ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{dxdy}{\sin^2(x)+\cos(y)}$$

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$$\Omega=\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\frac{dx}{\sin^2(x)+\cos(y)}\right)dy=\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\frac{\frac{dx}{\cos^2(x)}}{\frac{\sin^2(x)}{\cos^2(x)}+\frac{\cos(y)}{\cos^2(x)}}\right)dy=$$

$$=\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\frac{d(tanx)}{(tanx)^2+cosy(1+\tan^2(x))}\right)dy\overset{\tan(x)=t}{\cong}\int_0^{\frac{\pi}{2}}\left(\int_0^{\infty}\frac{dx}{cos(y)+(1+\cos(y))t^2}\right)dy=$$

$$=\int_0^{\frac{\pi}{2}}\frac{dy}{\sqrt{\cos(y)\,(1+\cos(y))}}\left(\int_0^{\infty}\frac{d\sqrt{\frac{1+\cos(y)}{\cos(y)}}t}{1+\left(\sqrt{\frac{1+\cos(y)}{\cos(y)}}t\right)^2}\right)dx=$$

$$\int_0^{\frac{\pi}{2}}\frac{dy}{\sqrt{\cos(y)\,(1+\cos(y))}}\left(arctan\sqrt{\frac{1+\cos(y)}{\cos(y)}}t\right)\Bigg|_0^{\infty}=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sqrt{\cos(y)(1+\cos(y))}}\right)dy=$$

$$\overset{\tan\left(\frac{y}{2}\right)=t}{\cong}\frac{\pi}{2}\int_0^1\frac{\left(\frac{2}{1+t^2}\right)}{\sqrt{\frac{1-t^2}{1+t^2}\cdot\frac{2}{1+t^2}}}dt==\frac{\pi}{2}\sqrt{2}\int_0^1\frac{1}{\sqrt{1-t^2}}dt=\frac{\pi}{2}\sqrt{2}\arcsin(1)=\frac{\pi^2}{2\sqrt{2}}$$