ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{y\sin(x)-1}{1+y\sin(x)}dxdy=2\left(Li_2\left(\frac{2-\pi-\sqrt{\pi^2-4}}{2}\right)+Li_2\left(\frac{2-\pi+\sqrt{\pi^2-4}}{2}\right)\right)$$

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$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{y\sin(x)-1}{1+y\sin(x)}dxdy=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}1-\frac{2}{1+y\sin(x)}dydx=$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-\frac{2}{\sin(x)}\ln\left(1+\frac{\pi}{2}\sin(x)\right)\right)dx=$$

$$=\frac{\pi^2}{4}-2\int_0^{\frac{\pi}{2}}\frac{\ln\left(1+\frac{\pi}{2}\sin(x)\right)}{\sin(x)}dx=\frac{\pi^2}{4}-2f\left(\frac{\pi}{2}\right)$$

$$Now,\ consider\ f(a)=\int_0^{\frac{\pi}{2}}\frac{\ln(1+a\sin(x))}{\sin(x)}dx\quad a\geq -1\Rightarrow\ f(0)=0$$

$$\Rightarrow f(1)=\int_0^{\frac{\pi}{2}}\frac{\ln(1+\sin(x))}{\sin(x)}dx\underbrace{=}_{\substack{x\to 2\tan^{-1}t\\ t=\tan\left(\frac{x}{2}\right)}}\int_0^1\frac{\ln\left(1+\frac{2t}{1+t^2}\right)}{\frac{2t}{1+t^2}}\frac{2}{1+t^2}dt=\int_0^1\frac{\ln\left(\frac{(1+t)^2}{1+t^2}\right)}{t}dt=$$

$$=2\int_0^1\frac{\ln(1+t)}{t}dt-\int_0^1\frac{\ln(1+t^2)}{t}dt=2\frac{\pi^2}{12}-\frac{1}{2}\int_0^1\frac{\ln(1+y)}{y}dy\ \ (by\ substituting\ y=t^2)$$

$$\Rightarrow f(1)=\frac{\pi^2}{6}-\frac{1}{2}\times\frac{\pi^2}{12}=\frac{\pi^2}{8}$$

$$f'(a)=\int_0^{\frac{\pi}{2}}\frac{1}{1+a\sin(x)}dx\underbrace{=}_{x\to\frac{\pi}{2}-x}\int_0^{\frac{\pi}{2}}\frac{1}{1+a\cos(x)}dx\underbrace{=}_{\substack{x\to 2\tan^{-1}t\\ t=\tan\left(\frac{x}{2}\right)}}\int_0^1\frac{2}{(1+a)+(1-a)t^2}dt$$

$$\Rightarrow f'(a)=\frac{2}{\sqrt{1-a^2}}\tan^{-1}\left(\sqrt{\frac{1-a}{1+a}}\right)=\frac{\cos^{-1}a}{\sqrt{1-a^2}}\quad for\ \ -1\le a\le 1$$

$$\Rightarrow f'(a)=\frac{2}{\sqrt{a^2-1}}\tanh^{-1}\left(\sqrt{\frac{a-1}{a+1}}\right)=\frac{\cosh^{-1}a}{\sqrt{a^2-1}}\quad for\quad a\ge 1$$

$$\Rightarrow f(a)=\frac{\pi^2}{8}-\frac{\left(\cos^{-1}a\right)^2}{2}\quad -1\le a\le 1\Rightarrow f(a)=\frac{\pi^2}{8}+\frac{\left(\cosh^{-1}a\right)^2}{2}\quad a\ge 1$$

$$\Rightarrow I=\frac{\pi^2}{4}-2f\left(\frac{\pi}{2}\right)=\frac{\pi^2}{4}-2\left(\frac{\pi^2}{8}+\frac{\left(\cosh^{-1}\left(\frac{\pi}{2}\right)\right)^2}{2}\right)=-\left(\cosh^{-1}\left(\frac{\pi}{2}\right)\right)^2=$$

$$=-\ln^2\left(\frac{\pi}{2}+\sqrt{\frac{\pi^2}{4}-1}\right)=-\ln^2\left(\frac{\pi+\sqrt{\pi^2-4}}{2}\right)$$

$$Let,\ \ z=\frac{\pi+\sqrt{\pi^2-4}}{2}\ \Rightarrow\ \frac{1}{z}=\frac{\pi-\sqrt{\pi^2-4}}{2}$$

$$Now,\ \ Li_2(1-z)+Li_2\left(1-\frac{1}{z}\right)=-\frac{\ln^2(z)}{2}\Rightarrow -\ln^2(z)=2\left(Li_2(1-z)+Li_2\left(1-\frac{1}{z}\right)\right)$$

$$\Rightarrow -\ln^2\left(\frac{\pi+\sqrt{\pi^2-4}}{2}\right)=2\left(Li_2\left(1-\frac{\pi+\sqrt{\pi^2-4}}{2}\right)+Li_2\left(1-\frac{\pi-\sqrt{\pi^2-4}}{2}\right)\right)$$

$$\Rightarrow I=2\left(Li_2\left(\frac{2-\pi-\sqrt{\pi^2-4}}{2}\right)+Li_2\left(\frac{2-\pi+\sqrt{\pi^2-4}}{2}\right)\right)$$