An Advanced Multi-Faceted Inequality: Jensen in Banach Spaces, Orlicz Norms, Transportation-Cost Bounds, and Girsanov Transforms

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Introduction

In this work, we present a *deeply integrated* inequality that exploits classical and modern techniques from:

- **Functional Analysis:** Jensen-/Clarkson-type inequalities in Banach spaces, duality arguments, and Orlicz norms.
- **Probability Theory:** Advanced tail bounds via Talagrand's transportation-cost (or TCI) inequalities, plus Markov-type arguments.
- **Stochastic Analysis:** Girsanov's theorem to *change the underlying measure* and bound exponential moments or path deviations.
- **Measure Theory:** Subtle integrability criteria ensuring that all transformations remain well-defined.

Our final statement (Theorem 3.1, Section 3) illustrates how to derive a *unified* control of (1) the expectation of a convex functional of a random sum and (2) the tail probability under a measure change. This orchestrates multiple ideas in one place to yield a powerful global bound.

1 Setting and Notation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{X_i\}_{i=1}^n$ be random variables taking values in a (potentially infinite-dimensional) Banach space B. Formally,

$$X_i : \Omega \to B, \quad B \text{ is a separable Banach space with norm } \|\cdot\|.$$

Define

$$S_n \;=\; \sum_{i=1}^n X_i \;\in\; B.$$

We assume each X_i belongs to some *Orlicz space* $L^\Phi(\Omega; B)$, where Φ is a Young function (for example, $\Phi(x) = \exp(x^\alpha) - 1$, $\alpha > 1$), ensuring

$$\mathbb{E}\big[\Phi(\|X_i\|)\big] < \infty.$$

Likewise for S_n.

Let $f : B \to [0, \infty)$ be *strictly convex* and possibly *Gâteaux-differentiable*, subject to the integrability condition $\mathbb{E}[f(S_n)] < \infty$.

1.1 Dual Pairing and Banach-Space Jensen

A key extension of Jensen's inequality to Banach spaces uses the dual pairing $\langle b, b^* \rangle$ for $b \in B$ and $b^* \in B^*$. If f is strongly convex with parameter $\kappa > 0$, then for $x, y \in B$,

$$f(y) \;\geq\; f(x) + \langle f'(x),\, y - x \rangle \;+\; \frac{\kappa}{2}\|y - x\|^2.$$

Such an inequality—when combined with Bochner integration—yields advanced Banach-space versions of Jensen's inequality.

2 Intermediate Results

We collect three lemmas. The first concerns *Orlicz norms* and Minkowski-type inequalities, the second is a *Talagrand (transportation-cost)* bound, and the third is a *Girsanov transform* lemma for measure changes.

2.1 Lemma 1: A Minkowski–Orlicz Inequality in Banach Spaces

Lemma 2.1. *Let* $X_1, \ldots, X_n \in L^\Phi(\Omega; B)$*, and define* $S_n = \sum_{i=1}^n X_i$*. Under standard convexity assumptions on* Φ *(dominating a quadratic near 0 and growing sufficiently fast for large arguments), there exists a constant* $C_\Phi > 0$ *such that*

$$\|S_n\|_{L^\Phi(B)} \;=\; \inf\Big\{\lambda > 0 \;\Big|\; \mathbb{E}\Big[\Phi\Big(\tfrac{\|S_n\|}{\lambda}\Big)\Big] \le 1\Big\} \;\le\; C_\Phi \sum_{i=1}^n \|X_i\|_{L^\Phi(B)}.$$

Sketch of Proof. Combine classical Orlicz-space geometry with the triangle inequality in $L^\Phi(B)$. In simpler cases (e.g. $\Phi(x) = x^\alpha/\alpha$ for $\alpha > 1$), one obtains a bound akin to

$$\|X_1 + \cdots + X_n\|_{L^\alpha(B)} \;\le\; n^{1-1/\alpha}\left(\|X_1\|_{L^\alpha(B)} + \cdots + \|X_n\|_{L^\alpha(B)}\right),$$

but in the general Orlicz setting we get a constant C_Φ that depends on Φ. □

2.2 Lemma 2: A Talagrand-Type Transportation-Cost Bound

Lemma 2.2 (Talagrand Bound)**.** *Suppose* $\mathbb{P}$ *satisfies a transportation-cost inequality (TCI) with constant* $\tau > 0$ *on* B*, i.e. for all probability measures* $\nu \ll \mathbb{P}$*,*

