ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \frac{(1-x)(1-x^2)(1-x^3)}{(1+x^2)\ln x}dx = \ln\left(\frac{18\pi^2}{5\varpi^4}\right)$$

where ϖ is Lemniscate Constant

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$$I = \int_0^1 \frac{(1-x)(1-x^2)(1-x^3)}{(1+x^2)\ln x}dx = \int_0^1 \frac{(1-x)}{\ln x}\left(x^3 - 2x - 1 + \frac{2(1+x)}{1+x^2}\right)dx =$$

$$= \int_0^1 \frac{-x^4 + x^3 + 2x^2 - x - 1}{\ln x}dx - 2\int_0^1 \frac{x^2-1}{(1+x^2)\ln x}dx$$

$$= -\int_0^1 \frac{x^4-1}{\ln x}dx + \int_0^1 \frac{x^3-1}{\ln x}dx + 2\int_0^1 \frac{x^2-1}{\ln x}dx - \int_0^1 \frac{x-1}{\ln x}dx - 2\int_0^1 \frac{x^2-1}{(1+x^2)\ln x}dx$$

$$= -H(5) + H(4) + 2H(3) - H(2) - 2K(2)$$

$$here,\ \ H(a) = \int_0^1 \frac{x^{a-1}-1}{\ln x}dx = \int_{x=0}^{1}\int_{t=1}^{a} x^{t-1}\,dtdx = \int_{t=1}^{a}\int_{x=0}^{1} x^{t-1}\,dxdt = \int_1^a \frac{dt}{t} = \ln(a)$$

$$\Rightarrow I = -\ln 5 + \ln 4 + 2\ln 3 - \ln 2 - 2K(2) = \ln\left(\frac{18}{5}\right) - 2K(2)$$

$$And,\ \ K(a) = \int_0^1 \frac{x^a - 1}{(1+x^2)\ln x}dx \ \Rightarrow K(0) = 0$$

$$\Rightarrow K'(a) = \int_0^1 \frac{x^a}{1+x^2}dx = \int_0^1 \frac{x^a - x^{a+2}}{1-x^4}dx \underbrace{=}_{x\to x^{\frac{1}{4}}} \frac{1}{4}\int_0^1 \frac{x^{\frac{a-3}{4}} - x^{\frac{a-1}{4}}}{1-x}dx =$$

$$= \frac{1}{4}\left(\psi\left(\frac{a+3}{4}\right) - \psi\left(\frac{a+1}{4}\right)\right)$$

$$\Rightarrow K(a) = \frac{1}{4}\int_0^a \left(\psi\left(\frac{a+3}{4}\right) - \psi\left(\frac{a+1}{4}\right)\right)da = \left[\ln\left(\frac{\Gamma\left(\frac{a+3}{4}\right)}{\Gamma\left(\frac{a+1}{4}\right)}\right)\right]_0^a = \ln\left(\frac{\Gamma\left(\frac{a+3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{a+1}{4}\right)\Gamma\left(\frac{3}{4}\right)}\right)$$

$$\Rightarrow K(2) = \ln\left(\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{3}{4}\right)}\right) = \ln\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\Gamma^2\left(\frac{3}{4}\right)}\right) = \ln\left(\frac{\Gamma^4\left(\frac{1}{4}\right)}{4\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(\frac{1}{4}\right)}\right) = \ln\left(\frac{\Gamma^4\left(\frac{1}{4}\right)}{4\left(\pi\sqrt{2}\right)^2}\right) = \ln\left(\frac{\Gamma^4\left(\frac{1}{4}\right)}{8\pi^2}\right)$$

$$Now,\ \varpi = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}} \Rightarrow \varpi^2 = \frac{\Gamma^4\left(\frac{1}{4}\right)}{8\pi} \Rightarrow K(2) = \ln\left(\frac{\varpi^2}{\pi}\right) \Rightarrow I = \ln\left(\frac{18}{5}\right) - 2\ln\left(\frac{\varpi^2}{\pi}\right) = \ln\left(\frac{18\pi^2}{5\varpi^4}\right)$$