ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\iint\limits_{[0,1]^2}\left(\frac{x+y}{\ln(xy)}-\tan^{-1}\left(\frac{y^2}{x}\right)\right)dxdy=\frac{1}{3\sqrt{2}}\sinh^{-1}(1)-\frac{1}{\sqrt{2}}\coth^{-1}(\sqrt{2})-\frac{13}{6}\ln(2)+\pi\left(\frac{1}{3\sqrt{2}}-\frac{1}{4}\right)$$

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$$\Omega=\underbrace{\iint\limits_{[0,1]^2}\frac{x+y}{\ln(xy)}dxdy}_{I_1}-\underbrace{\iint\limits_{[0,1]^2}\tan^{-1}\left(\frac{y^2}{x}\right)dxdy}_{I_2}=$$

$$I_1\underbrace{=}_{\substack{x=e^{-u}\\y=e^{-v}}}-\int_0^\infty\int_0^\infty\frac{(e^{-u}+e^{-v})}{u+v}e^{-u-v}dudv=-2\int_0^\infty\int_0^\infty\frac{e^{-2u-v}}{u+v}dudv=$$

$$=-2\int_0^\infty\left(\int_0^\infty\frac{e^{-2u-v}}{u+v}dv\right)du\underbrace{=}_{v\to ut}-2\int_0^\infty\left(\int_0^\infty\frac{e^{-(t+2)u}}{1+t}dt\right)du=$$

$$=-2\int_0^\infty\int_0^\infty\frac{e^{-(t+2)u}}{1+t}dudt=-2\int_0^\infty\frac{1}{(1+t)(2+t)}dt=-2\ln2$$

$$I_2=\int_0^1\left(\int_0^1\tan^{-1}\left(\frac{y^2}{x}\right)dx\right)dy\underbrace{=}_{x\to Xy^2}\int_0^1 y^2\left(\int_0^{\frac{1}{y^2}}\tan^{-1}\left(\frac{1}{X}\right)dX\right)dy=$$

$$\underbrace{=}_{y\to\frac{1}{Y}}\int_1^\infty\frac{1}{Y^4}\left(\int_0^{Y^2}\tan^{-1}\left(\frac{1}{X}\right)dX\right)dY=\int_1^\infty\frac{1}{Y^4}\left(\int_0^{Y^2}\frac{\pi}{2}-\tan^{-1}(X)dX\right)dY=$$

$$=\frac{\pi}{2}\int_1^\infty\frac{1}{Y^2}dY-\int_1^\infty\frac{F(Y^2)}{Y^4}dY=\frac{\pi}{2}-\int_1^\infty\frac{F(Y^2)}{Y^4}dY$$

$$\text{here, } F(Y^2)=\int_0^{Y^2}\tan^{-1}(X)dX\ \Rightarrow F(Y)=\int_0^Y\tan^{-1}(X)dX=Y\tan^{-1}(Y)-\frac{1}{2}\ln(1+Y^2)$$

$$\int_1^\infty\frac{F(Y^2)}{Y^4}dY\underbrace{=}_{IBP}\frac{1}{3}F(1)+\frac{1}{3}\int_1^\infty\frac{2Y}{Y^3}F'(Y^2)dY=\frac{1}{3}\left(\frac{\pi}{4}-\frac{\ln2}{2}\right)+\frac{2}{3}\int_1^\infty\frac{\tan^{-1}(Y^2)}{Y^2}dY$$

$$\Rightarrow I_2 = \frac{\pi}{2} - \frac{1}{3}\left(\frac{\pi}{4} - \frac{\ln 2}{2}\right) - \frac{2}{3}\int_1^\infty \frac{\tan^{-1}(Y^2)}{Y^2}dY$$

$$Now,\ \int_1^\infty \frac{\tan^{-1}(Y^2)}{Y^2}dY \underbrace{=}_{IBP} \frac{\pi}{4} + \int_1^\infty \frac{2}{1+Y^4}dY$$

$$\Rightarrow I_2 = \frac{\pi}{2} - \frac{1}{3}\left(\frac{\pi}{4} - \frac{\ln 2}{2}\right) - \frac{2}{3}\left(\frac{\pi}{4}\right) - \frac{2}{3}\int_1^\infty \frac{2}{1+Y^4}dY = \frac{\pi}{4} + \frac{\ln 2}{6} - \frac{2}{3}\int_1^\infty \frac{2}{1+Y^4}dY$$

$$\int_1^\infty \frac{2}{1+Y^4}dY = \int_1^\infty \frac{1+\frac{1}{Y^2} - \left(1-\frac{1}{Y^2}\right)}{Y^2+\frac{1}{Y^2}}dY = \int_1^\infty \frac{1+\frac{1}{Y^2}}{Y^2+\frac{1}{Y^2}}dY - \int_1^\infty \frac{1-\frac{1}{Y^2}}{Y^2+\frac{1}{Y^2}}dY =$$

$$\underbrace{=}_{\substack{u=Y-\frac{1}{Y}\\ v=Y+\frac{1}{Y}}} \int_0^\infty \frac{du}{u^2+2} - \int_2^\infty \frac{dv}{v^2-2} = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\left[\ln\left(\frac{v-\sqrt{2}}{v+\sqrt{2}}\right)\right]_2^\infty = \frac{\pi}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})$$

$$\Rightarrow I_2 = \frac{\pi}{4} + \frac{\ln 2}{6} - \frac{2}{3}\left(\frac{\pi}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})\right) = \frac{\pi}{4} - \frac{\pi}{3\sqrt{2}} + \frac{2}{3\sqrt{2}}\ln(1+\sqrt{2}) + \frac{\ln 2}{6}$$

$$\Rightarrow \Omega = I_1 - I_2 = -2\ln 2 - \left(\frac{\pi}{4} - \frac{\pi}{3\sqrt{2}} + \frac{2}{3\sqrt{2}}\ln(1+\sqrt{2}) + \frac{\ln 2}{6}\right) =$$

$$= -\frac{2}{3\sqrt{2}}\ln(1+\sqrt{2}) - \frac{13}{6}\ln(2) + \pi\left(\frac{1}{3\sqrt{2}} - \frac{1}{4}\right) =$$

$$= \frac{1}{3\sqrt{2}}\ln(1+\sqrt{2}) - \frac{1}{\sqrt{2}}\ln(1+\sqrt{2}) - \frac{13}{6}\ln(2) + \pi\left(\frac{1}{3\sqrt{2}} - \frac{1}{4}\right)$$

$$Now,\ \sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right) \quad and \quad \coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\Rightarrow \sinh^{-1}(1) = \ln(1+\sqrt{2}) = \coth^{-1}\sqrt{2}$$

$$\Rightarrow \Omega = \frac{1}{3\sqrt{2}}\sinh^{-1}(1) - \frac{1}{\sqrt{2}}\coth^{-1}\sqrt{2} - \frac{13}{6}\ln(2) + \pi\left(\frac{1}{3\sqrt{2}} - \frac{1}{4}\right)$$