ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \tan(x)\ln(1-\tan x)\,dx$$

Proposed by Fao Ler-Iraq

Solution by Pratham Prasad-India

$$I=\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \tan(x)\ln(1-\tan x)\,dx$$

$$I=\int_{\sqrt{2}-1}^{1}\frac{t\ln(1-t)}{1+t^2}dt=\int_{\sqrt{2}-1}^{1}\frac{t\ln(1-t)}{(t+i)(t-i)}dt$$

$$=\frac{1}{2}\int_{\sqrt{2}-1}^{1}\frac{\ln(1-t)}{(t+i)}dt+\frac{1}{2}\int_{\sqrt{2}-1}^{1}\frac{\ln(1-t)}{(t-i)}dt$$

$$=\frac{1}{2}\int_{\sqrt{2}-1+i}^{1+i}\frac{\ln(1+i-u)}{u}du+\frac{1}{2}\int_{\sqrt{2}-1-i}^{1-i}\frac{\ln(1-i-u)}{u}du$$

$$=\frac{1}{2}\int_{\sqrt{2}-1+i}^{1+i}\frac{\ln\left(1-\frac{u}{1+i}\right)}{u}du+\frac{1}{2}\ln(1+i)\ln\left(\frac{1+i}{\sqrt{2}-1+i}\right)+\frac{1}{2}\int_{\sqrt{2}-1-i}^{1-i}\frac{\ln\left(1-\frac{u}{1-i}\right)}{u}du$$
$$+\frac{1}{2}\ln(1-i)\ln\left(\frac{1-i}{\sqrt{2}-1-i}\right)$$

$$=\frac{1}{2}\int_{\frac{\sqrt{2}-1+i}{1+i}}^{1}\frac{\ln(1-u)}{u}du+\frac{1}{2}\ln(1+i)\ln\left(\frac{1+i}{\sqrt{2}-1+i}\right)+\frac{1}{2}\int_{\frac{\sqrt{2}-1-i}{1-i}}^{1}\frac{\ln(1-u)}{u}du$$
$$+\frac{1}{2}\ln(1-i)\ln\left(\frac{1-i}{\sqrt{2}-1-i}\right)=$$

$$=\frac{1}{2}Li_2\left(\frac{\sqrt{2}-1+i}{1+i}\right)+\frac{1}{2}\ln(1+i)\ln\left(\frac{1+i}{\sqrt{2}-1+i}\right)+\frac{1}{2}Li_2\left(\frac{\sqrt{2}-1-i}{1-i}\right)$$
$$+\frac{1}{2}\ln(1-i)\ln\left(\frac{1-i}{\sqrt{2}-1-i}\right)-Li_2(1)=$$

$$=\frac{1}{2}Li_2\left(\frac{\sqrt{2}-1+i}{1+i}\right)-\frac{1}{2}\ln(1+i)\ln\left(\frac{\sqrt{2}-1+i}{1+i}\right)+\frac{1}{2}Li_2\left(\frac{\sqrt{2}-1-i}{1-i}\right)$$
$$-\frac{1}{2}\ln(1-i)\ln\left(\frac{\sqrt{2}-1-i}{1-i}\right)-Li_2(1)=$$

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$$= \frac{1}{2}Li_2\left(\frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\right) - \frac{1}{2}\ln(1+i)\ln\left(\frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\right) + \frac{1}{2}Li_2\left(\frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}} - 1\right)\right)$$
$$- \frac{1}{2}\ln(1-i)\ln\left(\frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}} - 1\right)\right) - \zeta(2)$$

$$where:$$

$$\ln(1+i) = \frac{1}{2}\ln(2) + \ln\left(cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right) = \frac{1}{2}\ln(2) + \frac{i\pi}{4}$$

$$\ln(1+i) = \frac{1}{2}\ln(2) + \ln\left(cos\left(\frac{\pi}{4}\right) - isin\left(\frac{\pi}{4}\right)\right) = \frac{1}{2}\ln(2) - \frac{i\pi}{4}$$

$$\ln\left(\frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\right) = \ln\left(2 - \sqrt{2}\right) + iarctan\left(\sqrt{2} - 1\right)$$

$$\ln\left(\frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}} - 1\right)\right) = \ln\left(2 - \sqrt{2}\right) - iarctan\left(\sqrt{2} - 1\right)$$

$$Thus,$$

$$I = \frac{1}{2}Li_2\left(\frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\right) - \frac{1}{2}\ln(1+i)\ln\left(\frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\right) + \frac{1}{2}Li_2\left(\frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}} - 1\right)\right)$$
$$- \frac{1}{2}\ln(1-i)\ln\left(\frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}} - 1\right)\right) - \zeta(2)$$

$$I = \frac{\pi}{4}\arctan\left(\sqrt{2} - 1\right) - \frac{1}{2}\ln(2)\ln\left(2 - \sqrt{2}\right) + Re\left(Li_2\left(\frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\right)\right) - \zeta(2)$$

$$\boxed{\begin{gathered}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \tan(x)\ln(1 - tanx)\,dx = \\ = \frac{\pi}{4}\arctan\left(\sqrt{2} - 1\right) - \frac{1}{2}\ln(2)\ln\left(2 - \sqrt{2}\right) + Re\left(Li_2\left(\frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\right)\right) - \zeta(2)\end{gathered}}$$