ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$J=\int_0^{\frac{\pi}{4}} x\,tanx\ln(cosx)\,dx$$

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Solution by Pratham Prasad-India

$$J=\int_0^{\frac{\pi}{4}} x\,tanx\ln(cosx)\,dx=-\frac{1}{2}\int_0^{\frac{\pi}{4}} x\ \mathrm{d}(\ln^2(cosx))=-\frac{\pi}{32}ln^2(2)+\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln^2(cosx)dx$$

$$I=\int_0^{\frac{\pi}{4}}\ln^2(cosx)dx\ ,I=\frac{1}{4}\int_0^1\frac{\ln^2(1+x^2)}{1+x^2}dx\,.\ By\ Weierestras\ substitution:$$

$$I=\frac{1}{4}\int_0^1\frac{\ln^2\left(\frac{2(1+x^2)}{(1+x)^2}\right)}{1+x^2}dx\,,$$

$$I=\int_0^1\frac{\ln^2(1+x)}{1+x^2}dx$$
$$+\frac{1}{4}\int_0^1\frac{\ln^2(1+x^2)}{1+x^2}dx$$
$$+\frac{\ln^2(2)}{4}\int_0^1\frac{1}{1+x^2}dx-\int_0^1\frac{ln(1+x)\ln(1+x^2)}{1+x^2}dx\ -ln2\int_0^1\frac{\ln(1+x)}{1+x^2}dx$$
$$+\frac{ln2}{2}\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx$$

By putting results,

$$I=\left(-2Gln(2)-4\,Im\left(Li_3\left(\frac{1+i}{2}\right)\right)+\frac{7\pi^3}{64}+\frac{3}{16}\pi\ln^2(2)\right)$$
$$+\frac{1}{4}\left(-2Gln(2)+4\,Im\left(Li_3\left(\frac{1+i}{2}\right)\right)-\frac{7\pi^3}{96}+\frac{7}{8}\pi\ln^2(2)\right)+\frac{\ln^2(2)}{4}\left(\frac{\pi}{4}\right)$$
$$-\left(-\frac{5}{2}Gln(2)-4\,Im\left(Li_3\left(\frac{1+i}{2}\right)\right)+\frac{7\pi^3}{64}+\frac{3}{8}\pi\ln^2(2)\right)-\ln(2)\left(\frac{\pi}{8}\ln(2)\right)$$
$$+\frac{\ln(2)}{2}\left(\frac{\pi}{2}\ln(2)-C\right)$$

$$I=\frac{7\pi^3}{192}-\frac{G\ln(2)}{2}+\frac{5\pi\ln^2(2)}{16}-Im(Li_3(1-i))$$

$$J=-\frac{\pi}{32}ln^2(2)+\frac{1}{2}\left(\frac{7\pi^3}{192}-\frac{G\ln(2)}{2}+\frac{5\pi\ln^2(2)}{16}-Im\big(Li_3(1-i)\big)\right)$$

$$J=\frac{7\pi^3}{384}-\frac{G\ln(2)}{4}+\frac{\pi\ln^2(2)}{8}-\frac{1}{2}Im(Li_3(1-i))$$