ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1\int_0^1 \frac{(a-x^2y^2)dxdy}{(1+x)(1+y)(1+x^2y^2)}$$

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$$\int_0^1\int_0^1 \frac{(a-x^2y^2)dxdy}{(1+x)(1+y)(1+x^2y^2)}=$$

$$=\int_0^1\int_0^1 \frac{(a+1)dxdy}{(1+x)(1+y)(1+x^2y^2)}-\int_0^1\int_0^1 \frac{dxdy}{(1+x)(1+y)}$$

$$=(a+1)\int_0^1\int_0^1 \frac{dxdy}{(1+x)(1+y)(1+x^2y^2)}-\ln^2(2)$$

$$=(a+1)\int_0^1\left(\int_0^1 \frac{dx}{(1+x)(1+y)(1+x^2y^2)}\right)dy-\ln^2(2)$$

$$=(a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\int_0^1 \frac{\left((1+x^2y^2)+y^2(1-x)(1+x)\right)dx}{(1+x)(1+x^2y^2)}\right)dy-\ln^2(2)$$

$$=(a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\int_0^1 \frac{1}{1+x}+\frac{y^2}{1+x^2y^2}-\frac{y^2x}{1+x^2y^2}dx\right)dy-\ln^2(2)$$

$$=(a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\int_0^1 \frac{1}{1+x}dx+\int_0^1 \frac{y^2}{1+x^2y^2}dx-\int_0^1 \frac{y^2x}{1+x^2y^2}dx\right)dy-\ln^2(2)$$

$$=(a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\ln(2)+y\int_0^1 \frac{1}{1+x^2y^2}d(xy)-\frac{1}{2}\int_0^1 \frac{1}{1+x^2y^2}d(x^2y^2)\right)dy$$
$$-\ln^2(2)$$

$$=(a+1)\int_0^1 \frac{1}{(1+y)(1+y^2)}\left(\ln(2)+y\arctan(y)-\frac{1}{2}\ln(1+y^2)\right)dy-\ln^2(2)$$

$$=(a+1)\int_0^1 \frac{\ln(2)}{(1+y)(1+y^2)}dy+(a+1)\int_0^1 \frac{\mathrm{y}\arctan(\mathrm{y})}{(1+y)(1+y^2)}dy$$
$$-\frac{1}{2}(a+1)\int_0^1 \frac{\ln(1+y^2)}{(1+y)(1+y^2)}dy-\ln^2(2)$$

$$= \frac{(a+1)}{2}\ln(2)\int_0^1 \frac{1}{(1+y)}dy + \frac{(a+1)}{2}\ln(2)\int_0^1 \frac{1}{(1+y^2)}dy - \frac{(a+1)}{4}\ln(2)\int_0^1 \frac{2y}{(1+y^2)}dy$$
$$+\frac{1}{2}(a+1)\int_0^1 \frac{\arctan(y)}{(1+y^2)}dy + \frac{1}{2}(a+1)\int_0^1 \frac{y\arctan(y)}{(1+y^2)}dy$$
$$-\frac{1}{2}(a+1)\int_0^1 \frac{\arctan(y)}{(1+y)}dy - \frac{1}{4}(a+1)\int_0^1 \frac{\ln(1+y^2)}{(1+y^2)}dy$$
$$+\frac{1}{4}(a+1)\int_0^1 \frac{y\ln(1+y^2)}{(1+y^2)}dy - \frac{1}{4}(a+1)\int_0^1 \frac{\ln(1+y^2)}{(1+y)}dy - \ln^2(2)$$

$$= \frac{(a+1)}{2}\ln(2)\,I_1 + \frac{(a+1)}{2}\ln(2)\,I_2 - \frac{(a+1)}{4}\ln(2)\,I_3 + \frac{1}{2}(a+1)I_4 + \frac{1}{2}(a+1)I_5$$
$$-\frac{1}{2}(a+1)I_6 - \frac{1}{4}(a+1)I_7 + \frac{1}{4}(a+1)I_8 - \frac{1}{4}(a+1)I_9 - \ln^2(2)$$

$$= \frac{(a+1)}{2}\ln(2)\,(\ln(2)) + \frac{(a+1)}{2}\ln(2)\left(\frac{\pi}{4}\right) - \frac{(a+1)}{4}\ln(2)(\ln(2)) + \frac{1}{2}(a+1)\left(\frac{\pi^2}{32}\right)$$
$$+\frac{1}{2}(a+1)\left(\frac{1}{8}(4G-\pi\ln(2))\right) - \frac{1}{2}(a+1)\left(\frac{\pi}{8}\ln(2)\right)$$
$$-\frac{1}{4}(a+1)\left(\frac{\pi}{2}\ln(2)-G\right) + \frac{1}{4}(a+1)\left(\frac{1}{4}\ln^2(2)\right) - \frac{1}{4}(a+1)\left(\frac{3}{4}\ln^2(2)-\frac{\pi^2}{48}\right)$$
$$-\ln^2(2)$$

$$= \frac{1}{8}(a+1)\left(\pi\ln(2) + 2\ln^2(2) + \frac{\pi^2}{8} + 2G - \frac{\pi}{2}\ln(2) - \frac{\pi}{2}\ln(2) - \pi\ln(2) + 2G + \frac{1}{2}\ln^2(2)\right.$$
$$\left.-\frac{3}{2}\ln^2(2) + \frac{\pi^2}{24}\right) - \ln^2(2)$$

