2}$, equivalent to inequality $(t-1)^4 \geq 0$. Equality occurs if $t = 1$ that is , if $a_1 = a_2$.

As in the inequality in the statement all the component expressions are homogeneous, we can assume - without reducing the generality, that

$$\mathrm{A}_n[a] = 1 \quad (\Leftrightarrow \quad a_1 + a_2 + \ldots + a_n = n) \ .$$

Under this condition, the inequality to be demonstrated comes back to,

$$\mathrm{G}_n^n[a] \cdot \mathrm{Q}_n^2[a] \leq 1 \ \Leftrightarrow \ a_1 \cdot a_2 \cdot \ \ldots \ \cdot a_n \cdot (a_1^2 + a_2^2 + \ldots + a_n^2) \leq n \ . \qquad \mathbf{(9)}$$

Suppose by induction that $\mathrm{G}_{n-1}^{n-1}[a] \cdot \mathrm{Q}_{n-1}^2[a] \leq \mathrm{A}_{n-1}^{n+1}[a]$, **(10)**

with equality if and only if $a_1 = a_2 = \ldots = a_{n-1} \ (= \alpha)$.

Considering $x =: a_n$ and noticing that :

$$\mathrm{A}_n[a] = \frac{(n-1) \cdot \mathrm{A}_{n-1}[a] + x}{n} \quad , \qquad \mathbf{(11)}$$

$$\mathrm{Q}_n^2[a] = \frac{(n-1) \cdot \mathrm{Q}_{n-1}^2[a] + x^2}{n} \quad , \qquad \mathbf{(12)}$$

$$\mathrm{G}_n^n[a] = \mathrm{G}_{n-1}^{n-1}[a] \cdot x \quad , \qquad \mathbf{(13)}$$

then the inequality to be demonstrated (10) - is written equivalently ,

$$\mathrm{G}_{n-1}^{n-1}[a] \cdot x \cdot [(n-1) \cdot \mathrm{Q}_{n-1}^2[a] + x^2] \leq n \quad , \qquad \mathbf{(14)}$$

and the condition relation $\mathrm{A}_n = 1$, as $(n-1) \cdot \mathrm{A}_{n-1}[a] + x = n$. **(15)**

Regarding inequality (14) , we associate the following *Liouville*'s (*type*) *function* ,

$$\mathrm{L}(x) := \mathrm{G}_{n-1}^{n-1}[a] \cdot x \cdot [(n-1) \cdot \mathrm{Q}_{n-1}^2[a] + x^2] \ . \qquad \mathbf{(16)}$$

We have - using the induction hypothesis , respectively the *GM-AM inequality* ,

$$\mathrm{L}(x) = \mathrm{G}_{n-1}^{n-1}[a] \cdot x \cdot [(n-1) \cdot \mathrm{Q}_{n-1}^2[a] + x^2] = (n-1) \cdot \mathrm{G}_{n-1}^{n-1}[a] \cdot \mathrm{Q}_{n-1}^2[a] \cdot x + \mathrm{G}_{n-1}^{n-1}[a] \cdot x^3 \leq$$

$$\leq (n-1) \cdot \mathrm{A}_{n-1}^{n+1}[a] \cdot x + \mathrm{A}_{n-1}^{n-1}[a] \cdot x^3 \ . \qquad \mathbf{(17)}$$

With substitution $\mathrm{A}_{n-1} =: y$, from (15) we will have $x = n - (n-1)\,y$, so the last expression in (17) becomes ,

$$\Lambda(y) := (n-1)\cdot y^{n+1}\cdot\left[n-(n-1)y\right] + y^{n-1}\cdot\left[n-(n-1)y\right]^3 \quad . \tag{18}$$

After some routine calculations, we have ,

$$\Lambda'(y) = (n-1)y^{n-2}\left\{(n+1)y^2\left[n-(n-1)y\right] - (n-1)y^3 + \left[n-(n-1)y\right]^3 - 3y\left[n-(n-1)y\right]^2\right\} =$$

$$= (n-1)n\,y^{n-2}(1-y)\left[(n^2+n-2)y^2 - 2(n^2-3n+3)y + n^2\right] .$$

How $y = A_{n-1} > 0$, then the only acceptable solution of the equation $\Lambda'(y) = 0$ for $n \geqslant 2$ is $y = 1$, which is also the abscissa of the absolute maximum point of the function Λ .

