ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 \int_0^1 \frac{\ln(x)\ln(xy)}{1-xy}\,dxdy$$

Proposed by Aryan Desai-India

Solution by Pham Duc Nam-Vietnam

$$I = \int_0^1 \int_0^1 \frac{\ln(x)\ln(xy)}{1-xy}\,dxdy \overset{symmetry}{=} \int_0^1 \int_0^1 \frac{\ln(y)\ln(xy)}{1-xy}\,dxdy$$

$$2I = \int_0^1 \int_0^1 \frac{ln^2(xy)}{1-xy}\,dxdy \overset{symmetry}{=} -\int_0^1 \frac{ln^3(x)}{1-x}\,dx =$$

$$= -\sum_{n=0}^{\infty} \int_0^1 x^n\, ln^3(x)dx = 6\sum_{n=0}^{\infty} \frac{1}{(n+1)^4} = 6\zeta(4) = \frac{\pi^4}{30}$$