ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\psi=\int_0^{\frac{\pi}{2}}\frac{\ln(sinx)}{1+\cos^2(x)}dx$$

Proposed by Le Thu-Vietnam

Solved by Pratham Prasad-India

$$let,\frac{\pi}{2}-x=y$$

$$\psi=\int_0^{\frac{\pi}{2}}\frac{\ln(cosy)}{1+\sin^2(y)}dy,\qquad \psi=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\ln(1+\tan^2(y))}{1+2\tan^2(y)}\sec^2(y)\,dy$$

$$\psi=-\frac{1}{2}\int_0^{\infty}\frac{\ln(1+\epsilon^2)}{1+2\epsilon^2}d\epsilon,\qquad \psi=-\frac{1}{2}\int_0^{\infty}\frac{1}{1+2\epsilon^2}\int_0^1\frac{\epsilon^2}{1+\alpha\epsilon^2}d\alpha\,d\epsilon$$

$$\psi=-\frac{1}{2}\int_0^1\int_0^{\infty}\frac{1}{1+2\epsilon^2}\frac{\epsilon^2}{1+\alpha\epsilon^2}d\epsilon d\alpha$$

$$\psi=-\frac{1}{2}\int_0^1\frac{1}{2-\alpha}\int_0^{\infty}\frac{1}{1+\alpha\epsilon^2}-\frac{1}{1+2\epsilon^2}d\epsilon d\alpha$$

$$\psi=-\frac{1}{2}\int_0^1\frac{1}{2-\alpha}\int_0^{\infty}\frac{1}{1+\alpha\epsilon^2}d\epsilon d\alpha+\frac{1}{2}\int_0^1\frac{1}{2-\alpha}\int_0^{\infty}\frac{1}{1+2\epsilon^2}d\epsilon d\alpha$$

$$\psi=\frac{1}{2}\int_0^1\frac{1}{\alpha-2}\left(\frac{\pi}{2\sqrt{\alpha}}\right)d\alpha-\frac{1}{2}\int_0^1\frac{1}{\alpha-2}\left(\frac{\pi}{2\sqrt{2}}\right)d\alpha$$

$$\psi=\frac{\pi}{2}\int_0^1\frac{1}{\alpha-2}\left(\frac{1}{2\sqrt{\alpha}}\right)d\alpha-\frac{1}{2}\left(\frac{\pi}{2\sqrt{2}}\right)\int_0^1\frac{1}{\alpha-2}d\alpha$$

$$\psi=\frac{\pi}{2}\int_0^1\frac{1}{\beta^2-(\sqrt{2})^2}\,d\beta+\frac{1}{2}\left(\frac{\pi}{2\sqrt{2}}\right)\ln(2)$$

$$\psi=\frac{\pi}{4\sqrt{2}}\int_0^1\frac{1}{\beta-\sqrt{2}}\,d\beta-\frac{\pi}{4\sqrt{2}}\int_0^1\frac{1}{\beta+\sqrt{2}}\,d\beta+\frac{1}{2}\left(\frac{\pi}{2\sqrt{2}}\right)\ln(2)$$

$$\psi=\frac{\pi}{4\sqrt{2}}\ln\left(1-\frac{1}{\sqrt{2}}\right)-\frac{\pi}{4\sqrt{2}}\ln\left(1+\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{\pi}{2\sqrt{2}}\right)\ln(2)=\frac{\pi}{4\sqrt{2}}\ln\left(\frac{2(\sqrt{2}-1)}{\sqrt{2}+1}\right)=$$

$$=\frac{\pi}{4\sqrt{2}}\ln\left(2(\sqrt{2}-1)^2\right)=\frac{\pi}{2\sqrt{2}}\ln(2-\sqrt{2})$$