ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_1^{\infty}\frac{\ln(lnx)}{1-x+x^2}dx=\frac{2\pi}{\sqrt{3}}ln\Gamma\left(\frac{5}{6}\right)-\frac{\pi}{3\sqrt{3}}ln(2\pi)$$

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$$I=\int_1^{\infty}\frac{\ln(lnx)}{1-x+x^2}dx\underbrace{=}_{x\to\frac{1}{x}}\int_0^1\frac{\ln\left(ln\frac{1}{x}\right)}{x^2-x+1}dx=f'(1)$$

$$define,\ \ f(s)=\int_0^1\frac{\left(ln\frac{1}{x}\right)^{s-1}}{x^2-x+1}dx=\int_0^1\frac{\left(ln\frac{1}{x}\right)^{s-1}}{x^2-2xcos\left(\frac{\pi}{3}\right)+1}dx$$

$$\sum_{n=1}^{\infty}x^n\sin(n\theta)=Im\left(\sum_{n=0}^{\infty}x^ne^{in\theta}\right)=Im\left(\frac{1}{1-xe^{i\theta}}\right)=Im\left(\frac{1}{1-xcos\theta-ixsin\theta}\right)=\frac{xsin\theta}{1-2xcos\theta+x^2}$$

$$\Rightarrow\frac{1}{1-2xcos\theta+x^2}=\sum_{n=1}^{\infty}x^{n-1}\frac{\sin(n\theta)}{sin\theta}$$

$$\Rightarrow\frac{1}{x^2-2xcos\left(\frac{\pi}{3}\right)+1}=\sum_{n=1}^{\infty}x^{n-1}\frac{\sin\left(\frac{n\pi}{3}\right)}{sin\left(\frac{\pi}{3}\right)}=\frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}x^{n-1}\sin\left(\frac{n\pi}{3}\right)$$

$$\Rightarrow f(s)=\frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{3}\right)\int_0^1\left(ln\frac{1}{x}\right)^{s-1}x^{n-1}dx=\frac{2}{\sqrt{3}}\Gamma(s)\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)}{n^s}$$

$$\Rightarrow f'(s)=\frac{2}{\sqrt{3}}\Gamma'(s)\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)}{n^s}-\frac{2}{\sqrt{3}}\Gamma(s)\sum_{n=1}^{\infty}\frac{log_en}{n^s}\sin\left(\frac{n\pi}{3}\right)$$

$$\Rightarrow f'(1)=\frac{2}{\sqrt{3}}\Gamma'(1)\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{3}\right)}{n}-\frac{2}{\sqrt{3}}\Gamma(1)\sum_{n=1}^{\infty}\frac{log_en}{n}\sin\left(\frac{n\pi}{3}\right)$$

$$here,\ \ \Gamma'(1)=\Gamma(1)\psi(1)=-\gamma\qquad(\gamma\ is\ Euler-Mascheroni's\ constant)$$

$$\textit{And,} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n} = Im\left(-\ln\left(1-e^{\frac{i\pi}{3}}\right)\right) = -Im\left\{\ln\left(1-\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)\right)\right\}$$

$$= \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{3}\right)}{1-\cos\left(\frac{\pi}{3}\right)}\right) = \tan^{-1}\left(\cot\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{2}-\frac{\pi}{6} = \frac{\pi}{3}$$

***Now, from Kummer's series*:**

$$\frac{1}{2}\ln\left(\frac{\Gamma(x)}{\Gamma(1-x)}\right) = \left(\frac{1}{2}-x\right)\left(\gamma+\ln(2\pi)\right) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\log_e n}{n}\sin(2n\pi x)$$

$$\underbrace{\Rightarrow}_{x=\frac{1}{6}} \frac{1}{2}\ln\left(\frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)}\right) = \frac{1}{3}\left(\gamma+\ln(2\pi)\right) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\log_e n}{n}\sin\left(\frac{n\pi}{3}\right)$$

$$\frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)} = \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma^2\left(\frac{5}{6}\right)} = \frac{2\pi}{\Gamma^2\left(\frac{5}{6}\right)} \Rightarrow \frac{1}{2}\ln\left(\frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)}\right) = \frac{1}{2}\ln(2\pi) - \ln\Gamma\left(\frac{5}{6}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{\log_e n}{n}\sin\left(\frac{n\pi}{3}\right) = \frac{\pi}{2}\ln(2\pi) - \pi\ln\Gamma\left(\frac{5}{6}\right) - \frac{\pi}{3}\left(\gamma+\ln(2\pi)\right) = -\pi\ln\Gamma\left(\frac{5}{6}\right) - \frac{\pi}{6}\ln(2\pi) - \frac{\pi\gamma}{3}$$

$$\Rightarrow f'(1) = -\frac{2\gamma\pi}{3\sqrt{3}} - \frac{2}{\sqrt{3}}\left(-\pi\ln\Gamma\left(\frac{5}{6}\right) - \frac{\pi}{6}\ln(2\pi) - \frac{\pi\gamma}{3}\right) \Rightarrow I = \frac{2\pi}{\sqrt{3}}\ln\Gamma\left(\frac{5}{6}\right) - \frac{\pi}{3\sqrt{3}}\ln(2\pi)$$