ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx-3\int_0^1\frac{\ln(1-x)\,Li_2^2(1-x)}{1-x}dx$$

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Replacing $1-x$ *by* x *in the second integral,*

$$=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx-3\int_0^1\frac{\ln(x)\,Li_2^2(x)}{x}dx$$

Let,

$$A=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx\ ,B=\int_0^1\frac{\ln(x)Li_2^2(x)}{x}dx.\ Thus\ \boxed{I=A-3B}$$

$$A=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx$$

$$Applying\ Integration\ By\ parts, we\ get-A=\frac{1}{2}\int_0^1\frac{\ln^4(1-x)\,ln(x)}{x}dx$$

$$Replacing\ 1-x\ by\ x\ ,A=\frac{1}{2}\int_0^1\frac{\ln^4(x)\,ln(1-x)}{1-x}dx\ ,A=-\frac{1}{2}\int_0^1\ln^4(x)\sum_{n=1}^{\infty}x^nH_n\,dx$$

$$A=-\frac{1}{2}\sum_{n=1}^{\infty}H_n\int_0^1x^n\ln^4(x)\,dx,\ \ A=-\frac{1}{2}\sum_{n=1}^{\infty}H_n\int_0^1\frac{\partial^4}{\partial n^4}x^ndx$$

$$A=-\frac{1}{2}\sum_{n=1}^{\infty}H_n\frac{d^4}{dn^4}\int_0^1x^n\,dx,\qquad A=-\frac{1}{2}\sum_{n=1}^{\infty}H_n\frac{d^4}{dn^4}\left(\frac{1}{n+1}\right)$$

$$A=-12\sum_{n=1}^{\infty}H_n\left(\frac{1}{n+1}\right)^5,A=-12\left(\sum_{n=1}^{\infty}\frac{H_n}{n^5}-\zeta(6)\right),A=-12S_1+12\zeta(6)$$

$$S_1=\sum_{n=1}^{\infty}\frac{H_n}{n^5},define,\qquad J(q)=\int_0^1x^{n-1}Li_q(x)dx,\qquad J(q)=\frac{\zeta(q)}{n}-\frac{1}{n}\int_0^1x^{n-1}Li_{q-1}(x)dx$$

$$Applying\ Integration\ By\ parts, we\ get-J(q)=\frac{\zeta(q)}{n}-\frac{1}{n}\int_0^1x^{n-1}Li_{q-1}(x)dx$$

$$J(q)=\frac{\zeta(q)}{n}-\frac{1}{n}J(q-1),\qquad J(2)=\int_0^1x^{n-1}Li_2(x)dx$$

$$J(2)=\frac{\zeta(2)}{n}+\frac{1}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx,\qquad J(2)=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}$$

$$\textit{Thus by recursion}: J(q)=\frac{(-1)^{q-1}H_n}{n^q}-\sum_{k=1}^{q-1}\frac{(-1)^k}{n^k}\zeta(q-k+1)$$

$$\int_0^1 x^{n-1}Li_q(x)dx=\frac{(-1)^{q-1}H_n}{n^q}-\sum_{k=1}^{q-1}\frac{(-1)^k}{n^k}\zeta(q-k+1)$$

$$\textit{Setting } q=4 \textit{ and rearranging},$$

$$\frac{H_n}{n^4}=-\int_0^1 x^{n-1}Li_4(x)dx-\sum_{k=1}^{3}\frac{(-1)^k}{n^k}\zeta(4-k+1)$$

$$\frac{H_n}{n^5}=-\int_0^1\frac{x^{n-1}Li_4(x)}{n}dx-\sum_{k=1}^{3}\frac{(-1)^k}{n^{k+1}}\zeta(5-k)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=-\int_0^1\frac{Li_4(x)}{x}\sum_{n=1}^{\infty}\frac{x^n}{n}dx-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\int_0^1\frac{Li_4(x)}{x}\ln(1-x)dx-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\int_0^1\frac{\ln(1-x)}{x}\sum_{n=1}^{\infty}\frac{x^n}{n^4}dx-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^1 x^{n-1}\ln(1-x)dx-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=-\sum_{n=1}^{\infty}\frac{H_n}{n^5}-\sum_{k=1}^{3}(-1)^k\zeta(5-k)\zeta(k+1),\qquad \sum_{n=1}^{\infty}\frac{H_n}{n^5}=\zeta(2)\zeta(4)-\frac{1}{2}\zeta^2(3)+\zeta(2)\zeta(4)$$

$$\boxed{S_1=\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)},\qquad A=-12\left(\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)\right)+12\zeta(6)$$

$$A=-21\zeta(6)+6\zeta^2(3)+12\zeta(6),\qquad \boxed{A=\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx\;=6\zeta^2(3)-9\zeta(6)}$$

$$B=\int_0^1\frac{\ln(x)\,Li_2^2(x)}{x}dx,\qquad \textit{Applying Integration By parts, we get}-$$