$$= \frac{1}{8}(a+1)\left(-\pi\ln(2) + \ln^2(2) + \frac{\pi^2}{6} + 4G\right) - \ln^2(2)$$

$$I_1 = \int_0^1 \frac{1}{(1+y)}dy = [\ln(y+1)]_0^1 = \ln(2)$$

$$I_2 = \int_0^1 \frac{1}{(1+y^2)}dy = \arctan(1) = \frac{\pi}{4}$$

$$I_3 = \int_0^1 \frac{2y}{(1+y^2)}dy \overset{y^2=x}{\triangleq} \int_0^1 \frac{1}{(1+x)}dx \overset{x=y}{\triangleq} \int_0^1 \frac{1}{(1+y)}dy = I_1 = \ln(2)$$

$$I_4 = \int_0^1 \frac{\arctan(y)}{(1+y^2)}dy \overset{\arctan(y)=t}{\triangleq} \int_0^{\frac{\pi}{4}} t\,dt = \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{32}$$

$$I_5 = \int_0^1 \frac{y\arctan(y)}{(1+y^2)}dy = \left[\frac{1}{2}\arctan(y)\ln(1+y^2)\right]_0^1 - \frac{1}{2}\int_0^1 \frac{\ln(1+y^2)}{(1+y^2)}dy =$$

$$= \frac{\pi}{8}\ln(2) - \frac{1}{2}\int_0^1 \frac{\ln(1+y^2)}{(1+y^2)}dy = \frac{\pi}{8}\ln(2) - \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(sec^2(x))\,dx =$$

$$=\frac{\pi}{8}\ln(2)+\int_0^{\frac{\pi}{4}}\ln(cosx)dx=\frac{\pi}{8}\ln(2)+\frac{G}{2}-\frac{\pi}{4}\ln(2)=\frac{1}{8}(4G-\pi\ln(2))$$

$$I_5=\frac{\pi}{8}\ln(2)+\int_0^{\frac{\pi}{4}}\ln(cosx)dx=\frac{\pi}{8}\ln(2)+\frac{G}{2}-\frac{\pi}{4}\ln(2)=\frac{1}{8}(4G-\pi\ln(2))$$

$$I_6=\int_0^1\frac{\arctan(y)}{(1+y)}dy=[\ln(1+y)\arctan(y)]_0^1-\int_0^1\frac{\ln(1+y)}{1+y^2}dy$$

$$=\frac{\pi}{4}\ln(2)-\int_0^{\frac{\pi}{4}}\ln(siny+cosy)\,dy+\int_0^{\frac{\pi}{4}}\ln(cosy)\,dy$$

$$=\frac{\pi}{4}\ln(2)-\int_0^{\frac{\pi}{4}}\ln\left(\sqrt{2}\cos\left(\frac{\pi}{4}-y\right)\right)dy+\int_0^{\frac{\pi}{4}}\ln(cosy)dy$$

$$=\frac{\pi}{8}\ln(2)-\int_0^{\frac{\pi}{4}}\ln\left(\cos\left(\frac{\pi}{4}-y\right)\right)dy+\int_0^{\frac{\pi}{4}}\ln(cosy)dy$$

$$=\frac{\pi}{8}\ln(2)-\int_0^{\frac{\pi}{4}}\ln(\cos(y))\,dy+\int_0^{\frac{\pi}{4}}\ln(cosy)dy=\frac{\pi}{8}\ln(2)$$

$$I_6=\frac{\pi}{8}\ln(2)-\int_0^{\frac{\pi}{4}}\ln(\cos(y))\,dy+\int_0^{\frac{\pi}{4}}\ln(cosy)dy=\frac{\pi}{8}\ln(2)$$

$$I_7=\int_0^1\frac{\ln(1+y^2)}{(1+y^2)}dy=\int_0^{\frac{\pi}{4}}\ln(sec^2(x))\,dx=-2\int_0^{\frac{\pi}{4}}\ln(cosx)dx=\frac{\pi}{2}\ln(2)-G$$

$$I_8=\int_0^1\frac{y\ln(1+y^2)}{(1+y^2)}dy\overset{y^2=y}{\cong}\frac{1}{2}\int_0^1\frac{\ln(1+y)}{(1+y)}dy\overset{y^2=y}{\cong}\frac{1}{4}\ln^2(2)$$

$$I_9=\int_0^1\frac{\ln(1+y^2)}{(1+y)}dy$$

$$I_9=\int_0^1\frac{\ln(1+y^2)}{(1+y)}dy=\left[\ln(1+y)\ln(1+y^2)\right]_0^1-2\int_0^1\frac{y\ln(1+y)}{(1+y^2)}dy$$

$$=\ln^2(2)-2\int_0^1\frac{y\ln(1+y)}{(1+y^2)}dy=\ln^2(2)-2\left(\frac{\pi^2}{96}+\frac{1}{8}\log^2(2)\right)=\frac{3}{4}\ln^2(2)-\frac{\pi^2}{48}$$

$$I_9=\ln^2(2)-2\int_0^1\frac{y\ln(1+y)}{(1+y^2)}dy=\ln^2(2)-2\left(\frac{\pi^2}{96}+\frac{1}{8}\log^2(2)\right)=\frac{3}{4}\ln^2(2)-\frac{\pi^2}{48}$$