ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \frac{x\ln^2(x)}{1+x+x^2}dx$$

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$$Let\ I(a) = \int_0^1 \frac{x^a}{1+x+x^2}dx.\ Thus, I''(1) = \int_0^1 \frac{x\ln^2(x)}{1+x+x^2}dx$$

$$I(a) = \int_0^1 \frac{x^a(1-x)}{1-x^3}dx, \qquad I(a) = \int_0^1 (x^a - x^{a+1})\sum_{r=0}^{\infty} x^{3r}\,dx$$

$$I(a) = \sum_{r=0}^{\infty}\int_0^1 (x^{a+3r} - x^{a+1+3r})\,dx, \qquad I(a) = \sum_{r=0}^{\infty}\left(\frac{1}{3r+a+1} - \frac{1}{3r+a+2}\right)$$

$$I(a) = \frac{1}{3}\left(\psi\left(\frac{a+1}{3}\right) - \psi\left(\frac{a+2}{3}\right)\right), \qquad I'(a) = \frac{1}{9}\left(\psi_1\left(\frac{a+1}{3}\right) - \psi_1\left(\frac{a+2}{3}\right)\right)$$

$$I''(a) = \frac{1}{27}\left(\psi_2\left(\frac{a+1}{3}\right) - \psi_2\left(\frac{a+2}{3}\right)\right), \qquad I''(1) = \frac{1}{27}\left(\psi_2\left(\frac{2}{3}\right) - \psi_2(1)\right)$$

$$I''(1) = \frac{1}{27}\left(\frac{4\pi^3}{3\sqrt{3}} - 26\zeta(3) + 2\zeta(3)\right), \qquad I''(1) = \frac{1}{27}\left(\frac{4\pi^3}{3\sqrt{3}} - 24\zeta(3)\right)$$

$$\int_0^1 \frac{x\ln^2(x)}{1+x+x^2}dx = \frac{4\pi^3}{81\sqrt{3}} - \frac{8}{9}\zeta(3)$$