$$B=\int_0^1\frac{\ln^2(x)\ln(1-x)\,Li_2(x)}{x}dx,\ B=\int_0^1\frac{\ln^2(x)\ln(1-x)}{x}\sum_{n=1}^{\infty}\frac{x^n}{n^2}dx$$

$$B=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1\ln^2(x)\ln(1-x)\,x^{n-1}dx,\qquad B=\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{d^2}{dn^2}\int_0^1\ln(1-x)\,x^{n-1}\,dx$$

$$B=-\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{d^2}{dn^2}\left(\frac{H_n}{n}\right),\qquad B=-\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{d^2}{dn^2}\left(\frac{\gamma+\psi(n+1)}{n}\right)$$

$$B=-\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{2\gamma}{n^3}+\psi_2(n+1)-\frac{2}{n}\psi_1(n+1)+\frac{2}{n^3}\psi(n+1)\right)$$

$$B=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{2\zeta(3)}{n}+\frac{2\zeta(2)}{n^2}-\frac{2H_n}{n^3}-\frac{2H_n^{(2)}}{n^2}-\frac{2H_n^{(3)}}{n}\right),$$

$$B=\sum_{n=1}^{\infty}\left(\frac{2\zeta(3)}{n^3}+\frac{2\zeta(2)}{n^4}-\frac{2H_n}{n^5}-\frac{2H_n^{(2)}}{n^4}-\frac{2H_n^{(3)}}{n^3}\right)$$

$$B=2\zeta^2(3)+2\zeta(2)\zeta(4)-2S_1-2S_2-2S_3$$

$(S_1\,is\ calculated\ above)$

$$S_2=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}$$

$By\ Cauchy\ product\ on\ Li_2(x)\,and\ Li_3(x),$

$$Li_2(x)Li_3(x)=6\sum_{n=1}^{\infty}\frac{H_n}{n^4}x^n+3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}x^n+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}x^n-10Li_5(x)$$

$Divide\ both\ sides\ by\ x\ and\ integrate\ from\ 0\ to\ y,$

$$\int_0^y\frac{Li_2(x)Li_3(x)}{x}dx=6\sum_{n=1}^{\infty}\frac{H_n}{n^5}y^n+3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}y^n+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}y^n-10Li_6(y)$$

$$\frac{1}{2}Li_3^2(y)=6\sum_{n=1}^{\infty}\frac{H_n}{n^5}y^n+3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}y^n+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}y^n-10Li_6(y)$$

$Putting\ y=1,$

$$\frac{1}{2}Li_3^2(1)=6\sum_{n=1}^{\infty}\frac{H_n}{n^5}+3\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}-10Li_6(1)$$

$$\frac{1}{2}Li_3^2(1)=6S_1+3S_2+S_3-10Li_6(1)\qquad ---(1)$$

Now,

$$S_3=\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3},\qquad S_3=\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{1}{k^3}$$

Splitting the inner Sum,

$$S_3=\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{\infty}\frac{1}{k^3}-\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=n}^{\infty}\frac{1}{k^3}+\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=n}^{n}\frac{1}{k^3}$$

Switching the order of sum in the middle term,

$$S_3=\zeta^2(3)-\sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{k}\frac{1}{n^3}+\sum_{n=1}^{\infty}\frac{1}{n^6},\qquad \textit{Replacing k and n by each other,}$$

$$S_3=\zeta^2(3)-\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{1}{k^3}+\zeta(6),\qquad S_3=\zeta^2(3)-S_3+\zeta(6)$$

$$\boxed{S_3=\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}=\frac{1}{2}\left(\zeta^2(3)+\zeta(6)\right)}$$

Now Putting the Calculated Values Of S_1 and S_3 in equaltion (1) we get S_2 as –

$$\boxed{S_2=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=\zeta^2(3)-\frac{1}{3}\zeta(6)}$$

$$B=2\zeta^2(3)+2\zeta(2)\zeta(4)-2S_1-2S_2-2S_3$$

$$B=2\zeta^2(3)+2\zeta(2)\zeta(4)-2\left(\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)\right)-2\left(\zeta^2(3)-\frac{1}{3}\zeta(6)\right)-2\left(\frac{1}{2}\left(\zeta^2(3)+\zeta(6)\right)\right)$$

$$B=2\zeta(2)\zeta(4)-\frac{7}{2}\zeta(6)+\frac{2}{3}\zeta(6)-2\zeta(6),\qquad B=\frac{7}{2}\zeta(6)-\frac{7}{2}\zeta(6)+\frac{2}{3}\zeta(6)-2\zeta(6)$$

$$\boxed{B=\int_0^1\frac{\ln(x)\,Li_2^2(x)}{x}dx=\frac{-1}{3}\zeta(6)},\qquad I=A-3B$$

$$I=\left(6\zeta^2(3)-9\zeta(6)\right)-3\left(\frac{-1}{3}\zeta(6)\right),\quad I=6\zeta^2(3)-8\zeta(6)$$

$$\int_0^1\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx-3\int_0^1\frac{\ln(1-x)\,Li_2^2(1-x)}{1-x}dx=6\zeta^2(3)-8\zeta(6)$$