ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I=\int_0^1 Li_3(-x^2)(1-x^2)x+\frac{x\ln(1+x)}{x^2+1}\,dx$$

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Solution by Pratham Prasad-India

$$I=\frac{1}{2}\int_0^1 Li_3(-x)(1-x)dx+\int_0^1\frac{x\ln(1+x)}{x^2+1}\,dx$$

$$I=\frac{1}{2}\int_0^1 Li_3(-x)\,dx-\frac{1}{2}\int_0^1 xLi_3(-x)\,dx+\int_0^1\frac{x\ln(1+x)}{1+x^2}dx$$

$$\boxed{I=\frac{1}{2}A-\frac{1}{2}B+C}$$

$$Where,$$

$$A=\int_0^1 Li_3(-x)\,dx,\qquad A=Li_3(-1)-\int_0^1 x\left(\frac{Li_2(-x)}{x}\right)dx$$

$$A=Li_3(-1)-\int_0^1 Li_2(-x)dx,\qquad A=Li_3(-1)-Li_2(-1)+1-2\ln(2)$$

$$\boxed{A=-\frac{3}{4}\zeta(3)+\frac{\pi^2}{12}+1-2\ln(2)}$$

$$B=\int_0^1 xLi_3(-x)\,dx$$

$$B=\frac{1}{2}Li_3(-1)-\int_0^1\frac{x^2}{2}\left(\frac{Li_2(-x)}{x}\right)dx,\qquad B=\frac{1}{2}Li_3(-1)-\frac{1}{2}\int_0^1 xLi_2(-x)\,dx$$

$$B=\frac{1}{2}Li_3(-1)-\frac{1}{2}\left(\frac{1}{24}(3-\pi^2)\right),\qquad \boxed{B=-\frac{3}{8}\zeta(3)-\frac{1}{16}+\frac{\pi^2}{48}}$$

$$C=\int_0^1\frac{x\ln(1+x)}{1+x^2}dx,\qquad C=\int_0^1\frac{x^2}{1+x^2}\int_0^1\frac{1}{1+xy}dy\,dx$$

$$C=\int_0^1\int_0^1\frac{x^2}{(1+x^2)(1+xy)}dx\,dy$$

$$C=\int_0^1\frac{y}{2(1+y^2)}\ln(2)+\frac{\ln(1+y)}{y(y^2+1)}-\frac{\pi}{4(1+y^2)}dy$$

$$C=\frac{1}{4}\ln^2(2)+\int_0^1\frac{\ln(1+y)}{y(y^2+1)}dy-\frac{\pi^2}{16}$$

$$C=\frac{1}{4}\ln^2(2)+\int_0^1\frac{\ln(1+y)}{y}dy-\int_0^1\frac{y\ln(1+y)}{(y^2+1)}dy-\frac{\pi^2}{16}$$

$$C=\frac{1}{4}\ln^2(2)+\int_0^1\frac{\ln(1+y)}{y}dy-C-\frac{\pi^2}{16}$$

$$2C=\frac{1}{4}\ln^2(2)+\frac{\pi^2}{12}-\frac{\pi^2}{16},\qquad 2C=\frac{1}{4}\ln^2(2)+\frac{\pi^2}{48},\qquad \boxed{C=\frac{1}{8}\ln^2(2)+\frac{\pi^2}{96}}$$

Putting everything back in,

$$I=\frac{1}{2}A-\frac{1}{2}B+C$$

$$I=\frac{1}{2}\left(-\frac{3}{4}\zeta(3)+\frac{\pi^2}{12}+1-2\ln(2)\right)-\frac{1}{2}\left(-\frac{3}{8}\zeta(3)-\frac{1}{16}+\frac{\pi^2}{48}\right)+\left(\frac{1}{8}\ln^2(2)+\frac{\pi^2}{96}\right)$$

Simplifying,

$$I=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{24}+\frac{17}{32}-\ln(2)+\frac{1}{8}\ln^2(2)$$

$$\boxed{\int_0^1 Li_3(-x^2)(1-x^2)x+\frac{x\ln(1+x)}{x^2+1}\,dx=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{24}+\frac{17}{32}-\ln(2)+\frac{1}{8}\ln^2(2)}$$