So we will have $\Lambda(y) \leqslant \Lambda(1) = n$.

Equality $y = 1$ involves $\alpha = 1$ and with (15) results $x = a_n$, thus $a_1 = a_2 = \ldots = a_n (= 1)$.

Also, according to (16), we have $L(x) \leqslant \Lambda(y) \leqslant n$, which ends the proof by induction of inequality (13) .

2. *Remark*

The inequality (7) cannot be completed up to a double inequality - such as inequality (6) , because dual inequality , $G_n^2[a]\cdot Q_n^n[a] \geq A_n^{n+2}[a]$ cannot occur, as can be easily observed, if we take $a_1 \to 0$, $a_2 = \ldots = a_n = 1$.

With the notation $\varphi(a) := (\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n))$, for φ a function defined on $(0, \infty)$, we have the following inequality for the means : $G_n[a]$, $H_n[a^2]$, $H_n[a]$,

3. *Corollary*

For any $n \in \mathbb{N}$, $n \geqslant 2$, $a_k > 0$, $k = \overline{1,n}$, and vectors $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$, $a^2 = (a_1^2, a_2^2, \cdots, a_n^2) \in \mathbb{R}_+^n$, there is the inequality ,

$$G_n^n[a]\cdot H_n[a^2] \geq H_n^{n+2}[a] , \tag{19}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

Proof

Everything results from inequality (7) , by substitution of vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$, with vector $1/a = (1/a_1, 1/a_2, \ldots, 1/a_n) \in \mathbb{R}_+^n$, noticing that

$$A_n[1/a] = 1/H_n[a] ,\; G_n[1/a] = 1/G_n[a] ,\; Q_n^2[1/a] = 1/H_n[a^2] .$$

In the following we will give two refinements of the inequality of the classical means $A_n[a] \geqslant G_n[a]$.

4. *Proposition*

If $n \in \mathbb{N}$, $n \geqslant 2$ and $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$, then the following inequalities occur :

$$a) \qquad G_n[a] \leq G_n[a]\cdot\left(\frac{Q_n[a]}{A_n[a]}\right)^{\frac{2}{n}} \leq A_n[a] , \tag{20}$$

***b*)** $$\mathbf{G}_n[a] \le \mathbf{A}_n[a]\cdot\left(\frac{\mathbf{A}_n[a]}{\mathbf{Q}_n[a]}\right)^{\frac{2}{n}} \le \mathbf{A}_n[a]\,, \tag{21}$$

with equality if and only if $a_1 = a_2 = \ \dots\ = a_n$.

Proof

***a*)** Using inequality $\mathbf{A}_n[a] \leqslant \mathbf{Q}_n[a]$ and inequality from *Proposition* 1 , we have :

$$\mathbf{G}_n^n[a]\cdot\mathbf{A}_n^2[a] \le \mathbf{G}_n^n[a]\cdot\mathbf{Q}_n^2[a] \le \mathbf{A}_n^{n+2}[a] \;\Leftrightarrow\; \mathbf{G}_n^n[a] \le \frac{\mathbf{G}_n^n[a]\cdot\mathbf{Q}_n^2[a]}{\mathbf{A}_n^2[a]} \le \mathbf{A}_n^n[a]\,,$$

hence the inequality in the statement ;

