ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 \frac{\ln(y)Li_2(y)}{1-y}dy$$

Proposed by Aryan Desai-India

Solution by Pham Duc Nam-Vietnam

$$I=\int_0^1 \frac{\ln(y)Li_2(y)}{1-y}dy=-\int_0^1 \frac{xln(x)}{1-x}dx\left(\int_0^1 \frac{\ln(t)}{1-xt}dt\right)$$

$$I=\int_0^1 \ln(t)\,dt(\zeta(2)\frac{1}{1-t}-Li_2(t)(\frac{1}{1-t}+\frac{1}{t}))$$

$$I=\zeta(2)\int_0^1 \frac{\ln(t)}{1-t}dt-I-\int_0^1 \frac{\ln(t)Li_2(t)}{t}dt$$

$$I=-\frac{1}{2}\zeta^2(2)-\frac{1}{2}\left(\ln(t)\,Li_3(t)|_0^1-Li_4(t)|_0^1\right)$$

$$I=-\frac{1}{2}\zeta^2(2)+\frac{1}{2}Li_4(1)=\frac{1}{2}\zeta(4)-\frac{1}{2}\zeta^2(2)=-\frac{\pi^4}{120}$$