$$\mathcal{T}_c(\nu, \mathbb{P})^2 \;\le\; 2\tau\, \mathcal{H}(\nu \mid \mathbb{P}),$$

where $\mathcal{T}_c$ *is the* L^2*-Wasserstein cost for* $c(x, y) = \|x - y\|^2$*, and* $\mathcal{H}(\nu|\mathbb{P})$ *is the relative entropy. Then for any* 1*-Lipschitz function* $\varphi : B \to \mathbb{R}$*,*

$$\mathbb{P}\{\varphi(S_n) \ge m\} \;\le\; \exp\Big(-\frac{m^2}{2\tau}\Big).$$

Sketch of Proof. A standard argument: If φ is 1-Lipschitz, then $\varphi(S_n)$ has sub-Gaussian tails under $\mathbb{P}$ by the TCI. Detailed treatments appear in works of Talagrand, Bobkov–Götze, and others on measure concentration. □

Remark 2.3. *In finite dimensions, such a TCI often arises from log-concave measures or Gaussian measures. In infinite dimensions, one typically uses Gaussian-like or product measures with additional properties (e.g. a Bakry–Émery criterion for curvature).*

2.3 Lemma 3: A Girsanov-Type Argument for Changing Measures

Lemma 2.4 (Girsanov–Kusuoka–Stroock). *Assume* $(\Omega, \mathcal{F}, \mathbb{P})$ *supports a cylindrical Brownian motion* $(W_t)_{t\geq 0}$ *in a reflexive Banach space* B*. Let* θ_t *be an adapted* B^**-valued process with* $\int_0^T \|\theta_t\|_{B^*}^2 \, dt < \infty$ *a.s. Then there is an equivalent probability measure* $\mathbb{Q} \sim \mathbb{P}$ *on* $(\Omega, \mathcal{F})$ *under which*

$$\widetilde{W}_t \;=\; W_t - \int_0^t \theta_s \, ds$$

is a cylindrical Brownian motion in B*, and*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} \;=\; \exp\Big(\int_0^T \langle \theta_s, \, dW_s \rangle \;-\; \tfrac{1}{2}\int_0^T \|\theta_s\|_{B^*}^2 \, ds\Big).$$

Sketch of Proof. This is essentially the Girsanov theorem in Banach spaces (Kusuoka–Stroock extension). The density arises from the exponential martingale associated with θ_t. □

Remark 2.5. *The measure change lets us* tilt *or re-center the distribution of* S_n *(when it can be embedded in a stochastic process), thereby improving certain bounds (e.g. for large deviation estimates).*

3 Main Inequality and Proof

We now combine all these tools into one statement.

Theorem 3.1 (A Comprehensive Inequality). *Let* $X_1, \ldots, X_n$ *be* B*-valued random variables in an Orlicz space* $L^\Phi(\Omega; B)$*. Define* $S_n = \sum_{i=1}^n X_i$*. Let* $f : B \to [0, \infty)$ *be strictly convex, Gâteaux-differentiable, and satisfy* $\mathbb{E}[f(S_n)] < \infty$*. Suppose:*

(i) ***(Banach-Jensen)*** f *is strongly convex on* B*, i.e. for* $\kappa > 0$*,*

$$f(y) \;\geq\; f(x) + \langle f'(x), \, y - x \rangle + \frac{\kappa}{2}\|y - x\|^2, \quad x, y \in B.$$

(ii) ***(Orlicz-Minkowski)*** $\|S_n\|_{L^\Phi(B)} \leq C_\Phi \sum_{i=1}^n \|X_i\|_{L^\Phi(B)}$*, as in Lemma 2.1.*

(iii) ***(Transportation-Cost Inequality)*** *Under* $\mathbb{P}$, S_n *satisfies Lemma 2.2 with some constant* τ.

(iv) ***(Measure Tilt)*** *If necessary, one can apply Lemma 2.4 to shift the distribution of* S_n.

Then for any $m > 0$ *and any* 1*-Lipschitz* $\varphi : B \to \mathbb{R}$, *the following hold:*

(1) Lower Bound:

$$\mathbb{E}\big[f(S_n)\big] \;\ge\; f\big(\mathbb{E}[S_n]\big) \;+\; \frac{\kappa}{2}\,\mathbb{E}\big[\|S_n - \mathbb{E}[S_n]\|^2\big].$$

(2) Upper Bound:

$$\mathbb{E}\big[f(S_n)\big] \;\le\; n\,f\Big(\tfrac{1}{n}\,\|S_n\|_{L^\Phi(B)}\Big) \;\le\; n\,f\Big(\tfrac{C_\Phi}{n}\,\sum_{i=1}^{n}\|X_i\|_{L^\Phi(B)}\Big).$$

(3) Talagrand Tail:

$$\mathbb{P}\big\{\varphi(S_n) \ge m\big\} \;\le\; \exp\Big(-\frac{m^2}{2\tau}\Big).$$

(4) Tilted Measure: *There exists* $\mathbb{Q} \sim \mathbb{P}$ *under which* S_n *can be re-centered by some drift* θ *(via Girsanov), improving bounds when* S_n *is inconveniently centered under* $\mathbb{P}$.

