ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1\int_0^1 \frac{\arctan(x+y)}{x+y}\,dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution by Quadri Faruk Temitope-Nigeria

$$\textit{Using integration by parts, IBP we write:}$$

$$I=\int_0^1 [\ln(x+y)\,arctan(x+y)|_0^1-\int_0^1\frac{\ln(x+y)}{(x+y)^2+1}dy]dx$$

$$\textit{Let } I=A-B \textit{ where:}$$

$$A=\int_0^1 \ln(x+1)\arctan(x+1)\,dx;\ B=\int_0^1\int_0^1\frac{\ln(x+y)}{(x+y)^2+1}dxdy$$

$$\textit{Applying IBP again to } A, \textit{we get:}$$

$$A=[(x\ln(x)-x)\,arctan(x)|_1^2-\int_1^2\frac{x\ln(x)}{x^2+1}dx+\int_1^2\frac{x}{x^2+1}dx$$

$$\textit{After simplifications:}$$

$$A=2\ln(2)\arctan(2)-2\arctan(2)+\frac{\pi}{4}-\sum_{n=0}^{\infty}(-1)^n\int_1^2 x^{2n+1}\ln(x)\,dx+\left\{\frac{1}{2}\ln(x^2+1)\right\}_1^2$$

$$A=\pi\ln(2)-2\arctan\left(\frac{1}{2}\right)\ln(2)-\pi+2\arctan\left(\frac{1}{2}\right)+\frac{\pi}{4}-\frac{1}{4}Li_2(-4)-\frac{1}{2}\ln(2)\ln(5)+\frac{1}{2}\ln(2)-\frac{\pi^2}{48}$$

$$\textit{Using the identity: } \Im\left|\frac{i}{(x+y)+i}\right|=\frac{i}{(x+y)^2+1}\ \textit{we write:}$$

$$B=\Im\int_0^1\int_0^1\frac{\ln(x+y)}{(x+y)+i}dxdy$$

$$B=\Im(Li_2(i(x+1))-Li_2(ix)+\ln(1+x)\ln(-i(1+x-i))-\ln(x)\ln(-i(x+i))\,dx$$

$$B=-\left(-\frac{\pi}{2}+2arctan(2)-\frac{\ln(5)}{2}-2G+\pi\ln(2)-\frac{1}{2}\ln(2)\right)$$

$$I=A-B=2\Im Li_2\left(\frac{1}{2i}\right)-2G+\pi\ln(2)-\frac{\pi}{2}+2arctan\left(\frac{1}{2}\right)+\frac{\ln(5)}{2}-\ln(2)$$