ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))}dx$$

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$$Let's\ simplify\ the\ subintegral\ expression:$$

$$\frac{\sin(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))}=\frac{\sin(x)\,(1+\cos(x))}{\cos(x)(1+\sin(x))(1+\cos(x))}=\frac{\sin(x)}{\cos(x)\,(+\sin(x))}=$$

$$\frac{\tan(x)}{1+\sin(x)};\ \left\{Let: tan\frac{x}{2}=t\ .\ Then\ tan(x)=\frac{2t}{1-t^2};\ \sin(x)=\frac{2t}{1+t^2},\qquad dx=\frac{2}{1-t^2}dt\right\}$$

$$\int_0^{\frac{\pi}{4}}\frac{\sin(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))}dx=\int_0^{\frac{\pi}{4}}\frac{\tan(x)}{(1+\sin(x))}dx\overset{t=tan\frac{x}{2}}{\triangleq}\int_0^{tan\frac{\pi}{8}}\frac{4t}{(1-t)(1+t)^3}dt$$

$$From:\ \frac{4t}{(1-t)(1+t)^3}=\frac{A}{1-t}+\frac{B}{1+t}+\frac{C}{(1+t)^2}+\frac{D}{(1+t)^3}$$

$$A(1+t)^3+B(1-t)(1+t)^2+C(1-t)(1+t)+D(1-t)=4t$$

$$Let's\ simplify\ .\ We\ \ get$$

$$\begin{cases}A-B=0\\3A-B-C=0\\3A+B-D=0\\A+B+C+D=0\end{cases}\rightarrow\ \begin{cases}A-B=0\\7A+B=4\end{cases}\rightarrow\ A=\frac{1}{2};\ B=\frac{1}{2};\ \ C=1\,;D=-2$$

$$Then\ \ :\ \frac{4t}{(1-t)(1+t)^3}=\frac{\frac{1}{2}}{1-t}+\frac{\frac{1}{2}}{1+t}+\frac{1}{(1+t)^2}-\frac{2}{(1+t)^3}$$

$$\int_0^{tan\frac{\pi}{8}}\frac{4t}{(1-t)(1+t)^3}dt=\frac{1}{2}\int_0^{tan\frac{\pi}{8}}\frac{dt}{1-t}+\frac{1}{2}\int_0^{tan\frac{\pi}{8}}\frac{dt}{1+t}+\int_0^{tan\frac{\pi}{8}}\frac{dt}{(1+t)^2}-2\int_0^{tan\frac{\pi}{8}}\frac{dt}{(1+t)^3}=$$

$$\frac{1}{2}ln\left|\frac{1+t}{1-t}\right|\Bigg|_0^{tan\frac{\pi}{8}}-\frac{1}{(1+t)}\Bigg|_0^{tan\frac{\pi}{8}}+\frac{1}{(1+t)^3}\Bigg|_0^{tan\frac{\pi}{8}}=$$

$$\frac{1}{2}\left(ln\left|\frac{1+\sqrt{2}-1}{2-\sqrt{2}}\right|-\ln(1)\right)-\left(\frac{1}{1+\sqrt{2}-1}-1\right)+\left(\frac{1}{\left(1+\sqrt{2}-1\right)^2}-1\right)=$$

$$\frac{1}{2}\ln(1+\sqrt{2})-\frac{\sqrt{2}}{2}+1+\frac{1}{2}-1=\frac{1}{2}\left(\ln(1+\sqrt{2})-\sqrt{2}+1\right)$$