***b*)** Using inequality from *Proposition* 1 and inequality $\mathbf{A}_n[a] \leqslant \mathbf{Q}_n[a]$, we have :

$$\mathbf{G}_n^n[a]\cdot\mathbf{Q}_n^2[a] \le \mathbf{A}_n^{n+2}[a] = \mathbf{A}_n^n[a]\cdot\mathbf{A}_n^2[a] \le \mathbf{A}_n^n[a]\cdot\mathbf{Q}_n^2[a] \;\Leftrightarrow\; \mathbf{G}_n^n[a] \le \frac{\mathbf{A}_n^n[a]\cdot\mathbf{A}_n^2[a]}{\mathbf{Q}_n^2[a]} \le \mathbf{A}_n^n[a]\,,$$

hence the inequality in the statement.

5. *Corollary*

For any $n \in \mathbb{N}$, $n \geqslant 2$, $a_k > 0$, $k = \overline{1, n}$, and vector $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}_+^n$,

the expressions $\mathbf{G}_n[a]\cdot\left(\frac{\mathbf{Q}_n[a]}{\mathbf{A}_n[a]}\right)^{\frac{2}{n}}$, $\mathbf{A}_n[a]\cdot\left(\frac{\mathbf{A}_n[a]}{\mathbf{Q}_n[a]}\right)^{\frac{2}{n}}$ are *means* of numbers $a_1, a_2, \dots, a_n$.

Proof

The demonstration results from the previous sentence, because

$$\min_{1\le k\le n} a_k \le \mathbf{G}_n[a] \le \mathbf{G}_n[a]\cdot\left(\frac{\mathbf{Q}_n[a]}{\mathbf{A}_n[a]}\right)^{\frac{2}{n}} \le \mathbf{A}_n[a] \le \max_{1\le k\le n} a_k \,, \quad \text{etc.}$$

Similarly, we will highlight two refinements of the inequality of the classical means $\mathbf{H}_n[a] \leqslant \mathbf{G}_n[a]$.

6. *Proposition*

For any $n \in \mathbb{N}$, $n \geqslant 2$, $a_k > 0$, $k = \overline{1, n}$, and vectors $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}_+^n$, $a^2 = (a_1^2, a_2^2, \cdots, a_n^2) \in \mathbb{R}_+^n$, then the following inequalities occur :

***a*)** $$\mathbf{H}_n[a] \le \mathbf{G}_n[a]\cdot\left(\frac{\mathbf{H}_n[a^2]}{\mathbf{H}_n[a]}\right)^{\frac{2}{n}} \le \mathbf{G}_n[a]\,, \tag{22}$$

***b*)** $$\mathbf{H}_n[a] \le \mathbf{H}_n[a]\cdot\left(\frac{\mathbf{H}_n[a]}{\mathbf{H}_n[a^2]}\right)^{\frac{2}{n}} \le \mathbf{G}_n[a]\,, \tag{23}$$

with equality if and only if $a_1 = a_2 = \ \dots\ = a_n$.

Proof

Inequalities result from *Proposition 4*, by substitution of vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$, with vector $1/\boldsymbol{a} = (1/a_1, 1/a_2, \ldots, 1/a_n) \in \mathbb{R}_+^n$.

Also the demonstration can be obtained using the inequality from the relationship (19).

7. Corollary

For any $n \in \mathbb{N}$, $n \geqslant 2$, $a_k > 0$, $k = \overline{1, n}$, and vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$, $\boldsymbol{a}^2 = (a_1^2, a_2^2, \cdots, a_n^2) \in \mathbb{R}_+^n$, the expressions $\mathbf{G}_n[a] \cdot \left(\dfrac{\mathbf{H}_n[a^2]}{\mathbf{H}_n[a]} \right)^{\frac{2}{n}}$, $\mathbf{H}_n[a] \cdot \left(\dfrac{\mathbf{H}_n[a]}{\mathbf{H}_n[a^2]} \right)^{\frac{2}{n}}$ are *means* of numbers $a_1, a_2, \ldots, a_n$.

The ***Proof*** results from the previous *Proposition*.

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