Proof Outline. **(1) Lower Bound:** Strong convexity of f implies

$$f(y) \;\ge\; f(x) + \langle f'(x), y - x\rangle + \frac{\kappa}{2}\|y - x\|^2.$$

Choose $x = \mathbb{E}[S_n]$ and $y = S_n$. Taking expectations, the cross term vanishes and we get

$$\mathbb{E}[f(S_n)] \;\ge\; f(\mathbb{E}[S_n]) + \frac{\kappa}{2}\,\mathbb{E}\big[\|S_n - \mathbb{E}[S_n]\|^2\big].$$

(2) Upper Bound: By a Banach-space extension of Jensen (or a simpler pointwise inequality),

$$f(S_n) = f\Big(n \cdot \frac{S_n}{n}\Big) \;\le\; n\,f\Big(\frac{S_n}{n}\Big).$$

Taking expectation and applying an integrability argument (e.g. $\mathbb{E}[f(\|Y\|)] \leq f(\|Y\|_{L^\Phi(B)})$ for a suitable monotone f) gives

$$\mathbb{E}[f(S_n)] \;\leq\; n\, f\Big(\|S_n/n\|_{L^\Phi(B)}\Big).$$

Then Lemma 2.1 implies

$$\Big\|\frac{S_n}{n}\Big\|_{L^\Phi(B)} \;\leq\; \frac{C_\Phi}{n} \sum_{i=1}^{n} \|X_i\|_{L^\Phi(B)},$$

hence

$$\mathbb{E}[f(S_n)] \;\leq\; n\, f\Big(\tfrac{C_\Phi}{n} \sum_{i=1}^{n} \|X_i\|_{L^\Phi(B)}\Big).$$

(3) Talagrand Tail: From Lemma 2.2, any 1-Lipschitz φ yields

$$\mathbb{P}\{\varphi(S_n) \geq m\} \;\leq\; \exp\Big(-\tfrac{m^2}{2\tau}\Big).$$

(4) Tilted Measure: If we wish to recenter S_n or impose a drift, Lemma 2.4 shows that we can define θ_t and switch from $\mathbb{P}$ to $\mathbb{Q}$, under which S_n (embedded in a cylindrical Brownian motion) has a different distribution. This can tighten certain bounds or facilitate large deviation estimates.

All parts together give the claimed inequalities. □

4 Concluding Remarks and Generalizations

1. Extensions to Orlicz–Bochner Spaces. We can work in more general Orlicz–Bochner or Lorentz–Bochner spaces, provided the triangle inequality and integrability conditions hold.

2. Logarithmic Sobolev Inequalities. Talagrand's TCI can be replaced or complemented by a Gross-type logarithmic Sobolev inequality to handle sub-Gaussian or sub-exponential tails in both finite and infinite dimensions.

3. Boué–Dupuis Variational Methods. Beyond Girsanov, one can use Boué–Dupuis representations for exponential functionals (common in path-space large deviations). This is often combined with measure-tilting for path-dependent random variables.

4. Applications in Stochastic PDEs. In PDE contexts (e.g. the 2D Navier–Stokes equation with random forcing), the mild solution is split into a stationary Gaussian plus a perturbation. Girsanov-based arguments then alter the forcing to derive exponential moment or concentration estimates, interacting neatly with TCI methods (via Hamilton–Jacobi or Kantorovich PDE interpretations).

Final Summary. Theorem 3.1 merges:

- **Banach-space Jensen** (lower/upper bounds via strong convexity),
- **Orlicz-Minkowski** inequalities for sums in L^{Φ},
- **Talagrand TCI** for concentration of S_n,
- **Girsanov measure tilt** for refined distribution manipulations.

This combination exemplifies how modern probability, functional analysis, geometry of Banach spaces, and measure theory converge to produce a *single, multi-pronged inequality* bounding both the mean behavior and the tails of complicated Banach-valued sums.

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