ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \ln(1+\tanh(x)+\coth(x))\,dx$$

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Solution by Pratham Prasad-India

$$\int_0^1 \ln(1+\tanh(x)+\coth(x))\,dx =$$

$$= \int_0^1 \ln\left(1+\frac{e^{2x}-1}{e^{2x}+1}+\frac{e^{2x}+1}{e^{2x}-1}\right)dx = \int_0^1 \ln\left(\frac{3e^{4x}+1}{e^{4x}-1}\right)dx =$$

$$= \ln(3)+\int_0^1 \ln\left(e^{4x}+1/3\right)-\ln\left(e^{4x}-1\right)dx = \ln(3)+\int_0^1\int_{-1}^{1/3}\frac{1}{e^{4x}+a}\,da\,dx =$$

$$= \ln(3)+\int_{-1}^{\frac{1}{3}}\int_0^1 \frac{1}{e^{4x}+a}\,dx\,da = \ln(3)-\int_{-1}^{\frac{1}{3}}\frac{1}{4a}\int_0^1\frac{-4ae^{-4x}}{1+ae^{-4x}}\,dx\,da =$$

$$= \ln(3)-\frac{1}{4}\int_{-1}^{\frac{1}{3}}\frac{1}{a}\left(\ln\left(1+ae^{-4}\right)-\ln(1+a)\right)da = \ln(3)+\frac{1}{4}\left(Li_2\left(-\frac{a}{e^4}\right)-Li_2(-a)\right)\Bigg|_{-1}^{\frac{1}{3}} =$$

$$= \ln(3)+\frac{1}{4}Li_2\left(-\frac{1}{3e^4}\right)-\frac{1}{4}Li_2\left(\frac{1}{e^4}\right)-\frac{1}{4}Li_2\left(-\frac{1}{3}\right)+\frac{1}{4}Li_2(1) =$$

$$= \ln(3)+\frac{1}{4}Li_2\left(-\frac{1}{3e^4}\right)-\frac{1}{4}Li_2\left(\frac{1}{e^4}\right)-\frac{1}{4}Li_2\left(-\frac{1}{3}\right)+\frac{1}{4}\zeta(2)$$