ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega = \int_0^{\pi} \ln^2(1+cosx)\,dx$$

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$$\Omega = \int_0^{\pi} \ln^2(1+cosx)\,dx = \int_0^{\pi} \ln^2\left(2\cos^2\left(\frac{x}{2}\right)\right)dx \overset{y=\frac{x}{2}}{\triangleq} 2\int_0^{\frac{\pi}{2}} \ln^2\left(2\cos^2(y)\right)dy =$$

$$= 2\int_0^{\frac{\pi}{2}} (\ln(2)+2\ln(cosy))^2 dy$$

$$= \pi\ln^2(2) + 8\int_0^{\frac{\pi}{2}} (\ln(cosy))^2 dy + 8ln(2)\int_0^{\frac{\pi}{2}} \ln(cosy)\,dy$$

$$\overset{y=\frac{\pi}{2}-y}{\triangleq} \pi\ln^2(2) + 4\underbrace{\int_0^{\pi} (\ln(siny))^2 dy}_{I_2} + 8ln(2)\underbrace{\int_0^{\frac{\pi}{2}} \ln(siny)\,dy}_{I_1}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \ln(siny)\,dy, \quad I_1 \overset{y=\frac{\pi}{2}-y}{\triangleq} \int_0^{\frac{\pi}{2}} \ln(cosy)\,dy$$

$$2I_1 = \int_0^{\frac{\pi}{2}} \ln(sinycosy)\,dy, \quad 2I_1 = \int_0^{\frac{\pi}{2}} \ln(\sin(2y))\,dy - \frac{\pi}{2}\ln(2)$$

$$2I_1 = \frac{1}{2}\int_0^{\pi} \ln(\sin(x))\,dx - \frac{\pi}{2}\ln(2), \quad 2I_1 = \int_0^{\frac{\pi}{2}} \ln(\sin(x))\,dx - \frac{\pi}{2}\ln(2)$$

$$2I_1 = I_1 - \frac{\pi}{2}\ln(2), \quad I_1 = -\frac{\pi}{2}\ln(2)$$

$$I_2 = Re\int_0^{\pi} (\ln(siny))^2 dy, \quad I_2 \overset{y=\frac{\theta}{2}}{\triangleq} \frac{1}{2}Re\int_0^{2\pi} \left(\ln\left(\sin\left(\frac{\theta}{2}\right)\right)\right)^2 dy$$

$$I_2 = \frac{1}{2}Re\int_0^{2\pi} \left[\log\left(1-e^{i\theta}\right) - \ln(2) - \frac{i}{2}(\theta-\pi)\right]^2 dy$$

$$I_2 = \frac{1}{2} Re \int_0^{2\pi} \log^2(2) - \frac{1}{4}(\theta - \pi)^2 \, d\theta + \frac{1}{2} Re \int_0^{2\pi} -i\log\left(1 - e^{i\theta}\right)(\theta - \pi) d\theta$$

$$+ \frac{1}{2} Re \int_0^{2\pi} \log^2\left(1 - e^{i\theta}\right) - 2\log(2)\log\left(1 - e^{i\theta}\right) d\theta$$

$$I_2 = \frac{1}{2}\left(2\pi \log^2(2) - \frac{\pi^3}{6}\right) + \frac{1}{2}\left(\frac{\pi^3}{3}\right) + \frac{1}{2} Re \oint_{|z|=1} \left(-\frac{i}{z}\right) \left(\log^2(1 - z) - 2\log(2)\log(1 - z)\right) \, dz$$

$$I_2 = \frac{1}{2}\left(2\pi \log^2(2) + \frac{\pi^3}{6}\right)$$

$$\boxed{\int_0^{\pi} \ln^2(1 + cosx) \, dx = \pi \ln^2(2) + 2\pi\zeta(2)}$$