ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$J=\int_0^{\frac{\pi}{4}} x\,tanx\ln(cosx)\,dx$$

Proposed by Naren Bhandari-Nepal

Solution by Pratham Prasad-India

$$J=\int_0^{\frac{\pi}{4}} x\,tanx\ln(cosx)\,dx=-\frac{1}{2}\int_0^{\frac{\pi}{4}} x\ \mathrm{d}(\ln^2(cosx))=-\frac{\pi}{32}ln^2(2)+\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln^2(cosx)dx$$

$$I=\int_0^{\frac{\pi}{4}}\ln^2(cosx)dx$$

$$I=\int_0^{\frac{\pi}{4}}\ln^2(2cosx)dx-2\ln(2)\int_0^{\frac{\pi}{4}}\ln(cosx)\,dx-\frac{\pi}{4}\ln^2(2)$$

$$I=\int_0^{\frac{\pi}{4}}x^2dx+2\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$I=\frac{\pi^3}{192}+\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n^2}\sin\left(\frac{n\pi}{2}\right)-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$=\frac{\pi^3}{192}+Im\left(\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}}{n^2}i^n\right)-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$=\frac{\pi^3}{192}+Im\left(\sum_{n=1}^{\infty}\frac{(-i)^nH_{n-1}}{n^2}\right)-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$=\frac{\pi^3}{192}+Im\left(\zeta(3)-Li_3(1+i)+\ln(1+i)\,Li_2(1+i)+\frac{1}{2}\ln(-i)\ln^2(1+i)\right)$$
$$-2\ln(2)\left(\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\frac{\pi}{4}\ln^2(2)$$

$$=\frac{7\pi^3}{192}+\frac{5\pi}{16}\ln^2(2)-\frac{1}{2}Gln(2)+Im\left(Li_3(1-i)\right)$$

$$J=-\frac{\pi}{32}ln^2(2)+\frac{7\pi^3}{384}+\frac{5\pi}{32}\ln^2(2)-\frac{1}{4}Gln(2)+\frac{1}{2}Im\left(Li_3(1-i)\right)$$

$$J=\frac{7\pi^3}{384}+\frac{\pi}{8}\ln^2(2)-\frac{1}{4}Gln(2)+\frac{1}{2}Im\left(Li_3(1-i)\right)$$