ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^\infty \frac{\sqrt{x}}{1+e^{2x}}dx$$

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Solution by Pratham Prasad-India

$$\int_0^\infty \frac{\sqrt{x}}{1+e^{2x}}dx = \int_0^\infty \frac{\sqrt{x}e^{-2x}}{1+e^{-2x}}dx = \int_0^\infty \sqrt{x}\sum_{r=1}^{\infty}(-1)^{r-1}e^{-2rx}\,dx =$$

$$= \sum_{r=1}^{\infty}(-1)^{r-1}\int_0^\infty \sqrt{x}\,e^{-2rx}dx = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{(2r)^{\frac{3}{2}}}\int_0^\infty \sqrt{t}\,e^{-t}dt = \frac{1}{2^{\frac{3}{2}}}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{(r)^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right) =$$

$$= \frac{1}{2^{\frac{3}{2}}}\left(1-\frac{1}{\sqrt{2}}\right)\zeta\left(\frac{3}{2}\right)\left(\frac{\sqrt{\pi}}{2}\right) = \frac{\pi}{8}(\sqrt{2}-1)\zeta\left(\frac{3}{2}\right)$$